LARGE VISCOUS SOLUTIONS FOR SMALL DATA IN SYSTEMS OF CONSERVATION LAWS THAT CHANGE TYPE

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Abstract. We study a quadratic system of conservation laws with an elliptic region. The second order terms in the fluxes correspond to type IV in Schaeffer and Shearer classification. There exists a special singularity for the EDOs associated to traveling waves for shocks. In our case, this singularity lies on the elliptic boundary. We prove that high amplitude Riemann solutions arise from Riemann data with arbitrarily small amplitude in the hyperbolic region near the special singularity. For such Riemann data there is no small amplitude solution. This behavior is related to the bifurcation of one of the codimension-3 nilpotent singularities of planar ODEs studied by Dumortier, Roussarie and Sotomaior.

1. Introduction

A famous theorem of Lax [8] states that systems of n conservation laws with small data have Riemann solution consisting of n small waves, rarefactions or shocks, separated by constant states, under certain hypotheses. What happens if the hypotheses are violated? T.P. Liu [9] showed in 1974 that if the hypothesis of genuine nonlinearity fails, the rarefactions and shocks can join. Still, they form n wave groups separated by $n-1$ constant states.

In this work, we accept as physically admissible shocks that are the limits of traveling waves and we find an example of a system of two equations for which the Riemann solution consists of two shocks with $O(1)$ amplitude no matter how small the data is, provided it is close to a special point on the locus where the characteristic speeds coincide; this locus is the boundary of the elliptic region. Of course, our data does not admit local solutions.

Though our example occurs in a system with quadratic flux functions, such a point exists generically for systems that change from hyperbolic to elliptic type. This point is associated to a local bifurcation of the traveling wave ODE for conservation laws, studied by Dumortier, Roussarie and Sotomaior in [4]. At this point, which we call DRS point, three equilibria collapse and one of the four equilibria of the quadratic ODE stays away. In the classification of DRS, this is called an elliptic bifurcation. Thus the existence of large Riemann solutions for small data is generic.

Dumortier, Roussarie and Sotomaior studied three types of codimension-three bifurcations for planar vector fields: elliptic, saddle and focus. Azevedo, Marchesin, Plohr and Zumbrum in [1] proved that saddle bifurcation are associated to the existence of local Riemann solutions containing three waves for systems of two conservation laws. This solution has more waves than dimensions, and one of these waves is not a Lax wave. In [1] it was also proved that foci bifurcations do not occur for ODEs originating from systems of two quadratic conservation laws. Therefore, we conjecture that for such systems the consequences of violations of the Lax theorem hypotheses are understood now.

Section 2 is divided in two subsections; in the first one, we define rarefactions, shocks, the Rankine-Hugoniot set and its classification, and we present a version of the famous Lax theorem; Lax 1-shocks and 2-shocks are also defined. In the second subsection we define wave groups and wave speed compatibility; then, we present two new theorems on wave speed compatibility. The first one is on the composite locus defined by Liu [9] as the right states of rarefaction-shock pairs with no intermediate states: the composite curve forms an envelope for these shocks. The other result determines when a 1-shock can not be followed by a 2-shock. These results describe possible structures for non-local solutions.

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In Section 3 we present our main result, the existence of large viscous solutions for small data. We state this result in two theorems, the first for a left state at a special point on the boundary of the elliptic region, the second for the left state in a open set in the hyperbolic region; in both cases the right states lie in open sets in the hyperbolic region. Proofs are given in Section 4, which is divided in three subsections. In the first two subsections we prove the existence of Riemann problems with small data with shocks that are necessarily large. In the third subsection we prove that the Riemann problems for small data do not admit solutions with small rarefactions followed by shocks. In Section 4 we do not prove mathematically the existence of viscous profiles for the shocks, rather the existence of the viscous profiles is based in analytic and numerical considerations. An overview of the Riemann solution for any right state is presented in Section 5. In the Section 6 we present some remarks about our results.

2. Background

In this section we review some results for systems of two conservation laws in one space dimension and present two new results, Theorems 2.16 and 2.22, which provide us with typical parts of the global solutions of the Riemann problem (RP). These systems are partial differential equations of the form:

$$U_t + F(U)_x = 0, \tag{2.1}$$

where $U(x,t) = (u,v)^T \in \mathbb{R}^2$ for $x \in \mathbb{R}$ and $t \geq 0$, $F \in C^2(\mathbb{R}^2, \mathbb{R}^2)$. Smooth solutions of (2.1) satisfy $U_t + DF(U)U_x = 0$.

