IDENTIFYING QUADRIC BUNDLE STRUCTURES ON COMPLEX PROJECTIVE VARIETIES

CAROLINA ARAUJO

Abstract. In this paper we characterize smooth complex projective varieties that admit a quadric bundle structure on some dense open subset in terms of the geometry of certain families of rational curves.

1. Introduction

Let X be a smooth complex projective variety, and assume that X is uniruled, i.e., there exists a rational curve through every point of X. Then there exists an irreducible family of rational curves that sweeps out a dense open subset of X. Such a family is called a *covering* family of rational curves on X. A covering family of rational curves reflects a lot of the geometry of the ambient variety X, specially when this family is "minimal" in some sense. There are several different notions of "minimality". In this paper we consider *unsplit* and *minimal* covering families. A covering family H of rational curves on X is called *minimal* if, for a general point $x \in X$, the subfamily H_x of H parametrizing curves through x is proper. It is called *unsplit* if H itself is proper. Unsplit covering families of rational curves are very powerful tools in studying the geometry of uniruled varieties. However, not every uniruled variety carries such a family. Think, for instance, of a smooth quartic threefold. The family of conics on it is minimal but not unsplit. On the other hand, every uniruled complex projective variety X carries a minimal covering family of rational curves. For instance, one may fix an ample line bundle on X and take H to be a covering family of rational curves having minimal degree with respect to this fixed line bundle.

Fix H a minimal covering family of rational curves on X. Given a general point $x \in X$, let $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ denote the closed subset of the projectivized tangent space at x consisting of tangent directions at x to rational curves parametrized by H_x (see Definition 10). This is called the *variety of minimal rational tangents* at x associated to the family H. It was first introduced in [Mor79], where Mori used it to prove the Hartshorne conjecture, and it has found many applications within the theory of uniruled varieties, specially Fano varieties (see [Hwa01] for a survey). The variety of minimal rational tangents comes with a natural embedding into $\mathbb{P}(T_xX)$. The general philosophy is that many properties of X can be detected by studying this projective embedding. For instance, Theorem 1 below establishes that $\mathcal{C}_x = \mathbb{P}(T_xX)$ if and only if $X \cong \mathbb{P}^n$.

Theorem 1 ([CMSB02] - see also [Keb02a]). *Let X be a smooth complex projective n-dimensional variety, H a minimal covering family of rational curves on X, and $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ the associated variety of minimal rational tangents at $x \in X$. Suppose* *that $\mathcal{C}_x = \mathbb{P}(T_xX)$ for a general point $x \in X$. Then $X \cong \mathbb{P}^n$ and under this isomorphism H corresponds to the family of lines on $\mathbb{P}^n$.*

If $X \ncong \mathbb{P}^n$, then it is not possible to completely recover X from the embedding $\mathcal{C}_x \hookrightarrow \mathbb{P}(T_xX)$. Indeed, if X' is the blow up of X at a point, then H induces a minimal covering family of rational curves on X' whose variety of minimal rational tangents at a general point is projectively isomorphic to $\mathcal{C}_x$. On the other hand, if one imposes some extra condition on X, for instance the Picard number being 1, then in some cases X may be fully recovered from its variety of minimal rational tangents at a general point. This is illustrated by the next theorem.

Theorem 2 ([Mok05, Main Theorem]). *Let X and Y be smooth complex projective Fano varieties with Picard number 1, and suppose that Y is either a Hermitian symmetric space or a contact homogeneous manifold. Let H_X and H_Y be minimal covering families of rational curves on X and Y respectively, and $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ and $\mathcal{D}_y \subset \mathbb{P}(T_yY)$ the associated varieties of minimal rational tangents at general points $x \in X$ and $y \in Y$, respectively. Suppose that there is an isomorphism $\mathbb{P}(T_xX) \cong \mathbb{P}(T_yY)$ inducing an isomorphism $\mathcal{C}_x \cong \mathcal{D}_y$. Then $X \cong Y$, and under this isomorphism curves from H_X correspond to curves from H_Y.*

Notice that this theorem applies in particular to a smooth hyperquadric $Y = Q_n \subset \mathbb{P}^{n+1}$ with $n \geq 3$. In this case the only minimal covering family of rational curves on Y is the family of lines, and the associated variety of minimal rational tangents at any $y \in Y$ is a positive dimensional hyperquadric in $\mathbb{P}(T_yY)$.

