Constant mean curvature hypersurfaces in warped product spaces

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Abstract

We study hypersurfaces of constant mean curvature immersed into warped product spaces of the form $\mathbb{R} \times_\varrho \mathbb{P}^n$, where $\mathbb{P}^n$ is a complete Riemannian manifold. In particular, our study includes that of constant mean curvature hypersurfaces in product ambient spaces, that have been extensively studied recently. It also includes constant mean curvature hypersurfaces in the so called *pseudo-hyperbolic* spaces. If the hypersurface is compact, we show that the immersion must be a leaf of the trivial totally umbilical foliation $t \in \mathbb{R} \mapsto \{t\} \times \mathbb{P}^n$, generalizing previous results by Montiel [11]. We also extend a result of Guan and Spruck [7] from hyperbolic ambient space to the general situation of warped products. This extension allows us to give a slightly more general version of a result by Montiel [11], and to derive height estimates for compact constant mean curvature hypersurfaces with boundary in a leaf.

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1 Introduction

It is a classical result that a compact hypersurface embedded in Euclidean space with constant mean curvature must be a round sphere. Alexandrov [1] gave a proof of this fact by making a clever use of the maximum principle for elliptic partial differential equations. The now called Alexandrov's reflexion method works as well for hypersurfaces in Euclidean sphere and hyperbolic space, since its main

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requirement of having a large number of isometric reflexions is satisfied in such ambient spaces.

An attempt to extend the above result from constant sectional curvature manifolds to a larger class of Riemannian spaces should consider manifolds with a plenty of complete embedded constant mean curvature hypersurfaces. Such hypersurfaces play the role of the umbilical ones in spaces of constant sectional curvature. Then, one looks for geometric conditions on an immersed complete constant mean curvature hypersurface that force it to be one of those already classified. In space forms, one proves such classification results using the plenty of isometries of the space. Since here we consider more general ambient manifolds, one needs to develop an appropriate method of proof.

Montiel [11] observed that a natural class of manifolds to consider is that of warped products $M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$ where $\mathbb{P}^n$ is a complete n-dimensional Riemannian manifold, $\varrho : \mathbb{R} \to \mathbb{R}_+$ is a smooth function and the product manifold $\mathbb{R} \times \mathbb{P}^n$ is endowed with the complete Riemannian metric

$$\langle\ ,\ \rangle = \pi_{\mathbb{R}}^*(dt^2) + \varrho^2(\pi_{\mathbb{R}})\pi_{\mathbb{P}}^*(\langle\ ,\ \rangle_{\mathbb{P}}).$$

Here $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}}$ denote the projections onto the corresponding factor and $\langle\ ,\ \rangle_{\mathbb{P}}$ is the Riemannian metric on $\mathbb{P}^n$. Each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ (called here a *slice*) of the foliation $t \in \mathbb{R} \mapsto \mathbb{P}_t$ of M^{n+1} by complete hypersurfaces has constant mean curvature. Its mean curvature vector field is

$$\vec{H}_t = -\mathcal{H}(t)T$$

where $\mathcal{H}(t) = \varrho'(t)/\varrho(t)$ and $T = \partial/\partial t \in TM$. For further geometric inside, observe that $\mathcal{T} = \varrho T$ is a *closed conformal vector field* on M^{n+1}, that is, it satisfies

$$\bar{\nabla}_V \mathcal{T} = \varrho' V \quad \text{for any } \ V \in TM. \tag{1}$$

Here and elsewhere $\bar{\nabla}$ stands for the Levi-Civita connection in M^{n+1} and, by abuse of notation, we denote in the same way functions on $\mathbb{R}$ and their lift to M^{n+1}. In [11, §3] it is carefully shown that any Riemannian manifold M^{n+1} with a closed conformal vector field is locally isometric to a warped product manifold with one-dimensional factor. Furthermore, the isometry is global if M^{n+1} is complete and simply connected.

Extending the well-known *Mercator projection*, used in cartography to conformally project the two-dimensional sphere into the Euclidean plane [16, p.173] (see [13] for the hyperbolic case), we conformally transform the warped product space $\mathbb{R} \times_\varrho \mathbb{P}^n$ into a product space with factor $\mathbb{P}^n$. In fact, let τ: $\mathbb{R} \times \mathbb{P}^n \to \mathbb{J} \times \mathbb{P}^n$ be given by $\tau(t,x) = (s(t),x)$ where $\mathbb{J} = s(\mathbb{R})$ and

$$s(t) = s_0 - \int_0^t (1/\varrho(u))du.$$

Then τ is a reversing orientation isometry between M^{n+1} and $\mathbb{J} \times \mathbb{P}^n$ endowed with the conformal metric

$$\langle \, , \, \rangle = \lambda^2(s) \left(ds^2 + \langle \, , \, \rangle_{\mathbb{P}^n} \right), \tag{2}$$

where the conformal factor is $\lambda(s) = \varrho(t(s))$. Suppose that $\varrho(t)$ satisfies

$$\int_0^{+\infty} \frac{1}{\varrho} < +\infty \quad \text{and} \quad \int_{-\infty}^{0} \frac{1}{\varrho} = +\infty, \tag{3}$$

and take $s_0 = \int_0^{+\infty}(1/\varrho)$. Then, we have that $\mathbb{J} = \mathbb{R}_+$ and, therefore, $\mathbb{P}^n$ acts as a *boundary at infinite* of $\mathbb{R} \times_\varrho \mathbb{P}^n$, as does $\{0\} \times \mathbb{R}^n$ in $\mathbb{H}^{n+1}$, and the leaves $\mathbb{P}_t$ can be thought as horospheres in a fixed direction of $\mathbb{H}^{n+1}$.