Definition 2.1. The set of U in $\mathbb{R}^2$ where $DF(U)$ has:
i) two distinct real eigenvalues is called the *strictly hyperbolic region*;
ii) two distinct complex conjugate eigenvalues is called the *elliptic region*;
iii) one double real eigenvalue is called the *coincidence locus*.

In the strictly hyperbolic region the characteristic speeds of $DF(U)$ are ordered so that the lowest is called 1-speed, $\lambda_1(U)$, and the highest is called 2-speed, $\lambda_2(U)$. The corresponding eigenvectors are $\vec{r}_1(U)$ and $\vec{r}_2(U)$.

Definition 2.2. The set of U in $\mathbb{R}^2$ in the strictly hyperbolic region where $\nabla\lambda_i \cdot \vec{r}_i \neq 0$, for $i = 1, 2$, is classically called strictly hyperbolic genuinely nonlinear region [8]. In this work we utilize the acronym *shgnr*.

A RP is an initial value problem with constant data on the left and right hand sides of the origin, called U_L and U_R, that is

$$U(x,0) = U_L \text{ if } x < 0 \quad \text{and} \quad U(x,0) = U_R \text{ if } x > 0. \tag{2.2}$$

2.1. **Centered Waves.** The solutions of (2.1) and (2.2) are centered, i.e., for $t > 0$ they depend only on the speed $\xi = x/t$. So, smooth solutions, for $t > 0$, satisfy $(DF(U) - \xi)U_\xi = 0$.

Definition 2.3. The centered smooth solutions on the hyperbolic region are:
i) 1-rarefactions if $U_\xi = \vec{r}_1(U)$ and $\xi = \lambda_1(U)$;
ii) 2-rarefactions if $U_\xi = \vec{r}_2(U)$ and $\xi = \lambda_2(U)$.

We say that an i-rarefaction curve from U_0 is the set of U states on an i-rarefaction solution in Definition 2.3 satisfying $\lambda_i(U_0) \leq \lambda_i(U)$. Rarefaction curves parametrize rarefaction waves.

It is well known that nonlinear conservation laws lead to discontinuous solutions. Following Gel'fand [6] and Courant-Friedrichs [3], we require that the shocks are limits as $\epsilon \searrow 0$ of traveling waves $U(x,t) = \bar{U}(\eta)$, $\eta = (x - st)/\epsilon$, of the equation

$$U_t + F(U)_x = \epsilon U_{xx} \tag{2.3}$$

with $\lim_{\eta\to\pm\infty} U(\eta) = U_\pm$, i.e., we impose that the associated ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{2.4}$$

has an orbit "connecting" the equilibria U_- to U_+. In this case we say that the discontinuity is admissible, or that it has a viscous profile or that it forms a shock. So, if there is no orbit starting at U_- and finishing at U_+ we say that the discontinuity is inadmissible or that it has no viscous profile, and we do not call it shock. In particular, each discontinuity must satisfy the following two Rankine-Hugoniot (RH) conditions

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0, \tag{2.5}$$

where U_- and U_+ are, respectively, the left and right states of the discontinuity and s is its speed. We denote the shock by (U_-, U_+, s) or (U_-, U_+) and the discontinuity speed by $s(U_-, U_+)$ or just s.

Definition 2.4. The RH set for a fixed U_- is a one-dimensional set in U-space:

$$\mathcal{H}(U_-) = \left\{U_+ \in \mathbb{R}^2 : \exists s \in \mathbb{R} \text{ such that equation (2.5) holds}\right\}. \tag{2.6}$$

The RH set of U_- is typically formed by (possibly disconnected) curves with self-intersections at some points.

Motivated by Lax [8] and Conley-Smoller [2], we classify shocks with viscous profiles based on the type of the equilibria U_- and U_+. Important equilibria in our work are: (i) repellers, with two positive eigenvalues (or positive real part); (ii) saddles, with one positive and one negative eigenvalue; (iii) attractors, with two negative eigenvalues (or negative real part); (iv) repeller-saddles, with one positive eigenvalue and one zero eigenvalue; (v) saddle-attractors, with one negative eigenvalue and one zero eigenvalue. (In this work, unless specified otherwise, repeller-saddles and saddle-attractors are always non degenerate, see [12].)

Some shocks appearing in Riemann solutions are:

Definition 2.5. (i) 1-shocks: U_- is a repeller and U_+ is a saddle (1S in the figures); (ii) 2-shocks: U_- is a saddle and U_+ is an attractor (2S in the figures).

