FINITE DIFFERENCE SCHEMES WITH CROSS DERIVATIVES CORRECTORS FOR MULTIDIMENSIONAL PARABOLIC SYSTEMS

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Abstract. We propose finite difference schemes for multidimensional quasilinear parabolic systems whose main feature is the introduction of correctors which control the second-order terms with mixed derivatives. We show that with these correctors the schemes inherit physically relevant properties present at the continuous level, such as the existence of invariant domains and/or the nonincrease of the total amount of entropy. Applications to the compressible Navier-Stokes system are given.

1. Introduction

We are concerned with general $N \times N$ quasilinear parabolic systems of the form

$$\partial_t U + \sum_{\alpha=1}^{d} \partial_\alpha F^\alpha(U) = \sum_{\alpha,\beta=1}^{d} \partial_\alpha (B^{\alpha\beta}(U)\partial_\beta U). \tag{1.1}$$

An example of such systems used recurrently in this paper is the system of compressible Navier-Stokes equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p\,\mathrm{Id}) = \sum_{\alpha\beta} \partial_\alpha (A_{\alpha\beta}(U)\partial_\beta u), \tag{1.3}$$

$$\begin{aligned} &\partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{p}{\gamma-1}\right) + \operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2 + \frac{p}{\gamma-1} + p\right)u\right] \\ &\qquad = \sum_\alpha \partial_\alpha \left[u \cdot \sum_\beta A_{\alpha\beta}(U)\partial_\beta u + \kappa(U)\partial_\alpha T\right], \end{aligned} \tag{1.4}$$

where ρ, u, p, T represent, respectively, the density, velocity, pressure and temperature of the fluid, $\kappa(U) \ge 0$ is the heat conduction coefficient, $(A_{\alpha\beta})_{\alpha\beta}$ is a nonnegative symmetric tensor, $U = (\rho, q := \rho u, E := \rho|u|^2/2 + p/(\gamma-1))$, $\gamma > 1$, $p = \rho T$. The matrices $A_{\alpha\beta}$ are given by

$$A_{\alpha\beta}(U) = \mu(U)(\delta_{\alpha\beta}\,\mathrm{Id} + e_\beta \otimes e_\alpha) + \lambda(U) e_\alpha \otimes e_\beta, \qquad \mu \ge 0,\ \lambda + \frac{2}{d}\mu \ge 0, \tag{1.5}$$

1991 *Mathematics Subject Classification.* 65N06,35K55,35K65,35Q30,76D05,28D20.

Key words and phrases. finite difference schemes, quasilinear parabolic systems, cross derivatives, Navier-Stokes equations, invariant domains, entropy.

where e_α is the α-th element of the canonical basis of $\mathbb{R}^d$ and d is the space dimension. Here we shall assume that $\kappa(U) \equiv 0$ since it is a coefficient of a second-order term which does not involve cross derivatives, that are the main concern of this paper. Then the system takes the form (1.1) with the block matrices

$$B^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{A_{\alpha\beta}u}{\rho} & \dfrac{A_{\alpha\beta}}{\rho} & 0 \\ -\dfrac{u^t A_{\alpha\beta}u}{\rho} & \dfrac{u^t A_{\alpha\beta}}{\rho} & 0 \end{pmatrix}. \tag{1.6}$$

A well known computation making use of (1.2)-(1.4) leads to the following additional relation

$$\partial_t \left[-\rho \ln(\frac{p^{1/\gamma}}{\rho})\right] + \operatorname{div}\left[-\rho u \ln(\frac{p^{1/\gamma}}{\rho})\right] = -(1-\frac{1}{\gamma})\frac{1}{T}\sum_{\alpha\beta} A_{\alpha\beta}\partial_\beta u \partial_\alpha u \le 0. \tag{1.7}$$

If we denote $S = \ln(\frac{p^{1/\gamma}}{\rho})$, we can see that S is the thermodynamic entropy (modulo a multiplicative positive constant), and from (1.7), (1.2) we obtain

$$\partial_t S + u\cdot\nabla S = (1-\frac{1}{\gamma})\frac{1}{p}\sum_{\alpha\beta} A_{\alpha\beta}\partial_\beta u\partial_\alpha u. \tag{1.8}$$

Our main concern in this paper is the problem of how to design good finite difference schemes for systems of the type (1.1) if there are non-zero second-order terms with cross derivatives $\partial^2_{\alpha\beta},\ \alpha \neq \beta$. The presence of these terms prohibits the use of standard methods like dimensional splitting. Our strategy is then to design real multidimentional methods which contain correctors whose purpose is to stabilize the cross derivative terms. We show that our finite difference schemes preserve physically relevant properties present at the continuous level such as entropy decreasing and invariant regions, related to the entropy equations (1.7) and (1.8) in the case of Navier-Stokes system.

Since our main concern in this paper is related with the second-order space derivatives terms, from now on we shall disregard the terms with first-order space derivatives in (1.1) and consider systems of the type

$$\partial_t U = \sum_{\alpha,\beta=1}^{d} \partial_\alpha(B^{\alpha\beta}(U)\partial_\beta U). \tag{1.9}$$

The essence of our method of introducing cross derivatives correctors can be better understood through the following example. We would like to prescribe a finite difference scheme for the simplest parabolic equation presenting cross derivative, namely,

$$\partial_t u = \partial_x^2 u + \partial_y^2 u + 2\partial^2_{xy} u. \tag{1.10}$$

One is naturally tempted to try something like

$$u^{n+1}_{ij} = u^n_{ij} + \Big(\sigma^{xx}\Delta^+_x\Delta^-_x + \sigma^{yy}\Delta^+_y\Delta^-_y + \frac{1}{2}\sigma^{xy}\Delta^0_x\Delta^0_y\Big)u^n_{ij}, \tag{1.11}$$

where

$$\begin{aligned}
&\Delta_x^+ u_{ij} = u_{i+1\,j} - u_{ij}, \qquad \Delta_y^+ u_{ij} = u_{i\,j+1} - u_{ij},\\
&\Delta_x^- u_{ij} = u_{ij} - u_{i-1\,j}, \qquad \Delta_y^- u_{ij} = u_{ij} - u_{i\,j-1},\\
&\Delta_x^0 u_{ij} = u_{i+1\,j} - u_{i-1\,j}, \qquad \Delta_y^0 u_{ij} = u_{i\,j+1} - u_{i\,j-1},\\
&\sigma^{xx} = \frac{\Delta t}{\Delta x^2}, \qquad \sigma^{yy} = \frac{\Delta t}{\Delta y^2}, \qquad \sigma^{xy} = \frac{\Delta t}{\Delta x \Delta y}.
\end{aligned}$$

The problem with the above scheme is that it is not monotone due to the presence of the term $\Delta_x^0\Delta_y^0 u_{ij}^n$ which gives rise to the 'corner' terms $-u_{i-1\,j+1}^n$, and $-u_{i+1\,j-1}^n$ which have 'bad' signs and cannot be controlled by the 'central' terms appearing in $\Delta_x^+\Delta_x^- u_{ij}^n + \Delta_y^+\Delta_y^- u_{ij}^n$. In particular, it obviously does not satisfy the maximum principle, verified at the continuous level. Therefore, to fix this problem, we propose to add the corrector $\frac{1}{2}\sigma^{xy}\Delta_x^+\Delta_x^-\Delta_y^+\Delta_y^- u_{ij}^n$ to the above scheme, hence obtaining the corrected scheme

(1.12)
$$u_{ij}^{n+1} = u_{ij}^n + \left(\sigma^{xx}\Delta_x^+\Delta_x^- + \sigma^{yy}\Delta_y^+\Delta_y^- + \frac{1}{2}\sigma^{xy}\Delta_x^0\Delta_y^0 + \frac{1}{2}\sigma^{xy}\Delta_x^+\Delta_x^-\Delta_y^+\Delta_y^-\right)u_{ij}^n.$$

This corrected scheme, evidently still consistent, is now monotone: the bad sign corner terms produced by $\Delta_x^0\Delta_y^0 u$ are now controlled by similar terms appearing from $\Delta_x^+\Delta_x^-\Delta_y^+\Delta_y^- u$ now with a good sign; the other terms are controlled by $\Delta_x^+\Delta_x^- u + \Delta_y^+\Delta_y^- u$ (by assuming $\Delta x = \Delta y$). We hope this simple example elucidates the very realm of our concern in this paper.

The remaining of the paper is organized as follows. In Section 2, we provide some auxiliary results which will be useful in the next sections. In Section 3, we study the case of quasi-gradient systems and give applications to linear systems, to the Navier-Stokes system with $\lambda = \mu$ and to scalar equations. For the latter, we prove the convergence to a weak entropy solution to (1.9). Then, in Section 4, we analyze implicit schemes for systems like (1.1) in nonconservative form, and apply this analysis to the Navier-Stokes system with $\lambda \le \mu$.

2. Tools for invariant domains and entropy

In this section we state and prove some lemmas which will be used in the next sections in the discussion of invariant domains and entropy stability. We first recall the following result which is a particular case of [6].

Theorem 2.1 ([6]). *Let G_i, $i = 1, \dots, M$, be functions of class C^2 over an open convex domain $\mathcal{U} \subseteq \mathbb{R}^N$. Assume that 0 is a regular value for all G_i and $\Omega = \{U \in \mathcal{U} : G_i(U) < 0,\ i = 1, \dots, M\}$ is convex, with $\overline{\Omega} \subseteq \mathcal{U}$. Let $\varphi : \mathcal{U} \to \mathbb{R}^N$ be of class C^2 and such that $\varphi'(U)$ is diagonalizable with nonnegative eigenvalues for all $U \in \mathcal{U}$ and also*

(2.1) $$\varphi'(\omega)(T_\omega(\partial\Omega)) \subseteq T_\omega(\partial\Omega), \qquad \forall \omega \in \partial\Omega,$$

where, as usual, $T_\omega(S)$ denotes the tangent space to S at $\omega \in S$. Notice that (2.1) means that $\nu(\omega)$, the unity outer normal to $\partial\Omega$ at ω, is an eigenform of $\varphi'(\omega)$.

For any $\varepsilon > 0$, denote $\varphi_\varepsilon = \varphi + \varepsilon\,\mathrm{Id}$, where Id is the identity map, and suppose further that the following two conditions are satisfied:

- (P1) *If $\{U_n\}_{n\in\mathbb{N}}$ is a sequence in Ω with $|U_n| \to \infty$ then $|\varphi_\varepsilon(U_n)| \to \infty$,*
- (P2) *For any vector $\mathbf{e} \in \mathbb{R}^N$, $\sup_{U\in\Omega} \mathbf{e}\cdot U < +\infty$ implies $\sup_{U\in\Omega} \mathbf{e}\cdot\varphi_\varepsilon(U) < +\infty$.*

Then, for all $U \in \overline{\Omega}$ and $\omega \in \partial\Omega$, we have

$$\langle \nu(\omega), \varphi(U) - \varphi(\omega) \rangle \le 0, \tag{2.2}$$

where $\nu(\omega)$ denotes the unity outer normal to $\partial\Omega$ at ω.

Remark 2.1. We notice that properties (P1) and (P2) above are trivially satisfied in the case where $\overline{\Omega}$ is compact, recovering a result in [12].