When the Picard number of X is not 1, one can still use $\mathcal{C}_x$ to get information about X. The following is a structure theorem for varieties X for which $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX)$ for general $x \in X$.

Theorem 3 ([Ara06, Theorems 1.1 and 3.4]). *Let X be a smooth complex projective variety, H a minimal covering family of rational curves on X, and $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ the associated variety of minimal rational tangents at $x \in X$. Suppose that for a general point $x \in X$, $\mathcal{C}_x$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX)$.*

Then there exists a dense open subset $X^\circ \subset X$, a smooth quasi-projective variety Y° and a $\mathbb{P}^{d+1}$-bundle $\pi^\circ : X^\circ \to Y^\circ$ such that every curve from H meeting X° corresponds to a line on a fiber of π°. If H is unsplit, then one may take X° such that $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$.

In this paper we go one step further. We provide the following structure theorem for varieties X carrying an unsplit covering family of rational curves for which $\mathcal{C}_x$ is a positive dimensional hyperquadric in its linear span in $\mathbb{P}(T_xX)$ for general $x \in X$. By a *quadric bundle* we mean a flat projective morphism between quasi-projective varieties whose fibers are all isomorphic to irreducible and reduced (but not necessarily smooth) hyperquadrics.

Main Theorem. *Let X be a smooth complex projective variety admitting an unsplit covering family of rational curves H. Let $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ denote the variety of minimal rational tangents associated to H at a general point $x \in X$. Suppose $\mathcal{C}_x$ is a positive dimensional hyperquadric in its linear span in $\mathbb{P}(T_xX)$.*

Then there exists an open subset $X^\circ \subset X$, with $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$, a smooth quasi-projective variety Y° and a quadric bundle $\pi^\circ : X^\circ \to Y^\circ$ such that every curve from H meeting X° corresponds to a line on a fiber of π°.

Remarks 4. **(1)** The unsplitness assumption in the Main Theorem cannot be dropped. Indeed, start with a smooth complex projective variety Z admitting a quadric bundle structure $\pi : Z \to Y$, and let $\sigma \subset Z$ be a multi-section of π. Then let X be the blowup of Z along σ.
(2) In general one should not expect the quadric bundle $\pi^\circ : X^\circ \to Y^\circ$ obtained in the Main Theorem to extend to a quadric bundle on the whole of X. For instance, given $n \geq 6$, let X be the complete intersection in $\mathbb{P}^n \times \mathbb{P}^n$ of two general divisors of type $(1,1)$, and one general divisor of type $(0,2)$. Let $\pi : X \to \mathbb{P}^n$ be the first projection. The general fiber of π is isomorphic to a $(n-3)$-dimensional hyperquadric, but there are fibers of higher dimension.
(3) In the special case when H is a family of lines under some projective embedding of X, and $\dim \mathcal{C}_x > \left[\frac{\dim X}{2}\right]$, the statement of the Main Theorem follows from [BI08, Theorem 3.1].

This paper is organized as follows. In Section 2 we gather results about minimal and unsplit covering families of rational curves and their rationally connected quotients. In Section 3 we discuss the variety of minimal rational tangents associated to a minimal covering family of rational curves and the distribution defined by its linear span. In Section 4, we prove the Main Theorem.

Notation. Throughout this paper we work over the field of complex numbers. By a rational curve we always mean a projective rational curve. If V is a complex vector space, we denote by $\mathbb{P}(V)$ the projective space of 1-dimensional linear subspaces of V. Let X be a quasi-projective variety. By a general point of X, we mean a point in some dense open subset of X. If $f : X \to T$ is a proper morphism onto another quasi-projective variety and $t \in T$, we denote the fiber of f over t by X_t. We denote by $\rho(X/T)$ the relative Picard number of X over T. This is the dimension of the vector space $N_1(X/T)$ of 1-cycles on X with real coefficients generated by irreducible curves contracted by f, modulo numerical equivalence.

