Revealing Incomplete Financial Markets*

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Abstract

Incomplete financial markets without arbitrage opportunities are characterized by the existence of multiple risk neutral probabilities. In this context, a cost function describes the minimum amount necessary for superhedging strategies and it can be recovered from the worst monetary expectation among the whole set of risk neutral probabilities. This paper characterizes the class of non-linear functions, called *incomplete prices*, that can be viewed as a cost function of frictionless financial markets without arbitrage opportunities in the two periods framework. First, we obtain some criteria that allow to know if a given function is actually an incomplete price. Interestingly, a new role for prices is given because we can recover the market struture from any incomplete price, clarifying the understanding about the interdependence between the market structure and the functional form of incomplete prices. For instance, a financial markets with a Riesz subspace of attanaible claims is in fact a "partition market" of bets and such markets are revealed by an incomplete price given by a Choquet integral with respect to a particular concave capacity.

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1 Introduction

Since the Arrow's *Role of Securities* seminal paper the theory of equilibrium for markets in which both spot commodities and securities are traded is the fundamental scope for the study of basic problems of economic theory such as equilibrium existence[1], asset pricing and so on. The general equilibrium model assumes that the price of assets satisfies *equilibrium conditions* in a setting where many agents demand assets profiles in accordance with their preferences and their endowments. It provides the main elements for the study of financial market models given by the set of basic securities and the respective price system. A fundamental result says that for an economy with financial markets satisfying mild conditions, at equilibrium, financial markets must not offer arbitrage opportunities for any agent[2]. For a two period economy it implies the impossibility at equilibrium to realize positive net financial returns in the second period without spending at the initial period some amount of money on the asset market.

The *principle of no-arbitrage* can be viewed as the central principle of modern finance because it is the key for the determination of the *value* of the assets. As is well-known, no-arbitrage principle and the assumption of complete markets[3] enforce linear pricing rule: the cost of replication of any asset is given by the mathematical expectation of his payoffs under the unique risk neutral probability obtained by no-arbitrage principle. On the other hand, market incompleteness says that not all securities can be replicated by feasible portfolios on the market. Equivalently, while in a complete market every asset can be hedged perfectly, in the incomplete market case it is possible to stay on the safe side for many cases only by *superhedging strategies*, *i.e.*, with a portfolio strategy which generates payoffs across the states that are at least as large as the underlying contingent claim. A fundamental condition for incomplete markets without arbitrage opportunities is the *existence of multiple risk neutral probabilities.*

A *cost function* describes the minimum value necessary for the replication or a superreplication of any contingent claim, and a corresponding strategy is referred to as a *minimum-cost superhedging strategy.* An essential fact for the determination of cost functions is that the standard linear approach fails for any non attainable claim[4]. In this sense, a very known result says that the

[1] Arrow (1953) proposed this approach for the presence of a complete securities markets and used the results from Arrow and Debreu (1954) as well as McKenzie (1954) for the existence of equilibrium. However, as is widely accepted, incomplete market is a more natural and intuitive hypothesis (Magill and Quinzii (1996) and Magill and Shafer (1991) are basics references for general equilibrium analysis of incomplete markets, there it is possible to find the list of main contributions for incomplete markets theory. Föllmer and Shied (2004) provided a treatment of basics results in incomplete markets following the lines of finance theory).

[2] See, for instance, Florenzano (1999), page 18.

[3] Recall that a financial market is complete if the trading of basic assets reproduce any financial payoff, otherwise the financial market is incomplete.

[4] Some results show that this medothological problem is typical for some importants classes of assets, for example, a well known result from the work of Ross (1976) says that whenever the payoff of every *call* or *put option* can be replicated, the securities market must be complete.

set of risk-neutral probabilities plays an important role for the determination of a cost function: in fact, in presence of a fair risk-free security the cost of any contingent claim can be determinated by his maximum expected value with respect to all risk neutral probabilities. Hence cost functions satisfy conditions obtained by some characterizations existing in the literature (*e.g.*, Huber (1981), Gilboa and Schmeidler (1989), Chateauneuf (1991)). However, these properties are not sufficient for the characterizations of cost functions, for instance, as we will see in the Example 49 the epsilon-contaminated functions can never be a cost function of some market.

This paper characterizes the class of non-linear functions, called *incomplete prices*, that can be viewed as a cost functions of a frictionless financial market without arbitrage opportunities in the two periods framework. First, we obtain some criteria that allow to know if a given function is actually an incomplete price. More precisely, we consider the class $\mathcal{H}_0$ of all non-linear functions on the space of claims that is consistent with the characterization as commented below. Such functions C induces two interesting class of assets, namely, the set of unambiguous assets

$$F_C := \{Y : C(Y) + C(-Y) = 0\},$$

and the class of nonwasteful assets

$$L_C := \{Y : X > Y \Rightarrow C(X) > C(Y)\}.$$

We show that a function $C \in \mathcal{H}_0$ is an incomplete price if and only if $F_C = L_C$, so an incomplete price C must belong to $\mathcal{H}_0$ with the additional requirement that $F_C = L_C$. In a sense, such price rules fails to provide a precise information about the valuation of many claims. In fact, for any incomplete price the corresponding underlying market is characterized by a set of attainable claims given by the set of unambiguous assets which, by our characterization, is the same as the set of nonwasteful assets.

Interestingly, a new role for prices is given because we can recover the market struture from any incomplete price, clarifying the understanding about the interdependence between the market structure and the functional form of incomplete prices. For instance, a financial market with a Riesz subspace of attanaible claims is in fact a "partition market" of bets and such markets are revealed by an incomplete price given by a Choquet integral with respect to a particular concave capacity.

Also, Aliprantis and Tourky (2002) showed that if the number of securities is less than half the number of states of the world, then generically we have the absence of perfect replication of *any* option. Hence, the approach of finding the value of an option by reference to the prices of the primitive securities breaks down for any option. In another way, Baptista (2007) showed that (generically) if every *risk binary contingent claim* is non attainable then every option is non attainable.

2 Framework and Basic Results

Let $S = \{s_1, ..., s_n\}$ be a finite set of states of nature. At date one, one and only one state s will occur, and an asset $X \in \mathbb{R}^S$ bought at date $t = 0$ will deliver payoff $X(s)$ at date 1 if s occurs.

We assume that at date 0 agents can trade a finite number of assets $X_j \in \mathbb{R}^S$, $0 \le j \le m$, with respective prices q_j. Also, we suppose that[5]

$$X_0 = S^* := (1, ..., 1) \in \mathbb{R}^S$$

is the *riskless bond* and for sake of simplicity we suppose that $q_0 = 1$. A portfolio of an agent is identified with a vector $\theta = (\theta_0, \theta_1, ...\theta_m) \in \mathbb{R}^{m+1}$, where θ_j denotes the quantities of asset X_j possessed by the agent.

We recall that an arbitrage opportunity is a portfolio strategy with no cost that yields a strictly positive profit in some states and exposes no loss risk. The existence of such an arbitrage opportunity may be viewed as a kind of market inefficiency. The following definition establishes the basic properties of prices for efficient financial markets[6]:

Definition 1 *The market $\mathcal{M} = (X_j, q_j;\ 0 \le j \le m)$ is assumed to offer* no-arbitrage opportunity *(NAO) if for any portfolio $\theta \in \mathbb{R}^{m+1}$,*

$$\begin{aligned} \sum_{j=0}^{m} \theta_j X_j &> 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j > 0, \\ \sum_{j=0}^{m} \theta_j X_j &= 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j = 0. \end{aligned}$$

Denote by $F := \mathit{span}\,(X_0, X_1, ..., X_m)$ the *subspace of income transfers* or the *set of attainable claims*. Let 2^S be the field of all subsets of S and Δ the set of all probability measures on $(S, 2^S)$. A well known property says that[7]:

Remark 2 *The market $\mathcal{M} = (X_j, q_j;\ 0 \le j \le m)$ offers no arbitrage opportunity if and only if there exists a strictly positive probabilitity*[8] *$P_0 \in \Delta$ such that $E_{P_0}(X_j) = q_j$, $0 \le j \le m$.*

Since, in general, the probability measure P_0 above is not uniquely determined an important definition follows as:

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[5] For any $A \subset S$, we will denote by A^* the characteristic function of the *event* A:

$$\begin{aligned} A^* &: S \to \{0, 1\} \\ s &\in A^*(s) = 1 \text{ iff } s \in A. \end{aligned}$$

[6] We use the following notation: For $X \in \mathbb{R}^S$, $X > 0$ means that $X \ge 0$ (i.e., $X(s) \ge 0$ for any $s \in S$) and $X \ne 0$.

[7] A nice reference for the well know results used here is the chapter 1 of Föllmer and Schied (2004).

[8] Note that P_0 strictly positive means that $P_0(\{s\}) > 0$ for any $s \in S$. We are denoting $E_P(X)$ as the integral of the random variable X w.r.t. the probabilitiy P.

Definition 3 *The set*

$$\mathcal{Q} = \{P \in \Delta : E_P(X_j) = q_j, \forall j \in \{0, ..., m\}\},$$

is called the set of risk-neutral probabilities (or martingal measures)*.*

Note that the set $\mathcal{Q}$ of all risk-neutral probabilities describes the family of all probability measures that agree about the value of all basic assets. Remark 2 is known as the *fundamental theorem of asset pricing* and it says that $\mathcal{Q}$ is nonempty if and only if *NAO* is true.

As it is usual, the market $\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m)$ is complete if every claim $Y \in \mathbb{R}^S$ is attanaible, *i.e.*, $F = \mathbb{R}^S$. Otherwise, we say that the market $\mathcal{M}$ is incomplete. A basic fact says that, if the prices of basic securities satisfies the NAO property then completeness of financial market is equivalent to the equality $\mathcal{Q} = \{P_0\}$, where P_0 turns out to be the unique probability measure satisfying conditions of Remark 2.

Given a market $\mathcal{M} = (X_j, q_j, 0 \leq j \leq m)$, a simple and important result follows as:

Lemma 4 *Consider a market $\mathcal{M} = (X_j, q_j, 0 \leq j \leq m)$ without arbitrage oportunity, a claim $X \in F$ if and only if $E_P(X) = E_Q(X)$ for any $P, Q \in \mathcal{Q}$.*

Lemma 4 says that a claim is attanaible if and only if it satisfies the law of one price, *i.e.*, every risk neutral probability agrees about its monetary value.