There are two cases (after normalization) in which *all* slices have the *same* constant mean curvature $\mathcal{H}$. The first one is when $\mathcal{H}(t) = 0$ ($\varrho(t) = 1$), and the ambient space is just a Riemannian product $M^{n+1} = \mathbb{R} \times \mathbb{P}^n$. Constant mean curvature hypersurfaces in these spaces have been extensively studied in recent years. The second case is when $\mathcal{H}(t) = 1$ (either $\varrho(t) = e^t$ or $\varrho(t) = \cosh t$), and M^{n+1} belongs to the class of *pseudo-hyperbolic* manifolds defined in [17]. When $\varrho(t) = e^t$, the conformal factor in (2) is $\lambda(s) = 1/s$ and (3) is satisfied. Moreover, if $\mathbb{P}^n$ is Ricci flat then M^{n+1} is Einstein with negative Ricci curvature, and if $\mathbb{P}^n$ is flat then M^{n+1} is a negatively curved space form. Thus, for $\varrho(t) = e^t$ we deal with ambient spaces that have many ressemblaces with hyperbolic space $\mathbb{H}^{n+1}$.

Montiel's method of proof in [11] combines the use of two Minkowski-type formulas. In Corollary 7 he gives the following (Montiel's $1^{\underline{\text{st}}}$ result).

> *Let $\mathbb{P}^n$ be a compact manifold satisfying* $\mathrm{Ric}_{\mathbb{P}} > \sup_{\mathbb{R}}\{-\varrho^2\mathcal{H}'(t)\}$. *Then any compact orientable immersed constant mean curvature hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ that is locally a graph over $\mathbb{P}^n$ must be a slice.*

This result has the following consequences (Corollary 8) for the class of pseudo-hyperbolic ambient spaces (Montiel's $2^{\underline{\text{nd}}}$ result).

> (a) *Let $\mathbb{P}^n$ be compact with non-negative Ricci curvature. Then any compact constant mean curvature hypersurface in $\mathbb{R} \times_{e^t} \mathbb{P}^n$ that is locally a graph on $\mathbb{P}^n$ must be a slice.*
>
> (b) *Let $\mathbb{P}^n$ be compact with Ricci curvature satisfying* $\mathrm{Ric}_{\mathbb{P}} \geq -1$. *Then any compact constant mean curvature hypersurface in $\mathbb{R} \times_{\cosh t} \mathbb{P}^n$ that is locally a graph on $\mathbb{P}^n$ must be a slice.*

In Section 2 we compute the Laplacian of $\sigma \circ h \in \mathcal{C}^\infty(\Sigma)$, where h is the height function of an immersed hypersurface $\Sigma^n \mapsto \mathbb{R} \times_\varrho \mathbb{P}^n$ and $\sigma \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\sigma'(t) = \varrho(t)$. This yields a rather simple differential equation that has several applications for compact hypersurfaces. In particular, it allows a generalization of Montiel's $2^{\underline{\text{nd}}}$ result, for instance, by removing the assumption on the Ricci curvature. We also consider the case of complete hypersurfaces via the Omori-Yau maximum principle.

In Section 3 we extend a result of Guan and Spruck [7] from hyperbolic ambient space to the general situation studied in this paper. Such extension allows a slight generalization of Montiel's $1^{\underline{\text{st}}}$ result. Then we use our result to provide height estimates for compact constant mean curvature hypersurfaces with boundary contained in a slice of either a product or a pseudo-hyperbolic ambient space; thus extending results in [7] and [9]. Further applications for graphs with boundary are given in [2].

2 The $1^{\underline{\text{st}}}$ equation

In this section we compute a basic partial differential equation whose strength is based on its independence of the curvature tensor of the ambient space. We then derive several consequences, in particular, a generalization of Montiel's $1^{\underline{\text{st}}}$ result.

Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be an isometric immersion of an n-dimensional Riemannian manifold Σ^n; its *height function* $h \in \mathcal{C}^\infty(\Sigma)$ is defined as $h = \pi_\mathbb{R} \circ f$, where $\pi_\mathbb{R}$ denotes the projections onto the first factor.

Proposition 1 *Let $f\colon \Sigma^n \to M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$ be an isometric immersion with mean curvature vector field $\vec{H}$. If $\sigma(t) = \int_{t_0}^t \varrho(r)dr$, then*

$$\Delta\sigma(h) = n\varrho(h)(\mathcal{H}(h) + \langle \vec{H}, T\rangle) \tag{4}$$

where $\mathcal{H}(t) = \varrho'(t)/\varrho(t)$ and $T = \partial/\partial t \in TM$.

Proof: The gradient of $\pi_\mathbb{R} \in \mathcal{C}^\infty(M)$ is $\bar{\nabla}\pi_\mathbb{R} = T$, and thus the gradient of h is

$$\nabla h = (\bar{\nabla}\pi_\mathbb{R})^\top = T - \langle T, N\rangle N, \tag{5}$$

where by $(\ )^\top$ we mean taking the tangential component of a vector field along f and N is a (local) smooth unit normal vector field. It is a standard fact that the Levi-Civita connections of a warped product satisfies

$$\bar{\nabla}_V T = \mathcal{H}(V - \langle V, T\rangle T) \quad \text{for any } \ V \in TM. \tag{6}$$

It follows from (5) and (6) that

$$\bar{\nabla}_X \nabla h = \mathcal{H}(h)(X - \langle X, T\rangle T) - X(\langle T, N\rangle)N + \langle T, N\rangle AX \tag{7}$$

for any $X \in T\Sigma$. Here $AX = -\bar{\nabla}_X N$ denotes the second fundamental form of f with respect to N. Then, we get

$$\nabla_X \nabla h = (\bar{\nabla}_X \nabla h)^\top = \mathcal{H}(h)(X - \langle X, \nabla h\rangle \nabla h) + \langle T, N\rangle AX, \tag{8}$$

where ∇ stands for the Levi-Civita connection in Σ^n. It follows from here that the Laplacian of h is given by

$$\Delta h = \mathcal{H}(h)(n - \|\nabla h\|^2) + n\langle \vec{H}, T\rangle. \tag{9}$$

Since $\nabla\sigma(h) = \varrho(h)\nabla h$, we have that

$$\Delta\sigma(h) = \varrho(h)\Delta h + \varrho'(h)\|\nabla h\|^2 = n\varrho(h)(\mathcal{H}(h) + \langle \vec{H}, T\rangle),$$

and this concludes the proof. ∎

We first analyze the case of compact hypersurfaces (without boundary). Our first result is mostly technical because of the assumption on the immersion itself.