2. Families of rational curves and their rationally connected quotients

Let X be a complex projective variety. There is a scheme $\mathrm{RatCurves}^n(X)$ parametrizing rational curves on X. This scheme is constructed as the normalization of a certain subscheme of the Chow scheme $\mathrm{Chow}(X)$ parametrizing effective 1-cycles on X. More generally, given a proper morphism $f : X \to Y$ between complex quasi-projective varieties, there is a Y-scheme $\mathrm{RatCurves}^n(X/Y)$ parametrizing rational curves on X contained on fibers of f. As before, $\mathrm{RatCurves}^n(X/Y)$ is constructed as the normalization of a certain subscheme of the Chow scheme $\mathrm{Chow}(X/Y)$ parametrizing effective 1-cycles on X contracted by f. We refer to [Kol96, Chapters I and II] for details, constructions and proofs. See also [Deb01].

Definition 5. Let $f : X \to Y$ be a proper morphism between complex quasi-projective varieties. By a *family of rational curves on X over Y* we mean an irreducible component H of $\mathrm{RatCurves}^n(X/Y)$. In particular H is a normal quasi-projective variety. The universal properties of $\mathrm{RatCurves}^n(X/Y)$ yield universal family morphisms

$$\begin{array}{ccc} U & \xrightarrow{\eta} & X, \\ {\scriptstyle p}\downarrow & & \\ H & & \end{array}$$

where U is a normal quasi-projective variety, $p : U \to H$ is a $\mathbb{P}^1$-bundle, and $\eta : U \to X$ is the evaluation morphism. We say that H is a *covering family* if the image of η is dense in X.

Now assume that Y is a point and X is a complex projective variety, and hence $\mathrm{RatCurves}^n(X/Y) = \mathrm{RatCurves}^n(X)$. Let H be a covering family of rational curves on X. Given a point $x \in X$, we denote by H_x the subscheme of H parametrizing rational curves passing through x, and set $\mathrm{locus}(H_x) := \eta\big(p^{-1}(H_x)\big)$ (with the reduced scheme structure). We say that H is *unsplit* if it is proper. We say that it is *minimal* if, for a general point $x \in X$, H_x is proper.

Remark 6. Let $f : X \to Y$ be a proper morphism between complex quasi-projective varieties, and $y \in Y$ a point. Then there exists a natural map

$$\iota : \mathrm{RatCurves}^n(X_y) \to \mathrm{RatCurves}^n(X/Y).$$

Let H' be a family of rational curves on X_y, and let H be the irreducible component of $\mathrm{RatCurves}^n(X/Y)$ containing $\iota\big(H'\big)$. Then it may happen that $\iota^{-1}(H) \neq H'$. For instance, let $f : X \to Y$ be a family of quadric surfaces in $\mathbb{P}^3$, X_y a smooth fiber, and H' one of the two families of lines on X_y. If we choose $f : X \to Y$ suitably, then $\iota^{-1}(H)$ is the union of the two families of lines on X_y.

Let $C, C' \subset X_y$ be two curves parametrized by points of H. Then they are numerically equivalent on X. If the restriction morphism $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Q} \to \mathrm{Pic}(X_y)\otimes_{\mathbb{Z}}\mathbb{Q}$ is surjective, then they are also numerically equivalent on X_y.

This observation allows us to conclude in certain cases that $\iota^{-1}(H) = H'$. For instance, let $f : X \to Y$ be as above, and suppose that X_y is isomorphic to a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$, with $n \geq 3$. Let H' be the family of rational curves on X_y corresponding to lines on Q_n under the isomorphism $X_y \cong Q_n$. Then $\mathrm{Pic}(X_y) \cong \mathbb{Z}$, and thus the morphism $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}(X_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Moreover, any curve on Q_n that is numerically equivalent to a line is itself a line, and there is a unique family of lines on Q_n. Hence in this case $\iota^{-1}(H) = H'$.