Remark 2.2. As an illustration of a non compact case, take the map $\varphi(u,v) = (u + 1/v, -u + v)$, in the region $\Omega = \{(u,v) \in \mathcal{U} \,:\, G_1(u,v) < 0,\ G_2(u,v) < 0\}$, with $\mathcal{U} = \{(u,v) \,:\, v > 0\}$, $G_1(u,v) = -u - \log v$, $G_2(u,v) = -u + \log v$. We easily verify that condition (2.1) is satisfied and that $\varphi'(U)$ is diagonalizable with nonnegative eigenvalues for all $U \in \mathcal{U}$. Property (P1) is also immediately verified since $|U_n| \to \infty$ with $U_n = (u_n, v_n) \in \Omega$ implies $1 \le v_n \to \infty$ and so for the second coordinate of $\varphi + \varepsilon\,\mathrm{Id}$, namely, $\varphi_{2\varepsilon}(u,v) = -u + (1+\varepsilon)v$, we have $|\varphi_{2\varepsilon}(u_n, v_n)| \ge (1+\varepsilon)v_n - \log v_n \to \infty$. As for (P2), we observe that $\sup_{U\in\Omega} \mathbf{e}\cdot U < +\infty$, for some vector $\mathbf{e} = (e_1, e_2) \in \mathbb{R}^2$, implies necessarily that $e_2 < 0$ so $\mathbf{e}\cdot\varphi_\varepsilon(U) = e_1(1/v + (1+\varepsilon)u) + e_2(-u + (1+\varepsilon)v) = (e_1 u + e_2 v) + (e_1(1/v + \varepsilon u) + e_2(\varepsilon v + u)) = A(U) + B(U) < +\infty$, since by assumption $A(U) < +\infty$ and this implies also that $B(U) < +\infty$, because, for $U = (u,v) \in \Omega$, we have $|u| < \log v$, so that $B(U) < e_2\varepsilon v + C(\log v + 1/v)$, for a certain $C > 0$, which is certainly $< +\infty$, for $U \in \Omega$. We shall see later on other examples where $\overline{\Omega}$ is not compact.

Remark 2.3. The conclusion of Theorem 2.1 still holds for a map $\varphi : \mathcal{U} \to \mathbb{R}^N$ that can be written as $\varphi = \sum_{r=1}^{R} \alpha_r \varphi_r$ with $\alpha_r \ge 0$ and φ_r satisfy the assumptions of Theorem 2.1.

As a consequence of Theorem 2.1 and Remark 2.3 we have the following.

Lemma 2.1. *Let $\mathcal{U} \subseteq \mathbb{R}^N$ and Ω be as in Theorem 2.1. For each $J \in \mathbf{J}$, where $\mathbf{J}$ is a finite set, let $\varphi_J : \mathcal{U} \to \mathbb{R}^N$ be maps that can be written as linear combinations with nonnegative coefficients of maps satisfying the same hypotheses as φ in Theorem 2.1. Further, assume that*

$$\sum_{J\in\mathbf{J}} \varphi_J(U) = U, \qquad \text{for all } U \in \mathcal{U}. \tag{2.3}$$

Then, if $U_J \in \overline{\Omega}$ for all $J \in \mathbf{J}$, we have

$$\sum_{J\in\mathbf{J}} \varphi_J(U_J) \in \overline{\Omega}. \tag{2.4}$$

Proof. Applying (2.3) and Theorem 2.1, for all $\omega \in \partial\Omega$, we have

$$\langle \nu(\omega), \sum_{J\in\mathbf{J}} \varphi_J(U_J) - \omega \rangle = \sum_{J\in\mathbf{J}} \langle \nu(\omega), \varphi_J(U_J) - \varphi_J(\omega) \rangle \le 0, \tag{2.5}$$

which implies (2.4). □

The following lemma is a discrete version of a result previously established in [1], [2].

Lemma 2.2. *Let $\mathcal{U} \subseteq \mathbb{R}^N$ be an open convex set, and let $\varphi_J : \mathcal{U} \to \mathbb{R}^N$ be of class C^2, $J \in \mathbf{J}$, where $\mathbf{J}$ is a finite set. Let $\eta : \mathcal{U} \to \mathbb{R}$ be of class C^2, convex, and $q_J : \mathcal{U} \to \mathbb{R}$ be such that*

$$q_J'(U) = \eta'(U)\varphi_J'(U) \qquad \text{for all } U \in \mathcal{U}, J \in \mathbf{J}, \tag{2.6}$$

normalized so that

$$\sum_{J\in\mathbf{J}} q_J(0) = \eta(0). \tag{2.7}$$

Assume that each φ_J is η-dissipative in a (not necessarily convex) subset $\mathcal{U}_{stab}$ of $\mathcal{U}$, that is,

$$D_J(U,V) := q_J(U) - q_J(V) - \eta'(U)(\varphi_J(U) - \varphi_J(V)) \leq 0, \qquad U, V \in \mathcal{U}_{stab}. \tag{2.8}$$

Suppose further that

$$\sum_{J\in\mathbf{J}} \varphi_J(U) = U, \qquad \textit{for all } U \in \mathcal{U}. \tag{2.9}$$

Then, if $U_J \in \mathcal{U}_{stab}$ for all $J \in \mathbf{J}$ and $\sum_{J\in\mathbf{J}} \varphi_J(U_J) \in \mathcal{U}_{stab}$, we have

$$\eta\left(\sum_{J\in\mathbf{J}} \varphi_J(U_J)\right) \leq \sum_{J\in\mathbf{J}} q_J(U_J). \tag{2.10}$$

Proof. The equations (2.6) and (2.9) imply

$$\eta'(U) = \sum_{J\in\mathbf{J}} q_J'(U), \tag{2.11}$$

thus with (2.7) it gives

$$\eta(U) = \sum_{J\in\mathbf{J}} q_J(U). \tag{2.12}$$

Setting $U = \sum_{J\in\mathbf{J}} \varphi_J(U_J)$ and $V = U_J$ in (2.8), we obtain

$$q_J(U) - q_J(U_J) - \eta'(U)(\varphi_J(U) - \varphi_J(U_J)) \leq 0, \tag{2.13}$$

for $J \in \mathbf{J}$. Adding these inequalities with J running over $\mathbf{J}$ and using (2.9), we find

$$\sum_{J\in\mathbf{J}} q_J(U) \leq \sum_{J\in\mathbf{J}} q_J(U_J), \tag{2.14}$$

and thus with (2.12) we obtain (2.10). □

We also recall the following sufficient condition for dissipativity.

Lemma 2.3 ([1])**.** *Let $\mathcal{U} \subseteq \mathbb{R}^N$ be open and convex, and assume $\varphi : \mathcal{U} \to \mathbb{R}^N$ and $\eta : \mathcal{U} \to \mathbb{R}$ are of class C^2, with η convex, and $q : \mathcal{U} \to \mathbb{R}$ is such that*

$$q'(U) = \eta'(U)\varphi'(U), \qquad \textit{for all } U \in \mathcal{U}. \tag{2.15}$$

Assume further that $\mathcal{U}_{stab} \subseteq \mathcal{U}$, and that

$$\varphi'(U)^t\eta''(U) \quad \textit{is symmetric nonnegative for all } U \in \mathcal{U}_{stab}. \tag{2.16}$$

Then either of the following implies that φ is η-dissipative in $\mathcal{U}_{stab}$:

(i) *η is strictly convex and $\eta'(\mathcal{U}_{stab})$ is convex;*
(ii) *φ is a C^2 diffeomorphism over an open set containing $\mathcal{U}_{stab}$, and $\varphi(\mathcal{U}_{stab})$ is convex.*

The proof is obtained easily by looking at the function $\psi(t) = D(U, V(t))$, $0 \leq t \leq 1$, with $V(t)$ defined by $\eta'(V(t)) = (1-t)\eta'(U) + t\eta'(V)$ in case (i), or $\varphi(V(t)) = (1-t)\varphi(U) + t\varphi(V)$ in case (ii). The details can be found in [1].

Remark 2.4. The existence of q such that condition (2.15) holds is equivalent to the property that $\varphi'(U)^t\eta''(U)$ is symmetric for all $U\in\mathcal{U}$. Thus only the nonnegativity is involved in (2.16). In particular, if η is strictly convex (i.e. $\eta''(U)$ is positive definite), the symmetry means that $\varphi'(U)$ is self-adjoint for the scalar product defined by $\eta''(U)$. Thus in this case, condition (2.16) is equivalent to the property that $\varphi'(U)$ has nonnegative eigenvalues for $U\in\mathcal{U}_{\mathrm{stab}}$.

Finally, we shall need the following properties on symmetric bilinear forms.

Lemma 2.4. *Let φ,ψ be symmetric bilinear forms over $\mathbb{R}^p$ with $\varphi\geq 0$. Assume*

$$|\psi(x,x)|\leq\varphi(x,x),\qquad\forall x\in\mathbb{R}^p. \tag{2.17}$$

Then, for all $x,y\in\mathbb{R}^p$, we have

$$|\psi(x,y)|\leq\sqrt{\varphi(x,x)}\sqrt{\varphi(y,y)}. \tag{2.18}$$

Proof. The assumption (2.17) means that $\varphi\pm\psi\geq 0$. Thus we have

$$\begin{aligned}
|\psi(x,y)| &\leq \frac{1}{2}|\varphi(x,y)+\psi(x,y)|+\frac{1}{2}|\varphi(x,y)-\psi(x,y)|\\
&\leq \frac{1}{2}\sqrt{\varphi(x,x)+\psi(x,x)}\sqrt{\varphi(y,y)+\psi(y,y)}\\
&\quad+\frac{1}{2}\sqrt{\varphi(x,x)-\psi(x,x)}\sqrt{\varphi(y,y)-\psi(y,y)}\\
&\leq \frac{1}{4}\left(\varepsilon(\varphi(x,x)+\psi(x,x))+\frac{1}{\varepsilon}(\varphi(y,y)+\psi(y,y))\right)\\
&\quad+\frac{1}{4}\left(\varepsilon(\varphi(x,x)-\psi(x,x))+\frac{1}{\varepsilon}(\varphi(y,y)-\psi(y,y))\right)\\
&=\frac{1}{2}(\varepsilon\varphi(x,x)+\frac{1}{\varepsilon}\varphi(y,y)),\qquad\forall\varepsilon>0.
\end{aligned}$$

Now, taking $\varepsilon=\sqrt{\varphi(y,y)}/\sqrt{\varphi(x,x)}$, (2.18) follows. ☐

Lemma 2.5. *Let φ be a symmetric bilinear form over $\mathbb{R}^p$ with $\varphi\geq 0$. Let A be a linear operator on $\mathbb{R}^p$ and consider the bilinear form*

$$\psi(x,y)=\varphi(x,Ay),\quad x,y\in\mathbb{R}^p. \tag{2.19}$$

If A is diagonalizable with nonnegative eigenvalues and if ψ is symmetric, then ψ is nonnegative.

Proof. Let to the reader. ☐

3. Explicit conservative schemes for quasi-gradient systems

In this section, we consider systems (1.9) where the matrices $B^{\alpha\beta}(U)$ have the quasi-gradient form, to be defined below. This form enables to write down flux vector splitting type formulas for our schemes, that are written down in Subsection 3.1. Then we describe some applications. The first one is concerned with linear systems in Subsection 3.2. Then, we focus on the nonlinear Navier-Stokes system when $\lambda=\mu$ in Subsection 3.3, unfortunately we are only able to treat this case by our method. Finally, we analyze the case of the scalar equations in Subsection 3.4.

