

There are no C^1 -stable intersections of regular Cantor sets

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1. Introduction: The existence of stable intersections of regular Cantor sets is a fundamental tool to provide persistent examples of non-hyperbolic C^2 diffeomorphisms of surfaces, as did Newhouse ([N]), by means of the concept of thickness of a Cantor set. The thickness is a fractal invariant, which is continuous and positive in the C^2 (or even in the $C^{1+\alpha}$, $0 < \alpha < 1$) topology, such that, if the product of the thicknesses of two regular Cantor sets is larger than one and their support intervals intersect in a nontrivial way, then they have stable intersection (see [PT]). However Ures ([U]) showed that, in the C^1 topology, the thickness of regular Cantor sets is terribly discontinuous. Indeed, generic C^1 -regular Cantor sets have zero thickness. He also showed that two regular Cantor sets whose support intervals touch at one point cannot have extremal stable intersection (in the sense of [M]) in the C^1 topology and he used these results to show that C^1 -generic first homoclinic bifurcations present full upper density of hyperbolicity at the initial bifurcation parameter.

However, despite the discontinuity of the thickness in the C^1 topology, the Hausdorff dimension of regular Cantor sets is continuous and positive in the C^1 topology (and coincides with the limit capacity). On the other hand, it was showed in [MY] that generic pairs of regular Cantor sets in the C^2 (or $C^{1+\alpha}$) topology whose sum of Hausdorff dimensions is larger than one have translations which have stable intersection. Moreover, they have translations whose intersections have stably positive Hausdorff dimensions. This poses a more difficult problem: is it always possible to destroy intersections of regular Cantor sets by performing arbitrarily

small C^1 perturbations of them? The situation is particularly delicate when the intersection between the Cantor sets has positive Hausdorff dimension, which is a typical situation in the C^2 topology as seen before. We answer this question in the following

Theorem 1. *Given any pair (K, K') of regular Cantor sets, we can find, arbitrarily close to it in the C^1 topology, pairs (\tilde{K}, \tilde{K}') of regular Cantor sets with $\tilde{K} \cap \tilde{K}' = \emptyset$.*

We can conclude, as in Theorem I.1 of [M], that, for generic pairs (K, K') of C^1 -regular Cantor sets, the arithmetic difference $K - K' = \{x - y \mid x \in K, y \in K'\} = \{t \in \mathbb{R} \mid K \cap (K' + t) \neq \emptyset\}$ has empty interior (and so is a Cantor set).

Since stable intersections of Cantor sets are the main known obstructions to density of hyperbolicity for diffeomorphisms of surfaces, the previous result gives some hope of proving density of hyperbolicity in the C^1 topology for diffeomorphisms of surfaces.

The main technical difference between the C^1 case and the C^2 (or even $C^{1+\alpha}$) cases is the lack of bounded distortion of the iterates of ψ in the C^1 case, and this fact will be fundamental for the proof of the previous result.

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2. Proofs of the results:

We recall that K is a C^k -regular Cantor set, $k \geq 1$, if:

- i) there are disjoint compact intervals I_1, I_2, \dots, I_r such that $K \subset I_1 \cup \dots \cup I_r$ and the boundary of each I_j is contained in K ;
- ii) there is a C^k expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \dots \cup I_r$ such that $\psi(I_j)$ is the convex hull of a finite union of some intervals I_s satisfying:
 - ii.1) for each j , $1 \leq j \leq r$ and n sufficiently big, $\psi^n(K \cap I_j) = K$;
 - ii.2) $K = \bigcap_{n \in \mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \dots \cup I_r)$.

We say that $\{I_1, I_2, \dots, I_r\}$ is a *Markov partition* for K and that K is *defined* by ψ .

Given $s \in [1, k]$ and another regular Cantor set \tilde{K} , we say that \tilde{K} is close to K in the C^s topology if \tilde{K} has a Markov partition $\{\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_r\}$ such that the interval \tilde{I}_j has endpoints

close to the endpoints of I_j , for $1 \leq j \leq r$ and \tilde{K} is defined by a C^s map $\tilde{\psi}$ which is close to ψ in the C^s topology.

Given a $C^{3/2}$ -regular Cantor set K , we define the parameter $\lambda(K) = \max\{|\psi'(x)|, x \in \bigcup_{j=1}^r I_j\} > 0$, which depends continuously on K in the C^1 -topology.