Remark 2.6. Lax's famous shock inequalities arise from the observation that in the hyperbolic region the eigenvalues of the linearization of the ODE (2.4) at the equilibria $U_\pm$ are $\lambda_1(U_\pm) - s$ and $\lambda_2(U_\pm) - s$.

There are other discontinuities that typically have viscous profile, as it happens in our work. These shocks are:

Definition 2.7. i) over-compressive shocks (O-shocks): U_- is a repeller and U_+ is an attractor;
ii) left-characteristic 1-shocks (L1-shocks): U_- is a repeller-saddle and U_+ is a saddle;
iii) right-characteristic 2-shocks (R2-shocks): U_- is a saddle and U_+ is a saddle-attractor;
iv) left-characteristic over-compressive shocks (LO-shocks): U_- is a repeller-saddle and U_+ is an attractor;
v) right-characteristic over-compressive shocks (RO-shocks): U_- is a repeller and U_+ is a saddle-attractor.

We say that 1-rarefactions, 1-shocks and L1-shocks are 1-waves (or waves of the 1-family) while 2-rarefactions, 2-shocks and R2-shocks are 2-waves (or waves of the 2-family). We say also that a shock is left-characteristic if s equals an eigenvalue of $DF(U_-)$ and that it is right-characteristic if s equals an eigenvalue of $DF(U_+)$.

The following type of discontinuity helps us in locating L1 and R2-shocks:

Definition 2.8. Crossing discontinuities (X-disc.): U_- and U_+ are saddles.

Remark 2.9. In bifurcation theorems for ODEs, generically, there are no connections between saddles and this is the case in our work (see [1]). (When there exists a connection between the left and right saddles the X-disc. is known as transitional or under-compressive shock.)

Each point of the RH set $\mathcal{H}$ is classified according to the Definitions 2.5, 2.7 and 2.8. Typically, there are connected parts in $\mathcal{H}$ consisting of 1-shocks, 2-shocks, O-shocks and inadmissible discontinuities (such as X-disc.). Similarly, there are isolated points in $\mathcal{H}$ representing characteristic shocks. The characteristic shocks in the Definition 2.7 separate parts of $\mathcal{H}$ with distinct types defined by Table 1. For left-characteristic shocks the speed increases from type I to type II; note that $s < \lambda_1(U_L)$ in type I and $s > \lambda_1(U_L)$ in type II. At right-characteristic shocks the speed s is critical (see Bethe-Wendroff [15]).

Type I	Characteristic shock	Type II
1-shock	L1-shock	X-disc.
O-shock	LO-shock	2-shock
1-shock	RO-shock	O-shock
X-disc.	R2-shock	2-shock

TABLE 1. A characteristic shock appears separating type I and II parts of $\mathcal{H}$.

We now state a consequence of Lax's classical theorem for systems of two equations with smooth fluxes in a sufficiently small neighborhood N with closure $\bar{N} \subset B$, an open region in the *shgnr*. For the complete statement of Lax's Theorem see [8] or [13]; for admissibility see [2].

Theorem 2.10. *Given any U_L and U_R in N, there exist two arcs, tangent to $\vec{r}_1$ at U_L and to $\vec{r}_2$ at U_R, which intersect transversally at U_M in B. These arcs parameterize shock or rarefaction waves, see Fig. 1. The 1-wave curve segment from U_L to U_M followed by the 2-wave curve segment from U_M to U_R parameterize the unique solution of the RP with data U_L, U_R.*

Corollary 2.11. Let U_M be the middle state of the RP with data U_L, U_R in Theorem 2.10. Then $|U_M - U_L| \searrow 0$ (and $|U_M - U_R| \searrow 0$) as $|U_R - U_L| \searrow 0$.

Remark 2.12. The neighborhood N must be small enough to ensure that the curves from any U_L, U_R emanating from N also lie in the *shgnr*, including their intersections U_M.

2.2. **Wave Groups and Compatibility.** A global solution of a RP may contain several centered waves, i.e., rarefactions and shocks. We use the convention of describing the solution from the left to the right, i.e., with increasing speed. A sequence of two waves, w_a followed by w_b, is compatible if the speed of w_a is less than or equal to the speed of w_b. A compatible sequence of i-waves is an i-wave group or an i-group. The speed of an i-rarefaction at U (in the hyperbolic region) is greater than the speed of an i-shock with $U_- = U$ and is smaller than the speed of an i-shock with $U_+ = U$; so, only characteristic i-shocks can join the i-rarefactions in order to form i-groups. We present two such wave groups, which are important for this paper:

Definition 2.13. (i) 1-RS: 1-rarefaction from U_L to U_M followed by a L1-shock from U_M to U_R;
(ii) 2-SR: R2-shock from U_L to U_M followed by a 2-rarefaction from U_M to U_R.