3 Cost Functions of Incomplete Markets

Market incompleteness says that not all securities can be replicated by feasible portfolios on the market, or equivalently, while in a complete market every asset can be hedged perfectly, in the incomplete market case it is possible to stay on the safe side for many cases only by *superhedging strategies*, *i.e.*, with a portfolio strategy which generates payoffs across the states that are at least as large as the underlying contingent claim: Consider a non attainable claim X, a *superhedging strategy* or *superreplication* of X[9] is any portfolio $\theta \in \mathbb{R}^{m+1}$ such that

$$\sum_{j=0}^{m} \theta_j X_j \geq X.$$

So, it is natural to view the cost of a non attainable claim as the lowest possible price of a superreplication of X. Summing up, we obtain the following definition:

Definition 5 *For any claim $X \in \mathbb{R}^S$, the cost of X is given by*

$$C(X) = inf \left\{ \sum_j \theta_j q_j : \sum_j \theta_j X_j \geq X \right\}.$$

[9] For instance, the existence of superhedging strategies for any non attanaible claim follows from the existence of the riskless bond.

Moreover, C will be called the cost function of the market $\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m)$.

Remark 6 *It is worth noticing that under NAO assumption the cost of any attanaible claim X trivially writes:*

$$C(X) = \sum_{j=0}^{m} \theta_j q_j,$$

for any portfolio $\theta = (\theta_0, \theta_1, ...\theta_m)$ such that $X = \sum_{j=0}^{m} \theta_j X_j$. Moreover, the NAO condition says that C is a strictly positive functional on F, in fact, following the notation of Remark 2, note that

$$C(X) = E_{P_0}(X), \text{ for any } X \in F.$$

The set of risk-neutral probabilities $\mathcal{Q}$ plays an important role for the determination of a cost function and it is a well-known property that can be enunciated as:

Remark 7 *For a market $\mathcal{M}$ offering no-arbitrage opportunity, the cost function satisfies, for any $X \in \mathbb{R}^S$:*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

From Remark 7 a cost function is necessarily the maximum of expectations with respect to a given family of probabilities for which characterizations exist in the literature (*e.g.*, Huber (1981), Gilboa and Schmeidler (1989) and Chateauneuf (1991)). Based on these characterizations, C must be[10]:

1. subadditive, *i.e.*,
$$C(X+Y) \leq C(X) + C(Y), \forall X, Y \in \mathbb{R}^S;$$
2. Positively affinely homogeneous, *i.e.*,
$$C(\alpha X + kS^*) = \alpha C(X) + k, \forall X \in \mathbb{R}^S, \forall k \in \mathbb{R}, \forall \alpha \geq 0;$$
3. Monotone, *i.e.*,
$$X \geq Y \Rightarrow C(X) \geq C(Y), \forall X, Y \in \mathbb{R}^S.$$

Remark 8 *As is well-known, any function with these three properties is Lipschitz continuous*[11]*.*

[10] The same characterization is the key for the representation of coherent risk measures as introduced by Artzner et al. (1999).

[11] In fact, we should to consider the supnorm $X \mapsto \|X\|_\infty := \max_{s \in S} |X(s)|$ and we have

$$|C(X) - C(Y)| \leq \|X - Y\|_\infty, \text{ for all } X, Y \in \mathbb{R}^S.$$

However, these conditions are not sufficient for the characterizations of cost functions as will be clear in the next sections[12].

Building on the well-known properties discussed in Remarks 2 and 7, a Lemma about cost functions is naturrally derived:

Lemma 9 *The mapping $C : X \in \mathbb{R}^S \to C(X) \in \mathbb{R}$ is the cost function of a frictionless financial market of securities without arbitrage opportunities if:*

1) *There exist $X_0,\ X_1, ...,\ X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 such that: $E_{P_0}(X_j) = C(X_j)$, $0 \leq j \leq m$;*

2) *Denoting, $\mathcal{Q} := \{P \in \Delta : E_P(X_j) = C(X_j),\ 0 \leq j \leq m\}$, then $\forall X \in \mathbb{R}^S$:*

$$C(X) = \max_{P\in\mathcal{Q}} E_P(X).$$

So, in this case C is the cost function of the market $\mathcal{M} = \{X_j, q_j\}_{j=0}^m$, where $q_j := E_{P_0}(X_j)$.

4 Incomplete Markets from Incomplete Prices

4.1 Incomplete Prices: Definition and an Auxiliary Result

We denote by $\mathcal{H}_0$ the family of all *subadditive, positively affinely homogeneus* and *monotone* functions $C : \mathbb{R}^S \to \mathbb{R}$. Also, we denote by $\mathcal{H}$ the family of all function in $\mathcal{H}_0$ that are *strictly positive*. The class $\mathcal{H}_0$ describes the natural candidate to be a cost function of a frictionless market with securities without arbitrage opportunity. So, next we introduce our terminology for functions that in fact are cost functions of some incomplete market.

Definition 10 *We say that a function $C : \mathbb{R}^S \to \mathbb{R}$ is an incomplete price if C is a cost function induced from some incomplete market $\mathcal{M} = (X_j, q_j, 0 \leq j \leq m)$.*

Remark 11 *We note that if $C \in \mathcal{H}$ is linear then it is trivial that C is a cost function of the complete market $\mathcal{M} = (X_j, q_j, 0 \leq j \leq m)$, where span $\{X_j\}_{j=0}^m = \mathbb{R}^S$ and $q_j = C(X_j)$.*

Our main goal is to give a full characterization of incomplete prices and to describe certain classes of incomplete prices related to some specific types of incompleteness of financial markets. Some natural questions follows as:

- How to recognise that a particular function is an incomplete price?
- How to derive the underlying market from a given incomplete price?
- What special properties of incomplete prices could recover some particular and important market strutures? (e.g., Arrow markets of securities with the riskless bond).

[12] As we will see, by the no-arbitrage principle, a cost function must be *strictly positive*: $X > 0 \Rightarrow C(X) > 0$. However, adding this condition to the classical conditions mentioned above we still have a set of necessary but not sufficient conditions.

Remark 12 *We know that* $C \in \mathcal{H}_0$ *if and only if there exists a nonempty, closed and convex set set* $\mathcal{K} \subset \Delta$ *such that for any* $X \in \mathbb{R}^S$*,*

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

(See for instance Huber (1981)). Moreover, we note that $C \in \mathcal{H}$ *if and only if there exists a strictly positive probability* $P_0 \in \mathcal{K}$[13]*.*

Given a function $C \in \mathcal{H}_0$, we define the set of *unambiguous assets* as,

$$F_C := \left\{ Y \in \mathbb{R}^S : C(Y) + C(-Y) = 0 \right\}.$$

In fact, taking C as the rule for the determination of asset prices, the family of claims F_C describes the assets for which there is no pricing distinction between a selling position or a buying position. The set of probabilities that agree about the expected value of all unambiguous assets is given by,

$$Q_C := \{ P \in \Delta : \ E_P(Y) = C(Y), \text{ for any } Y \in F_C \}.$$

A first elementary fact says that:

Lemma 13 *Given a function* $C \in \mathcal{H}_0$*, the set of unambiguous assets* F_C *is a linear subspace.*

Recall that by Lemma 4 in any market without arbitrage opportunities a claim is attainable if and only if any risk neutral probability agrees about its monetary value. So, it is intuitive that if C is an incomplete price then the subspace of unambiguous assets is equal to the subspace of attanaible claims, in fact:

Lemma 14 *If* $C : \mathbb{R}^S \to \mathbb{R}$ *is an incomplete price then* $F_C = F$*.*

An auxiliary characterization of incomplete prices follows as:

Theorem 15 *Let* $C : \mathbb{R}^S \to \mathbb{R}$ *be given, then* (i) *is equivalent to* (ii)*:*
(i) C *is an incomplete price;*
(ii) C *is a strictly positive positive linear form on* F_C *and*

$$C(X) = \max_{P \in \mathcal{Q}_C} E_P(X).$$

Furthemore, under (i) *and* (ii) F_C *is the set of attanaible claims and* $\mathcal{Q}_C$ *is the set of risk-neutral probabilities of the underlying market.*

[13] In fact, if $C \in \mathcal{H}_0$ is strictly positive then $C(\{s_i\}^*) > 0, \forall i \in \{1, ..., n\}$. Hence, for every state $s_i \in S$ there exists a probability $P_i \in \mathcal{K}$ such that $E_{P_i}(\{s_i\}^*) > 0$, since $\mathcal{K}$ is convex we obtain that it is possible to find a strictly positive probability in $\mathcal{K}$. For the converse, by assumption there exists a strictly positive probability $P_0 \in \mathcal{K}$, hence if $X > 0$

$$C(X) \geq E_{P_0}(X) \geq \max_{s \in S} P_0(\{s\}) X(s) > 0.$$

The examples below illustrate the usefulness of the criterion given by Theorem 15 .

Example 16 *Consider* $S = \{s_1, s_2, s_3\}$ *and*

$$\begin{array}{rcl} C & : & \mathbb{R}^3 \to \mathbb{R} \\ X & \mapsto & C(X) = \max\{E_{P_1}(X), E_{P_2}(X)\}, \end{array}$$

where $P_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ *and* $P_2 = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$*. Hence, denoting* $X(s_k) = x_k$*,* $k = 1, 2, 3$ *:*

$$C(X) = \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3, \text{ if } x_1 \geq x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3, \text{ if } x_1 < x_2 \end{cases}.$$

Note that: (a) $C(S^*) = 1$*; (b)* $C(X) = -C(-X)$ *if and only if* $x_1 = x_2$*, and then* F_C *is a linear subspace; (c) on* F_C *we have that* $C(X) = \frac{3}{4}x_1 + \frac{1}{4}x_3 = \frac{3}{4}x_2 + \frac{1}{4}x_3$*, which implies that* C *is a strictly positive linear form on* F_C*; (d) Note that we may take* $\{X_0, X_1\} \equiv \{(1,1,1), (0,0,1)\}$ *as a basis of* F_C*, where* $C(X_0) = 1$*,* $C(X_1) = \frac{1}{4}$ *and*

$$\mathcal{Q}_C = \left\{ \left(p_1, p_2, \frac{1}{4}\right) : \ p_1, p_2 \geq 0 \text{ and } p_1 + p_2 = \frac{3}{4} \right\}.$$