3.1. **A family of schemes for quasi-gradient systems.** Systems (1.9) are defined to have the quasi-gradient form if the matrices $B^{\alpha\beta}(U)$ in (1.9) are scalar multiples of jacobian matrices of maps $\Phi_{\alpha\beta}:\mathcal{U}\subseteq\mathbb{R}^N\to\mathbb{R}^N$ of class C^3, that is,

$$B^{\alpha\beta}(U)=\mu^{\alpha\beta}(U)\Phi'_{\alpha\beta}(U),\qquad \mu^{\alpha\beta}\in C^2(\mathcal{U}). \tag{3.1}$$

Before writing down our schemes, let us introduce the following notations. For $I\in\mathbb{Z}^d$ and $\phi:\mathbb{Z}^d\to\mathbb{R}^N$,

$$\begin{aligned}\Delta_\alpha^+\phi_I&=\phi_{I+e_\alpha}-\phi_I,\\ \Delta_\alpha^-\phi_I&=\phi_I-\phi_{I-e_\alpha},\\ \Delta_\alpha^0\phi_I&=\phi_{I+e_\alpha}-\phi_{I-e_\alpha},\end{aligned} \tag{3.2}$$

where e_α is the α's basis vector of $\mathbb{R}^d$. We easily deduce the relations

$$\Delta_\alpha^-(\nu_{I'}\Delta_\alpha^+\phi_I)=\nu_{I'}\phi_{I+e_\alpha}+\nu_{I'-e_\alpha}\phi_{I-e_\alpha}-(\nu_{I'}+\nu_{I'-e_\alpha})\phi_I, \tag{3.3}$$

$$\begin{aligned}\Delta_\alpha^0(\nu_I\Delta_\beta^0\phi_I)=\;&\nu_{I+e_\alpha}\phi_{I+e_\alpha+e_\beta}+\nu_{I-e_\alpha}\phi_{I-e_\alpha-e_\beta}\\ &-\nu_{I+e_\alpha}\phi_{I+e_\alpha-e_\beta}-\nu_{I-e_\alpha}\phi_{I-e_\alpha+e_\beta},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Delta_\alpha^+\Delta_\alpha^-(\nu_I\Delta_\beta^+\Delta_\beta^-\phi_I)&=\Delta_\alpha^+\Delta_\alpha^-(\nu_I\phi_{I+e_\beta})+\Delta_\alpha^+\Delta_\alpha^-(\nu_I\phi_{I-e_\beta})\\ &\qquad-2\Delta_\alpha^+\Delta_\alpha^-(\nu_I\phi_I)\\ &=\nu_{I+e_\alpha}\phi_{I+e_\alpha+e_\beta}+\nu_{I-e_\alpha}\phi_{I-e_\alpha+e_\beta}-2\nu_I\phi_{I+e_\beta}\\ &\quad+\nu_{I+e_\alpha}\phi_{I+e_\alpha-e_\beta}+\nu_{I-e_\alpha}\phi_{I-e_\alpha-e_\beta}-2\nu_I\phi_{I-e_\beta}\\ &\quad-2\nu_{I+e_\alpha}\phi_{I+e_\alpha}-2\nu_{I-e_\alpha}\phi_{I-e_\alpha}+4\nu_I\phi_I.\end{aligned} \tag{3.5}$$

We consider a timestep $\Delta t>0$ and space lengths $\Delta x_1,\dots,\Delta x_d>0$, and we denote

$$\sigma^{\alpha\beta}=\frac{\Delta t}{\Delta x_\alpha\Delta x_\beta},\qquad \alpha,\beta=1,\dots,d. \tag{3.6}$$

We consider $B^{\alpha\beta}(U)$ satisfying (3.1), such that for some coefficients $\tilde\mu^{\alpha\beta}(U)$, $\alpha\neq\beta$,

$$\tilde\mu^{\alpha\beta}(U)\geq|\mu^{\alpha\beta}(U)|,\qquad \forall\alpha\neq\beta, \tag{3.7}$$

and

$$\mu^{\alpha\alpha}(U)\geq\max_{\beta\neq\alpha}\tilde\mu^{\alpha\beta}(U),\tilde\mu^{\beta\alpha}(U),\qquad \forall\alpha. \tag{3.8}$$

Also, we assume that, for $\alpha\neq\beta$, there exist functions $\tilde\Phi_{\alpha\beta}:\mathcal{U}\to\mathbb{R}^N$ such that, for all $U\in\mathcal{U}$,

$$\tilde\Phi_{\alpha\beta}(U)=\tilde\Phi_{\beta\alpha}(U),\qquad \forall\alpha\neq\beta, \tag{3.9}$$

all the $\Phi'_{\alpha\beta}(U)$ and $\tilde\Phi'_{\alpha\beta}(U)$ are simultaneously symmetrizable, that is, there exists a scalar product in $\mathbb{R}^N$, $(\ |\ )_U$, depending on U, for which they are self-adjoint, and

$$\tilde\Phi'_{\alpha\beta}(U)\geq\pm\Phi'_{\alpha\beta}(U),\qquad \forall\alpha\neq\beta, \tag{3.10}$$

and also

$$\sigma^{\alpha\alpha}\Phi'_{\alpha\alpha}(U)\geq\sum_{\beta\neq\alpha}\sigma^{\alpha\beta}\tilde\Phi'_{\alpha\beta}(U),\qquad \forall\alpha, \tag{3.11}$$

where the inequalities in (3.10) and (3.11) are understood in the sense of self-adjoint operators. We notice that these inequalities are independent of the choice of the particular symmetrizing scalar product. Finally, we assume the following CFL condition,

$$\mathrm{Id} - \sum_{\alpha} 2\sigma^{\alpha\alpha}\bar{\mu}^{\alpha\alpha}\Phi'_{\alpha\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta}\underline{\tilde{\mu}}^{\alpha\beta}\tilde{\Phi}'_{\alpha\beta} \geq 0, \tag{3.12}$$

where $\mu_I^{\alpha\beta} = \mu^{\alpha\beta}(U_I^n)$ and

$$\bar{\mu}^{\alpha\alpha} = \sup_{I\in\mathbb{Z}^d} \mu_I^{\alpha\alpha}, \quad \text{and} \quad \underline{\tilde{\mu}}^{\alpha\beta} = \inf_{I\in\mathbb{Z}^d} \tilde{\mu}_I^{\alpha\beta}.$$

We propose the following discretization for (1.9),

$$\begin{aligned} U_I^{n+1} = U_I^n &+ \sum_{\alpha} \sigma^{\alpha\alpha}\Delta_\alpha^- \mu_{I+e_\alpha/2}^{\alpha\alpha}\Delta_\alpha^+\Phi_{\alpha\alpha}(U_I^n) \\ &+ \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}\Delta_\alpha^0\mu_I^{\alpha\beta}\Delta_\beta^0\Phi_{\alpha\beta}(U_I^n) + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}\Delta_\alpha^+\Delta_\alpha^-\tilde{\mu}_I^{\alpha\beta}\Delta_\beta^+\Delta_\beta^-\tilde{\Phi}_{\alpha\beta}(U_I^n), \end{aligned} \tag{3.13}$$

where

$$\mu_{I+e_\alpha/2}^{\alpha\alpha} = \frac{1}{2}(\mu_I^{\alpha\alpha} + \mu_{I+e_\alpha}^{\alpha\alpha}). \tag{3.14}$$

Then according to (3.1) this is obviously a conservative consistent second-order in space scheme for (1.9), and we have the following stability results.

Theorem 3.1. *Let $\mathcal{U}$, Ω satisfy the hypotheses of Theorem 2.1 and $\Phi_{\alpha\beta}$, $\tilde{\Phi}_{\alpha\beta} : \mathcal{U} \to \mathbb{R}^N$ be of class C^2 and satisfy* (3.7)-(3.12), *and* (2.1) *for all α, β. Further, assume that the maps*

$$\begin{aligned} &\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}, \quad \Psi_\alpha := \sigma^{\alpha\alpha}\Phi_{\alpha\alpha} - \sum_{\beta\neq\alpha}\sigma^{\alpha\beta}\tilde{\Phi}_{\alpha\beta}, \\ &\Psi := \mathrm{Id} - \sum_{\alpha} 2\sigma^{\alpha\alpha}\bar{\mu}^{\alpha\alpha}\Phi_{\alpha\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta}\underline{\tilde{\mu}}^{\alpha\beta}\tilde{\Phi}_{\alpha\beta} \end{aligned} \tag{3.15}$$

satisfy properties (P1) and (P2) of Theorem 2.1.
Then $\overline{\Omega}$ is an invariant domain for (3.13), *that is, if $U_I^n \in \overline{\Omega}$, for all $I \in \mathbb{Z}^d$, then $U_I^{n+1} \in \overline{\Omega}$, for all $I \in \mathbb{Z}^d$.*

Theorem 3.2. *Let $\mathcal{U} \subseteq \mathbb{R}^N$ be open and convex, and $\Phi_{\alpha\beta}$, $\tilde{\Phi}_{\alpha\beta} : \mathcal{U} \to \mathbb{R}^N$ be of class C^2 and satisfy* (3.7)-(3.12). *Assume $\eta : \mathcal{U} \to \mathbb{R}$ is convex and C^2, and that there exist functions $q_{\alpha\beta}$ and $\tilde{q}_{\alpha\beta}$ satisfying* (2.6) *with φ_J replaced by $\Phi_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha\beta}$, respectively. Further, assume that the maps*

$$\begin{aligned} &\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}, \quad \Psi_\alpha := \sigma^{\alpha\alpha}\Phi_{\alpha\alpha} - \sum_{\beta\neq\alpha}\sigma^{\alpha\beta}\tilde{\Phi}_{\alpha\beta}, \\ &\Psi := \mathrm{Id} - \sum_{\alpha} 2\sigma^{\alpha\alpha}\bar{\mu}^{\alpha\alpha}\Phi_{\alpha\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta}\underline{\tilde{\mu}}^{\alpha\beta}\tilde{\Phi}_{\alpha\beta} \end{aligned} \tag{3.16}$$

are η-dissipative in $\mathcal{U}_{stab} \subseteq \mathcal{U}$, and that $U_I^n \in \mathcal{U}_{stab}$ for $I \in \mathbb{Z}^d$ and $U_I^{n+1} \in \mathcal{U}_{stab}$.

Then we have the discrete entropy inequality

$$\eta(U_I^{n+1}) \le \eta(U_I^n) + \sum_\alpha \sigma^{\alpha\alpha}\Delta_\alpha^- \mu_{I+e_\alpha/2}^{\alpha\alpha}\Delta_\alpha^+ q_{\alpha\alpha}(U_I^n) + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}\Delta_\alpha^0 \mu_I^{\alpha\beta}\Delta_\beta^0 q_{\alpha\beta}(U_I^n) + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}\Delta_\alpha^+\Delta_\alpha^-\tilde{\mu}_I^{\alpha\beta}\Delta_\beta^+\Delta_\beta^-\tilde{q}_{\alpha\beta}(U_I^n). \tag{3.17}$$

Proof of Theorem 3.1. We are going to apply Lemma 2.1. By collecting the terms corresponding to the argument U_J^n for each relevant J, the scheme (3.13) can be written as

$$U_I^{n+1} = \sum_{J\in\mathbf{J}} \varphi_J(U_J^n), \qquad \mathbf{J} = \{I, I\pm e_\alpha, I\pm e_\alpha\pm e_\beta;\ \alpha,\beta = 1,\dots,d, \alpha\neq\beta\}. \tag{3.18}$$

First of all, we remark that for any U,

$$U = \sum_{J\in\mathbf{J}} \varphi_J(U), \tag{3.19}$$

this follows from the property that if U_I^n is constant, then U_I^{n+1} also takes the same value. In order to obtain our result by applying Lemma 2.1, we just have to prove that each of the φ_J can be written as a linear combination with nonnegative coefficients of maps satisfying the same hypotheses as φ in Theorem 2.1. By assumption, the maps $\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}$, Ψ_α and Ψ satisfy the hypotheses of Theorem 2.1. Let us first remark that $\tilde{\Phi}_{\alpha\beta}$ and $\Phi_{\alpha\alpha}$ are linear combination with nonnegative coefficients of these maps,

$$\begin{aligned} \tilde{\Phi}_{\alpha\beta} &= \frac{1}{2}(\tilde{\Phi}_{\alpha\beta} + \Phi_{\alpha\beta}) + \frac{1}{2}(\tilde{\Phi}_{\alpha\beta} - \Phi_{\alpha\beta}),\\ \Phi_{\alpha\alpha} &= \frac{1}{\sigma^{\alpha\alpha}}\left(\Psi_\alpha + \sum_{\beta\neq\alpha}\sigma^{\alpha\beta}\tilde{\Phi}_{\alpha\beta}\right). \end{aligned} \tag{3.20}$$