Definition 7 (The H-rationally connected quotient). Let X be a normal complex projective variety. Suppose that X admits an unsplit covering family of rational curves H. We define an equivalence relation $\sim_H$ on X as follows:

$$x \sim_H y \iff x \text{ and } y \text{ can be connected by a chain of curves from } H.$$

By [Kol96, IV.4.16] (see also [Cam92]), there exists a proper surjective morphism $\pi^\circ : X^\circ \to Y^\circ$ from a dense open subset of X onto a normal quasi-projective variety whose fibers are $\sim_H$-equivalence classes. We call this map the *H-rationally connected quotient of* X. When Y° is a point we say that X is H-rationally connected.

Remarks 8. (1) If X is smooth, then, by [ADK07, Lemma 2.2], there is a morphism $\pi^\circ : X^\circ \to Y^\circ$ as above with the additional properties that $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$, Y° is smooth, and π° is an equidimensional proper morphism with irreducible and reduced fibers (see also [BCD07, Proposition 1] and [AW01, 3.1, 3.2]).

(2) If X is smooth and H_x is irreducible for general $x \in X$, then the general fiber of π° is a smooth Fano variety with Picard number 1 by [ADK07, Propopsition 2.3].

We end this section with a simple but useful observation.

Proposition 9. *Let $f : X \to C$ be a proper morphism from a normal quasi-projective variety onto a smooth quasi-projective curve. Suppose that the general fiber of f is a smooth Fano variety with Picard number 1, and that every fiber of f is irreducibe. Then $\rho(X/C) = 1$.*

Proof. Let H be a covering family of rational curves on X over C. Let $C^\circ \subset C$ be a dense open subset such that, for every $c \in C^\circ$, X_c is a smooth Fano variety with Picard number 1, and X_c contains a curve parametrized by some point of H. We note that, for every $c \in C^\circ$, $\mathrm{Pic}(X_c) \cong \mathbb{Z}$ and $N_1(X_c) = \mathbb{R} \cdot [\ell]$, where $[\ell] \in N_1(X_c)$ is the class of a curve $\ell \subset X_c$ parametrized by a point of H. Set $X^\circ = f^{-1}(C^\circ)$, and $f^\circ = f|_{X^\circ} : X^\circ \to C^\circ$.

First we observe that $\mathrm{Pic}(X^\circ) = (f^\circ)^* \mathrm{Pic}(C^\circ) \oplus \mathbb{Z} \cdot L^\circ$, where $L^\circ \in \mathrm{Pic}(X^\circ)$ is (f°)-ample. Indeed, let F denote a general fiber of f°, $r : \mathrm{Pic}(X^\circ) \to \mathrm{Pic}(F) \cong \mathbb{Z}$ the natural restriction map, and suppose that $M \in \mathrm{Pic}(X^\circ)$ is such that $r(M) = 0$. Then $M \cdot \ell = 0$ for every curve $\ell \subset X^\circ$ parametrized by some point of H. Thus $M|_{X_c} \cong \mathcal{O}_{X_c}$ for every $c \in C^\circ$. Therefore $(f^\circ)_* M$ is an invertible sheaf on C° (see for instance [Har77, III.12.9]), and there is an isomorphism $(f^\circ)^*(f^\circ)_* M \to M$, i.e., $M \in (f^\circ)^* \mathrm{Pic}(C^\circ)$.

Let L be an element of $\mathrm{Pic}(X)$ extending L°. Since all fibers of f are irreducible, $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = (f^* \mathrm{Pic}(C) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \mathbb{Q} \cdot L$ by [Har77, II.6.5]. From this it follows that $\rho(X/C) = 1$. $\square$

3. The variety of minimal rational tangents

Let X be a smooth complex projective uniruled variety, H a minimal covering family of rational curves on X, and $x \in X$ a general point.

Definition 10. We define the map $\tau_x : H_x \dashrightarrow \mathbb{P}(T_x X)$ by sending a curve that is smooth at x to its tangent direction at x. We define the *variety of minimal rational tangents* associated to H at x to be the closure of the image of τ_x in $\mathbb{P}(T_x X)$, and denote it by $\mathcal{C}_x$. Notice that $\mathcal{C}_x$ may be reducible.