Proposition 2 *Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact hypersurface such that either*

$$\|\vec{H}\| \leq \mathcal{H} \circ h \quad \textit{or} \quad \|\vec{H}\| \leq -\mathcal{H} \circ h \tag{10}$$

holds along Σ^n. Then $\mathbb{P}^n$ is compact and $f(\Sigma^n)$ is a slice.

Proof: At any point of Σ^n we have by Cauchy-Schwarz that

$$\mathcal{H}(h) - \|\vec{H}\| \leq \mathcal{H}(h) + \langle \vec{H}, T\rangle \leq \mathcal{H}(h) + \|\vec{H}\|.$$

By assumption the function $\mathcal{H}(h) + \langle \vec{H}, T\rangle$ does not change sign. It follows from (4) that $\Delta\sigma(h)$ does not change sign either. But being Σ^n compact, the divergence theorem gives $\Delta\sigma(h) = 0$, and hence $\sigma(h)$ must be constant (that is, any subharmonic or superharmonic function on a compact Riemannian manifold without boundary must be constant; this property is used several times in the paper). Since $\sigma'(t) = \varrho(t) > 0$, we conclude that h itself must be constant. ∎

Notice that (10) implies that the function $\mathcal{H} \circ h \in \mathcal{C}^\infty(\Sigma^n)$ does not change sign, and says just that in the minimal case. It is thus natural (and convenient) to assume that $\mathcal{H} \in \mathcal{C}^\infty(\mathbb{R})$ does not change sign, instead of involving the immersion f in the hypothesis. Geometrically, the fact that $\mathcal{H}$ does not change sign means that the mean curvature vectors of *all* slices $\mathbb{P}_t$ point in the same direction.

Next corollary of Proposition 2 states the analogous in $\mathbb{R} \times_\varrho \mathbb{P}^n$ of the non-existence of compact hypersurfaces that are either minimal in $\mathbb{R}^{n+1}$ and $\mathbb{R} \times \mathbb{H}^n$ or with mean curvature function $0 \leq H \leq 1$ in $\mathbb{H}^{n+1}$. The case $\mathcal{H}(t) \leq 0$ can be reduced to the case $\mathcal{H}(t) \geq 0$ by changing the orientation of the factor $\mathbb{R}$.

Proposition 3 *Assume $\mathcal{H}(t) \geq 0$ and set $\mathcal{H}_0 = \inf_{\mathbb{R}} \mathcal{H}(t)$. A compact hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ with mean curvature function $0 \leq H \leq \mathcal{H}_0$ only occurs if $\mathbb{P}^n$ is compact and, then, it is any slice $\mathbb{P}_{t_0}$ where $\mathcal{H}(t_0) = \mathcal{H}_0$.*

Proof: From Proposition 2 we have that $\mathbb{P}^n$ must be compact and that $H = \mathcal{H}(t_0)$ for some $t_0 \in \mathbb{R}$. Thus $H = \mathcal{H}_0$ by assumption. ∎

A submanifold $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ is called *two-sided* if its normal bundle is trivial, i.e., there is a globally defined unit normal vector field. For instance, every hypersurface with non-zero constant mean curvature is trivially two-sided. Then, we define the smooth *angle function* $\Theta\colon \Sigma^n \to [-1, 1]$ by

$$\Theta(p) = \langle N(p), T \rangle$$

where N denotes the global normal field.

If f is locally a graph over $\mathbb{P}^n$ (i.e., transversal to T) then either $\Theta < 0$ or $\Theta > 0$ along Σ^n. Thus, asking Θ not to change sign is a weaker assumption than being a local graph. Notice that $\Theta^2 = 1$ if and only if Σ^n is a slice (see (11) below).

From now on, every time the angle function of a two-sided hypersurface does not change sign, then the orientation N is chosen so that $\Theta \leq 0$, and then the mean curvature function is $H = \langle \vec{H}, N \rangle$.

Montiel observed that if $\mathbb{P}^n$ is compact and if the mean curvature of the slices is non decreasing ($\mathcal{H}'(t) \geq 0$) then any compact constant mean curvature graph over $\mathbb{P}^n$ must be a slice (see [11, Remark 6]). To see this, he compares the hypersurface with slices and then invokes the maximum principle. The following theorem generalizes such result as well as Corollary 8 in [11] (Montiel's $2^{\underline{\mathbf{nd}}}$ result).

Theorem 4 *Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature H. Assume that $\mathcal{H}'(t) \geq 0$ and that the angle function Θ does not change sign. Then $\mathbb{P}^n$ is compact and $f(\Sigma^n)$ is a slice.*

Proof: Let $p_{\min}, p_{\max} \in \Sigma^n$ be such that

$$h(p_{\min}) = \underline{h} := \min_{\Sigma} h \quad \text{and} \quad h(p_{\max}) = \bar{h} := \max_{\Sigma} h.$$

Therefore, $\nabla h(p_{\min}) = 0$ and $\nabla h(p_{\max}) = 0$. We have from (5) that

$$\|\nabla h\|^2 = 1 - \Theta^2, \tag{11}$$

and therefore

$$\Theta(p_{\min}) = \pm 1 \quad \text{and} \quad \Theta(p_{\max}) = \pm 1. \tag{12}$$

Moreover, (9) gives

$$\begin{cases} \Delta h(p_{\min}) = n(\mathcal{H}(\underline{h}) + \langle \vec{H}(p_{\min}), T\rangle) \geq 0 \\ \\ \Delta h(p_{\max}) = n(\mathcal{H}(\bar{h}) + \langle \vec{H}(p_{\max}), T\rangle) \leq 0. \end{cases}$$

Hence,

$$-\langle \vec{H}(p_{\min}), T\rangle \leq \mathcal{H}(\underline{h}) \quad \text{and} \quad \mathcal{H}(\bar{h}) \leq -\langle \vec{H}(p_{\max}), T\rangle. \tag{13}$$