Then, for $J = I$, we have according to (3.3)-(3.5)

$$\varphi_I(U) = U - \sum_\alpha \sigma^{\alpha\alpha}(\mu_{I-e_\alpha/2}^{\alpha\alpha} + \mu_{I+e_\alpha/2}^{\alpha\alpha})\Phi_{\alpha\alpha}(U) + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \sigma^{\alpha\beta}\tilde{\mu}_I^{\alpha\beta}\tilde{\Phi}_{\alpha\beta}(U), \tag{3.21}$$

thus $\varphi_I = \Psi + \sum_\alpha C^\alpha \Phi_{\alpha\alpha} + \sum_{\alpha\neq\beta} D^{\alpha\beta}\tilde{\Phi}_{\alpha\beta}$ for some $C^\alpha \ge 0$ and $D^{\alpha\beta} \ge 0$, and φ_I admits a decomposition as asserted. Similarly, for $J = I \pm e_\alpha$, from (3.3)-(3.5) we obtain

$$\begin{aligned} \varphi_{I\pm e_\alpha} &= \sigma^{\alpha\alpha}\mu_{I\pm e_\alpha/2}^{\alpha\alpha}\Phi_{\alpha\alpha} - \sum_{\beta\neq\alpha}\frac{\sigma^{\alpha\beta}}{2}(\tilde{\mu}_{I\pm e_\alpha}^{\alpha\beta}\tilde{\Phi}_{\alpha\beta} + \tilde{\mu}_I^{\beta\alpha}\tilde{\Phi}_{\beta\alpha})\\ &= \mu_{I\pm e_\alpha/2}^{\alpha\alpha}\Psi_\alpha + \sum_{\beta\neq\alpha}\frac{\sigma^{\alpha\beta}}{2}(2\mu_{I\pm e_\alpha/2}^{\alpha\alpha} - \tilde{\mu}_{I\pm e_\alpha}^{\alpha\beta} - \tilde{\mu}_I^{\beta\alpha})\tilde{\Phi}_{\alpha\beta}, \end{aligned} \tag{3.22}$$

which shows according to (3.14), (3.8) that $\varphi_{I\pm e_\alpha}$ also admits a decomposition as asserted. Now, for $J = I \pm e_\alpha \pm e_\beta$, $\alpha \neq \beta$, we have

$$\varphi_{I+e_\alpha+e_\beta} = \frac{\sigma^{\alpha\beta}}{4}\left(\mu^{\alpha\beta}_{I+e_\alpha}\Phi_{\alpha\beta} + \mu^{\beta\alpha}_{I+e_\beta}\Phi_{\beta\alpha} + \tilde{\mu}^{\alpha\beta}_{I+e_\alpha}\tilde{\Phi}_{\alpha\beta} + \tilde{\mu}^{\beta\alpha}_{I+e_\beta}\tilde{\Phi}_{\beta\alpha}\right), \tag{3.23}$$

$$\varphi_{I-e_\alpha-e_\beta} = \frac{\sigma^{\alpha\beta}}{4}\left(\mu^{\alpha\beta}_{I-e_\alpha}\Phi_{\alpha\beta} + \mu^{\beta\alpha}_{I-e_\beta}\Phi_{\beta\alpha} + \tilde{\mu}^{\alpha\beta}_{I-e_\alpha}\tilde{\Phi}_{\alpha\beta} + \tilde{\mu}^{\beta\alpha}_{I-e_\beta}\tilde{\Phi}_{\beta\alpha}\right), \tag{3.24}$$

$$\varphi_{I-e_\alpha+e_\beta} = \frac{\sigma^{\alpha\beta}}{4}\left(-\mu^{\alpha\beta}_{I-e_\alpha}\Phi_{\alpha\beta} - \mu^{\beta\alpha}_{I+e_\beta}\Phi_{\beta\alpha} + \tilde{\mu}^{\alpha\beta}_{I-e_\alpha}\tilde{\Phi}_{\alpha\beta} + \tilde{\mu}^{\beta\alpha}_{I+e_\beta}\tilde{\Phi}_{\beta\alpha}\right), \tag{3.25}$$

$$\varphi_{I+e_\alpha-e_\beta} = \frac{\sigma^{\alpha\beta}}{4}\left(-\mu^{\alpha\beta}_{I+e_\alpha}\Phi_{\alpha\beta} - \mu^{\beta\alpha}_{I-e_\beta}\Phi_{\beta\alpha} + \tilde{\mu}^{\alpha\beta}_{I+e_\alpha}\tilde{\Phi}_{\alpha\beta} + \tilde{\mu}^{\beta\alpha}_{I-e_\beta}\tilde{\Phi}_{\beta\alpha}\right). \tag{3.26}$$

Using (3.7) and (3.10) it is easy to obtain the asserted decompositions for each of the four maps above. This concludes the proof of Theorem 3.1. □

Proof of Theorem 3.2. This is obtained very much like the previous one except that now we apply Lemma 2.2. Indeed, the decomposition (3.18) is still valid with (3.21), (3.22) and (3.23)-(3.26). The property (3.19) holds also, thus we only have to check that each φ_J is η-dissipative in $\mathcal{U}_{\rm stab}$. Since the φ_J are linear combinations with nonnegative coefficients of the maps $\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}$, Ψ_α and Ψ that are η-dissipative in $\mathcal{U}_{\rm stab}$ by assumption, the hypotheses of Lemma 2.2 are satisfied, and (2.10) (or more precisely (2.14) if (2.7) fails) translates into (3.17), which proves the result. □

To finish this subsection we make the link of our conditions (3.1), (3.9)-(3.12) with sharp conditions for invariance in the continuous case [4],[11] for equation (1.9). Our conditions indeed imply the (degenerate) parabolicity of (1.9), and the B-convexity of the functions G_i.

Lemma 3.1. *Assume the hypotheses of Theorem 3.1 are satisfied. Then, for all $U \in \mathcal{U}$ we have*

$$\sum_{\alpha\beta} \xi_\alpha \xi_\beta B^{\alpha\beta}(U) \geq 0 \qquad \forall \xi \in \mathbb{R}^d, \tag{3.27}$$

in the sense of the symmetrizing scalar product, where $B^{\alpha\beta}$ is defined by (3.1). *Moreover, for all $\omega \in \partial\Omega$, and $a_1, \dots, a_d \in T_\omega(\partial\Omega)$, we have*

$$\sum_{\alpha,\beta} G_i''(\omega)(a_\alpha, B^{\alpha\beta}(\omega)a_\beta) \geq 0. \tag{3.28}$$

Proof. With (3.7)-(3.11) and noticing that $(\sigma^{\alpha\beta})^2 = \sigma^{\alpha\alpha}\sigma^{\beta\beta}$, we compute omitting the argument U to simplify

$$\begin{aligned}
\sum_{\alpha\beta} \xi_\alpha\xi_\beta B^{\alpha\beta} &= \sum_\alpha \xi_\alpha^2 \mu^{\alpha\alpha}\Phi'_{\alpha\alpha} + \sum_{\alpha\neq\beta} \xi_\alpha\xi_\beta \mu^{\alpha\beta}\Phi'_{\alpha\beta} \\
&\geq \sum_\alpha \xi_\alpha^2 \mu^{\alpha\alpha}\Phi'_{\alpha\alpha} - \sum_{\alpha\neq\beta} \frac{1}{2}\left(\frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}}\xi_\alpha^2 + \frac{\sigma^{\alpha\beta}}{\sigma^{\beta\beta}}\xi_\beta^2\right)\tilde{\mu}^{\alpha\beta}\tilde{\Phi}'_{\alpha\beta} \\
&\geq \sum_\alpha \xi_\alpha^2 \mu^{\alpha\alpha}\Phi'_{\alpha\alpha} - \sum_{\alpha\neq\beta} \frac{1}{2}\left(\frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}}\xi_\alpha^2\mu^{\alpha\alpha} + \frac{\sigma^{\alpha\beta}}{\sigma^{\beta\beta}}\xi_\beta^2\mu^{\beta\beta}\right)\tilde{\Phi}'_{\alpha\beta}
\end{aligned}$$

$$= \sum_{\alpha} \xi_\alpha^2 \mu^{\alpha\alpha} \Phi'_{\alpha\alpha} - \sum_{\alpha\neq\beta} \frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}} \xi_\alpha^2 \mu^{\alpha\alpha} \tilde{\Phi}'_{\alpha\beta}$$
$$= \sum_{\alpha} \frac{\xi_\alpha^2 \mu^{\alpha\alpha}}{\sigma^{\alpha\alpha}} \left(\sigma^{\alpha\alpha} \Phi'_{\alpha\alpha} - \sum_{\beta,\beta\neq\alpha} \sigma^{\alpha\beta} \tilde{\Phi}'_{\alpha\beta} \right) \geq 0.$$

Then, for the second property, notice that since Ω is convex, we have $G_i{}''(\omega)(a,a) \geq 0$ for any $a \in T_\omega(\partial\Omega)$. The assumptions (2.1) for $\Phi'_{\alpha\beta}(\omega)$ and $\tilde{\Phi}'_{\alpha\beta}(\omega)$ means that $T_\omega(\partial\Omega)$ is stable by any $\Phi'(\omega)$ for $\Phi = \Phi_{\alpha\beta}$ or $\Phi = \tilde{\Phi}_{\alpha\beta}$. In other words,

$$G_i'(\omega)\Phi'(\omega) = p_i(\omega) G_i'(\omega) \quad \text{for all } \omega \in \partial\Omega, \tag{3.29}$$

for some scalar $p_i(\omega)$. Differentiating this identity leads to

$$G_i''(\omega)d\omega \cdot \Phi'(\omega) + G_i'(\omega)\Phi''(\omega)d\omega = p_i'(\omega)d\omega\, G_i'(\omega) + p_i(\omega) G_i''(\omega)d\omega, \tag{3.30}$$

and therefore

$$G_i''(\omega)(a, \Phi'(\omega)b) + G_i'(\omega)\Phi''(\omega)(a,b) = p_i(\omega) G_i''(\omega)(a,b), \quad a, b \in T_\omega(\partial\Omega). \tag{3.31}$$

In particular, we deduce the symmetry condition

$$G_i{}''(\omega)(a, \Phi'(\omega)b) = G_i{}''(\omega)(b, \Phi'(\omega)a) \quad \text{for all } \omega \in \partial\Omega, \quad a, b \in T_\omega(\partial\Omega). \tag{3.32}$$

This enables to use Lemmas 2.4 and 2.5 (recall that the restriction to a stable subspace of a diagonalizable operator is also diagonalizable). With (3.7)-(3.11), we compute (omitting the argument ω to simplify)

$$\sum_{\alpha,\beta} G_i{}''(a_\alpha, B^{\alpha\beta} a_\beta)$$
$$= \sum_{\alpha} G_i{}''(a_\alpha, B^{\alpha\alpha} a_\alpha) + \sum_{\alpha\neq\beta} G_i{}''(a_\alpha, \mu^{\alpha\beta} \Phi'_{\alpha\beta} a_\beta)$$
$$\geq \sum_{\alpha} G_i{}''(a_\alpha, B^{\alpha\alpha} a_\alpha)$$
$$\quad - \sum_{\alpha\neq\beta} \left(G_i{}''(a_\alpha, \tilde{\mu}^{\alpha\beta} \tilde{\Phi}'_{\alpha\beta} a_\alpha) \right)^{1/2} \left(G_i{}''(a_\beta, \tilde{\mu}^{\alpha\beta} \tilde{\Phi}'_{\alpha\beta} a_\beta) \right)^{1/2}$$
$$\geq \sum_{\alpha} G_i{}''(a_\alpha, B^{\alpha\alpha} a_\alpha) - \frac{1}{2} \sum_{\alpha\neq\beta} \left[\frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}} G_i{}''(a_\alpha, \tilde{\mu}^{\alpha\beta} \tilde{\Phi}'_{\alpha\beta} a_\alpha) \right.$$
$$\left. + \frac{\sigma^{\alpha\beta}}{\sigma^{\beta\beta}} G_i{}''(a_\beta, \tilde{\mu}^{\alpha\beta} \tilde{\Phi}'_{\alpha\beta} a_\beta) \right]$$
$$\geq \sum_{\alpha} G_i{}''(a_\alpha, \mu^{\alpha\alpha} \Phi'_{\alpha\alpha} a_\alpha) - \sum_{\alpha\neq\beta} \frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}} \mu^{\alpha\alpha} G_i{}''(a_\alpha, \tilde{\Phi}'_{\alpha\beta} a_\alpha)$$
$$= \sum_{\alpha} \mu^{\alpha\alpha} G_i{}'' \left(a_\alpha, \Phi'_{\alpha\alpha} a_\alpha - \sum_{\beta,\beta\neq\alpha} \frac{\sigma^{\alpha\beta}}{\sigma^{\alpha\alpha}} \tilde{\Phi}'_{\alpha\beta} a_\alpha \right) \geq 0,$$

which is the desired inequality. □

Remark 3.1. Following the same argument, one easily proves that for all $U \in \mathcal{U}$

$$\sum_{\alpha\beta} \big(V_\alpha | B^{\alpha\beta}(U) V_\beta\big)_U \geq 0 \qquad \forall\, V_1, \ldots, V_d \in \mathbb{R}^N, \tag{3.33}$$

which is stronger than (3.27) (take $V_\alpha = \xi_\alpha V$ for any $V \in \mathbb{R}^N$). The inequality (3.33) is also valid in case of Theorem 3.2, with $(|)_U$ defined by $\eta''(U)$. This ensures the entropy inequality

$$\partial_t \eta(U) - \sum_{\alpha,\beta=1}^{d} \partial_\alpha(\eta'(U) B^{\alpha\beta}(U) \partial_\beta U) \leq 0, \tag{3.34}$$

for any suitable solution to (1.9).