Remarks 11. **(1)** Let $\tilde{H}_x$ be the normalization of H_x. By [Kol96, II.1.7, II.2.16], $\tilde{H}_x$ is a smooth projective variety. Moreover, the map $\tilde{\tau}_x : \tilde{H}_x \to \mathcal{C}_x$ induced by $\tau_x : H_x \dashrightarrow \mathcal{C}_x$ is the normalization morphism by [Keb02b] and [HM04]. This implies in particular that $\mathcal{C}_x$ cannot be a singular cone in $\mathbb{P}(T_x X)$ (see [Ara06, Lemma 4.3]).
(2) If $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_x X)$, then the intersection of any two irreducible components of $\mathcal{C}_x$ is empty by [ADK07, Lemma 2.8] (see also [Hwa07, Proposition 2.2]).

In the next section, we will investigate varieties X for which $\mathcal{C}_x$ is a positive dimensional hyperquadric in its linear span in $\mathbb{P}(T_x X)$. Remarks 11 above imply that in this case $\mathcal{C}_x$ is a smooth hyperquadric in its linear span in $\mathbb{P}(T_x X)$.

Our next goal is to explain how the variety of minimal rational tangents defines a distribution on X, and provide conditions under which this distribution is integrable. First let us recall the definition of distribution and Frobenius' criterion for integrability.

12 (**Distributions**)**.** Let X be a smooth complex projective variety. A *distribution* on X is a subbundle of the tangent bundle of a dense open subset X° of X, $E^\circ \hookrightarrow X^\circ$. We regard two distributions as being the same if they agree on the open subset where both are defined. The subbundle $E^\circ \hookrightarrow T_{X^\circ}$ can be extended to a subsheaf E of T_X. By abuse of notation we denote this distribution by $E \hookrightarrow T_X$.

We say that the distribution $E \hookrightarrow T_X$ is *integrable* if through every point $x \in X^\circ$ there is a complex analytic manifold $M \subset X^\circ$ such that $E_y = T_y M$ for every $y \in M$.

The Lie bracket defines a section of $Hom(\wedge^2 E^\circ, T_{X^\circ}/E^\circ)$, which we call the *Frobenius bracket tensor of* E,

$$[,] \ : \ \wedge^2 E^\circ \ \longrightarrow \ T_{X^\circ}/E^\circ.$$

By Frobenius' Theorem, the distribution $E \hookrightarrow T_X$ is integrable if and only if the Frobenius bracket tensor of E vanishes identically.

Now we fix notation and gather assumptions that will be used for the rest of the paper.

Notation 13. **(1)** Let X be a smooth complex projective variety, and H a minimal covering family of rational curves on X. Let $\mathcal{C}_x$ be the variety of minimal rational tangents associated to H at a general point $x \in X$, and denote by $\mathcal{E}_x$ its linear span in $\mathbb{P}(T_x X)$. Then we obtain a distribution $E \hookrightarrow T_X$ by setting E_x to be the linear subspace of $T_x X$ corresponding to $\mathcal{E}_x \subset \mathbb{P}(T_x X)$ for general $x \in X$. We denote by

$$[,]_x \ : \ \wedge^2 E_x \ \longrightarrow \ T_x X/E_x,$$

the Frobenius bracket tensor of E at a general point x, and by C_x the cone in E_x corresponding to $\mathcal{C}_x \subset \mathcal{E}_x$.

(2) Let V be a complex vector space, and $C \subset V$ a cone such that $\mathbb{P}(C)$ is an algebraic set in $\mathbb{P}(V)$. We define $\hat{C}$ to be the cone in $\wedge^2 V$ consisting of elements of the form $u \wedge v$, where u and v generate a linear subspace of V that is tangent to the cone C at a smooth point. Given $\omega \in \wedge^2 V^*$, we say that C is *isotropic with respect to* ω if $\omega(\xi) = 0$ for every $\xi \in \hat{C}$.

Remark 14. If V and C are as in Notation 13.2, $\omega \in \wedge^2 V^*$ is nondegenerate (i.e., a symplectic form), and C is isotropic with respect to ω, then $\dim C \le \frac{\dim V}{2}$.