Before we proceed, for later use notice that the proof of (13) only uses that Σ^n is compact. From (12), (13) and $\mathcal{H}' \geq 0$, we obtain

$$-\Theta(p_{\min})\, H(p_{\min}) \leq \mathcal{H}(\underline{h}) \leq \mathcal{H}(\bar{h}) \leq -\Theta(p_{\max})\, H(p_{\max}).$$

By assumption $\Theta(p_{\min}) = \Theta(p_{\max}) = \operatorname{sign}\Theta$, and hence

$$-H \operatorname{sign}\Theta \leq \mathcal{H}(\underline{h}) \leq \mathcal{H}(\bar{h}) \leq -H \operatorname{sign}\Theta.$$

It follows that $\mathcal{H} \circ h = -H \operatorname{sign}\Theta$. We obtain from (4) that

$$\Delta\sigma(h) = n\varrho(h)H(\Theta - \operatorname{sign}\Theta),$$

and thus $\Delta(\sigma \circ h)$ does not change sign. Therefore, $\sigma \circ h$ and hence h itself must be constant. ∎

We have two useful corollaries of the proof of Theorem 4. For instance, dropping the assumption that $\mathcal{H}' \geq 0$ we still have the following result.

Proposition 5 *Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided hypersurface such that Θ does not change sign. Then, we have*

$$\min_\Sigma H \leq \mathcal{H}(\min_\Sigma h) \quad \text{and} \quad \max_\Sigma H \geq \mathcal{H}(\max_\Sigma h).$$

Proof: We have that $\Theta(p_{\min}) = \Theta(p_{\max}) = -1$ since we are always choosing $\Theta \leq 0$. It follows from (13) that

$$\min_\Sigma H \leq H(p_{\min}) \leq \mathcal{H}(\min_\Sigma h) \quad \text{and} \quad \mathcal{H}(\max_\Sigma h) \leq H(p_{\max}) \leq \max_\Sigma H,$$

and this concludes the proof. ∎

Our second result is for minimal immersions.

Proposition 6 *Assume that $\varrho''(t) \geq 0$. A compact minimal hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ only occurs if $\mathbb{P}^n$ is compact and then it is any slice $\mathbb{P}_{t_0}$ where $\mathcal{H}(t_0) = 0$.*

Proof: From (13) we have $\varrho'(\bar{h}) \leq 0 \leq \varrho'(\underline{h})$. Then $\varrho''(t) \geq 0$ yields $\varrho' \circ h = 0$, and hence $\mathcal{H} \circ h = 0$. The proof follows from Proposition 2. ∎

To extend the preceding results from compact to complete submanifolds we use the following well-known Omori-Yau maximum principle [18].

Lemma 7 *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in \mathcal{C}^\infty(M)$ is bounded from below, then there exists a sequence of points $\{p_j\} \in M$ such that*

$$\lim_{j\to\infty} u(p_j) = \inf_M u, \quad \|\nabla u(p_j)\| < 1/j \quad \textit{and} \quad \Delta u(p_j) > -1/j.$$

Remark 8 The Omori-Yau maximum principle (thus our next result) holds under the weaker assumption (see [3])

$$\mathrm{Ric}_M \geq -C(1 + r^2 \log^2(r+2))$$

where r is the distance function in M to a fixed point and C a positive constant.

Next Theorem is the analogous of Theorem 4 for complete hypersurfaces contained in a slab.

Theorem 9 *Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a two sided complete hypersurface of constant mean curvature H, with Ricci curvature bounded from below and*

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n$$

where $t_1, t_2 \in \mathbb{R}$ are finite. Assume that $\mathcal{H}'(t) > 0$ almost everywhere and that the angle function Θ does not change sign. Then $f(\Sigma^n)$ is a slice.

Proof: By Lemma 7 using (9) and (11) there exists a sequence $\{p_j\} \in \Sigma^n$ such that

$$\lim_{j\to\infty} h(p_j) = \underline{h} := \inf h > -\infty, \qquad \|\nabla h(p_j)\|^2 = 1 - \Theta^2(p_j) < (1/j)^2$$

and

$$\Delta h(p_j) = \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) + nH(p_j)\Theta(p_j) > -1/j.$$

The last equation gives

$$-nH(p_j)\Theta(p_j) < 1/j + \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2).$$

Since $\lim_{j\to\infty}\Theta(p_j) = \operatorname{sign}\Theta$ by the second equation, it follows that

$$-\operatorname{sign}\Theta \lim_{j\to+\infty} H(p_j) \leq \mathcal{H}(\underline{h}). \tag{14}$$

Similarly, applying Lemma 7 to $-h$ yields a sequence $\{q_j\} \in \Sigma^n$ such that

$$\mathcal{H}(\bar{h}) \leq -\operatorname{sign}\Theta \lim_{j\to+\infty} H(q_j) \tag{15}$$

where $\bar{h} := \sup h < \infty$. We obtain from (14), (15) and our assumptions that

$$-\operatorname{sign}\Theta H \leq \mathcal{H}(\underline{h}) \leq \mathcal{H}(\bar{h}) \leq -\operatorname{sign}\Theta H,$$

and since $\mathcal{H}'(t) > 0$ almost everywhere, we conclude that $\underline{h} = \bar{h}$. ∎

For complete hypersurfaces we have the following version of Proposition 5.

Proposition 10 *Let $f\colon \Sigma^n \to \mathbb{R}\times_\varrho \mathbb{P}^n$ be a two-sided complete hypersurface with Ricci curvature bounded from below and contained in a slab. Assume that the angle function Θ does not change sign. Then,*

$$\inf_\Sigma H \leq \mathcal{H}(\inf_\Sigma h) \quad \text{and} \quad \sup_\Sigma H \geq \mathcal{H}(\sup_\Sigma h).$$