We may associate for each $j \leq r$ a gap $U_j \subset I_j$ of K , which is determined by the combinatorics of (K, ψ) in the following way: we take the smallest $m_j \geq 1$ such that $\psi^{m_j}(I_j)$ is the convex hull of the union of more than one interval of the Markov partition of K , say of $I_{s_j} \cup I_{s_j+1} \cup \dots \cup I_{s_j+l_j}$, and, if V_j is the gap between I_{s_j} and I_{s_j+1} , we set $U_j := \psi^{-m_j}(V_j)$. We define a parameter $a(K)$ in the following way: given $n \in \mathbb{N}$, $j \leq r$ and \tilde{J} a connected component of $\psi^{-n}(I_j)$, we define $a(\tilde{J}) = |U|/|\tilde{J}|$, where $U \subset \tilde{J}$ is the gap of K such that $\psi^n(U) = U_j$, and we define $a(K) = \inf_{\tilde{J}} \{a(\tilde{J}); \tilde{J} \text{ connected component of } \psi^{-n}(I_j), \exists n \in \mathbb{N}, j \leq r\} > 0$. Notice that $a(K)$ depends continuously on K in the $C^{3/2}$ topology.

In the next lemma we will exploit the lack of bounded distortion in order to produce a very distorted geometry (with very large gaps) near some subsets of a C^1 -regular Cantor set.

Lemma 1. *Let K be a C^2 -regular Cantor set. Let $c(K) = 2\lambda(K)/a(K)$. Then, given $\varepsilon > 0$, let $n_0 = \lceil c(K) \log \varepsilon^{-1}/\varepsilon \rceil$. Then, for any $\delta > 0$ and for any compact set $X \subset K$ satisfying*

- i) $\psi^j|_X$ is injective for $0 \leq j \leq n_0$, where $\psi: K \rightarrow K$ is the expansive function which defines K .
- ii) $\psi^i(X) \cap \psi^j(X) = \emptyset$, for $0 \leq i < j \leq n_0$,

we can find a covering of K formed by intervals (J_i) of its construction (i.e. intervals which are connected components of $\psi^{-n}(I_j)$ for some $n \in \mathbb{N}$, $j \leq r$) which have size smaller than δ satisfying the following properties: let D be the union of the intervals J_i for all i and the intervals $\psi^j(J_i)$, $1 \leq j \leq n_0$, for the intervals J_i which intersect X . There is a Cantor set \tilde{K} in the ε -neighbourhood of K in the C^1 -topology with $a(\tilde{K}) \geq a(K)$ such that all connected components of D are still intervals of the construction of \tilde{K} and $J_i \cap X \neq \emptyset \Rightarrow \tilde{K} \cap J_i$ has a gap V_i with $|V_i| \geq (1 - \varepsilon)|J_i|$.

Proof: Let $A = \{i \mid J_i \cap X \neq \emptyset\}$, where $\{J_i\}$ is the covering of K by the connected components of $\psi^{-N}(\bigcup_{j=1}^r I_j)$, for some large N so that, in particular, $|J_i| < \delta$, for all i .

Since X is compact, if N is large enough, we have by the hypothesis of the lemma, $\psi^j|_{\tilde{X}}$ injective for $0 \leq j \leq n_0$ and $\psi^i(\tilde{X}) \cap \psi^j(\tilde{X}) = \emptyset$ for $0 \leq i < j \leq n_0$, where $\tilde{X} := \bigcup_{i \in A} J_i$.

We may perform, as in Lemma II.2.1 of [M], a small change in ψ in the $C^{3/2}$ topology in such a way that the restrictions of ψ to the intervals $\psi^j(J_i)$ with $i \in A$, $0 \leq j < n_0$ become affine; we change ψ just in these intervals and in the gaps attached to them.

Now we will make small C^1 perturbations on the restriction of ψ to the intervals $\psi^j(J_i)$ with $i \in A$, $0 \leq j < n_0$. We will begin changing ψ in the intervals $\psi^{n_0-1}(J_i)$, then in the intervals $\psi^{n_0-2}(J_i)$ and so on, in order to make the proposition of a gap in each of these intervals grow in such a way that the size of each of the two remaining intervals is multiplied by $1 - \frac{2a\varepsilon}{3\lambda}$, where $\lambda = \lambda(K)$ and $a = a(K)$.