Proposition 15 ([HM98, Proposition 9])**.** *Let X, H, $\mathcal{C}_x$, $\mathcal{E}_x$, $E \hookrightarrow T_X$, $[,]_x$ and $\hat{C}_x$ be as in Notation 13. Then the kernel of $[,]_x$ contains $\hat{C}_x$.*

In [HM98, Proposition 9] X is assumed to be a Fano variety, but this condition is not used in the proof.

The next sufficient condition for integrability of $E \hookrightarrow T_X$ is a straightforward consequence of Frobenius' Theorem, Proposition 15 and the following linear algebra lemma.

Proposition 16. *Let X, H, $\mathcal{C}_x$, $\mathcal{E}_x$ and $E \hookrightarrow T_X$ be as in Notation 13. If $\mathcal{C}_x$ is an irreducible hypersurface in $\mathcal{E}_x$ for general $x \in X$, then $E \hookrightarrow T_X$ is integrable.*

Lemma 17. *Let V, C and $\hat{C} \subset \wedge^2 V$ be as in Notation 13.2. If C is an irreducible nonlinear cone of codimension 1 in V, then $\hat{C}$ is nondegenerate in $\wedge^2 V$.*

Proof. Suppose that $\hat{C}$ is degenerate in $\wedge^2 V$. Then there is a nonzero element $\omega \in \wedge^2 V^*$ such that C is isotropic with respect to ω. Set

$$Q = \ker \omega = \left\{ v \in V \ \middle| \ \omega(v, u) = 0 \text{ for every } u \in V \right\},$$

and consider the natural projection $\pi : V \to V/Q$. The form ω induces a symplectic form ω_π on V/Q. Moreover the cone $\pi(C)$ is isotropic with respect to ω_π, and thus $\dim \pi(C) \leq \frac{\dim\ V/Q}{2}$ by Remark 14. Since C is nondegenerate, $\pi(C) \neq 0$, and since ω_π is nonzero, $\pi(C) \neq V/Q$. Therefore $\pi(C)$ has codimension 1 in V/Q. So we conclude that $\dim\ V/Q = 2$ and $\dim \pi(C) = 1$. Since $\pi(C)$ is an irreducible cone, it must be a 1-dimensional linear subspace in V/Q. Thus C is a hyperplane in V, which contradicts the nonlinearity of C. □

When X is a smooth Fano variety with Picard number 1, we have the following necessary condition for integrability of $E \hookrightarrow T_X$.

Proposition 18 ([Hwa98, Proposition 2]). *Let X, H and $\mathcal{C}_x$ be as in Notation 13, and assume moreover that X is a Fano variety with Picard number 1. Let $D \hookrightarrow T_X$ be a distribution on X such that $\mathcal{C}_x \subset \mathbb{P}(T_xD)$ for general $x \in X$. Then $D \hookrightarrow T_X$ is integrable if and only if $D_x = T_xX$ for general $x \in X$.*

Lemma 19. *Let X, H, $\mathcal{C}_x$, $\mathcal{E}_x$ and $E \hookrightarrow T_X$ be as in Notation 13. Let $\pi^\circ : X^\circ \to Y^\circ$ be the H-rationally connected quotient of X, and assume that the general fiber of π° is a Fano variety with Picard number 1. If $E \hookrightarrow T_X$ is integrable, then it coincides with the distribution defined by π°.*

Proof. Let F a general fiber of π°, and let $x \in F$ be a general point. Then $\text{locus}(H_x) \subset F$. The projectivized tangent cone to $\text{locus}(H_x)$ at x contains $\mathcal{C}_x$. Hence $\mathcal{C}_x \subset \mathbb{P}(T_xF)$, and thus $\mathcal{E}_x \subset \mathbb{P}(T_xF)$. Therefore, since the cokernel of $T_F \hookrightarrow T_X|_F$ is torsion free, the inclusion $E|_F \hookrightarrow T_X|_F$ factors through an inclusion $E|_F \hookrightarrow T_F$. If $E \hookrightarrow T_X$ is an integrable distribution on X, then $E|_F \hookrightarrow T_F$ is an integrable distribution on F.