Proof: Since $\operatorname{sign}\Theta = -1$, we obtain from (14) and (15) that

$$\inf_\Sigma H \leq \lim_{j\to+\infty} H(p_j) \leq \mathcal{H}(\inf_\Sigma h) \quad \text{and} \quad \mathcal{H}(\sup_\Sigma h) \leq \lim_{j\to+\infty} H(q_j) \leq \sup_\Sigma H,$$

and this concludes the proof. ∎

Any function $u \in \mathcal{C}^\infty(\mathbb{P})$ determines an entire graph $\Gamma(u)$ over $\mathbb{P}^n$ by the map $f_u\colon \mathbb{P}^n \to \mathbb{R}\times\mathbb{P}^n$ defined as $f_u(q) = (u(q), q)$. It is a standard fact that the equation for the mean curvature function H of $\Gamma(u)$ is

$$\operatorname{Div} \frac{Du}{\sqrt{1+\|Du\|^2}} = -nH, \tag{16}$$

where Du denotes the gradient of $u \in \mathcal{C}^\infty(\mathbb{P})$ and Div the divergence on $\mathbb{P}^n$. If $\mathbb{P}^n$ is compact, it follows easily from (16) that any entire graph in $\mathbb{R}\times\mathbb{P}^n$ whose mean curvature H does not change sign is necessarily minimal. As $\mathcal{H} = 0$, from (9) it follows that the height function u is harmonic on the compact $\Gamma(u)$, and thus the graph must be a slice.

Extending a result due to Heinz ($n = 2$) it was proved independently by Chern [4] and Flanders [5] that any entire graph in Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal. A beautiful argument due to Salavessa [15] shows that, for a complete non-compact $\mathbb{P}^n$, an entire graph in $\mathbb{R}\times\mathbb{P}^n$ with constant mean curvature H is minimal provided that the Cheeger constant $\mathfrak{h}(\mathbb{P})$ of $\mathbb{P}^n$ vanishes. To see this, recall that

$$\mathfrak{h}(\mathbb{P}) = \inf_D \frac{\text{area}\,(\partial D)}{\text{area}\,(D)}$$

where $D\subset\mathbb{P}^n$ is any compact domain with smooth boundary. Integrating (16) over D and using the divergence theorem, we obtain

$$n\,\text{area}\,(D)\min_D H \le n\int_D H dA_{\mathbb{P}} = \oint_{\partial D}\frac{\langle Du,\nu\rangle}{\sqrt{1+\|Du\|^2}}ds \le \text{area}\,(\partial D)$$

and, similarly, $n\,\text{area}\,(D)\max_D \ge -\text{area}\,(\partial D)$. We thus have,

$$\inf_{\mathbb{P}} H \le \frac{1}{n}\frac{\text{area}\,(\partial D)}{\text{area}\,(D)} \quad \text{and} \quad \sup_{\mathbb{P}} H \ge -\frac{1}{n}\frac{\text{area}\,(\partial D)}{\text{area}\,(D)},$$

and hence,

$$\inf_{\mathbb{P}} H \le \frac{1}{n}\mathfrak{h}\,(\mathbb{P}) \quad \text{and} \quad \sup_{\mathbb{P}} H \ge -\frac{1}{n}\mathfrak{h}\,(\mathbb{P}).$$

In particular, when $\mathfrak{h}(\mathbb{P})=0$ we obtain $\inf_{\mathbb{P}} H \le 0 \le \sup_{\mathbb{P}} H$. Then, if H is constant it must vanish.

As a consequence of Proposition 10 we have the following result for graphs in Riemannian products.

Corollary 11 *Let $\Gamma(u)$ be an entire graph over $\mathbb{P}^n$ determined by $u\in\mathcal{C}^\infty(\mathbb{P})$. If u is bounded and if the Ricci curvature of $\Gamma(u)$ is bounded from below then the mean curvature function of the graph satisfies*

$$\inf_\Gamma H \le 0 \le \sup_\Gamma H.$$

In particular, if H is constant then the graph must be minimal.

For other results of this type see Corollary in [14, p. 445] and Theorem 2 in [8]. Examples of entire graphs in the product space $\mathbb{R}\times\mathbb{H}^2$ of constant mean curvature $H\in(0,\frac{1}{2}]$ with u bounded only on one side where given in [12].

To conclude this section we consider the case of parabolic submanifold, where by *parabolic* we mean that any subharmonic function on the submanifold, bounded from above, must be constant.

Proposition 12 *Let* $f\colon \Sigma^n \to \mathbb{R}\times_\varrho \mathbb{P}^n$ *be an isometric immersion. Assume that* Σ^n *is parabolic and that either*

(i) $h \le \bar{h} < +\infty$ *and* $\|\vec{H}\| \le \mathcal{H}\circ h$, *or*

(ii) $h \ge \underline{h} > -\infty$ *and* $\|\vec{H}\| \le -\mathcal{H}\circ h$.

Then $f(\Sigma^n)$ *is a slice.*

Proof: In case (i) using (4) we have $\Delta\sigma(h) \ge 0$, and the proof follows since Σ^n is parabolic and $\sigma(h) \le \sigma(\bar{h})$. Case (ii) is analogous using $-\sigma$. ∎

3 The $2^{\underline{\text{nd}}}$ equation

We already reached several conclusions for hypersurfaces whose mean curvature is smaller than the mean curvature of slices. To get rid of this restriction, we assume a bound on the normalized Ricci curvature of $\mathbb{P}^n$ and we introduce a partial differential equation coming from the Codazzi equation.

The following result extends Theorems 1.2 and 2.2 in [7] proved for hypersurfaces in hyperbolic space (we will explain in which sense in Remark 15).