Remark 2.14. The wave groups 1-SR and 2-RS are defined similarly. There exist groups with more than two i-waves, see [9]. However, they do not appear in this paper.

The next lemma establishes which compatible shocks can follow a 1-rarefaction.

Lemma 2.15. *Let U_L be in the shgnr. Let U_M^* be the family of states that lie on the 1-rarefaction from U_L. If $U_R^* \in \mathcal{H}(U_M^*)$, the sequence of a 1-rarefaction from U_L to U_M^* followed by a shock (U_M^*, U_R^*, s) is:*
i) compatible if (U_M^*, U_R^*, s) is a 2-shock, a L1-shock or a LO-shock;
ii) incompatible if (U_M^*, U_R^*, s) is a 1-shock or a O-shock.

Proof. The speed of the 1-rarefaction at U_M^* is $\lambda_1(U_M^*)$ and:
A) for 2-shocks $\lambda_1(U_M^*) < s$, for L1-shocks and LO-shocks $\lambda_1(U_M^*) = s$, so (i) holds;
B) for O-shocks and 1-shocks $s < \lambda_1(U_M^*)$, so (ii) holds. □

The locus of right states for a rarefaction from U_L followed by a shock with no intermediate state forms the composite wave and was already studied in [9] and [5]. The following new result shows that the composite curve is the envelope of RH loci.

Theorem 2.16. *Let U_L and U_M^* be defined as in the Lemma 2.15. Assume that there exists a L1 (or LC) shock $(U_L, U_R, \lambda_1(U_L))$; this shock joins two distinct parts of $\mathcal{H}(U_L)$ of types I and II as described in Table 1. Then, by continuity, for each U_M^* close to U_L, there also exists a left-characteristic L1 (or LC) shock $(U_M^*, U_R^*, \lambda_1(U_M^*))$ that joins two parts of $\mathcal{H}(U_M^*)$ of the same types I and II.*

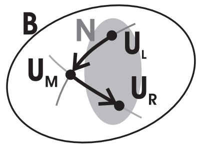

FIGURE 1. The transverse set of curves near U_L and the middle point U_M of the Riemann problem solution with data U_L, U_R.

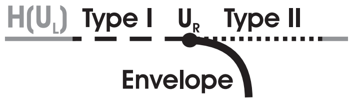

FIGURE 2. The Type I part of $\mathcal{H}(U_L)$ and the envelope join differentiably.

We have:

i) The family of states U_R^* forms a curve that is the envelope of the family of curves $\mathcal{H}(U_M^*)$;

ii) The envelope starts at U_R and is tangent to $\mathcal{H}(U_L)$ at U_R. The type I part of $\mathcal{H}(U_L)$ and the envelope join differentiably at U_R; the type II part of $\mathcal{H}(U_L)$ and the envelope form a cusp at U_R, see Fig. 2.

Proof. Differentiating (2.5) with $U_- = U_L$, $U_+ = U_R$ we obtain

$$\big(DF(U_R) - sI\big)dU_R - (U_R - U_L)ds = 0. \tag{2.7}$$

For L1 and LO-shocks $s(U_L, U_R)$ is not an eigenvalue of $DF(U_R)$, so the implicit function theorem establishes that $\mathcal{H}(U_L)$ is a curve near U_R. By continuity, for U_M^* close to U_L, $\mathcal{H}(U_M^*)$ is also a curve near U_R^*.

The states U_R^* are solutions of $F(U_R^*) - F(U_M^*) - \lambda_1(U_M^*)(U_R^* - U_M^*) = 0$, so

$$\big(DF(U_R^*) - \lambda_1(U_M^*)I\big)dU_R^* - \big(DF(U_M^*) - \lambda_1(U_M^*)I + (U_R^* - U_M^*)\nabla\lambda_1(U_M^*)\cdot\big)dU_M^* = 0$$

is obtained by differentiation. Taking dU_M^* equal to the eigenvector $\vec{r}_1(U_M^*)$ normalized so that $\nabla\lambda_1(U_M^*)\cdot\vec{r}_1(U_M^*) = 1$ in order to follow the direction of increasing speed along the 1-rarefaction, this equation becomes

$$\big(DF(U_R^*) - \lambda_1(U_M^*)I\big)dU_R^* - (U_R^* - U_M^*) = 0. \tag{2.8}$$

So, as U_M^* moves along the 1-rarefaction with velocity $\vec{r}_1(U_M^*)$ the corresponding U_R^* moves with velocity $\vec{e}^* = dU_R^*$ that satisfies Equation (2.8) and U_R^* form the composite wave. Since the tangent vector to the curve $\mathcal{H}(U_M^*)$ at U_R^* satisfies Equation (2.8), the curve $\mathcal{H}(U_M^*)$ is tangent to the composite curve at each corresponding U_R^*, so the composite curve is the envelope of the family of curves $\mathcal{H}(U_M^*)$, i.e., (i) holds.