Remark 3.2. The assumption of simultaneous symmetrizability of the $B^{\alpha\beta}$ is quite restrictive for applications (as we see in Subsection 3.3). A more natural assumption would be that $(B^{\alpha\beta})^* = B^{\beta\alpha}$, where the adjoint is with respect to some scalar product $(|)_U$ depending on U.

3.2. **Linear systems.** The most simple case of a quasi-gradient system is that of a linear system with constant coefficients. In this case (1.9) may be written as

$$\partial_t U = \sum_{\alpha,\beta=1}^{d} B^{\alpha\beta} \partial^2_{\alpha\beta} U, \tag{3.35}$$

where $B^{\alpha\beta}$ are constant matrices. Then, it is not restrictive to assume that

$$B^{\alpha\beta} = B^{\beta\alpha}, \tag{3.36}$$

and the quasi-gradient form (3.1) is obviously satisfied with $\mu^{\alpha\beta} = 1$ and $\Phi_{\alpha\beta}(U) = B^{\alpha\beta} U$. Let us now write down the assumptions of Subsection 3.1. We take $\tilde{\mu}^{\alpha\beta} = 1$, thus we have to assume that there exist matrices $\tilde{B}^{\alpha\beta}$, $\alpha \neq \beta$, satisfying

$$\tilde{B}^{\alpha\beta} = \tilde{B}^{\beta\alpha} \qquad \forall \alpha \neq \beta, \tag{3.37}$$

the matrices $B^{\alpha\beta}$ and $\tilde{B}^{\alpha\beta}$ are simultaneously symmetrizable and

$$\tilde{B}^{\alpha\beta} \geq \pm B^{\alpha\beta} \qquad \forall \alpha \neq \beta, \tag{3.38}$$

$$\sigma^{\alpha\alpha} B^{\alpha\alpha} \geq \sum_{\beta\neq\alpha} \sigma^{\alpha\beta} \tilde{B}^{\alpha\beta}, \qquad \forall \alpha, \tag{3.39}$$

and the CFL condition

$$\sum_{\alpha} 2\sigma^{\alpha\alpha} B^{\alpha\alpha} \leq \mathrm{Id} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta} \tilde{B}^{\alpha\beta}. \tag{3.40}$$

Notice that the optimal choice for (3.38) is $\tilde{B}^{\alpha\beta} = |B^{\alpha\beta}|$, which gives the main stability condition

$$\sigma^{\alpha\alpha} B^{\alpha\alpha} \geq \sum_{\beta\neq\alpha} \sigma^{\alpha\beta} |B^{\alpha\beta}| \qquad \forall \alpha. \tag{3.41}$$

We then have the following discretization for (3.35),

$$U_I^{n+1} = U_I^n + \sum_\alpha \sigma^{\alpha\alpha} B^{\alpha\alpha} \Delta_\alpha^- \Delta_\alpha^+ U_I^n + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4} B^{\alpha\beta} \Delta_\alpha^0 \Delta_\beta^0 U_I^n + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4} \tilde{B}^{\alpha\beta} \Delta_\alpha^+ \Delta_\alpha^- \Delta_\beta^+ \Delta_\beta^- U_I^n. \tag{3.42}$$

Denote by $(U|V)$ the scalar product symmetrizing all the $B^{\alpha\beta}$ and $\tilde{B}^{\alpha\beta}$ for all α, β (i.e. these operators are self-adjoint for this scalar product), and define $|U|^2 = (U|U)$. Theorem 3.2 implies the following stability property for the scheme (3.42),

$$\sum_{I\in\mathbb{Z}^d} |U_I^{n+1}|^2 \leq \sum_{I\in\mathbb{Z}^d} |U_I^n|^2. \tag{3.43}$$

Indeed, if we set $\eta(U) = |U|^2/2$, $q_{\alpha\beta}(U) = (U|B^{\alpha\beta}U)/2$ and $\tilde{q}_{\alpha\beta}(U) = (U|\tilde{B}^{\alpha\beta}U)/2$, we see that these functions satisfy the hypotheses of Theorem 3.2, where (3.37)-(3.40) correspond to (3.9)-(3.12). Thus we get

$$\eta(U_I^{n+1}) \leq \eta(U_I^n) + \sum_\alpha \sigma^{\alpha\alpha} \Delta_\alpha^- \Delta_\alpha^+ q_{\alpha\alpha}(U_I^n) + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4} \Delta_\alpha^0 \Delta_\beta^0 q_{\alpha\beta}(U_I^n) + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4} \Delta_\alpha^+ \Delta_\alpha^- \Delta_\beta^+ \Delta_\beta^- \tilde{q}_{\alpha\beta}(U_I^n), \tag{3.44}$$

and summing up over all $I \in \mathbb{Z}^d$ we immediately obtain (3.43).

It is also interesting to notice that the scheme (3.42) defines a self-adjoint operator $(U_I^n)_I \mapsto (U_I^{n+1})_I$ with respect to the scalar product induced by $(|)$. To see that, we compute

$$\sum_I \sum_\alpha (\sigma^{\alpha\alpha} B^{\alpha\alpha} \Delta_\alpha^- \Delta_\alpha^+ U_I | V_I) = \sum_I \sum_\alpha (U_I | \sigma^{\alpha\alpha} B^{\alpha\alpha} \Delta_\alpha^- \Delta_\alpha^+ V_I),$$

$$\sum_I \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} (\frac{\sigma^{\alpha\beta}}{4} B^{\alpha\beta} \Delta_\alpha^0 \Delta_\beta^0 U_I | V_I) = \sum_I \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} (U_I | \frac{\sigma^{\alpha\beta}}{4} B^{\alpha\beta} \Delta_\alpha^0 \Delta_\beta^0 V_I),$$

$$\sum_I \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} (\frac{\sigma^{\alpha\beta}}{4} \tilde{B}^{\alpha\beta} \Delta_\alpha^+ \Delta_\alpha^- \Delta_\beta^+ \Delta_\beta^- U_I | V_I) = \sum_I \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} (U_I | \frac{\sigma^{\alpha\beta}}{4} \tilde{B}^{\alpha\beta} \Delta_\alpha^+ \Delta_\alpha^- \Delta_\beta^+ \Delta_\beta^- V_I),$$

where we have omitted the superscript n, and thus we conclude that

$$\sum_I (U_I^{n+1} | V_I) = \sum_I (U_I | V_I^{n+1}), \tag{3.45}$$

which proves the assertion.

Remark 3.3. The main stability condition (3.41) that we have to assume within our approach does not allow to treat all parabolic systems. In order to see this, consider the case $d = 2$ with U scalar. Then $B^{\alpha\beta} \in \mathbb{R}$ for $\alpha, \beta = 1, 2$, and the degenerate parabolic nature of (1.9) writes

$$B^{11} \geq 0, \quad B^{22} \geq 0, \quad (B^{12})^2 \leq B^{11} B^{22}, \tag{3.46}$$

while (3.41) writes

$$\sigma^{11} B^{11} \geq \sigma^{12} |B^{12}|, \quad \sigma^{22} B^{22} \geq \sigma^{12} |B^{12}|. \tag{3.47}$$

Obviously (3.47) implies (3.46) since $(\sigma^{12})^2 = \sigma^{11}\sigma^{22}$. In case $\sigma^{11}B^{11} = \sigma^{22}B^{22}$ the two sets of conditions are equivalent, otherwise not. One could argue then that we can satisfy this relation by choosing the suitable ratio $\Delta x_1/\Delta x_2$. But this is not possible in the nonlinear case when $B^{\alpha\beta}(U)$ depend on U.

3.3. **Navier-Stokes equations: case $\lambda = \mu$.** When $\lambda = \mu$ the system of Navier-Stokes equations (1.2)-(1.5) becomes a quasi-gradient system. Indeed, in this case the matrices $A_{\alpha\beta}$ are symmetric and yet $A_{\alpha\beta} = A_{\beta\alpha}$, for all α, β, namely,

$$A_{\alpha\beta}(U) = \mu(U)(\delta_{\alpha\beta}\,\mathrm{Id} + e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha) \equiv \mu(U)\Lambda_{\alpha\beta}. \tag{3.48}$$

Therefore, we have

$$B^{\alpha\beta}(U) = \mu(U)\begin{pmatrix} 0 & 0 & 0 \\ -\frac{\Lambda_{\alpha\beta}u}{\rho} & \frac{\Lambda_{\alpha\beta}}{\rho} & 0 \\ -\frac{u^t\Lambda_{\alpha\beta}u}{\rho} & \frac{u^t\Lambda_{\alpha\beta}}{\rho} & 0 \end{pmatrix}, \tag{3.49}$$

and it is not difficulty to see that

$$B^{\alpha\beta}(U) = \mu(U)\Phi'_{\alpha\beta}(U), \tag{3.50}$$

with

$$\Phi_{\alpha\beta}(U) = \begin{pmatrix} 0 \\ \Lambda_{\alpha\beta}u \\ \frac{1}{2}u^t\Lambda_{\alpha\beta}u \end{pmatrix}. \tag{3.51}$$

We then define, for $\alpha \neq \beta$,

$$\tilde{\Phi}_{\alpha\beta}(U) = \begin{pmatrix} 0 \\ \tilde{\Lambda}_{\alpha\beta}u \\ \frac{1}{2}u^t\tilde{\Lambda}_{\alpha\beta}u \end{pmatrix}, \tag{3.52}$$

with

$$\tilde{\Lambda}_{\alpha\beta} = e_\alpha \otimes e_\alpha + e_\beta \otimes e_\beta. \tag{3.53}$$

We take $\mathcal{U} = \{(\rho, q, E) \in \mathbb{R}^{d+2} \,:\, \rho > 0,\ E > |q|^2/2\rho\}$, and

$$\Omega = \{U \in \mathcal{U} \,:\, 0 < \rho_* < \rho < \rho^*,\ S > S_0\}. \tag{3.54}$$

We recall that, as a function of (ρ, q, E), S is given by

$$S = \frac{1}{\gamma}\ln\Big(\frac{2E\rho - |q|^2}{2\rho^{\gamma+1}/(\gamma-1)}\Big). \tag{3.55}$$

By taking $G = -S + S_0$, we see that $\mathcal{U}$ and Ω satisfy the assumptions of Theorem 2.1, indeed it is well known (see [11]) that the functions G defined in this way are quasiconvex, that is, the regions $S \geq S_0$ are convex, $S_0 \in \mathbb{R}$.