Given a two-sided hypersurface $f\colon \Sigma^n \to \mathbb{R}\times_\varrho \mathbb{P}^n$, we fix an orientation N and define $\phi \in \mathcal{C}^\infty(\Sigma^n)$ by

$$\phi = \sigma(h)H + \varrho(h)\Theta. \tag{17}$$

Theorem 13 *Let* $f\colon \Sigma^n \to \mathbb{R}\times_\varrho \mathbb{P}^n$ *be a two-sided hypersurface of constant mean curvature. If the angle function* Θ *does not change sign and the Ricci curvature of* $\mathbb{P}^n$ *satisfies*

$$\mathrm{Ric}_{\mathbb{P}} \ge \sup_{\mathbb{R}}\{-\varrho^2\mathcal{H}'(t)\}, \tag{18}$$

then ϕ *is subharmonic.*

Proof: The Codazzi equation of $f\colon \Sigma^n \to \mathbb{R}\times_\varrho \mathbb{P}^n$ is

$$(\bar{R}(X,Y)N)^\top = (\nabla_Y A)X - (\nabla_X A)Y, \tag{19}$$

where $\bar{R}$ denotes the curvature tensor of $\mathbb{R}\times_\varrho \mathbb{P}^n$. It follows from (1) that

$$\nabla\langle N, \mathcal{T}\rangle = -A(\mathcal{T}^\top) = -\varrho(h)A(\nabla h), \tag{20}$$

where $\mathcal{T} = \varrho T$. Therefore, using (8) and (19) we conclude from (20) that

$$\begin{aligned}\nabla_X \nabla \langle N, \mathcal{T}\rangle &= -\varrho'(h)\langle X, \nabla h\rangle A\nabla h - \varrho(h)(\nabla_X A)(\nabla h) - \varrho(h)A(\nabla_X \nabla h)\\ &= -\varrho(h)(\nabla_{\nabla h} A)X - \varrho(h)(\bar{R}(\nabla h, X)N)^{\top} - \varrho'(h)AX - \langle N, \mathcal{T}\rangle A^2 X.\end{aligned}$$

Let $\overline{\mathrm{Ric}}$ be the Ricci tensor of $M^{n+1} = \mathbb{R}\times_\varrho \mathbb{P}^n$. Using $\mathrm{tr}(\nabla_Z A) = \langle \nabla\, \mathrm{tr} A, Z\rangle$, we obtain

$$\Delta\langle N, \mathcal{T}\rangle = -n\varrho(h)\langle \nabla H, \nabla h\rangle + \varrho(h)\,\overline{\mathrm{Ric}}(N, \nabla h) - n\varrho'(h)H - \langle N, \mathcal{T}\rangle \|A\|^2. \quad (21)$$

The curvature tensor of M^{n+1} expressed in terms of the curvature tensor of $\mathbb{P}^n$ is

$$\begin{aligned}\bar{R}(U,V)W &= R_{\mathbb{P}}(\hat{U}, \hat{V})\hat{W} - \mathcal{H}^2(\langle V, W\rangle U - \langle U, W\rangle V)\\ &\quad + \mathcal{H}'\langle W, T\rangle(\langle U, T\rangle V - \langle V, T\rangle U) - \mathcal{H}'(\langle V, W\rangle\langle U, T\rangle - \langle U, W\rangle\langle V, T\rangle)T,\end{aligned}$$

where we denote $\hat{U} = \pi_{\mathbb{P}*}U$. Then, the Ricci tensor of M^{n+1} can be given in terms of the Ricci tensor of $\mathbb{P}^n$, namely,

$$\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}_{\mathbb{P}}(\hat{V}, \hat{W}) - (n\mathcal{H}^2 + \mathcal{H}')\langle V, W\rangle - (n-1)\mathcal{H}'\langle V, T\rangle\langle W, T\rangle. \quad (22)$$

Thus,

$$\overline{\mathrm{Ric}}(N, X) = \mathrm{Ric}_{\mathbb{P}}(\hat{N}, \hat{X}) - (n-1)\mathcal{H}'(h)\Theta\langle X, \nabla h\rangle$$

for any $X \in T\Sigma$. Since $T = \nabla h + \Theta N$, then $(\nabla h)^* = -\Theta N^*$, where $(\ \ )^*$ means taking the $\mathbb{P}^n$-component of a vector field in TM. Thus,

$$\overline{\mathrm{Ric}}(N, \nabla h) = -(n-1)\Theta\left(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2\right) \quad (23)$$

where, as usual, $(n-1)\mathrm{Ric}_{\mathbb{P}}(\,\cdot\,) = \mathrm{Ric}_{\mathbb{P}}(\,\cdot\,,\,\cdot\,)$. Since the mean curvature H is constant, we conclude from (4), (21), (23) and $\varrho(h)\Theta = \langle N, \mathcal{T}\rangle$ that

$$\Delta\phi = -\varrho(h)\Theta\left\{\|A\|^2 - nH^2 + (n-1)\left(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2\right)\right\}. \quad (24)$$

From (18) and $\|\hat{N}\|^2_{\mathbb{P}} = \varrho^{-2}(h)\|\nabla h\|^2$ we obtain

$$\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2 \geq 0, \quad (25)$$

and the proof follows using that $\|A\|^2 \geq nH^2$. ∎

Remark 14 It follows easily from (22) that (18) is equivalent to

$$\overline{\mathrm{Ric}}(X) \geq \overline{\mathrm{Ric}}(T) \quad \text{for all} \quad X \in TM.$$

In other words, the direction T must be of least Ricci curvature.

Remark 15 Given a vertical graph over $\mathbb{R}^n$ in $\mathbb{H}^{n+1}$ with constant mean curvature, the result in [7, §2] asserts that the mean curvature function computed with respect to the underlying Euclidean metric is subharmonic. To see that the preceding result extends the one in [7] we consider the case of pseudo-hyperbolic ambient spaces $\mathbb{R}\times_{e^t}\mathbb{P}^n$. Then (18) reduces to $\mathrm{Ric}_{\mathbb{P}}\geq 0$ (this holds for $\mathbb{H}^{n+1}$ since $\mathbb{P}^n=\mathbb{R}^n$) and the subharmonic function ϕ takes the simple form

$$\phi = e^h(H+\Theta). \tag{26}$$

It turns out that ϕ is also the mean curvature of the hypersurface when computed in the product metric of $\mathbb{R}_+\times\mathbb{P}^n$. In fact, a straightforward computation yields that the mean curvature function $\hat{H}$ of $\hat{f}=\tau\circ f\colon \Sigma^n\to\mathbb{J}\times\mathbb{P}^n$ is $\hat{H}=\varrho(h)H+\varrho'(h)\Theta$. Then observe that $\hat{H}=\phi$ if and only if $\varrho(t)=e^t$.

Montiel's 1st result [11, Corollary 7], in our case is an easy consequence of our Theorem 13.