Now, note that if $x_1 < x_2$

$$\max_{P \in \mathcal{Q}_C} E_P(X) = \frac{3}{4}x_2 + \frac{1}{4}x_3 < \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = C(X),$$

which allows us to conclude that C *is not an incomplete price. An interesting fact is that this kind of functional appears as a particular case of* insurance functionals *in Castagnoli, Maccheroni and Marinacci (2002). So, in this case, the insurance market admits some frictions (e.g., transactions costs).*

Example 17 *Again, consider a case with three states of nature and the function* $C : \mathbb{R}^3 \to \mathbb{R}$ *that satisfies:*

$$C(X) = \begin{cases} x_3, \text{ if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2), \text{ if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}$$

Note that: (a) $C(S^*) = 1$*; (b)* $C(X) = -C(-X)$ *if and only if* $x_1 + x_2 - 2x_3 = 0$*, hence* F_C *is a linear subspace; (c) on* F_C *we have that* $C(X) = x_3 = \frac{1}{2}(x_1 + x_2)$*, which implies that* C *is a strictly positive linear form on* F_C*; (d) Note that we may take* $\{X_0, X_1\} \equiv \{(1,1,1), (2,0,1)\}$ *as a basis of* F_C*, where* $C(X_0) = 1$*,* $C(X_1) = 1$ *and*

$$\mathcal{Q}_C = \left\{ (p, p, 1 - 2p) : \ 0 \leq p \leq \frac{1}{2} \right\}.$$

It turns out that:

$$\max_{0 \leq p \leq \frac{1}{2}} (px_1 + px_2 + (1 - 2p)x_3) = C(X),$$

hence C *is an incomplete price, in fact,* C *is the cost function of the market* $\mathcal{M} = ((1,1,1), (2,0,1);\ 1, 1)$*.*

4.2 Incomplete Prices and Nonwasteful Assets

Now, we introduce a fundamental notion for the characterization of incomplete prices. For motivation, suppose that C is a *potential* incomplete price and consider the case where there are two assets X and Y such that $Y > X$ and $C(X) = C(Y)$. If X and Y are available for an investor and he chooses X then he incurs into a *payoff wasteful* because spending the same amount of money the payoff stream promised by Y is at least equal to the payoff promised by X and for some state Y delivers a strictly bigger payment. The behavior of a reasonable investor would be to never choose claims that imply a payoff wasteful unless he believes that the event $\{s \in S : Y(s) > X(s)\}$ is a miracle[14].

Definition 18 *Let $\mathbb{R}^S$ be the set of claims and $C : \mathbb{R}^S \to \mathbb{R}$ a function in $\mathcal{H}_0$. We say that a contigent claim X is wasteful asset if there exists a contingent claim $Y > X$ such that $C(Y) = C(X)$.*

A nonwasteful asset is a contingent claim with the property that if some payoff assigned to a state by the claim is replaced by a better payoff, then the resulting contingent claim is strictly more expensive than the original one. So, an investor behavior that seems reasonable is never consistence with a choice of some wasteful asset.

Given a function $C \in \mathcal{H}_0$, we denote by L_C the set of all nonwasteful assets[15], *i.e.*,

$$L_C := \left\{X \in \mathbb{R}^S : Y > X \ \Rightarrow C(Y) > C(X)\right\}.$$

Now we are able to derive an interesting result saying that a *potential incomplete price* is actually an incomplete price if and only if the respective sets of unambiguous assets and of nonwasteful assets coincides.

Theorem 19 *C is an incomplete price if and only if $C \in \mathcal{H}_0$ and $L_C = F_C$.*

An immediate useful corollary follows as,

Corollary 20 *C is an incomplete price if and only if $C \in \mathcal{H}$ and $L_C \subset F_C$.*

Remark 21 *We note that for any $C \in \mathcal{H}$ we have that $F_C \subset L_C$. In fact, consider X s.t. $C(X) = -C(-X)$, for any $Y > X$ we obtain that $C(Y) - C(X) = C(Y) + C(-X) \geq C(Y - X) > 0$.*

Remark 22 *The fact that in Example 16, C is not an incomplete price, although it belongs to $\mathcal{H}$, can be easily shown by exhibiting some $X \in L_C$, which does not belong to F_C, in fact $X = (1, 0, 0)$ does the trick.*

[14] But such beliefs are not consistent with the setting where every simple bet $\{s\}^*$ has positive cost.

[15] In the context of decision theory under ambiguity, Lehrer (2007) provided a representation for preferences using a similar notion called *fat-free acts*.

4.3 Markets of $\{0,1\}$-Securities

Arrow (1963) introduced the notion of contingent markets where agents can trade promisses concerning the future uncertainty realizations. A wide class of assets used is known as Arrow securities characterized by a promisse on a particular state of nature $s \in S$, *i.e.*, in a financial market the set of possible Arrow securities is given by $\mathbb{A} := \left\{\{s\}^{*} : s \in S\right\}$[16]. Given an event A, the $\{0,1\}$-security A^{*} is also often called a *bet on the event* A. For the classes of markets with only $\{0,1\}$-securities and the bond a natural characterization (see Definition 5 and Lemma 9) of cost functions follows as:

Definition 23 *We say that the mapping $C : \mathbb{R}^{S} \to \mathbb{R}$ is the cost function of a frictionless market of $\{0,1\}$-securities without arbitrage opportunities if C satisfies the conditions of Lemma 9 under the additional condition saying that there exists a collection of events $S, B_1, ..., B_m$ such that $X_0 = S^{*}$ and $X_j = B_j^{*}$ for any $j \in \{1, ..., m\}$.*

Definition 24 *Given an incomplete price C, if its underlying market is a market of $\{0,1\}$-securities we say that C is a $\{0,1\}$-incomplete price.*

Following the notation used in the previous discussion about incomplete prices, given a subadditive, positively affinely homogeneous, monotone and normalized function $C : \mathbb{R}^{S} \to \mathbb{R}$ we induced the set function

$$\begin{array}{rcl} \mu_C & : & 2^{S} \to [0,1] \\ A & \mapsto & \mu_C(A) := C(A^{*}) . \end{array}$$

Therefore, we define the set of unambiguous events by

$$\mathcal{E}_{\mu_C} = \left\{ B \in 2^{S} : \mu_C(A) + \mu_C(A^{c}) = 1 \right\},$$

which induces the following set of probabilities

$$\mathcal{Q}_{\mu_C} = \left\{ P \in \Delta : \ P(B) = \mu_C(B), \forall B \in \mathcal{E}_{\mu_C} \right\}$$

and finally the linear subspace generated by $\mathcal{E}_{\mu_C}$:

$$F_{\mathcal{E}_{\mu_C}} := span \left\{ B^{*} : B \in \mathcal{E}_{\mu_C} \right\}.$$

Lemma 25 *Let C be an incomplete price and let $B \subset S$, then the two following assertions are equivalent:*
(ii) $B \in \mathcal{E}_{\mu_C}$, i.e., *$B$ is an unambiguous event*
(ii) $B^{*} \in F$, i.e., *B^{*} is an attainable claim.*

The previous lemma says that a bet on the event A is attainable if and only if the event A is an unambiguous event. It suggests that we may interpret the lack of some bets on the financial market as a consequence of a vague information concerning the likelihood of some events.

[16] Of course, markets with only Arrow securities is a very particular case of markets with $\{0,1\}$-securities.

Remark 26 *Given a subadditive, positively affinely homogeneous, monotone and normalized function* $C:\mathbb{R}^S\to\mathbb{R}$*, we obtain that* $\left\{B^*: B\in\mathcal{E}_{\mu_C}\right\}\subset F_C$*: in fact, if* B *is such that* $\mu_C(B)+\mu_C(B^c)=1$ *then*

$$\begin{aligned} C(B^*)+C(-B^*) &= C(B^*)+C\left((B^c)^*-S^*\right)= \\ \max_{P\in\mathcal{Q}}P(B)+\max_{P\in\mathcal{Q}}\left(P(B^c)-1\right) &= \mu_C(B)+\mu_C(B^c)-1=0. \end{aligned}$$

So, every porfolio with assets that are bets on unambiguous events are attainable. Moreover, $\mathcal{Q}_{_C}\subset\mathcal{Q}_{_{\mu_C}}$.

Theorem 27 *Let* $C:\mathbb{R}^S\to\mathbb{R}$ *be given, then* (i) *is equivalent to* (ii)*:*

(i) C *is a* $\{0,1\}$*-incomplete price;*

(ii) There exists a strictly positive probability P_0 *belonging to* $\mathcal{Q}_{_{\mu_C}}$ *and for any contingent claim* X*,*

$$C(X)=\max_{P\in\mathcal{Q}_{\mu_C}}E_P(X).$$

Furthemore, under (i) *and* (ii) $F_{\mathcal{E}_{\mu_C}}$ *is the set of attanaible claims and* $\mathcal{Q}_{_{\mu_C}}$ *is the set of risk-neutral probabilities of the underlying market.*

Example 28 *Consider the case with four states of nature and the function* $C:\mathbb{R}^4\to\mathbb{R}$ *that satisfies:*

$$C(X)=\begin{cases}\frac{3}{8}x_1+\frac{3}{8}x_2+\frac{3}{8}x_3, \text{ if } x_2+x_4\geq x_1+x_3\\ \frac{3}{8}x_1+\frac{5}{8}x_3, \text{ if } x_2+x_4< x_1+x_3\end{cases}$$

Note that computing μ_C*, we have*

$$\begin{aligned} \mu_C(\emptyset) &= 0, \mu_C(\{s_1\})=\frac{3}{8}, \mu_C(\{s_2\})=\frac{3}{8}, \\ \mu_C(\{s_3\}) &= \frac{5}{8}, \mu_C(\{s_4\})=\frac{3}{8}, \mu_C(\{s_1,s_2\})=\frac{3}{8}, \\ \mu_C(\{s_1,s_3\}) &= 1, \mu_C(\{s_1,s_4\})=\frac{3}{8}, \mu_C(\{s_2,s_3\})=\frac{5}{8}, \\ \mu_C(\{s_2,s_4\}) &= \frac{6}{8}, \mu_C(\{s_3,s_4\})=\frac{5}{8}, \mu_C(\{s_1,s_2,s_3\})=1, \\ \mu_C(\{s_1,s_3,s_4\}) &= 1, \mu_C(\{s_1,s_2,s_4\})=\frac{6}{8}, \\ \mu_C(\{s_2,s_3,s_4\}) &= 1, \mu_C(S)=1. \end{aligned}$$

which entails that

$$\mathcal{E}_{\mu_C}=\{\emptyset,S,\{s_1,s_2\},\{s_1,s_4\},\{s_2,s_3\},\{s_3,s_4\}\},$$

and

$$\mathcal{Q}_{_{\mu_C}}=\left\{\left(\frac{3}{8}-p,p,\frac{5}{8}-p,p\right):0\leq p\leq\frac{3}{8}\right\}\ni\left(\frac{2}{8},\frac{1}{8},\frac{4}{8},\frac{1}{8}\right)>0;$$

Also, since $C(X)=\max_{P\in\mathcal{Q}_{\mu_C}}E_P(X)$*,* C *is a* $\{0,1\}$*-incomplete price, moreover,* C *is the cost function of* $\mathcal{M}=\left(S^*,\{s_1,s_2\}^*,\{s_2,s_3\}^*;\ 1,\frac{3}{8},\frac{5}{8}\right)$.