By hypothesis F is a smooth Fano variety with Picard number 1. Consider the natural map $\iota : \text{RatCurves}^n(F) \to \text{RatCurves}^n(X)$. There is an irreducible component H_F of $\iota^{-1}(H)$ that is a minimal covering family of rational curves on F. Moreover, the variety of minimal rational tangents associated to H_F at a general point $x \in F$ is contained in $\mathcal{E}_x \subset \mathbb{P}(T_xF)$. So we can apply Proposition 18, and conclude that if $E|_F \hookrightarrow T_F$ is integrable, then $E_x = T_xF$ for F a general fiber of π° and x a general point of F. I.e., $E \hookrightarrow T_X$ coincides with the distribution defined by π°. □

4. Quadric bundles

We start this section by recalling the following characterization of quadric bundles due to Fujita. We say that a quadric bundle $\pi : X \to Y$ is a *geometric quadric bundle* if there exists a vector bundle V on Y such that X embeds into $\mathbb{P}(V)$ over Y as a divisor of relative degree 2.

Proposition 20 ([Fuj75, Corollary 5.5]). *Let X and Y be irreducible and reduced complex analytic spaces, and $\pi : X \to Y$ a proper and flat morphism whose fibers are all irreducible and reduced. Suppose that the general fiber of π is isomorphic to a hyperquadric $Q_n \subset \mathbb{P}^{n+1}$, and that there exists a π-ample line bundle L on X such that $L|_{X_t} \cong \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_{Q_n}$ for general $t \in Y$. Then π is a geometric quadric bundle.*

Proposition 21. *Let X be a smooth complex quasi-projective variety, B a smooth complex quasi-projective curve, and $\pi : X \to B$ a proper morphism with irreducible* *fibers. Suppose that the general fiber of π is isomorphic to a hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ with $n \geq 3$. Then π is a geometric quadric bundle.*

Proof. We want to apply Proposition 20 to $\pi : X \to B$. First we notice that, since X and B are smooth and π is proper and equidimensional, π is flat by [Gro66, 6.1.5]. Moreover, the morphism π admits a section $C \subset X$ by [GHS03], and thus its fibers are all reduced. So, in order to prove the lemma, we only need to produce a π-ample line bundle L on X whose restriction to a general fiber of π is isomorphic to $\mathcal{O}_{\mathbb{P}^{n+1}}(1)|_{Q_n}$.

Let F be a general fiber of π, and $H_F \subset \mathrm{RatCurves}^n(F)$ the unsplit covering family of rational curves on F corresponding to lines on Q_n under the isomorphism $F \cong Q_n$. Let H be the irreducible component of $\mathrm{RatCurves}^n(X/B)$ containing the image of H_F under the natural map $\iota : \mathrm{RatCurves}^n(F) \to \mathrm{RatCurves}^n(X/B)$. Since $\mathrm{RatCurves}^n(X/B)$ has countably many components and F is a general fiber of π, H is a covering family of rational curves on X over B. Moreover, by Remark 6, $\iota^{-1}(H) = H_F$. Consider the universal family morphisms (see Definition 5):

$$\begin{array}{ccc} U & \xrightarrow{\eta} & X. \\ {\scriptstyle p}\big\downarrow & & \\ H & & \end{array}$$

Let D be the unique irreducible component of the closure of $\eta\Big(p^{-1}\big(p(\eta^{-1}C)\big)\Big)$ in X that dominates B (with the reduced scheme structure). By construction, D is a Cartier divisor on X, and its restriction to a general fiber of π is a member of $\big|\mathcal{O}_{\mathbb{P}^{n+1}}(1)|_{Q_n}\big|$. By Proposition 9, $\rho(X/B) = 1$, and thus D is π-ample. $\square$

Remark 22. The condition $n \geq 3$ in Proposition 21 cannot be relaxed. Indeed, a smooth quadric surface Q_2 can be holomorphically deformed into any Hirzebruch surface $\mathbb{F}_a$ for a even.