We assume that $0 \leq \mu_* \leq \mu(U) \leq \mu^* < +\infty$, for $U \in \Omega$, for some numbers μ_*, μ^*, and we take $\mu^{\alpha\beta} = \tilde{\mu}^{\alpha\beta} = \mu$.

Lemma 3.2. *Let Λ be a $d \times d$ matrix, and define the $(d+2) \times (d+2)$ matrix*

$$Q_\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ -\Lambda u & \Lambda & 0 \\ -u^t\Lambda u & u^t\Lambda & 0 \end{pmatrix}, \tag{3.56}$$

for some $u \in \mathbb{R}^d$. Then, $Q_\Lambda^* = Q_{\Lambda^t}$, where the adjoint is with respect to the scalar product defined by the $(d+2) \times (d+2)$ positive definite symmetric matrix

$$C = a\, f_1^t f_1 + b\, f_2^t f_2 + c\, M^t M, \tag{3.57}$$

with $a, b, c > 0$ and

$$f_1 = (1,0,0)\,, \qquad f_2 = \big(|u|^2/2, -u^t, 1\big)\,, \qquad M = (-u, \mathrm{Id}_d, 0)\,. \tag{3.58}$$

In particular, if $\Lambda^t = \Lambda$, Q_Λ is self-adjoint, thus is diagonalizable with eigenvalues those of Λ to which we add twice the value 0. Moreover, Q_Λ is nonnegative (for the scalar product defined by C) if and only if Λ is nonnegative (for the usual scalar product).

Proof. The required identity $Q_\Lambda^* = Q_{\Lambda^t}$ means that $Q_\Lambda^t C = C Q_{\Lambda^t}$. Let us check this relation for each of the matrices $f_1^t f_1$, $f_2^t f_2$, and $M^t M$ of the decomposition (3.57). Indeed $Q_\Lambda^t f_1^t = 0$ for any Λ, thus by transposition and changing Λ into Λ^t we have $f_1 Q_{\Lambda^t} = 0$, which proves the first identity. The same cancellation occurs for f_2, and finally since

$$M^t M = \begin{pmatrix} |u|^2 & -u^t & 0 \\ -u & \mathrm{Id}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.59}$$

we get

$$Q_\Lambda^t M^t M = M^t M Q_{\Lambda^t} = \begin{pmatrix} u^t \Lambda^t u & -u^t \Lambda^t & 0 \\ -\Lambda^t u & \Lambda^t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.60}$$

The remaining assertions are trivial. $\square$

By applying the lemma, we deduce that $\Phi'_{\alpha\beta}$ and $\tilde{\Phi}'_{\alpha\beta}$ are simultaneously symmetrizable, and we can translate the nonnegativity conditions (3.10)-(3.12) into

$$\tilde{\Lambda}_{\alpha\beta} \pm \Lambda_{\alpha\beta} \geq 0, \qquad \sigma^{\alpha\alpha} \Lambda_{\alpha\alpha} - \sum_{\beta \neq \alpha} \sigma^{\alpha\beta} \tilde{\Lambda}_{\alpha\beta} \geq 0, \tag{3.61}$$

$$\rho\, \mathrm{Id}_d - \sum_\alpha 2\sigma^{\alpha\alpha} \bar{\mu}^{\alpha\alpha} \Lambda_{\alpha\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta} \underline{\tilde{\mu}}^{\alpha\beta} \tilde{\Lambda}_{\alpha\beta} \geq 0. \tag{3.62}$$

Because of the choice (3.53), the first condition in (3.61) is satisfied. Let us take $\sigma^{\alpha\beta} = \sigma$, for all α, β. Then the second condition reduces to $\mathrm{Id} + 2 e_\alpha \otimes e_\alpha - \sum_{\beta\neq\alpha} (e_\alpha \otimes e_\alpha + e_\beta \otimes e_\beta) \geq 0$, i.e. $(4-d) e_\alpha \otimes e_\alpha \geq 0$. Therefore, it is satisfied if $d \leq 4$. The last condition (3.62) holds if σ is sufficiently small.

Then, we have to check condition (2.1). For $\omega \in \partial\Omega$, according to (3.54) the normal $\nu(\omega)$ is either proportional to $\rho'(U)$ or to $S'(U)$. We compute

$$\rho'(U) = (1,0,0), \qquad S'(U) = h(U)(\square, -u^t, 1), \tag{3.63}$$

where $h(U) > 0$ and the form of $\square$ is immaterial. Thus for any Λ we have $\nu(\omega) Q_\Lambda = 0$, and in particular $\nu(\omega)\Phi'_{\alpha\beta} = \nu(\omega)\tilde{\Phi}'_{\alpha\beta} = 0$. This proves that (2.1) holds for all functions $\Phi_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha\beta}$.

We can also easily verify that the functions $\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}$, Ψ_α and Ψ satisfy properties (P1) and (P2). Indeed, property (P1) follows in all cases because, after adding $\varepsilon\,\mathrm{Id}$ to these maps, the last coordinate of the resulting maps equals εE plus a nonnegative term. Since $|U_n| \to \infty$, with $U_n \in \Omega$, implies $E_n \to \infty$, property (P1) is verified for all the maps $+\varepsilon\,\mathrm{Id}$, as required. Property (P2) follows in a similar manner once we observe that if $\mathbf{e} \in \mathbb{R}^{d+2}$ is such that $\sup_{U\in\Omega} \mathbf{e}\cdot U < +\infty$, then, necessarily, $e_{d+2} < 0$. This finishes the checking of the assumptions of Theorem 3.1. Concerning Theorem 3.2, define

$$\eta(U) = -\rho S, \tag{3.64}$$

where S is given in (3.55). It is well-known that η is strictly convex over $\mathcal{U}$, and since $\eta'(U)$ is a combination of $\rho'(U)$ and $S'(U)$ computed in (3.63), we see as above that

$$\eta'(U)\Phi'_{\alpha\beta}(U) = \eta'(U)\tilde{\Phi}'_{\alpha\beta}(U) = 0, \qquad \forall\alpha,\beta. \tag{3.65}$$

Therefore, the condition (2.6) is satisfied for $\Phi_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha\beta}$, with $q_{\alpha\beta} = \tilde{q}_{\alpha\beta} = 0$ for all α, β. The maps $\tilde{\Phi}_{\alpha\beta} \pm \Phi_{\alpha\beta}$, Ψ_α and Ψ are η-dissipative in $\mathcal{U}_{\text{stab}}$ (take $\mathcal{U}_{\text{stab}} = \Omega$ for example), a typical computation is

$$D_{\tilde{\Phi}_{\alpha\beta}}(U,V) = -\eta'(U)(\tilde{\Phi}_{\alpha\beta}(U) - \tilde{\Phi}_{\alpha\beta}(V)) = -\frac{1}{2}\rho h(U)(u-v)^t\tilde{\Lambda}_{\alpha\beta}(u-v) \le 0. \tag{3.66}$$

Collecting all the above facts, Theorems 3.1 and 3.2 can be applied, and we get the following.

Theorem 3.3. *Consider the scheme* (3.13) *applied to the system* (1.2)-(1.5) *where we remove the hyperbolic part on the left-hand side, with* $\kappa \equiv 0$, $\lambda = \mu$ *and* $d \le 4$, *where* $\Phi_{\alpha\beta}$ *and* $\tilde{\Phi}_{\alpha\beta}$ *are given by* (3.51) *and* (3.52), *and* $\sigma^{\alpha\beta} = \sigma$ *for all* α, β, *with* $\sigma > 0$ *sufficiently small so that* (3.62) *holds. Then, the domains* $S \ge S_0$ *are invariant, for any* $S_0 \in \mathbb{R}$, *and for* $\eta(U)$ *given by* (3.64) *we have*

$$\eta(U_I^{n+1}) \le \eta(U_I^n). \tag{3.67}$$

We remark that since $\rho_I^{n+1} = \rho_I^n$, the inequality (3.67) becomes $S_I^{n+1} \ge S_I^n$. In the present case of the Navier-Stokes system, the entropy inequality and the invariance of the domains $S \ge S_0$ are strongly correlated.

3.4. **Scalar equations.** In the case where (1.9) is a scalar equation we may write it in the form

$$\partial_t U = \sum_{\alpha,\beta} \partial^2_{\alpha\beta}\Phi_{\alpha\beta}(U), \tag{3.68}$$

where $\Phi_{\alpha\beta}$ are real valued smooth functions. Clearly, we may assume

$$\Phi_{\alpha\beta} = \Phi_{\beta\alpha}. \tag{3.69}$$

Therefore, we are in the case of a quasi-gradient system, with $\mu^{\alpha\beta} \equiv 1$. We then take also $\tilde{\mu}^{\alpha\beta} \equiv 1$, and we assume that there exist $\tilde{\Phi}_{\alpha\beta}$ satisfying (3.9)-(3.11), where the $\sigma^{\alpha\beta}$ are chosen so that (3.12) is satisfied.

From the proof of Theorem 3.1 we see that, in the present scalar case, (3.13) defines a monotone scheme, that is, if we write

$$U_I^{n+1} = G\left[(U_J^n)_{J\in\mathbf{J}}\right], \tag{3.70}$$

then the function $G\left[(U_J)_{J\in\mathbf{J}}\right]$ is a monotone nondecreasing function of each argument U_J, $J \in \mathbf{J}$. Following [5] we also write (3.70) in the form

$$U^{n+1} = \overrightarrow{G}(U^n), \tag{3.71}$$

where $U^n = (U_I^n)_{I\in\mathbb{Z}^d}$. The following properties easily follow from monotonicity (see [5], Proposition 3.1).

Proposition 3.1 ([5]). *The operator $U \mapsto \overrightarrow{G}(U)$ defined above has the following properties:*

(1) $U \le V$ *(i.e. $U_I \le V_I$ for all $I \in \mathbb{Z}^d$) implies $\overrightarrow{G}(U) \le \overrightarrow{G}(V)$;*

(2) $\inf_{J\in\mathbb{Z}^d} U_J \le \overrightarrow{G}(U)_I \le \sup_{J\in\mathbb{Z}^d} U_J$, *for all $I \in \mathbb{Z}^d$;*

(3) $\|\overrightarrow{G}(U) - \overrightarrow{G}(V)\|_{L^1(\Delta)} \le \|U - V\|_{L^1(\Delta)}$, *if $U, V \in L^1(\Delta)$;*

(4) $\|\overrightarrow{G}(U)\|_{BV(\Delta)} \le \|U\|_{BV(\Delta)}$.

Here, following [5], we denote

$$\|U\|_{L^1(\Delta)} := \sum_{I\in\mathbb{Z}^d} |U_I|\Delta_1 \dots \Delta_d, \tag{3.72}$$

$U \in L^1(\Delta)$ if the above sum is finite, and

$$\|U\|_{BV(\Delta)} := \sum_{\alpha=1}^{d} \frac{1}{\Delta x_\alpha} \|\Delta_\alpha^+ U\|_{L^1(\Delta)}. \tag{3.73}$$

We recall the following variant of Kružkov's [9] L^1 time continuity lemma (see e.g. [7, 8, 9] for a proof).