A direct consequence of Theorem 27 is the characterization of markets of $\{0,1\}$-securities through the notion of nonwasteful assets and span of the bets on the unambiguous events, as in the following corollary:

Corollary 29 *C is a $\{0,1\}$-incomplete price if and only if $C \in \mathcal{H}_0$ and $L_C = F_{\mathcal{E}_{\mu_C}}$*

In order to obtain an alternative characterization of $\{0,1\}$- incomplete prices, now we introduce some useful notation and definitions:

Definition 30 *$\mu : 2^S \to [0,1]$ is a capacity if,*
(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$,
(ii) $A \supseteq B \Rightarrow \mu(A) \geq \mu(B)$,
Moreover, μ is concave if for any $A, B \in 2^S$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

Remark 31 *Consider a subadditive, positively affinely homogeneous, monotone and normalized function $C : \mathbb{R}^S \to \mathbb{R}$ and the induced set-function μ_C on 2^S as we made previously. It is simple to see that μ_C is a capacity.*

Definition 32 *The anticore of a capacity μ is defined by*

$$acore(\mu) := \left\{P \in \Delta : P(A) \leq \mu(A), \forall A \in 2^S\right\}.$$

Remark 33 *It is well known that any concave capacity μ on 2^S has the following representation:*

$$\mu(E) = \max_{P \in \mathcal{K}} P(E),$$

for some nonempty, convex and closed set of probabilities $\mathcal{K}$ (actually, $\mathcal{K} = acore(\mu)$). See, for example, Chateauneuf and Jaffray (1989). But the converse is not true (examples can be found in Schmeidler (1972) or Huber and Strassen (1973)).

In a complete market setting a bet on the event A can be priced by the unique risk neutral probability, denoted by P_0, and in this case the price of the bet on the event A is given by $P_0(A)$, i.e., there is no ambiguity concerning the price of the bet on the event A. On the other hand, if there exists ambiguity concerning the price of some event A we have implicitly assumed an incomplete market struture with respective set of multiple risk neutral probabilities $\mathcal{Q}$, and in this case

$$\mu(A) := \max_{P \in \mathcal{Q}} P(A),$$

is the lowest cost associated to a superhedging strategy against the bet on the event A. Note that $\mu(A) + \mu(A^c) > 1$, *i.e.*, due to the pricing rule incompleteness the sum of the cost of the bets on the events A and A^c is more expensive than the cost of the riskless bond.

Definition 34 *The outer capacity of* μ*, denoted by* μ^**, is defined by:*

$$A \in 2^S \mapsto \mu^*(A) = \min\{\mu(B) : B \in \mathcal{E}_\mu \text{ and } A \subset B\},$$

where $\mathcal{E}_\mu = \left\{B \in 2^S : \mu(B) + \mu(B^c) = 1\right\}$.

Remark 35 *Given a capacity* μ *on* 2^S*, since* $\mu^* \geq \mu$ *clearly* $acore(\mu_C) \subset acore(\mu_C^*)$.

Theorem 36 *Let* $C : \mathbb{R}^S \to \mathbb{R}$ *be given, then (i) is equivalent to (ii):*
(i) C *is a* $\{0,1\}$*-incomplete price;*
(ii) C *satisfies,*
(a) $acore(\mu_C)$ *contains a strictly positive probability* P_0*,*
(b) $acore(\mu_C) = acore(\mu_C^*)$*,*
(c) For any contingent claim X*,*

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X).$$

Furthemore, under (i) and (ii) $F_{\mathcal{E}_{\mu_C}}$ *is the set of attanaible claims and* $acore(\mu_C)$ *is the set* $\mathcal{Q}$ *of risk-neutral probabilities of the underlying market*

Example 37 *Consider the same function as in Example 17 given by*

$$C(X) = \begin{cases} x_3, \text{ if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2), \text{ if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}.$$

We already proved that C *is an incomplete price. Note that for any* $A \neq \emptyset$*,*

$$\mu_C(A) \in \left\{\frac{1}{2}, 1\right\} \text{ with } \mu_C(A) = \frac{1}{2} \text{ iff } A \in \{\{s_1\}, \{s_2\}\},$$

which implies that $\mathcal{E}_{\mu_C} = \{\emptyset, S\}$*, hence for any* $A \neq \emptyset$*,we have that* $\mu_C^*(A) = 1$ *and* $acore(\mu_C^*) = \Delta$*. Since* $\delta_{\{s_1\}} \notin acore(\mu_C)$ *we obtain that*

$$acore(\mu_C) \neq acore(\mu_C^*).$$

Hence, C *is not a* $\{0,1\}$*-incomplete price.*

Now, we study the possibility of incomplete prices to be a Choquet integral, which is the natural extention of the usual integral for capacities.

Definition 38 *Let* $C : \mathbb{R}^S \to \mathbb{R}$ *be given, then* C *is a Choquet integral if*
(a) μ_C *defined by* $\mu_C(A) = C(A^*)$ *for any* $A \in 2^S$ *is a capacity,*
(b) For any $X \in \mathbb{R}^S$*,* $C(X) = \int X d\mu_C$ *where,*

$$\int X d\mu_C := \int_{-\infty}^{0} [\mu_C(\{X \geq t\}) - 1] \, dt + \int_{0}^{\infty} \mu_C(\{X \geq t\}) \, dt$$

We will see that the possibility of incomplete prices as Choquet integral is related to some strong condition on the set of attainable claims, so we present the following well known definition,

Definition 39 *A Riesz subspace of $\mathbb{R}^S$ is a linear subspace F of $\mathbb{R}^S$ such that $X, Y \in F$ implies that $X \vee Y \in F$ and $X \wedge Y \in F$.*

Lemma 40 *If an incomplete price C is a Choquet integral then the induced capacity μ_C is concave and the subspace F of attainable claims is a Riesz-space.*

Definition 41 *A "partition market" is a market without arbitrage opportunities with only $\{0,1\}$-securities $X_j := B_j^*$ where $\{B_j\}_{j=1}^m$ is a partition of the state space S.*

So, it is natural to say that the corresponding cost function is a cost function of a "partition market", and by Lemma 9 the mapping $C : \mathbb{R}^S \to \mathbb{R}$ is the cost function of a "partition market" if and only if there exist a list of events $B_1, ..., B_m \in 2^S$ forming a partition of S, and there exists a striclty positive probability P_0 on 2^S such that $P_0(B_j) = C(B_j^*)$, for any $j \in \{1, ..., m\}$, and $\forall X \in \mathbb{R}^S$

$$C(X) = \max \{E_P(X) : P(B_j) = \mu_C(B_j), \forall j \in \{1, ..., m\}\}.$$

Definition 42 *A $\{0,1\}$-incomplete price is a "partition incomplete price" if its underlying market is a "partition market".*

So, we obtain the following characterization,

Theorem 43 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then the following assertions are equivalent:*

(i) C is an incomplete price which is a Choquet integral;

(ii) C is a "partition incomplete price";

(iii) There exists a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s);$$

(iv) μ_C is concave, $\mu_C = \mu_C^*$, there exists at least a strictly positive probability $P_0 \in acore(\mu_C)$, and $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X),$$

(v) C satisfies,

(a) $\mathcal{E}_{\mu_C}$ is a Boolean algebra[17]*,*

[17] A family $\mathcal{E}$ of subsets of S is called a Boolean algebra if $\mathcal{E}$ contains S, it is closed for (finite) intersection and complement.

(b) There exists a strictly positive probability P_0 belonging to $\mathcal{Q}_{\mu_C}$,
(c) For any contingent claim X,

$$C(X) = \max_{P \in \mathcal{Q}_{\mu_C}} E_P(X).$$

In any case, the set of attainable claims is generated by the P_0-atoms[18] *of the Boolean algebra $\mathcal{E}_{\mu_C}$ and the set of all risk neutral probabilities is given by acore (μ_C).*

An immediate corollary follow as:

Corollary 44 *An incomplete price C is a Choquet integral if and only if its underlying set of attanaible claims is a Riesz space.*

We note that the class of incomplete prices that can be written as a Choquet integral is linked to financial markets where *derivative markets* (in the sense of Aliprantis, Brown and Werner (2000)) are complete. A *derivative contingent claim* is any contingent claim that has the same payoff in states in which the payoffs of all securities are the same. A restatement of the result due to Ross (1976) provided by Aliprantis, Brown and Werner (2000) says that derivative markets are complete if and only if the vector space of attanaible claims is a Riesz subspace. Hence, by the previous proposition we have that *Choquet incomplete prices* describe the minimum-cost of superreplication in markets where derivative markets are complete.

Example 45 *The most two simple examples of cost functions follows from the complete market case and the* "most incomplete" *market case under the existence of the bond. The first case is the market characterized by a probability measure $P \in \Delta$ such that*

$$C_P(X) := E_P(X) \text{ for any claim } X,$$

and in this case we have in fact a standard price rule or a "complete price", i.e., any linear function $C \in \mathcal{H}$ induces a complete markets without arbitrage opportunities.

The second case, on the other hand, presents as available trade only the riskless asset 1_S. Of course, for any claim X

$$C_{\max}(X) = \max_{s \in S} X(s)$$

which is a very special case of incomplete price.

Moreover, we have the following market space

$$F_{C_P} = \mathbb{R}^S \text{ and } F_{C_{\max}} = \{k 1_S : k \in \mathbb{R}\}.$$

[18] Let $\mathcal{E}$ a Boolean algebra of subsets of S and P a probability measure over E, we say that an event $E \in \mathcal{E}$ is a P-atom if $P(E) > 0$ and for any $F \in \mathcal{E}$ such that $F \subset E$, $P(F) = P(E)$ or $P(F) = 0$. If P is strictly positive on the finite Boolean algebra $\mathcal{E}$, E is a P-atom iff $P(E) > 0$ and if $F \subset E$ and $F \neq \emptyset$ then $F \notin \mathcal{E}$.