Theorem 16 *Let $f\colon\Sigma^n\to M^{n+1}=\mathbb{R}\times_\varrho\mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature. Assume that (18) holds and that the angle function Θ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$ or M^{n+1} has constant sectional curvature and Σ^n is a geodesic hypersphere. The latter case cannot occur if we assume that the inequality in (18) is strict.*

Proof: We know by Theorem 13 that ϕ is a subharmonic function on Σ^n, but being Σ^n compact that implies that ϕ is constant. Then (24) gives

$$\Theta\left(\|A\|^2-nH^2+(n-1)(\mathrm{Ric}_{\mathbb{P}}(\hat{N})+\mathcal{H}'(h)\|\nabla h\|^2)\right)=0. \tag{27}$$

We claim that $\mathcal{U}=\{p\in\Sigma^n:\Theta(p)=0\}$ has empty interior. To see this, assume on the contrary that $\mathcal{U}$ contains a non-empty open subset $\mathcal{V}$ of Σ^n. On $\mathcal{V}$ the function $\sigma(h)H=\phi$ is constant and, if $H\neq 0$, then $\sigma(h)$ and, equivalently, h is constant. But this is not possible, since $\|\nabla h\|^2=1-\Theta^2=1$ on $\mathcal{V}$. Therefore, it must be $H=0$, and then $\varrho(h)\Theta=\phi$ is constant on Σ^n. Since it vanishes on $\mathcal{V}$ it must vanish on all of Σ^n. Hence $\mathcal{U}=\Sigma^n$, but this is not possible because $\Theta^2=1$ at least where h attains its extrema. Summing up, $\mathcal{U}$ has empty interior. Then (27) implies that

$$\|A\|^2-nH^2+(n-1)(\mathrm{Ric}_{\mathbb{P}}(\hat{N})+\mathcal{H}'(h)\|\nabla h\|^2)=0,$$

that is,

$$\|A\|^2-nH^2=0 \tag{28}$$

and

$$\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2 = 0. \tag{29}$$

Equality (28) means that f is totally umbilical. Moreover, we observe that Montiel's reasoning in his proof of Corollary 7 in [11] also applies here, and allows us to conclude that the case where f is totally umbilical (but not a slice) can only occur if M^{n+1} has constant sectional curvature and Σ^n is a geodesic hypersphere.

Finally, when inequality in (18) is strict, then (29) is equivalent to $\hat{N}(p) = 0$ at any $p \in \Sigma^n$, that is, $\nabla h = 0$ on Σ^n, and hence $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$. ∎

One can use Theorem 13 to obtain height estimates for constant mean curvature hypersurfaces with (nonempty) boundary contained in a slice. Next result is an extension of the height estimates for vertical graphs in $\mathbb{R} \times \mathbb{P}^2$ in [9].

Theorem 17 *Let $f\colon \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a compact hypersurface of constant mean curvature $H > 0$ and nonempty boundary $\partial\Sigma^n \subset \mathbb{P}_0$. Assume*

$$\mathrm{Ric}_{\mathbb{P}} \geq \frac{n}{n-1}\alpha \quad \textit{for some} \quad \alpha \leq 0,$$

$\Theta \leq 0$ and $H^2 \geq |\alpha|$. Then, we have $f(\Sigma^n) \subset \left[0, \frac{H}{H^2 - |\alpha|}\right] \times \mathbb{P}^n$.

Proof: By assumption, $(n-1)\mathrm{Ric}_{\mathbb{P}}(\hat{N}) \geq n\alpha\|\hat{N}\|^2_{\mathbb{P}} \geq n\alpha$ because of $\alpha \leq 0$ and $\|\hat{N}\|^2_{\mathbb{P}} = \|\nabla h\|^2 \leq 1$. Consider $\psi \in \mathcal{C}^\infty(\Sigma^n)$ defined as

$$\psi = \phi + \frac{\alpha}{H}h = \frac{H^2 - |\alpha|}{H}h + \Theta.$$

Using (9) and (24), we have

$$\begin{aligned} \Delta\psi &= -\Theta(\|A\|^2 - nH^2 + (n-1)\mathrm{Ric}_{\mathbb{P}}(\hat{N}) - n\alpha) \\ &\geq -\Theta((n-1)\mathrm{Ric}_{\mathbb{P}}(\hat{N}) - n\alpha), \end{aligned}$$

and thus ψ is subharmonic on Σ^n. The maximum principles yields

$$\frac{H^2 - |\alpha|}{H}h - 1 \leq \frac{H^2 - |\alpha|}{H}h + \Theta = \psi \leq \max_{\partial\Sigma}\psi = \max_{\partial\Sigma}\Theta \leq 0,$$

and hence $0 \leq h \leq H/(H^2 - |\alpha|)$. ∎

For our next result we first recall a well known tangency principle. Let Σ_1^n and Σ_2^n be two hypersurfaces in an arbitrary Riemannian manifold N^{n+1} that are tangent at a common point p_0. Fix a normal vector η_0 at p_0 and locally parametrize both hypersurfaces in a neighborhood U of zero in $T_{p_0}\Sigma_1 = T_{p_0}\Sigma_2$ by means of the exponential map of N^{n+1} as follows:

$$\varphi_j(x) = \exp_{p_0}(x + \mu_j(x)\eta_0), \quad j = 1, 2,$$

where $\mu_j \in \mathcal{C}^\infty(U)$ are well determined functions satisfying $\mu_j(0) = 0$. One says that Σ_1^n lies above Σ_2^n in a neighborhood of p_0 if $\mu_1(x) \geq \mu_2(x)$ in a neighborhood of zero. This is equivalent to require that the geodesics of N^{n+1} normal to the hypersurface $\exp_{p_0}(U)$ in a neighborhood of p_0 in the orientation determined by η_0 intercept Σ_2^n before Σ_1^n. The following fact is well-known (cf. [6]).

> *Let Σ_1^n and Σ_2^n be hypersurfaces as above with constant mean curvature satisfying $H_{\Sigma_1} \leq H_{\Sigma_2}$ with respect to η_0. Then Σ_1^n and Σ_2^n coincide in a neighborhood of p_0.*

The following general result has no assumption on the curvature of $\mathbb{P}^n$ and is of independent interest.