Lemma 3.3. *Let Ω be an open subset of $\mathbb{R}^d$ and assume that there exist positive constants c_1, c_2 and c_3 such that the function $U : [0,T] \times \Omega \to \mathbb{R}$ satisfies $\|U(t,\cdot)\|_{L^\infty(\Omega)} \le c_1$ and $TV_\Omega(U(t,\cdot)) \le c_2$ for all $t \in [0,T]$, and that U is weakly Lipschitz continuous with respect to t in the sense that*

$$\left| \int_\Omega \big(U(t_2,x) - U(t_1,x)\big)\phi(x)\,dx \right| \le c_3(t_2 - t_1) \sum_{|\gamma|\le m} \|\partial^\gamma \phi\|_{L^\infty(\Omega)}, \tag{3.74}$$

for all $\phi \in C_c^m(\Omega)$, $0 \le t_1 \le t_2 \le T$, where we have used the multi-indice notation $\gamma = (\gamma_1, \dots, \gamma_d)$, $|\gamma| = \gamma_1 + \dots + \gamma_d$, $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}$. Then there exits a constant C, depending in particular on c_1, c_2 and c_3, such that the following L^1 time Hölder continuity holds:

$$\big\|U(t_2,\cdot) - U(t_1,\cdot)\big\|_{L^1(\Omega)} \le C(t_2 - t_1)^{1/(m+1)}, \quad 0 \le t_1 \le t_2 \le T. \tag{3.75}$$

Together with (3.68) we consider an initial condition

$$U(t,x)|_{t=0} = U_0(x), \qquad x \in \mathbb{R}^d, \tag{3.76}$$

with

$$U_0 \in L^\infty \cap L^1(\mathbb{R}^d). \tag{3.77}$$

We then consider the scheme (3.13) with

$$U_I^0 = \frac{1}{|R_I|} \int_{R_I} U_0(x)\,dx, \qquad I = (I_1, \dots, I_d) \in \mathbb{Z}^d, \tag{3.78}$$

where

$$R_I := \Pi_{\alpha=1}^d ((I_\alpha - 1/2)\Delta x_\alpha, (I_\alpha + 1/2)\Delta x_\alpha). \tag{3.79}$$

All the above discussion on the application of the scheme (3.13) to the scalar equation (3.68) leads to the following result.

Theorem 3.4. *Consider the scalar equation* (3.68) *with the symmetry condition* (3.69)*, and assume there exist* $\tilde{\Phi}_{\alpha\beta}$ *satisfying* (3.9)-(3.11)*, where the coefficients* $\sigma^{\alpha\beta} = \Delta t/(\Delta x_\alpha \Delta x_\beta)$ *are chosen so that* (3.12) *is satisfied, and* $\mu^{\alpha\beta} = \tilde{\mu}^{\alpha\beta} \equiv 1$. *Let the scheme* (3.13) *be applied to* (3.78)-(3.79)*, and define*

$$U^{\Delta t}(t,x) := \frac{t_{n+1}-t}{\Delta t} U_I^n + \frac{t-t_n}{\Delta t} U_I^{n+1}, \qquad t \in [t_n, t_{n+1}),\ x \in R_I,\ I \in \mathbb{Z}^d. \tag{3.80}$$

Then, as $\Delta t, \Delta x_\alpha \to 0$, $\alpha = 1, \dots, d$, *we have up to a subsequence* $U^{\Delta t}(t,x) \to U(t,x)$ *in* $C([0,T], L^1_{loc}(\mathbb{R}^d))$ *for any* $T > 0$. *The limit* $U(t,x)$ *is a weak entropy solution to* (3.68), (3.76)*, that is, it satisfies* (3.68) *in the sense of distributions,* $U(t,\cdot) \to U_0(\cdot)$ *in* $L^1_{loc}(\mathbb{R}^d)$ *as* $t \to 0$*, and it satisfies the entropy inequalities*

$$\partial_t \left(\eta(U)\right) - \sum_{\alpha,\beta} \partial^2_{\alpha\beta} \left(q_{\alpha\beta}(U)\right) \le 0, \tag{3.81}$$

for any η *convex and* C^2*, with* $q'_{\alpha\beta} = \eta' \Phi'_{\alpha\beta}$.

Proof. Because of Proposition 3.1 (3), it suffices to consider the case where $U_0 \in BV(\mathbb{R}^d)$. Indeed, any $U_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ may be arbitrarily approximated in $L^1(\mathbb{R}^d)$ by functions $U_0^n \in BV(\mathbb{R}^d)$, and thus the corresponding $U^{n,\Delta t}$ will be arbitrarily close to $U^{\Delta t}$. Therefore, let us assume $U_0 \in BV(\mathbb{R}^d)$. Then, (4) of Proposition 3.1 implies that $TV_x[U^{\Delta t}(t,\cdot)] \le TV[U_0]$, for all $t > 0$. We observe that (3.13) allows to obtain (3.74) with $m = 2$. Hence, Lemma 3.3 implies that $U^{\Delta t}$ satisfies (3.75), with $m = 2$ and C independent of Δt. We therefore conclude the compactness of $U^{\Delta t}$ in $C([0,T], L^1_{loc}(\mathbb{R}^d))$, and after extraction of a subsequence, $U^{\Delta t}(t,x) \to U(t,x)$ in $C([0,T], L^1_{loc}(\mathbb{R}^d))$ for any $T > 0$. Then, given $\phi \in C_0^\infty((0,\infty) \times \mathbb{R}^d)$, let

$$\phi_I^n := \frac{1}{\Delta t |R_I|} \int_{(t_n, t_{n+1}) \times R_I} \phi(t,x)\, dt\, dx. \tag{3.82}$$

As usual, multiplying (3.13) by ϕ_I^n, summing for $I \in \mathbb{Z}^d$, $n \in \mathbb{N}$, dividing by Δt, applying summation by parts, we obtain

$$\sum_{I,n} U_I^n \frac{\phi_I^n - \phi_I^{n-1}}{\Delta t} + \sum_{I,n} \sum_{\alpha} \Phi_{\alpha\alpha}(U_I^n) \frac{\Delta_\alpha^+ \Delta_\alpha^- \phi_I^n}{(\Delta x_\alpha)^2} + \sum_{I,n} \sum_{\alpha \ne \beta} \Phi_{\alpha\beta}(U_I^n) \frac{\Delta_\alpha^0 \Delta_\beta^0 \phi_I^n}{4 \Delta x_\alpha \Delta x_\beta}$$
$$+ \sum_{I,n} \sum_{\alpha \ne \beta} \tilde{\Phi}_{\alpha\beta}(U_I^n) \frac{\Delta_\alpha^+ \Delta_\alpha^- \Delta_\beta^+ \Delta_\beta^- \phi_I^n}{4 \Delta x_\alpha \Delta x_\beta} = 0.$$

Multiplying the above equation by $\Delta t \Delta x_1 \dots \Delta x_d$, using obvious approximations for ϕ, and denoting $h = \max_\alpha \Delta x_\alpha$, we obtain

$$\int_{(0,\infty) \times \mathbb{R}^d} \{U^{\Delta t} \phi_t + \sum_{\alpha,\beta} \Phi_{\alpha\beta}(U^{\Delta t}) \partial^2_{\alpha\beta} \phi\}\, dt\, dx + o(\Delta t) + o(h) = 0, \tag{3.83}$$

and passing to the limit this proves that $U(t,x)$ satisfies (3.68) in the sense of distributions. For the entropy inequality, we apply Theorem 3.2, and the discrete entropy inequality (3.17) enables to conclude as above, using a test function $\phi \in C_0^\infty((0,\infty) \times \mathbb{R}^d)$, $\phi \ge 0$. ☐

Remark 3.4. It is not known if the entropy inequalities (3.81) ensure the uniqueness of the solution. However, with further entropy dissipation estimates, uniqueness is known, see [3], [10]. But here it is not clear if such entropy dissipation estimates can be obtained.

4. Implicit schemes for systems in nonconservative form

In this section we consider systems in the nonconservative form

$$\partial_t U = \sum_{\alpha,\beta} B^{\alpha\beta}(U)\partial^2_{\alpha\beta}U + H(U,\nabla_x U), \tag{4.1}$$

where $\mathcal{U}$ is a convex open subset of $\mathbb{R}^N$, $B^{\alpha\beta}:\mathcal{U}\to\mathbb{M}_{N\times N}$ and $H:\mathcal{U}\times\mathbb{M}_{N\times d}\to\mathbb{R}^N$ are smooth maps, and $\mathbb{M}_{s\times s'}$ denotes the space of $s\times s'$ real matrices. In the interesting cases the system (4.1) is parabolic, maybe degenerate. Following the general philosophy of the present paper, we are interested in prescribing a finite difference approximation for (4.1) which preserves invariant domains that are present at the continuous level. Therefore, we assume that $G:\mathcal{U}\to\mathbb{R}$ is a function of class C^3 such that $G'(U)\neq 0$, for all $U\in\mathcal{U}$, and $\Omega_r=\{U\in\mathcal{U}\,:\,G(U)<r\}$ is convex for all $r\in\mathbb{R}$. We also assume that

$$G'(U)B^{\alpha\beta}(U) = p_{\alpha\beta}(U)G'(U), \qquad \forall U\in\mathcal{U}, \tag{4.2}$$

for some smooth scalar functions $p_{\alpha\beta}$. It is known (see [4, 11]) that (4.2) together with

$$\sum_{\alpha\beta} p_{\alpha\beta}(U)\xi_\alpha\xi_\beta \geq 0, \qquad \forall U\in\mathcal{U},\ \xi\in\mathbb{R}^d, \tag{4.3}$$

and

$$G'(U)H(U,M)\leq 0, \qquad \forall U\in\mathcal{U}, M\in\mathbb{M}_{N\times d} \text{ such that } G'(U)M=0, \tag{4.4}$$

imply that the regions $\overline{\Omega}_r$ are positively invariant for the flow generated by the system (4.1), as long as the solution takes its values in $\mathcal{U}$. We look for an analogous result at the discrete level.

We assume that there exist smooth maps $\tilde{B}^{\alpha\beta}:\mathcal{U}\to\mathbb{M}_{N\times N}$, for $\alpha\neq\beta$, such that

$$G'(U)\tilde{B}^{\alpha\beta}(U) = \tilde{p}_{\alpha\beta}(U)G'(U), \qquad \forall U\in\mathcal{U}, \tag{4.5}$$

and

$$\pm p_{\alpha\beta}(U)\leq \tilde{p}_{\alpha\beta}(U), \qquad \forall U\in\mathcal{U}. \tag{4.6}$$

Using still the notations (3.2) and (3.6), we propose the following second-order in space locally implicit finite difference approximation scheme for (4.1),

$$\begin{aligned} U_I^{n+1} = U_I^n &+ \sum_\alpha \sigma^{\alpha\alpha}B^{\alpha\alpha}(U_I^{n+1})\Delta_\alpha^+\Delta_\alpha^- U_I^n \\ &+ \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}B^{\alpha\beta}(U_I^{n+1})\Delta_\alpha^0\Delta_\beta^0 U_I^n + \Delta t\, H\left(U_I^{n+1},\frac{\Delta^0 U_I^n}{2\Delta x}\right) \\ &+ \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \frac{\sigma^{\alpha\beta}}{4}\tilde{B}^{\alpha\beta}(U_I^{n+1})\Delta_\alpha^+\Delta_\alpha^-\Delta_\beta^+\Delta_\beta^- U_I^n, \end{aligned} \tag{4.7}$$

where $\Delta^0 U_I^n/2\Delta x \equiv (\Delta_\alpha^0 U_I^n/2\Delta x_\alpha)_{\alpha=1}^d$. On the onset, we assume that U_J^n, $J\in\mathbf{J}=\{I, I\pm e_\alpha, I\pm e_\alpha\pm e_\beta;\ \alpha,\beta=1,\dots,d,\alpha\neq\beta\}$, $\Delta t,\Delta x_\alpha$, $\alpha=1,\dots,d$, are such that there exists U_I^{n+1} satisfying (4.7), and we also need the following CFL condition,

$$\sum_\alpha 2\sigma^{\alpha\alpha}p_{\alpha\alpha}(U) < 1 + \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \sigma^{\alpha\beta}\tilde p_{\alpha\beta}(U), \qquad \forall U\in\mathcal{U}. \tag{4.8}$$

Instead of (4.4) we impose the following stronger condition. We assume that there exist smooth functions $K_\alpha:\mathcal{U}\times\mathbb{M}_{N\times d}\to\mathbb{R}$, $\alpha=1,\dots,d$, such that

$$G'(U)H(U,M)\le \sum_\alpha K_\alpha(U,M)G'(U)M_\alpha, \qquad \forall M=(M_\alpha)_{\alpha=1}^d\in\mathbb{M}_{N\times d}, \forall U\in\mathcal{U}. \tag{4.9}$$

The following result shows that under the above conditions, the scheme (4.7) preserves domains that are (locally) invariant for (4.1).