One of the most well studied class of markets are the *Arrow securities markets*. For these markets strutures the following definition is very natural for our analysis:

Definition 46 *Given a market $\mathcal{M} = \{X_j, q_j;\ 0 \leq j \leq m\}$ without arbitrage oportunities, we say that a state $s^* \in S$ is a Arrow state if $\{s^*\}$ is an unambiguous set. We denote by E_0 the union of all Arrow state,*

$$E_0 = \bigcup_{\{s^*\} \in \mathcal{E}} \{s^*\}.$$

Example 47 *Consider the following incomplete price*

$$C_A(X) = \sum_{s \in E_0} X(s)\, Q(\{s\}) + Q(E_o^c) \underset{s \in E_o^c}{max} X(s),$$

where $Q(E) \in (0,1)$. Note that the cost of betting on the event E is given by the following capacity,

$$\mu_{C_A}(E) = \begin{cases} Q(E),\ E \subseteq E_0 \\ Q(E \cap E_0) + Q(E_0^c),\ \textit{otherwise.} \end{cases}$$

One possible underlying market of securities for this incomplete price is the market of Arrow securities and one bond given by:

$$\mathcal{M} = \left\{1_S, \left(1_{\{s_k\}}\right)_{k=1,\ldots,K}; 1, (q_k)_{k=1,..,K}\right\},$$

were E_0 is the set of all Arrow states and $q_k = Q(\{s_k\})$. We dub C_A as an "Arrow incomplete price".

Example 48 *Now, we give an example of a market of $\{0,1\}$-securities for which the corresponding cost function is not a Choquet integral*

$$\mathcal{M} = \left\{1_S, 1_{\{s_1,s_2\}}, 1_{\{s_2,s_3\}}; 1, q_1, q_2\right\}, \textit{ where } q_1, q_2 > 0 \textit{ and } q_1 + q_2 < 1\}.$$

For the incomplete price C related to this market we obtain a capacity μ_C where

$$\mu_C(\{s_1, s_2, s_3\}) + \mu_C(\{s_2\}) = (q_1 + q_2) + (q_1 \wedge q_2),$$

and

$$\mu_C(\{s_1, s_2\}) + \mu_C(\{s_2, s_3\}) = q_1 + q_2,$$

i.e., *μ_C is not concave. Moreover, the set of unambiguous events*

$$\mathcal{E}_{\mu_C} = \{\emptyset, S, \{s_1, s_2\}, \{s_3, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}\},$$

is not a Boolean algebra because the event $\{s_2\} = \{s_1, s_2\} \cap \{s_2, s_3\}$ does not belong to $\mathcal{E}_{\mu_C}$.

Example 49 *An example of Choquet integral that is not an incomplete price is the Choquet integral w.r.t. an epsilon-contaminated concave capacity. For instance, consider a strictly positive probability $Q \in \Delta$, a level $\varepsilon \in (0,1)$, and the following capacity:*

$$\lambda(A) = \begin{cases} (1-\varepsilon) Q(A) + \varepsilon, \ A \neq \emptyset \\ 0, \ A = \emptyset. \end{cases}.$$

Note that acore $(\lambda) = \{(1-\varepsilon) Q(A) + \varepsilon P : P \in \Delta\}$. Consider the function $C : \mathbb{R}^S \ni X \to C(X) = \int X d\lambda$. In fact, for any contingent claim X it is true that

$$C(X) = (1-\varepsilon) E_Q(X) + \varepsilon \max X(S).$$

The set of unambiguous events if given by $\mathcal{E}_\lambda = \{\emptyset, S\}$ and by Theorem 43,

$$C(X) = \max X(S),$$

and, of course, it is possible if and only if $\varepsilon = 1$. Also, note that $L_C = \mathbb{R}^S$ and $F_C = \{\alpha 1_S : \alpha \in \mathbb{R}\}$. Hence, for any $Q \in \Delta$ and $\varepsilon \in (0,1)$ the set $\{(1-\varepsilon) Q(A) + \varepsilon P : P \in \Delta\}$ can not be a set of all risk neutral probabilites of some frictionless market.

5 Appendix

Proof of Lemma 4[19]:

That $X \in F$ implies that all risk measures agree is obvious. In order to prove the reverse implication, assume that $X \notin F$ and $P(X) = Q(X)$ for any $P, Q \in \mathcal{Q}$, *i.e.*, the law of one price is true for some non-attanaible claim.

First, we note that:

$$C(X) = \min \{C(Y) : Y \geq X \text{ and } Y \in F\}.$$

In fact, by the NAO assumption there exists a strictly positive probabililty P_0 such that such that $C(Y) = E_{P_0}(Y)$ for any $Y \in F$. For any $n \in \{1, 2, ...\}$ consider the attanaible claim Y^n such that $E_{P_0}(Y^n) \leq C(X) + n^{-1}$. Hence, for any $s \in S$

$$Y^n(s) \leq P_0(\{s\})^{-1} \left(C(X) + n^{-1}\right) \leq (C(X) + 1) \max_{s \in S} P_0(\{s\})^{-1} =: k$$

therefore $Y^n \leq kS^*$ for any $n \geq 1$. Clearly,

$$C(X) = \inf \{C(Y) : X \leq Y \ \leq kS^* \text{and } Y \in F\},$$

and since $\{Y \in F : X \leq Y \ \leq kS^*\}$ is compact and C is continuous (by Remark 8) we obtain that the min can be substituted to inf in the definition of C.

[19] For sake of completeness we give a proof of this result. For the case of a general state space see, for instance, Föllmer and Shied (2004), chapter 1.

Hence, given $X \in \mathbb{R}^S \backslash F$ there exists $Y_0 \in F$ such that $Y_0 > X$ and $C(X) = E_{P_0}(Y_0)$. So, we have that $E_{P_0}(Y_0) > E_{P_0}(X)$. Now, as it is true that

$$C(X) = \sup_{P \in \mathcal{Q}} E_P(X),$$

and we suppose that $E_P(X) = E_Q(X)$ for any $P, Q \in \mathcal{Q}$, it turns out that $E_{P_0}(X) = C(X)$, hence

$$C(X) = E_{P_0}(Y_0) > E_{P_0}(X) = C(X),$$

a contradiction. ⊠

Proof of Lemma 13:

First, consider $Y \in F_C$ and $\lambda \in \mathbb{R}_+$, since C is positively homogeneous we have that $C(\lambda Y) = \lambda C(Y)$ and $C(\lambda(-Y)) = \lambda C(-Y)$, then $C(\lambda Y) + C(-\lambda Y) = 0$, *i.e.*, $\lambda Y \in F_C$. If $\lambda < 0$, by the definition $-Y \in F_C$ and then $(-\lambda)(-Y) \in F_C$, *i.e.*, $\lambda Y \in F_C$.

Now, if $Y, Z \in F_C$, since C is subadditive

$$\begin{aligned} C(Y+Z) &\leq C(Y) + C(Z), \text{ and} \\ C(-(Y+Z)) &\leq C(-Y) + C(-Z), \end{aligned}$$

hence, adding these two inequalities

$$0 = C(0) \leq C(Y+Z) + C(-(Y+Z)) \leq 0,$$

i.e., $Y + Z \in F_C$. ⊠

Proof of Lemma 14:

Since $E_P(X) = C(X)$ for any $X \in F$ and for any $P \in \mathcal{Q}$ clearly $F \subset F_C$.

Conversely, let $X \in F_C$, since for any $P \in \mathcal{Q}$,

$$E_P(X) \leq C(X) \text{ and } E_P(-X) \leq C(-X),$$

and

$$E_P(X) + E_P(-X) = 0 = C(X) + C(-X),$$

we obtain that $E_P(X) < C(X)$ is impossible for any $P \in \mathcal{Q}$, *i.e.*, for all $X \in F_C$ the mapping $P \mapsto \Phi_X(P) := E_P(X)$ is constant over $\mathcal{Q}$, and by Lemma 4, $X \in F$. ⊠

Proof of Theorem 15:

$(i) \Rightarrow (ii)$

By our assumption, there exists $X_0,\ X_1, ...,\ X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 on 2^Ssuch that $E_{P_0}(X_j) = C(X_j)$, $0 \leq j \leq m$. Moreover, $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : E_P(X_j) = C(X_j),\ 0 \leq j \leq m\}$.

Now, note that no-arbitrage principle implies that C is a strictly positive linear form on F; actually, by Remark 2 there exists a strictly positive probability P_0 such that $\forall Y \in F$, $C(Y) = E_{P_0}(Y)$. By Lemma 14 we know that $F = F_C$, hence C is a strictly positive linear form on F_C.

Since $\mathcal{Q}_C$ and $\mathcal{Q}$ are nonempty, closed and convex set of probabilities, remains to show that $\mathcal{Q}_C = \mathcal{Q}$. If $P \in \mathcal{Q}$ we know that $C(Y) = E_P(Y)$ for any $Y \in F$, since $F = F_C$ we obtain that $P \in \mathcal{Q}_C$. Now, $P \in \mathcal{Q}_C$ says that $C(Y) = E_P(Y)$ for any $Y \in F_C$. Again, since $F = F_C$ entails that

$$F_C = span(X_0, ..., X_m),$$

in particular, $C(X_j) = E_P(X_j)$ for any $j \in \{0, 1, ..., m\}$, *i.e.*, $P \in \mathcal{Q}$.

$(ii) \Rightarrow (i)$

Since $S^* \in F_C$, let us consider $X_0,\ X_1, ..., X_m$, with $X_0 = S^*$, a basis of the linear subspace F_C. We intend to show that C is a cost function with respect to this family of securities $X_0,\ X_1, ..., X_m$.