Proposition 18 *Let $f\colon \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided constant mean curvature hypersurface with nonempty boundary $f(\partial\Sigma) \subset \mathbb{P}_\tau$, and whose angle function Θ does not change sign. Then, we have:*

(i) If $H \leq \inf_{[\tau,+\infty)} \mathcal{H}$ then $f(\Sigma^n) \subset (-\infty, \tau] \times \mathbb{P}^n$.

(ii) If $H \geq \sup_{(-\infty,\tau]} \mathcal{H}$ then $f(\Sigma^n) \subset [\tau, +\infty) \times \mathbb{P}^n$.

In particular, if $\mathcal{H}'(t) \geq 0$ and $H = \mathcal{H}(\tau)$ then $f(\Sigma^n) \subset \mathbb{P}_\tau$.

Proof: Assume $H \leq \inf_{[\tau,+\infty)} \mathcal{H}$ but that $h \leq \tau$ does not hold. Hence, we obtain

$$\max_\Sigma h = h(p_0) = \tau_0 > \tau$$

at some interior point p_0 of Σ^n. Take $\Sigma_1 = \Sigma^n$, $\Sigma_2 = \mathbb{P}_{\tau_0}$, and hence $\Sigma_1 \neq \Sigma_2$. Observe that Σ_1 and Σ_2 are tangent at the common point p_0, and that Σ_1 lies above Σ_2 with respect to the common normal $\eta_0 = -T$ at p_0. Since

$$H_{\Sigma_1} = H \leq \inf_{[\tau,+\infty)} \mathcal{H} \leq \mathcal{H}(\tau_0) = H_{\Sigma_2},$$

by the tangency principle we would get that Σ_1 and Σ_2 coincide in some open neighborhood of p_0. This is in contradiction to $\Sigma_1 \neq \Sigma_2$. The proof for the case $H \geq \sup_{(-\infty,\tau]} \mathcal{H}$ is similar. ∎

The following consequence of Proposition 18 extends Proposition 2.3 in [10].

Corollary 19 *Let $f\colon \Sigma^n \to \mathbb{R}\times_{e^t}\mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature with nonempty boundary $f(\partial\Sigma) \subset \mathbb{P}_\tau$, and whose angle function Θ does not change sign. Then, we have:*

1. *$H \leq 1$ if and only if $h \leq \tau$.*
2. *$H \geq 1$ if and only if $h \geq \tau$ on Σ^n.*

In particular, $H = 1$ if and only if $f(\Sigma^n) \subset \mathbb{P}_\tau$.

To conclude we extend Theorem 3.3 in [10], that applies to graphs in hyperbolic space $\mathbb{H}^{n+1}$, to graphs in pseudo-hyperbolic manifolds. For the case of pseudo-hyperbolic space with $\varrho(t) = e^t$ we have the following.

Theorem 20 *Let $f\colon \Sigma^n \to \mathbb{R}\times_{e^t}\mathbb{P}^n$ be a compact hypersurface of constant mean curvature $H \notin [0,1)$ and nonempty boundary $f(\partial\Sigma^n) \subset \mathbb{P}_\tau$. Assume $\mathrm{Ric}_{\mathbb{P}} \geq 0$ and that the angle function Θ does not change sign. Set $C = \log(H/H-1)$. Then, we have:*

(i) If $H < 0$, then $f(\Sigma^n) \subset [\tau + C, \tau] \times \mathbb{P}^n$.

(ii) If $H > 1$, then $f(\Sigma^n) \subset [\tau, \tau + C] \times \mathbb{P}^n$.

(iii) If $H = 1$ then $f(\Sigma^n) \subset \mathbb{P}_\tau$.

Proof: By the preceding result, we have $f(\Sigma^n) \subset (-\infty, \tau] \times \mathbb{P}^n$ if $H < 0$, that $f(\Sigma^n) \subset [\tau, +\infty) \times \mathbb{P}^n$ if $H > 1$, and that $f(\Sigma^n) \subset \mathbb{P}_\tau$ if $H = 1$. From the maximum principle applied to the subharmonic function ϕ given by (26), we obtain

$$e^h(H-1) \leq e^h(H+\Theta) \leq \max_{\partial\Sigma} e^h(H+\Theta) = e^\tau(H + \max_{\partial\Sigma}\Theta) \leq e^\tau H,$$

and the proof follows easily. ∎

Finally, for the case of pseudo-hyperbolic space with $\varrho(t) = \cosh t$ we obtain the following.

Theorem 21 *Let $f\colon \Sigma^n \to \mathbb{R}\times_{\cosh t}\mathbb{P}^n$ be a compact hypersurface of constant mean curvature H and nonempty boundary $f(\partial\Sigma^n) \subset \mathbb{P}_0$. Assume $\mathrm{Ric}_{\mathbb{P}} \geq -1$ and that the angle function Θ does not change sign. Set $\tanh C = 1/H$. Then, we have:*

(i) If $H < -1$, then $f(\Sigma^n) \subset [C, 0] \times \mathbb{P}^n$.

(ii) If $H > 1$, then $f(\Sigma^n) \subset [0, C] \times \mathbb{P}^n$.

(iii) If $H = 0$ *then* $f(\Sigma^n) \subset \mathbb{P}_0$*.*

Proof: By Proposition 18, we have $f(\Sigma^n) \subset (-\infty, 0] \times \mathbb{P}^n$ if $H < 0$, that $f(\Sigma^n) \subset [0, +\infty) \times \mathbb{P}^n$ if $H > 0$, and that $f(\Sigma^n) \subset \mathbb{P}_0$ if $H = 0$. Now $\sigma(t) = \sinh t$, and from the maximum principle applied to the subharmonic function ϕ given by (17), we obtain

$$H \sinh h - \cosh h \leq \phi \leq \max_{\partial\Sigma} \phi = \max_{\partial\Sigma} \Theta \leq 0,$$

that is, $H \tanh h \leq 1$. Then, when $H < -1$ this gives $\tanh h \geq 1/H$, and when $H > 1$ this yields $\tanh h \leq 1/H$. ∎

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