Theorem 4.1. *Let $\mathcal{U}$, G, Ω_r, $r\in\mathbb{R}$, be as above and assume that conditions* (4.2), (4.5)-(4.9) *hold. Further, assume that*

$$\frac12\Delta x_\alpha\left|K_\alpha\left(U,\frac{\Delta^0U_I^n}{2\Delta x}\right)\right|\le p_{\alpha\alpha}(U)-\sum_{\beta\neq\alpha}\frac{\Delta x_\alpha}{\Delta x_\beta}\,\frac{\tilde p_{\alpha\beta}(U)+\tilde p_{\beta\alpha}(U)}{2}, \qquad \forall\alpha,\ \forall U\in\mathcal{U}. \tag{4.10}$$

Then $U_I^{n+1}\in\overline{\Omega}_r$ provided $U_J^n\in\overline{\Omega}_r$ for all $J\in\mathbf{J}$.

Proof. We argue by contradiction. Thus, assume the statement is false, that is, $U_J^n\in\overline{\Omega}_r$ for all $J\in\mathbf{J}$, but $G(U_I^{n+1})>r$. Denote $\omega=U_I^{n+1}$,

$$\phi_J=G'(\omega)(U_J^n-\omega), \qquad J\in\mathbf{J},$$

and

$$M=\frac{\Delta^0U_I^n}{2\Delta x}.$$

Multiplying (4.7) by $G'(\omega)$ and using (4.2) and (4.5), we obtain

$$\begin{aligned}0=\phi_I&+\sum_\alpha\sigma^{\alpha\alpha}p_{\alpha\alpha}(\omega)\Delta_\alpha^+\Delta_\alpha^-\phi_I+\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\sigma^{\alpha\beta}}{4}p_{\alpha\beta}(\omega)\Delta_\alpha^0\Delta_\beta^0\phi_I\\ &+\Delta t\,G'(\omega)H(\omega,M)+\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\sigma^{\alpha\beta}}{4}\tilde p_{\alpha\beta}(\omega)\Delta_\alpha^+\Delta_\alpha^-\Delta_\beta^+\Delta_\beta^-\phi_I,\end{aligned}$$

which together with (4.9) gives

$$\begin{aligned}0\le\phi_I&+\sum_\alpha\sigma^{\alpha\alpha}p_{\alpha\alpha}(\omega)\Delta_\alpha^+\Delta_\alpha^-\phi_I+\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\sigma^{\alpha\beta}}{4}p_{\alpha\beta}(\omega)\Delta_\alpha^0\Delta_\beta^0\phi_I\\ &+\Delta t\sum_\alpha K_\alpha(\omega,M)\frac{\Delta_\alpha^0\phi_I}{2\Delta x_\alpha}+\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\sigma^{\alpha\beta}}{4}\tilde p_{\alpha\beta}(\omega)\Delta_\alpha^+\Delta_\alpha^-\Delta_\beta^+\Delta_\beta^-\phi_I\\ &:=\sum_{J\in\mathbf{J}}c_J\phi_J.\end{aligned}$$

Now, we show that $c_I > 0$, while $c_J \geq 0$ for all other $J \in \mathbf{J}$. Indeed, we have

$$c_I = 1 + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sigma^{\alpha\beta}\tilde{p}_{\alpha\beta}(\omega) - \sum_{\alpha} 2\sigma^{\alpha\alpha}p_{\alpha\alpha}(\omega),$$

thus $c_I > 0$ by (4.8). For $c_{I\pm e_\alpha}$ we have

$$c_{I\pm e_\alpha} = \sigma^{\alpha\alpha}p_{\alpha\alpha}(\omega) - \sum_{\beta\neq\alpha} \frac{\sigma^{\alpha\beta}}{2}\left(\tilde{p}_{\alpha\beta}(\omega) + \tilde{p}_{\beta\alpha}(\omega)\right) \pm \frac{\Delta t}{2\Delta x_\alpha} K_\alpha(\omega, M),$$

and then $c_{I\pm e_\alpha} \geq 0$ by (4.10). Finally, we have for $\alpha \neq \beta$

$$c_{I+e_\alpha+e_\beta} = c_{I-e_\alpha-e_\beta} = \frac{\sigma^{\alpha\beta}}{4}(p_{\alpha\beta}(\omega) + p_{\beta\alpha}(\omega) + \tilde{p}_{\alpha\beta}(\omega) + \tilde{p}_{\beta\alpha}(\omega)),$$

$$c_{I-e_\alpha+e_\beta} = c_{I+e_\alpha-e_\beta} = \frac{\sigma^{\alpha\beta}}{4}(-p_{\alpha\beta}(\omega) - p_{\beta\alpha}(\omega) + \tilde{p}_{\alpha\beta}(\omega) + \tilde{p}_{\beta\alpha}(\omega)),$$

and hence, by (4.6), $c_{I\pm e_\alpha \pm e_\beta} \geq 0$. Since $\phi_J < 0$ for all $J \in \mathbf{J}$, we have a contradiction and the proof is finished. □

Remark 4.1. Similarly as in Lemma 3.1, under the conditions of Theorem 4.1, the degenerate parabolicity property (4.3) holds. Indeed, we have

$$\begin{aligned}
\sum_{\alpha\beta} p_{\alpha\beta}\xi_\alpha\xi_\beta &= \sum_\alpha p_{\alpha\alpha}\xi_\alpha^2 + \sum_{\alpha\neq\beta} p_{\alpha\beta}\xi_\alpha\xi_\beta \\
&\geq \sum_\alpha p_{\alpha\alpha}\xi_\alpha^2 - \sum_{\alpha\neq\beta}\tilde{p}_{\alpha\beta}|\xi_\alpha\xi_\beta| \\
&\geq \sum_\alpha p_{\alpha\alpha}\xi_\alpha^2 - \frac{1}{2}\sum_{\alpha\neq\beta}\tilde{p}_{\alpha\beta}\left(\frac{\Delta x_\alpha}{\Delta x_\beta}\xi_\alpha^2 + \frac{\Delta x_\beta}{\Delta x_\alpha}\xi_\beta^2\right) \\
&= \sum_\alpha \xi_\alpha^2\left(p_{\alpha\alpha} - \sum_{\beta\neq\alpha}\frac{\Delta x_\alpha}{\Delta x_\beta}\frac{\tilde{p}_{\alpha\beta} + \tilde{p}_{\beta\alpha}}{2}\right) \geq 0,
\end{aligned}$$

where in the last inequality we used (4.10).

4.1. **Application to the Navier-Stokes equations.** We consider the system of Navier-Stokes equations (1.2)-(1.5) with $\kappa \equiv 0$, where we ommit the first-order terms on the left-hand side, and write the second-order terms in nonconservative form

$$\partial_t U = \sum_{\alpha\beta} B^{\alpha\beta}(U)\partial^2_{\alpha\beta}U + \sum_{\alpha\beta}(B^{\alpha\beta})'(U)(\partial_\alpha U, \partial_\beta U), \tag{4.11}$$

where $B^{\alpha\beta}(U)$ is given by (1.6). Thus we have (4.1) with H given by

$$H(U, M) = \sum_{\alpha\beta}(B^{\alpha\beta})'(U)(M_\alpha, M_\beta). \tag{4.12}$$

We define for $\alpha \neq \beta$

$$\tilde{A}_{\alpha\beta}(U) = \frac{\mu(U) + \lambda(U)}{2}(e_\alpha \otimes e_\alpha + e_\beta \otimes e_\beta), \tag{4.13}$$

and

$$\tilde{B}^{\alpha\beta}(U) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\tilde{A}_{\alpha\beta}u}{\rho} & \frac{\tilde{A}_{\alpha\beta}}{\rho} & 0 \\ -\frac{u^t\tilde{A}_{\alpha\beta}u}{\rho} & \frac{u^t\tilde{A}_{\alpha\beta}}{\rho} & 0 \end{pmatrix}. \tag{4.14}$$

We take $G = -S = -\ln(p^{1/\gamma}/\rho)$. The quasiconvexity of $-S$ is obvious since $-S < -S_0$ if and only if $-\rho S + \rho S_0 < 0$, and $\eta = -\rho S$ is convex, this is a well known fact (see [11]). As in Lemma 3.2 we have for all U

$$-S'(U)B^{\alpha\beta}(U) = 0, \qquad -S'(U)\tilde{B}^{\alpha\beta}(U) = 0, \tag{4.15}$$

thus conditions (4.2), (4.5), (4.6) are satisfied with $p_{\alpha\beta}(U) = \tilde{p}_{\alpha\beta}(U) \equiv 0$. Also, from (4.1) we have $\partial_t S(U) = S'(U)H(U, \nabla_x U)$, and identifying with (1.8) we get that

$$-S'(U)H(U, \nabla_x U) = -(1 - \frac{1}{\gamma})\frac{1}{p}\sum_{\alpha\beta} A_{\alpha\beta}\partial_\beta u \partial_\alpha u \le 0, \tag{4.16}$$

hence $-S'(U)H(U, M) \le 0$ for all U and M. Therefore the condition (4.9) is also satisfied with $K_\alpha(U, M) \equiv 0$. Conditions (4.8) and (4.10) are then trivially satisfied. As a consequence we can apply Theorem 4.1, and we deduce that as soon as we can find $U_I^{n+1} \in \mathcal{U}$, the domains $S \ge r$ are locally invariant by the scheme (4.7) for the Navier-Stokes system (4.11), (1.6), (1.5).

We remark that in the above discussion we did not use the definition (4.13), and we could have taken instead $\tilde{A}_{\alpha\beta} = 0$. It may lead to the incorrect conclusion that, in the particular case of the Navier-Stokes system, correctors are not really necessary. In fact, the above analysis does not take into account the global stability (like entropy inequality), which is necessary in practice. Such an analysis can be performed by linearization. Let us consider the linearization of (4.11) around a constant state U_0, namely,

$$\partial_t U = \sum_{\alpha\beta} B^{\alpha\beta}(U_0)\partial^2_{\alpha\beta}U, \tag{4.17}$$

with $B^{\alpha\beta}(U_0)$ given by (1.6). Then, the linear part of the scheme (4.7), with $B^{\alpha\beta}(U_I^{n+1})$ and $\tilde{B}^{\alpha\beta}(U_I^{n+1})$ replaced by $B^{\alpha\beta}(U_0)$ and $\tilde{B}^{\alpha\beta}(U_0)$, corresponds to the scheme proposed in Subsection 3.2 for the linear system (4.17), with correctors $\tilde{B}^{\alpha\beta}(U_0)$ defined as above. Then, we see from the definitions and the remarks made in Subsection 3.3 that the choice (4.13)-(4.14) ensures that the stability condition (3.38) is satisfied, while (3.39) reads (whenever $\sigma^{\alpha\beta} = \sigma$)

$$\mu\,\mathrm{Id} + (\mu+\lambda)e_\alpha \otimes e_\alpha - \sum_{\beta\ne\alpha}\frac{\mu+\lambda}{2}(e_\alpha\otimes e_\alpha + e_\beta\otimes e_\beta) \ge 0, \tag{4.18}$$

i.e.

$$\frac{\mu-\lambda}{2}\,\mathrm{Id} + \frac{\mu+\lambda}{2}(4-d)e_\alpha\otimes e_\alpha \ge 0. \tag{4.19}$$

This it is satisfied if

$$d \le 4 \qquad \text{and} \qquad \lambda(U) \le \mu(U). \tag{4.20}$$

Under these assumptions, and if the CFL condition (3.40) is satisfied, the linearized scheme is L^2 stable for the scalar product defined by $\eta''(U_0)$, with $\eta(U) = -\rho S(U)$.

Acknowledgements

The research of H. Frid was partially supported by CNPq through the grants 352871/96-2, 479416/2001-0, and FAPERJ through the grant E-26/151.890/2000, E-26/152.192/2002.

The main part of this work was achieved while F. Bouchut was visiting IMPA in 2002, supported by the France-Brazil agreement for mathematical research.

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