By our assumption the restriction $C\mid_{F_C}$ of C on the linear subspace F_C of the Euclidian space $\mathbb{R}^S$ is a strictly positive linear form, hence it admits a strictly positive linear extension $\overline{C}\mid_{F_C}$ on $\mathbb{R}^S$ (see, for instance, Gale (1960)). Clearly, it is true that $\overline{C}\mid_{F_C}(S^*) = 1$, therefore there exists a striclty positive probability P_0 on $\left(S, 2^S\right)$ such that $E_{P_0}(X) = \overline{C}\mid_{F_C}(X)$, for any $X \in \mathbb{R}^S$; in particular, $E_{P_0}(X_j) = \overline{C}\mid_{F_C}(X_j) = C(X_j)$, $0 \leq j \leq m$. So, the condition 1) of Lemma 9 is satisfied. Recalling that $F = span(X_0, ..., X_m)$, by our construction F_C is the set of attanaible claims. The proof of (ii) implies (i) will be completed if we prove that C satisfies condition 2) of Lemma 9, or equally, that $\mathcal{Q}_C = \mathcal{Q}$, where $\mathcal{Q}$ is the set of risk neutral probabilities. By definition,

$$\mathcal{Q}_C := \{P \in \Delta :\ E_P(Y) = C(Y), \text{ for any } Y \in F_C\},$$

wich is nonempty because we saw that there exists a strictly positive probability $P_0 \in \mathcal{Q}_C$.

Since for any $j \in \{0, 1, ..., m\}$ the security X_j is unambiguous, we obtain that every probability $P \in \mathcal{Q}_C$ is a risk-neutral probability for the market $\mathcal{M} = (X_j, q_j := C(X_j), 0 \leq j \leq m)$[20]. Remains to prove that every risk-neutral probability belongs to $\mathcal{Q}_C$. In fact, let $P \in \mathcal{Q}$ and $Y \in F_C$, *i.e.*,

$$E_P(X_j) = C(X_j),\ 0 \leq j \leq m,$$

and there exists $\lambda_0, \lambda_1, ..., \lambda_m \in \mathbb{R}$ such that

$$Y = \sum_{j=0}^{m} \lambda_j X_j.$$

[20] By the existence of the strictly positive probability P_0, the financial market $\mathcal{M}$ is a market of securities with no-arbitrage opportunity.

Since the restriction $C\mid_{F_C}$ of C on the linear subspace F_C is a linear mapping,

$$\begin{aligned} E_P(Y) &= E_P\left(\sum_{j=0}^{m}\lambda_j X_j\right) = \sum_{j=0}^{m}\lambda_j E_P(X_j) = \\ \sum_{j=0}^{m}\lambda_j E_P(X_j) &= \sum_{j=0}^{m}\lambda_j C(X_j) = C\left(\sum_{j=0}^{m}\lambda_j X_j\right) = C(Y). \end{aligned}$$

henceforth,

$$E_P(Y) = C(Y)\ ,\ \text{for any } Y \in F_C,$$

this entails that $P \in \mathcal{Q}_C$, which completes the proof. $\boxtimes$

Proof of Theorem 19:

($\Rightarrow$) The fact that $C \in \mathcal{H}_0$ is immediate.

It remains to show that $L_C = F_C$. Suppose that $X \in F_C$, since C is a cost function we know that there exists a strictly positive probability P_0 such that $C(X) \geq E_{P_0}(X)$, $\forall X \in \mathbb{R}^S$ and $C(X) = E_{P_0}(X), \forall X \in F_C$. Hence, if $Y > X$ it comes that $C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X)$.

Now, suppose that $X \in L_C$ then by definition $Y > X \Rightarrow C(Y) > C(X)$. Suppose that $X \notin F_C = F$ (Lemma 14), since

$$C(X) = \min\{C(Y) : Y \geq X \text{ and } Y \in F_C\} \overset{(X \notin F_C)}{=} \min\{C(Y) : Y > X \text{ and } Y \in F_C\}$$

there exists $Z \in F_C$ such $Z > X$ and $C(Z) = C(X)$, a contradiction.

($\Leftarrow$) Since $C \in \mathcal{H}$ we know that there exists a nonempty, closed and convex set $\mathcal{K} \subset \Delta$ such that for any $X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

By Theorem 15 and Remark 12 it is enough to show that C is strictly positive and $\mathcal{K} = \mathcal{Q}_C$.

Consider $X > 0$, since $0 \in F_C$ and $F_C = L_C$ we obtain that $C(X) > C(0) = 0$, therefore C is strictly positive, so $C \in \mathcal{H}$. The inclusion $\mathcal{K} \subset \mathcal{Q}_C$ is simple: Consider $P \in \mathcal{K}$, if $P \notin \mathcal{Q}_C$ then there exists $X \in F_C$ such that $E_P(X) < C(X) = -C(-X)$, hence $E_P(-X) > C(-X) = \max_{P \in \mathcal{K}} E_P(-X)$, a contradiction.

So we need to show that $\mathcal{Q}_C \subset \mathcal{K}$, or equally that $\mathcal{K} \subsetneq \mathcal{Q}_C$ is impossible. Assume that there exists $P_1 \in \mathcal{Q}_F$ such that $P_1 \notin \mathcal{K}$. Then through the classical strict separation theorem (see, for instance, Dunford and Schwartz (1958)) there exists a contingent claim X_0 such that

$$E_{P_1}(X_0) > \max_{P \in \mathcal{K}} E_P(X_0) = C(X_0).$$

If we prove that there exists $Y \in F_C,\ Y \geq X_0$ such that $C(X_0) = C(Y)$, this will entail a contradiction, since

$$E_{P_1}(X_0) > C(X_0) = C(Y) = E_{P_1}(Y) \geq E_{P_1}(X_0).$$

So it is enough to show that for any contingent claim X, setting

$$E_X := \left\{ Y \in \mathbb{R}^S : Y \geq X \text{ and } C(Y) = C(X) \right\},$$

there exists $Y \in F_C \cap E_X$.

This result is obvious if $X \in F_C$, so let us assume that $X \notin F_C$. First, since $C \in \mathcal{H}$ by Remark 12 we know that $\mathcal{K}$ contains at least a strictly positive probability P_0.

Let us now prove that E_X is bounded from above, otherwise there would exist a sequence $\{Y_k\}_{k \geq 1}$, $Y_k \in E_X$, $\forall k \geq 1$ and $s_0 \in S$ such that $\lim_k Y_k(s_0) = +\infty$. But

$$\begin{aligned} \lim_k C(Y_k) &\geq \lim_k E_{P_0}(Y_k) = \lim_k \sum_{s \in S} P_0(s) Y_k(s) \\ &\geq \sum_{s \neq s_0} P_0(s) X(s) + \lim_k P_0(s_0) Y_k(s_0) = \infty, \end{aligned}$$

contradicting $C(Y_k) = C(X)$, $\forall k \geq 1$.

Let us now show that E_X has a maximal element for the partial preorder $\geq$ on $\mathbb{R}^S$. Thanks to Zorn´s lemma we just need to prove that every chain $(Y_\lambda)_{\lambda \in \Phi}$ in E_X has an upper bound. Define Y by

$$Y(s) := \sup_{\lambda \in \Phi} Y_\lambda(s), \forall s \in S,$$

E_X bounded from above implies that $Y \in \mathbb{R}^S$. It remains to check that $C(Y) = C(X)$, let $\varepsilon > 0$ be given, and let $s_i \in S$, hence there exists $\lambda_i \in \Phi$ such that $Y(s_i) \leq Y_{\lambda_i}(s_i) + \varepsilon$, since $(Y_\lambda)_{\lambda \in \Phi}$ is a chain there exists $n \geq 1$ and $\widetilde{\lambda} \in \{\lambda_1, ..., \lambda_n\}$ such that $Y_{\widetilde{\lambda}} \leq Y \leq Y_{\widetilde{\lambda}} + \varepsilon$, therefore $C\left(Y_{\widetilde{\lambda}}\right) \leq C(Y) \leq C\left(Y_{\widetilde{\lambda}}\right) + \varepsilon$, since $C\left(Y_{\widetilde{\lambda}}\right) = C(X)$ it turns out that $C(Y) = C(X)$. Let now Y_0 be a maximal element of E_X, the proof will be completed if we show that $Y_0 \in F_C$. From the hypothesis $F_C = L_C$, it is enough to show that $Y_0 \in L_C$. Let Y_1 be an arbitrary contingent claim such that $Y_1 > Y_0$, since Y_0 is a maximal element in E_X , it comes that $Y_1 \notin E_X$, but $Y_1 > X$, therefore $C(Y_1) > C(X) = C(Y_0)$, so $Y_0 \in L_C$ which completes the proof. ⊠

Proof of Corollary 20:

In fact, it is enough to show that for any $C \in \mathcal{H}_0$, $L_C = F_C$ iff C is strictly positive and $L_C \subset F_C$. Note that, as in the previous proof $F_C \subset L_C$ implies that C is strictly positive. For the converse, by Remark 12 we know that there exists a strictly positive probability P_0 such that $C(X) \geq E_{P_0}(X)$ for any contingent claim X and $C(Y) = E_{P_0}(Y)$ for any unambiguous asset Y. Let $X \in F_C$ and consider a contigent claim $Y > X$. Hence,

$$C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X),$$

i.e., $C(Y) > C(X)$, so $F_C \subset L_C$. ⊠

Proof of Lemma 25:

We recall that from Lemma 4, $X \in F$ iff $E_P(X) = E_Q(X)$, for any $P, Q \in \mathcal{Q}$.

$(i) \Rightarrow (ii)$ Let $B \in \mathcal{E}_{\mu_C}$, we need to show that $P(B) = Q(B)$, for any $P, Q \in \mathcal{Q}$. Assume that there exists $P_1, P_2 \in \mathcal{Q}$ such that $P_1(B) > P_2(B)$. Hence,

$$1 = C(B) + C(B^c) = \max_{P \in \mathcal{Q}} P(B) + \max_{P \in \mathcal{Q}} P(B^c) > P_2(B) + \max_{P \in \mathcal{Q}} P(B^c),$$

that is, $P_2(B^c) > \max_{P \in \mathcal{Q}} P(B^c)$, but $P_2 \in \mathcal{Q}$ hence the contradiction $P_2(B^c) > P_2(B^c)$. Therefore, $B^* \in F$.

$(ii) \Rightarrow (i)$ Let $B^* \in F$, hence $P(B) = Q(B)$, for any $P, Q \in \mathcal{Q}$ and therefore $\mu_C(B^c) = P_0(B)$, but $S^* \in F$ implies also $S^* - B^* \in F$ and then $\mu_C(B^c) = P_0(B^c)$, and clearly $\mu_C(B) + \mu_C(B^c) = 1$, *i.e*, $B \in \mathcal{E}_{\mu_C}$. $\boxtimes$

Proof of Theorem 27:

$(i) \Rightarrow (ii)$ Our assumption says that there exist $B_0,\ B_1, ...,\ B_m \in 2^S$ with $B_0 = S$ and a striclty positive probability P_0 on 2^S such that$P_0(B_j) = C\left(B_j^*\right)$, for any $j \in \{0, 1, ..., m\}$ and $\forall\ X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P(B_j) = C\left(B_j^*\right),\ 0 \leq j \leq m\}$.