More precisely, if $\psi^j(J_i) = [r, s]$, $i \in A$, $0 \leq j < n_0$ is some of these intervals and m is such that $\psi^m(J_i) = I_l$, let $\tilde{U} \subset J_i$ such that $\psi^m(\tilde{U}) = U_l$ and let $\psi^j(\tilde{U}) = (u, v) \subset [r, s]$. Writing $\psi|_{[r, s]}(x) = \lambda x + t$, we consider the affine map $\tilde{\psi}|_{[r, s]}$ given by

$$\tilde{\psi}|_{[r, s]}(x) = \begin{cases} \tilde{\lambda}(1 - \frac{2a\varepsilon}{3\lambda})^{-1}(x - r) + \tilde{\lambda}r + t, & \text{if } x \in [r, r + (1 - \frac{2a\varepsilon}{3\lambda})(u - r)] \\ \tilde{\lambda}(1 - \frac{2a\varepsilon}{3\lambda})^{-1}(x - s) + \tilde{\lambda}s + t, & \text{if } x \in [s - (1 - \frac{2a\varepsilon}{3\lambda})(s - v), s] \end{cases}$$

and we extend $\tilde{\psi}$ to $[r, s]$ in such a way that $\tilde{\psi}|_{[r, s]}$ is a C^1 function. Notice that the image $\tilde{\psi}((r + (1 - \frac{2a\varepsilon}{3\lambda})(u - r), s - (1 - \frac{2a\varepsilon}{3\lambda})(s - v))) = \psi((u, v))$ of the gap remains the same. The size of the new gap in $[r, s]$ is $v - u + \frac{2a\varepsilon}{3\lambda}(s - v + u - r) < v - u + \frac{2a\varepsilon}{3\lambda}(s - r) < (1 + \frac{2\varepsilon}{3\lambda})(v - u)$. In particular, it is not difficult to see that we may construct such a function $\tilde{\psi}$ with $\|\tilde{\psi} - \psi\|_{C^1} < \varepsilon$.

Finally, the total proportion of the complement of the new gap $V_i = \tilde{U}$ for the modified ψ (indeed $\tilde{\psi}|_{J_i}^{-n_0}(\psi^{n_0}(\tilde{U}))$) is at most $(1 - \frac{2a\varepsilon}{3\lambda})^{n_0}(1 - a) \leq (1 - \frac{2a\varepsilon}{3\lambda})^{\frac{2\lambda}{a\varepsilon} \log \varepsilon^{-1}} \cdot (1 - a) < \varepsilon^{4/3} < \varepsilon$. It is not difficult to see that after these perturbations we will also have $a(\tilde{K}) \geq a(K)$ (indeed in the non-affine part of the dynamics we are only increasing the proportion of some gaps, and in the affine and local part of the dynamics the proportion of the gaps is preserved), which concludes the proof of the Lemma. \square

Lemma 2. *Given a C^2 -regular Cantor set K' , for a residual set of C^2 -regular Cantor sets K , if $k = \lfloor (1 - HD(K'))^{-1} \rfloor + 1$ then $\bigcap_{j=1}^k A_j = \emptyset$, where $A_j = F_j((K \cap K') \cap P_j)$, $1 \leq j \leq k$ and*

$(F_j, P_j), 1 \leq j \leq k$ are distinct elements of $\{(\psi^r|_I, I); r \in \mathbb{N} \text{ and } I \text{ is a maximal interval of the construction of } K \text{ where } \psi^r \text{ is injective}\}$.

Proof: It is enough to show that, for generic C^2 -regular Cantor sets K , given distinct sets $A_j = \psi^{r_j}((K \cap K') \cap P_j), 1 \leq j \leq k$, where P_j is a maximal interval of the construction of K such that $\psi^{r_j}|_{P_j}$ is injective, we have $\bigcap_{j=1}^k \psi^{r_j}(K \cap K' \cap P_j) = \emptyset$. So, we will, from now on, fix the branches $\psi^{r_j}|_{P_j}$ that we will consider.