Proof of the Main Theorem. Let the notation be as in Notation 13. By Remarks 11, $\mathcal{C}_x$ is a positive dimensional smooth hyperquadric in $\mathcal{E}_x$. In particular both $\mathcal{C}_x$ and H_x are irreducible.

Let $\pi^\circ : X^\circ \to Y^\circ$ be the H-rationally connected quotient of X, where Y° is smooth, $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$, and π° is an equidimensional proper morphism with irreducible and reduced fibers (see Remark 8.1). The general fiber F of π° is a smooth Fano variety with Picard number 1 by Remark 8.2. Set $n = \dim F$.

By Proposition 16, $E \hookrightarrow T_X$ is integrable. Hence, by Lemma 19, it coincides with the distribution defined by π°. Therefore the general fiber F of π° admits an unsplit family of rational curves whose associated variety of minimal rational tangents at a general point $x \in F$ is a positive dimensional smooth hyperquadric in $\mathbb{P}(T_xF)$. By Theorem 2, $F \cong Q_n$, with $n \geq 3$.

Let $B \subset Y^\circ$ be a curve obtained as a complete intersection of general very ample divisors on Y°. Set $X_B = \pi^{-1}(B)$, and $\pi_B = \pi|_{X_B} : X_B \to B$. By Bertini's Theorem, both B and X_B are smooth. Moreover, the general fiber of π_B is isomorphic to Q_n, with $n \geq 3$, and every fiber is irreducible. Thus π_B is a (geometric) quadric bundle by Proposition 21. Therefore the locus S of Y° over which π° is not a quadric bundle has codimension at least 2. By replacing Y° with $Y^\circ \setminus S$ we get the desired statement. $\square$

Acknowledgements. I would like to thank Giuseppe Borrelli and Stéphane Druel for many fruitful discussions about some of the topics that appear in this paper. Partial financial support has been provided by CNPq – Brazil.

References

[ADK07] C. Araujo, S. Druel, and S. Kovács, *Cohomological characterizations of projective spaces and hyperquadrics*, pre-print math.AG/0707.4310. (To appear in Invent. Math.), 2007.

[Ara06] C. Araujo, *Rational curves of minimal degree and characterizations of projective spaces*, Math. Ann. **335** (2006), no. 4, 937–951.

[AW01] M. Andreatta and J. A. Wiśniewski, *On manifolds whose tangent bundle contains an ample subbundle*, Invent. Math. **146** (2001), no. 1, 209–217.

[BCD07] L. Bonavero, C. Casagrande, and S. Druel, *On covering and quasi-unsplit families of rational curves*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 45–76.

[BI08] M. C. Beltrametti and P. Ionescu, *On manifolds swept out by high dimensional quadrics*, Math. Z. (2008), To appear.

[Cam92] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 539–545.

[CMSB02] K. Cho, Y. Miyaoka, and N. I. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds*, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 1–88.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[Fuj75] T. Fujita, *On the structure of polarized varieties with Δ-genera zero*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 103–115.

[GHS03] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic).

[Gro66] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[HM98] J.-M. Hwang and N. Mok, *Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation*, Invent. Math. **131** (1998), no. 2, 393–418.

[HM04] ———, *Birationality of the tangent map for minimal rational curves*, Asian J. Math. **8** (2004), no. 1, 51–64.

[Hwa98] J.-M. Hwang, *Stability of tangent bundles of low-dimensional Fano manifolds with Picard number* 1, Math. Ann. **312** (1998), no. 4, 599–606.

[Hwa01] ———, *Geometry of minimal rational curves on Fano manifolds*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 335–393.

[Hwa07] ———, *Deformation of holomorphic maps onto Fano manifolds of second and fourth Betti numbers 1*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 3, 815–823.

[Keb02a] S. Kebekus, *Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 147–155.

[Keb02b] ———, *Families of singular rational curves*, J. Algebraic Geom. **11** (2002), no. 2, 245–256.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer-Verlag, Berlin, 1996.

[Mok05] N. Mok, *Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents*, pre-print 09 of IMR Preprint Series, 2005.

[Mor79] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), no. 3, 593–606.

Carolina Araujo: IMPA, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brazil
E-mail address: caraujo@impa.br