Let us to now prove that there exists a strictly positive probability P_0 belonging $\mathcal{Q}_{\mu_C}$. From Lemma 4 we know that if $B^* \in F$ then $P(B) = P_0(B)$ for any $P \in Q$, hence $\mu_C(B) = P_0(B)$. Since by Lemma 25 $B \in \mathcal{E}_{\mu_C}$ if and only if $B^* \in F$, it turns out that $P_0(B) = \mu_C(B)$, $\forall B \in \mathcal{E}_{\mu_C}$.

Now we need to show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$. Note that by Theorem 15 and Remark 26 we have that $\mathcal{Q} = \mathcal{Q}_C \subset \mathcal{Q}_{\mu_C}$. For the other inclusion, taking $P \in \mathcal{Q}_{\mu_C}$ and B_j let us show that $P(B_j) = C\left(B_j^*\right)$. Since B_j^* is an attainable claim (in fact, it is a basic asset) by Lemma 25 we know that $B_j \in \mathcal{E}_{\mu_C}$, hence $P \in \mathcal{Q}_{\mu_C}$ so $\mathcal{Q}_{\mu_C} \subset \mathcal{Q}$.

So $\mathcal{Q}_{\mu_C}$ is actually the set of risk-neutral probabilities of the initial market. Remains to prove that $F_{\mathcal{E}_{\mu_C}} = F$. In fact, Lemma 25 says that $B \in \mathcal{E}_{\mu_C} \Leftrightarrow B^* \in F$, hence $F_{\mathcal{E}_{\mu_C}} = \mathit{span}\left\{B^* : B \in \mathcal{E}_{\mu_C}\right\} = \mathit{span}\{B^* : B^* \in F\} = F$.

$(ii) \Rightarrow (i)$ Since $B_0 = S^*$, let us consider the finite family of all unambiguous events $B_0, B_1, ..., B_m$. By assumption there exists a strictly positive probability P_0 such that $P(B_j) = C\left(B_j^*\right),\ 0 \leq j \leq m$. The proof will be completed if we show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$ and $F_{\mathcal{E}_{\mu_C}} = F$, where $\mathcal{Q}$ and F refer to the previous defined market of $\{0,1\}$-securities $\mathcal{M} = (B_0^*, B_1^*, ..., B_m^*; 1, \mu_C(B_1^*), ..., \mu_C(B_m^*))$. But this is straightforward by the equality $\mathcal{E}_{\mu_C} = \{B_0, B_1, ..., B_m\}$. $\boxtimes$

Proof of Theorem 36:

$(i) \Rightarrow (ii)$ From the Definition 9 we have that for any $A \subset S$,

$$\mu_C(A) = C(A^*) = \max_{P \in \mathcal{Q}} P(A),$$

hence μ_C is an anti-exact capacity and the $\mathit{acore}(\mu_C)$ contains at least one strictly positive probability, namely P_0.

Let us now show that

$$C(X)=\max_{P\in acore(\mu_C)} E_P\left(X\right),\ \forall X\in\mathbb{R}^S.$$

Note that it is enough to show that $\mathcal{Q}=acore\left(\mu_C\right)$:

Consider $P\in acore\left(\mu_C\right)$, hence $P\left(B_j\right)\leq\mu_C\left(B_j\right),0\leq j\leq m$. But, in fact, B_j is unambiguous (Lemma 25) which entails $\mu_C\left(B_j\right)+\mu_C\left(B_j^c\right)=1$. Also, $P\left(B_j^c\right)\leq\mu_C\left(B_j^c\right),0\leq j\leq m$ and then

$$P\left(B_j\right)+P\left(B_j^c\right)=1=\mu_C\left(B_j\right)+\mu_C\left(B_j^c\right)=1,$$

allows us to obtain $P\left(B_j\right)=\mu_C\left(B_j\right),0\leq j\leq m$, *i.e.*, $P\in\mathcal{Q}$.

Now, setting $P\in Q$ and $A\subset S$, since our assumption says that

$$\mu_C\left(A\right)=\max_{P\in\mathcal{Q}}P\left(A\right),$$

clearly $P\left(A\right)\leq\mu_C\left(A\right)$, *i.e.*, $P\in acore\left(\mu_C\right)$.

For (b) it is enough to show that $acore\left(\mu_C^*\right)\subset acore\left(\mu_C\right)$, or else from the previous idendity $\mathcal{Q}=acore\left(\mu_C\right)$ that $acore\left(\mu_C^*\right)\subset\mathcal{Q}$. So let $P\in acore\left(\mu_C\right)$ and let B_j be chosen. By definition of μ_C^*, one has $\mu_C^*\left(B_j\right)=\mu_C\left(B_j\right)$ therefore $P\left(B_j\right)\leq\mu_C^*\left(B_j\right)$ implies $P\left(B_j\right)\leq\mu_C\left(B_j\right)$; as we notice before $B_j\in\mathcal{E}_{\mu_C}$, hence $\mu_C^*\left(B_j\right)=\mu_C\left(B_j\right)$ and $P\left(B_j^c\right)\leq\mu_C^*\left(B_j^c\right)$ implies $P\left(B_j^c\right)\leq\mu_C\left(B_j^c\right)$ from $P\left(B_j\right)+P\left(B_j^c\right)=1=\mu_C\left(B_j\right)+\mu_C\left(B_j^c\right)$, it turns out that $P\left(B_j\right)\leq\mu_C\left(B_j\right)$.

$(ii)\Rightarrow(i)$ We need to prove that there exist $B_0,\ B_1,...,\ B_m\in2^S$ with $B_0=S$ and a striclty positive probability P_0 on 2^S such that $P_0\left(B_j\right)=C\left(B_j^*\right),$ for any $j\in\{0,1,...,m\}$ and $\forall\ X\in\mathbb{R}^S$,

$$C(X)=\max_{P\in\mathcal{Q}}E_P(X),$$

where $\mathcal{Q}=\{P\in\Delta:P\left(B_j\right)=C\left(B_j^*\right),\ 0\leq j\leq m\}$.

Note that C is well defined since $acore\left(\mu_C\right)\neq\emptyset$ (by assumption (a)) and compact, moreover for any $A\subset S$

$$\mu_C\left(A\right)=C\left(A^*\right)=\max_{P\in acore(\mu_C)}P\left(A\right).$$

Clearly $B_0:=S\in\mathcal{E}_{\mu_C}$, and $\mathcal{E}_{\mu_C}$ is formed with a finite number of events $B_0,B_1,...,B_m$. Note that for any $B\in\mathcal{E}_{\mu_C}$ and for any $P\in acore\left(\mu_C\right)$ it is true that $P\left(B\right)=\mu_C\left(B\right)$: actually $P\in acore\left(C\right)$ implies that $P\left(B\right)\leq\mu_C\left(B\right)$, $P\left(B^c\right)\leq\mu_C\left(B^c\right)$ and $P\left(B\right)+P\left(B^c\right)=1=\mu_C\left(B_j\right)+\mu_C\left(B_j^c\right)$, gives the desired equality (note that its implies that $\mathcal{Q}\supset acore\left(\mu_C\right)$). Since, by hypothesis there exists a strictly positive probability $P_0\in acore\left(\mu_C\right)$, it turns out that the first requirement is satisfied. So the formula

$$C(X)=\max_{P\in\mathcal{Q}}E_P(X),$$

holds for any $X \in \mathbb{R}^S$ if and only if $\mathcal{Q} = acore\,(\mu_C)$. Just above we proved that $\mathcal{Q} \supset acore\,(\mu_C)$. By our assumption (b) we only have to show that $\mathcal{Q} \subset acore\,(\mu_C^*)$. Let $P \in \mathcal{Q}$ and $A \subset S$, from the definition of μ_C^* we have that there exists $B \in \mathcal{E}_{\mu_C}$ such that $A \subset B$ and $\mu_C^*\,(A) = \mu_C\,(B)$, hence

$$P\,(A) \leq P\,(B) = C\,(B) = \mu_C^*\,(A)\,,$$

i.e., $P \in acore\,(\mu_C^*)$.

Futhermore, under (i) and (ii) $acore\,(\mu_C)$ is the set of risk-neutral probabilities and by Theorem 27 $F_{\mathcal{E}_{\mu_C}}$ is the set of attanaible claims. ⊠

Proof of Lemma 40:

First, we note that from Proposition 3 given by Schmeidler (1986) we have that if C is a subadditive Choquet integral with respect to the capacity μ_C then μ_C is a concave capacity.

Let now prove that F is a Riesz space:

Let $X, Y \in F$, then by Lemma 4 we have that for any $P \in \mathcal{Q}$, $E_P\,(X) + E_P\,(Y) = C(X) + C\,(Y)$. Since C is a Choquet Integral with respect to a concave capacity, it turns out that[21]

$$C\,(X) + C\,(Y) \geq C\,(X \vee Y) + C\,(X \wedge Y)\,.$$

Therefore, using the previous equality

$$E_P\,(X \vee Y) + E_P\,(X \wedge Y) = E_P\,(X) + E_P\,(Y) \geq C\,(X \vee Y) + C\,(X \wedge Y)\,.$$

But $E_P\,(X \vee Y) \leq C\,(X \vee Y)$ and $E_P\,(X \wedge Y) \leq C\,(X \wedge Y)$ for any $P \in \mathcal{Q}$. Hence, $E_P\,(X \vee Y) = C\,(X \vee Y)$ and $E_P\,(X \wedge Y) = C\,(X \wedge Y)$ for any $P \in \mathcal{Q}$ which implies by Lemma 4 that $X \vee Y$ and $X \wedge Y$ belongs to F. ⊠

Proof of Theorem 43:

Before the proof of Theorem **43** we need one important lemma:

Lemma 50 [22] *Let F be a Riesz subspace of $\mathbb{R}^n$ containing the unit vector $1_{\mathbb{R}^n} = (1, ..., 1) \in \mathbb{R}^n$ then F is a "partition" linear subspace of $\mathbb{R}^n$,* i.e.*, up to a permutation:*

$$x \in F \textit{ iff } x = (x_1, ...x_1, ..., x_j, ..., x_j, ..., x_m, ..., x_m)\,.$$

Proof: The proof is by induction on the cardinality $\#S$ of $S \geq 1$. Clearly the result is true if $\#S = 1$, now assume that the result is true for $\#S = k$ and let us show that it remains true for $\#S = k + 1$.