Typically, $K \cap K'$ does not contain any preperiodic point of ψ (indeed, there is only a countable number of them in K , and so almost all translations of K' do not intersect them). In this case, given a point $x \in \bigcap_{j=1}^k A_j$, with $x = \psi^{r_j}(y_j), 1 \leq j \leq k$ with $y_j \in K \cap K' \cap P_j, \forall j \leq k$, we have that the points y_j are all distinct (indeed, if $y_i = y_j$ with $i \neq j$ then $r_i \neq r_j$, since A_i and A_j are distinct, so x is periodic and the points $y_i, y_j \in K \cap K'$ are preperiodic). We may find disjoint intervals $\hat{P}_j(x) \subset P_j$ of the construction of K with $y_j \in \hat{P}_j(x), 1 \leq j \leq k$, and take $\varepsilon_x > 0$ and a neighbourhood \mathcal{V}_x of K in the C^1 topology such that, for $\tilde{K} \in \mathcal{V}_x$, if $\tilde{P}_j, \tilde{\hat{P}}_j(x)$ and $\tilde{\psi}$ denote the continuations of the intervals $P_j, \hat{P}_j(x)$ and of the map ψ which defines K respectively, we have $\bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{P}_j) \cap (x - \varepsilon_x, x + \varepsilon_x) \subset \bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{\hat{P}}_j(x))$. We may take a finite covering of the compact set $\bigcap_{j=1}^k A_j$ by intervals $(x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}), 1 \leq i \leq m$. We denote by $\tilde{\mathcal{V}}$ the neighbourhood $\bigcap_{i=1}^m \mathcal{V}_{x_i}$ of K in the C^1 topology. We will assume (by reducing $\tilde{\mathcal{V}}$, if necessary) that, for any $\tilde{K} \in \tilde{\mathcal{V}}, \bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{P}_j) \subset \bigcup_{i=1}^m (x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i})$, so $\bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{P}_j) \subset \bigcup_{i=1}^m \bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{\hat{P}}_j(x_i))$. It is enough to show that, for each $i \leq m$, there is an open and dense set $\tilde{\mathcal{V}}_i \subset \tilde{\mathcal{V}}$ such that, for any $\tilde{K} \in \tilde{\mathcal{V}}_i$, we have $\bigcap_{j=1}^k \tilde{\psi}^{r_j}((\tilde{K} \cap K') \cap \tilde{\hat{P}}_j(x_i)) = \emptyset$. Since the sets $\psi^{r_j}((\tilde{K} \cap K') \cap \tilde{\hat{P}}_j(x_i))$ are compact, the above condition is clearly open, so it is enough to prove that it is dense in $\tilde{\mathcal{V}}$.

Now, since the intervals $\tilde{\hat{P}}_j(x_i), 1 \leq j \leq k$ are disjoint, we may consider families of perturbations $\tilde{K}_{t_1, \dots, t_k} \in \tilde{\mathcal{V}}$ of K defined by maps $\tilde{\psi}_{t_1, \dots, t_k}$ which form a family of perturbations of $\tilde{\psi}$ depending on k small parameters $t_1, t_2, \dots, t_k \in (-\delta, \delta)$, for some (small) $\delta > 0$ for which $\tilde{\psi}_{t_1, \dots, t_k}^{r_j}|_{\tilde{\hat{P}}_j(x_i)} = \tilde{\psi}^{r_j}|_{\tilde{\hat{P}}_j} + t_j, \forall t_1, t_2, \dots, t_k \in (-\delta, \delta)$ (it is enough to make suitable perturbations of $\tilde{\psi}$ in the disjoint intervals $\tilde{\hat{P}}_j(x_i), 1 \leq j \leq k$). Since the limit capac-

ities (box dimensions) of the sets $\tilde{\psi}^{r_j}(K' \cap \tilde{P}_j)$ are bounded by $HD(K')$, $\prod_{j=1}^k \tilde{\psi}^{r_j}(K' \cap \tilde{P}_j)$ has limit capacity bounded by $k \cdot HD(K') < k - 1$, and so linear projections of it in \mathbb{R}^{k-1} have zero Lebesgue measure. Since $\{(t_1, \dots, t_k) \in \mathbb{R}^k \mid \bigcap_{j=1}^k (\psi^{r_j}(K' \cap \tilde{P}_j) + t_j) \neq \emptyset\} = \{(t, t+s_1, \dots, t+s_{k-1}) \mid (s_1, \dots, s_{k-1}) = D_k(\prod_{j=1}^k \psi^{r_j}(K' \cap \tilde{P}_j))\}$, where D_k is the linear map given by $D_k(x_1, \dots, x_k) := (x_2 - x_1, x_3 - x_1, \dots, x_k - x_1)$, for almost all (t_1, \dots, t_k) the intersection is empty, which implies the result. \square

Lemma 3. *Let (K, K') be a pair of C^2 -regular Cantor sets and $B = \{(\psi^r|_I, I); r \in \mathbb{N} \text{ and } I \text{ is a maximal interval of the construction of } K \text{ where } \psi^r \text{ is injective}\}$. Given any $m \geq 1$, the following holds: for any fixed distinct elements $(\psi^{r_j}|_{P_j}, P_j), 1 \leq j \leq m$ of B and for any $\eta > 0$ there is a C^2 -regular Cantor set \tilde{K} at a distance smaller than η from K in the C^1 topology and with $a(\tilde{K}) > a(K) - \eta$ such that $\bigcap_{j=1}^m \tilde{\psi}^{r_j}(\tilde{K} \cap K' \cap P_j) = \emptyset$, where $\tilde{\psi}$ is the map which defines \tilde{K} .*

Proof: Since the quantity $a(K)$ depends continuously on K in the $C^{3/2}$ -topology (and, in particular, in the C^2 -topology), Lemma 2 implies that the above statement is true for $m \geq k = \lfloor (1 - HD(K'))^{-1} \rfloor + 1$. We will argue by backward induction on m : we will show that, if $q \geq 1$ and the statement of Lemma 3 is true for every $m \geq q + 1$ then it is true for q .

Let (K, K') be a pair of C^2 -regular Cantor sets, $\eta \in (0, 1)$ and $(\psi^{r_j}|_{P_j}, P_j), 1 \leq j \leq q$ fixed distinct elements of B . Let $X = \bigcap_{j=1}^q \psi^{r_j}(K \cap K' \cap P_j)$. Let $\varepsilon = a(K)a(K')\eta/5\lambda(K')$ and let $N_0 = \lceil 2c(K) \log \varepsilon^{-1}/\varepsilon \rceil$, where $c(K) = 2\lambda(K)/a(K)$. For each pair (i, j) with $0 \leq i < j \leq N_0$, the intersection $\psi^i(X) \cap \psi^j(X)$ can be written as a finite union of intersections of at least $q + 1$ sets of the form $\psi^r(K \cap K' \cap I)$, where $(\psi^r|_I, I) \in B$. By the induction hypothesis (applied several times to make a sequence of small perturbations of the first Cantor set, one for which one of the intersections mentioned above), we may approximate K by a C^2 -regular Cantor set \check{K} at a distance smaller than $\eta/2$ from K in the C^1 topology with $a(\check{K}) > \max\{a(K) - \eta/2, a(K)/2\}$ and $c(\check{K}) < 4\lambda(K)/a(K)$, such that, if $\check{X} = \bigcap_{j=1}^q \check{\psi}^{r_j}(\check{K} \cap K' \cap \check{P}_j)$, where $\check{\psi}$ is the map which defines \check{K} , then the sets $\check{\psi}^j(\check{X}), 0 \leq j \leq N_0$ are pairwise disjoint. So, if $\check{Y} := (\check{\psi}^{r_1}|_{\check{P}_1})^{-1}(\check{X}) = (\check{K} \cap K' \cap \check{P}_1) \cap \bigcap_{j=2}^q (\check{\psi}^{r_1}|_{\check{P}_1})^{-1}(\check{\psi}^{r_j}(\check{K} \cap K' \cap \check{P}_j))$, then the sets

$\check{\psi}^j(\check{Y}), 0 \leq j \leq N_0$ are pairwise disjoint. Indeed, if $\check{\psi}^i(\check{Y}) \cap \check{\psi}^j(\check{Y}) \neq \emptyset$, with $0 \leq i < j \leq N_0$, we would have $\emptyset \neq \check{\psi}^{r_1}(\check{\psi}^i(\check{Y})) \cap \check{\psi}^{r_1}(\check{\psi}^j(\check{Y})) = \check{\psi}^i(\check{\psi}^{r_1}(\check{Y})) \cap \check{\psi}^j(\check{\psi}^{r_1}(\check{Y})) = \check{\psi}^i(\check{X}) \cap \check{\psi}^j(\check{X}) = \emptyset$, a contradiction.