So let F be a subspace of $\mathbb{R}^{k+1}$ contaning $1_{\mathbb{R}^{k+1}}$, and let G be defined by[23]:

$$G := \left\{y = (x_1, ..., x_k) \in \mathbb{R}^k : \exists x_{k+1} \text{ s.t. } (y, x_{k+1}) \in F\right\}.$$

[21] See, for instance, Huber (1981) pages 260 and 261.

[22] For sake of completeness we give a direct proof of this result, which in fact has been obtained independently by Polyrakis (1996, 1999).

[23] For $y = (x_1, ..., x_k) \in \mathbb{R}^k$ and $x_{k+1} \in \mathbb{R}$ we use the following notation:

$$(y, x_{k+1}) := (x_1, ..., x_k, x_{k+1}) \in \mathbb{R}^{k+1}.$$

It is straighforward to check that G is a Riesz-subspace of $\mathbb{R}^k$ containing $1_{\mathbb{R}^k}$, therefore by the induction hypothesis and up to a permutation $y \in G$ is equivalent to $y = (x_1, ..., x_1, ...x_j, ..., x_j, ..., x_m, ..., x_m)$ where $x_j \in \mathbb{R}$, $1 \leq j \leq m$. Clearly, if $x \in F$ then $x \in \widetilde{G} \oplus \widetilde{H}$ the direct sum of the linear subspaces of $\mathbb{R}^{k+1}$ given by

$$\begin{aligned} \widetilde{G} &= \left\{(y, 0) \in \mathbb{R}^{k+1} : y \in G\right\} \\ \widetilde{H} &= \left\{(0, ..., 0, x_{k+1}) \in \mathbb{R}^{k+1} : x_{k+1} \in \mathbb{R}\right\}. \end{aligned}$$

Therefore, $\dim F \leq \dim \widetilde{G} \oplus \widetilde{H} = m + 1$. It is also immediate to see that $\dim F \geq m$: in fact, $y \in G$ is equivalent to

$$y = \sum_{j=1}^{m} x_j V_j^*,$$

where each $V_j^* \in \mathbb{R}^k$, *i.e.*, $V_j \subset \{1, ..., k\}$, and $\{V_1^*, ..., V_m^*\}$ is a basis of G. Let $z_j \in \mathbb{R}$ be such that $\left(V_j^*, z_j\right) \in F$, $1 \leq j \leq m$; it is immediate to see that $\{\{V_1^*\}, ..., \{V_m^*\}\}$ linearly independent in G implies $\{\{V_1^*, z_1\}, ..., \{V_m^*, z_m\}\}$ linearly independent in F, hence $\dim F \geq m$.

Two cases have to be examined:

1) $\dim F = m + 1$: Clearly since $F \subset \widetilde{G} \oplus \widetilde{H}$, this implies that $F = \widetilde{G} \oplus \widetilde{H}$ and F is a "partition" space.

2) $\dim F = m$: In such a case since $\left\{W_j^* := \left\{V_j^*, z_j\right\}, 1 \leq j \leq m\right\}$ is linearly independent in F, $\left\{W_j^* :, 1 \leq j \leq m\right\}$ is a basis of F. Hence, we obtain that $x \in F$ if and only if there exists x_j, $1 \leq j \leq m$ such that $x = \sum_{j=1}^{m} x_j W_j^*$, in particular,

$$x_{k+1} = \sum_{j=1}^{m} x_j z_j, \qquad (\Gamma).$$

So, it remains to show that there exists $j_0 \in \{1, ..., m\}$ such that for any $x \in F$ it is possible to write $x = \sum_{j=1}^{m} x_j V_j^* + x_{j_0}$. Note that is enough to show that all the z_j's are equal to zero except $z_{j_0} = 1$. Since $1_{\mathbb{R}^{k+1}} \in F$ by the above property (Γ), we obtain that $\sum_{j=1}^{m} z_j = 1$.

Now take $j \neq i$, $j, i \in \{1, ..., m\}$. Since F is a Riesz space, $W_j^*, W_i^* \in F$ implies that $W_j^* \wedge W_i^* \in F$, but $W_j^* \wedge W_i^* = \left((V_j \cap V_i)^*, z_j \wedge z_i\right)$ and $V_j \cap V_i = \emptyset$, hence by property (Γ) we obtain that $0 = \sum_{j=1}^{m} x_j z_j = z_j \wedge z_i$, therefore $z_j \geq 0$. On the other hand, the Riesz space struture implies also that $W_j^* \vee W_i^* \in F$, but $W_j^* \vee W_i^* = (1_{\mathbb{R}^{k+1}}, z_j \vee z_i)$, hence by property (Γ) we obtain that $z_j \vee z_i = z_j + z_i$.

Summing up, we have

$$\begin{aligned}\sum_{j=1}^{m} z_j &= 1, \text{ therefore for any } j \neq i,\ j, i \in \{1, ..., m\} : \\ z_j \wedge z_i &= 0 \text{ and } \ z_j \vee z_i = z_j + z_i;\end{aligned}$$

this implies that there exists a unique $j_0 \in \{1, ..., m\}$ such that $z_{j_0} = 1$ and for any $j \in \{1, ..., m\} \setminus \{j_0\}$ it is true that $j = 0$, the desired result. $\boxplus$

Now we can start the proof of Theorem 43:

(i) $\Rightarrow$ *(ii)* By Lemma 40 we know that the set of attanaible claims F is a Riesz subspace of $\mathbb{R}^S$ containing the riskless bond S^*. Therefore, by Lemma 50 we obtain that F is a "partition" linear subspace of $\mathbb{R}^S$, hence C is the cost function of a "partition" market of $\{0, 1\}$-securities without arbitrage opportunities.

(ii) $\Rightarrow$ *(iii)* By assumption we have a partition $\{B_1, ..., B_m\}$ of the state space S and a strictly positive probability P_0 such that $P_0(B_j) = C\left(B_j^*\right)$ for any $j \in \{1, ..., m\}$. Recall that,

$$\mathcal{Q} = \{P \in \Delta : P(B_j) = P_0(B_j), 1 \leq j \leq m\}$$

and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

hence since $E_P(X) = \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s)$. Now, denote by Q the risk neutral probability such that for any $j \in \{1, ..., m\}$,

$$Q(B_j) = Q(\{\widehat{s} \in B_j : X(\widehat{s}) = \max X(B_j)\}).$$

Hence,

$$\begin{aligned}C(X) &= \max_{P \in \mathcal{Q}} \left\{ \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \\ &= \sum_{j=1}^{m} \max_{P \in \mathcal{Q}} \left\{ \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \sum_{j=1}^{m} Q(B_j) \max X(B_j).\end{aligned}$$

Which allows us to write,

$$C(X) = \sum_{j=1}^{m} P_0(B_j) \max_{s \in B_j} X(s).$$

(iii) $\Rightarrow$ *(i)* By our assumption we have that there exists a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S and such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s).$$

Hence,

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j),$$

and it is well know that

$$C(X) = \int X d\mu_C,$$

which completes this part of the proof.

Note that we proved that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(ii) \Rightarrow (iv)$ From Theorem 36, it remains to prove that C is concave and that $\mu_C = \mu_C^*$. Take $A \subset S$, since $(ii) \Leftrightarrow (iii)$, it comes from (iii) that

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j);$$

since $P_0(B_j) > 0$ and $\sum P_0(B_j) = 1$, as it is well-known μ_C is a plausibility function (*i.e.*, the dual of a belief function), hence μ_C is concave.

Remains to show that $\mu_C^* \leq \mu_C$. From Nehring (1999), we know that μ_C is concave implies that $\mathcal{E}_{\mu_C}$ is a Boolean algebra; let us show that it entails that μ_C^* is concave: Let A_1, A_2 be subsets of S, by definition of μ_C^* there exist $B_1 \supset A_1$ and $B_2 \supset A_2$, $B_i \in \mathcal{E}_{\mu_C}$ such that $\mu_C^*(A_i) = \mu_C(B_i)$, $i = 1, 2$. Hence, $\mu_C^*(A_1) + \mu_C^*(A_2) = \mu_C(B_1) + \mu_C(B_2) \geq \mu_C(B_1 \cup B_2) + \mu_C(B_1 \cap B_2)$. Since $B_1 \cup B_2, B_1 \cap B_2 \in \mathcal{E}_{\mu_C}$, $B_1 \cup B_2 \supset A_1 \cup A_2$ and $B_1 \cap B_2 \supset A_1 \cap A_2$, it turns out that $\mu_C^*(A_1) + \mu_C^*(A_2) \geq \mu_C^*(A_1 \cup A_2) + \mu_C^*(A_1 \cap A_2)$. Let $A \subset S$, μ_C^* concave implies that there exists a probability $P \in acore(\mu_C^*)$, but Theorem 36 guarantees that $acore(\mu_C) = acore(\mu_C^*)$ hence $P \in acore(\mu_C)$, therefore:

$$\mu_C^*(A) = P(A) \leq \mu_C(A),$$

which completes this part of the proof.

$(iv) \Rightarrow (v)$ Note that (a) comes from μ_C concave and the previously quoted result of Nehring (1999).

$(v) \Rightarrow (ii)$ By hyphotesis, there exists a strictly positive probability $P_0 \in \mathcal{Q}_{\mu_C}$ and $\mathcal{E}_{\mu_C}$ is a Boolean algebra. Let $\{B_1, ..., B_m\}$ be the collection of P_0-atoms of the Boolean algebra $\mathcal{E}_{\mu_C}$, hence $\{B_1, ..., B_m\}$ is a partition of S. Of course, $P_0(B_j) = C(B_j^*)$, for any $j \in \{1, ..., m\}$ and $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$. For $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$, note that if $P \in \Delta$ is such that $P(B_j) = \mu_C(B_j)$ for any $j \in \{1, ..., m\}$ then if $B \in \mathcal{E}_{\mu_C}$ and $B \notin \{B_1, ..., B_m\}$ hence there exists $\Lambda \subset \{1, ..., m\}$ such that $B = \cup_{j \in \Lambda} B_j$, therefore $P(B) = \sum_{j \in \Lambda} P(B_j) = \sum_{j \in \Lambda} \mu_C(B_j) = \mu_C(\cup_{j \in \Lambda} B_j) = \mu_C(B)$. Hence, C is a cost function of a "partition market". ⊠

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