So, \check{Y} satisfies the hypothesis of Lemma 1 for ε (since $N_0 = \lceil 2c(K) \log \varepsilon^{-1}/\varepsilon \rceil \geq \lceil c(\check{K}) \log \varepsilon^{-1}/\varepsilon \rceil$), and thus the conclusion of Lemma 1 holds: for any $\delta > 0$, we can find a covering of \check{K} formed by intervals (\check{J}_i) of its construction which have size smaller than δ satisfying the following properties: let D be the union of the intervals \check{J}_i for all i and the intervals $\check{\psi}^j(\check{J}_i), 1 \leq j \leq n_0$, for the intervals \check{J}_i which intersect \check{Y} . There is a Cantor set \overline{K} in the ε -neighbourhood of \check{K} in the C^1 -topology with $a(\overline{K}) \geq a(\check{K})$ such that all connected components of D are still intervals of the construction of \overline{K} and $\check{J}_i \cap \check{Y} \neq \emptyset \Rightarrow \overline{K} \cap \check{J}_i$ has a gap V_i with $|V_i| \geq (1 - \varepsilon)|\check{J}_i|$. Now, we can apply a C^1 diffeomorphism $(\eta/2)$ -close to the identity in the C^1 topology to \overline{K} in order to make it disjoint from K' . Indeed, let $(\check{J}_i)^{(1)}$ and $(\check{J}_i)^{(2)}$ be the connected components of $\check{J}_i \setminus V_i$. We will make small independent translations of these intervals (if they do intersect K') in the following way: if such an interval $(\check{J}_i)^{(s)}$ intersects K' , take an interval J' of the construction of K' intersecting it whose size belongs to the interval $(|(\check{J}_i)^{(s)}|/a(K'), \lambda(K')|(\check{J}_i)^{(s)}|/a(K')]$; the gaps attached to the ends of this interval have size larger than $|(\check{J}_i)^{(s)}|$ so we can apply a translation of it of size at most $\lambda(K')|(\check{J}_i)^{(s)}|/a(K')$ whose image is contained in one of these gaps. These translations can be performed all together (for all i, s) by a diffeomorphism at a C^1 distance to the identity smaller than $\eta/2$, since the gaps attached to the intervals $(\check{J}_i)^{(s)}$ have size at least $a(\check{K})|\check{J}_i|$ and the size of the translations is at most $\lambda(K')|(\check{J}_i)^{(s)}|/a(K') < \varepsilon \lambda(K')|\check{J}_i|/a(K') = a(\check{K})|\check{J}_i|\eta/5$. If we denote by \tilde{K} the image of \check{K} by this diffeomorphism (which is a regular Cantor set defined by a map $\tilde{\psi}$ conjugated to $\overline{\psi}$ by the diffeomorphism that we applied to \overline{K}) we will have $(\tilde{K} \cap \check{J}_i) \cap K'$ empty, $\forall i$, so $\tilde{Y} := (\tilde{K} \cap K' \cap \check{P}_1) \cap \bigcap_{j=2}^q (\tilde{\psi}^{r_1}|_{\check{P}_1})^{-1} \tilde{\psi}^{r_j}(\tilde{K} \cap K' \cap \check{P}_j) = \emptyset$, and, applying $\tilde{\psi}^{r_1}$, we get $\tilde{X} := \bigcap_{j=1}^q \tilde{\psi}^{r_j}(\tilde{K} \cap K' \cap \check{P}_j) = \tilde{\psi}^{r_1}(\tilde{Y}) = \emptyset$, and we are done. \square

Proof of Theorem 1. Lemma 3 for $m = 1$ implies that generically $K \cap K' = \emptyset$. It follows that, for each $r \in \mathbb{Q}$, $\{(K, K') \mid r \notin K - K'\} = \{(K, K') \mid K \cap (K' + r) = \emptyset\}$ is residual, and thus $\{(K, K') \mid (K - K') \cap \mathbb{Q} = \emptyset\}$ is residual. So, generically, for (K, K') pair of C^1 -regular Cantor sets, $K - K'$ has empty interior, and so is a Cantor set. \square

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