Taking $U_M^* = U_L$, $U_R^* = U_R$ and $dU_R^* = \vec{e}$ Equation (2.8) becomes

$$(DF(U_R) - \lambda_1(U_L)I)\vec{e} - (U_R - U_L) = 0. \tag{2.9}$$

For $s = \lambda_1(U_L)$, $dU_R = \vec{e}$ and $ds = 1$, equation (2.7) becomes (2.9). So, the vector $\vec{e}$ points in the direction of increasing speed along $\mathcal{H}(U_L)$. The shock speed increases from the type I to the type II shocks, so that (ii) holds. □

The Theorem 2.16 describes two possible structures for nonlocal solutions.

Corollary 2.17. After the 1-rarefaction the only compatible shocks of the 1-family are left-characteristic 1-shocks (L1-shocks) which lie exactly on the envelope and form the so called 1-composite wave (see Fig. 3.a).

Corollary 2.18. After the 1-rarefaction the compatible LO-shocks form an envelope that is a boundary for Riemann solutions. This envelope separates a region that is reached by a sequence of a 1-rarefaction followed by a 2-shocks from another region that is not reached by such kind of sequence (see Fig. 3.b).

Remark 2.19. Note that after a 1-wave a small 2-wave is always compatible, so the envelope mentioned in Corollary 2.17 (the 1-composite wave) does not form a boundary for Riemann solutions.

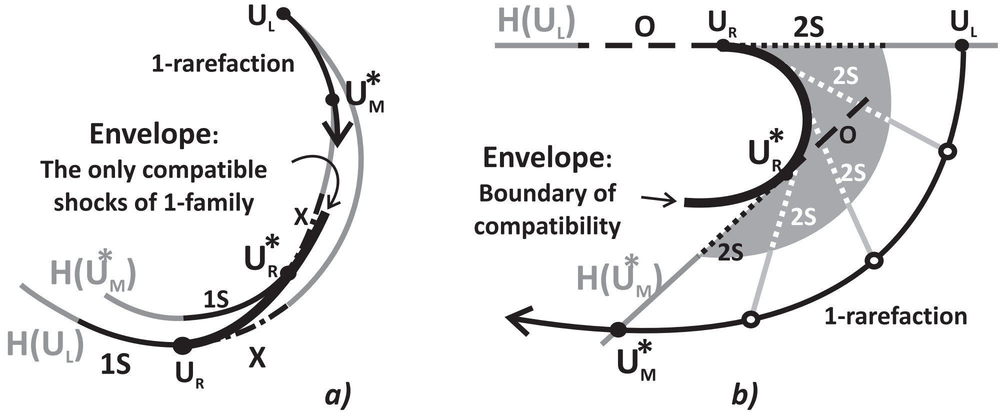

FIGURE 3. The envelope of $\mathcal{H}(U_M^*)$. After an 1-rarefaction a) L1-shocks are compatible while 1-shocks are incompatible; b) 2-shocks and LO-shocks are compatible while O-shocks are incompatible.

Remark 2.20. A left-characteristic 2-shock, or L2-shock, is a shock such that U_- is a saddle-attractor and U_+ is an attractor. If $\mathcal{H}(U_L)$ contains a L2-shock and U_M^* denote the states on the 2-rarefaction from U_L then an envelope is also formed by the $\mathcal{H}(U_M^*)$ representing 2-composite waves. This result holds by an argument similar to that in Theorem 2.16.

Generally, a 1-shock can be followed by a 2-shock without violating the speed compatibility condition. However, this is not always true.

Lemma 2.21. *Let U_M be in the shgnr. There exists a 2-shock part of $\mathcal{H}(U_M)$ having U_M as beginning point; this 2-shock part is called $\mathcal{S}_2$. If $U_R \in \mathcal{S}_2$ then (see Figure 4): (i) $\mathcal{S}_2$ is divided in two pieces by U_R, one adjacent to U_M, $\mathcal{S}_2^{ad}$, and other away from U_M, $\mathcal{S}_2^{aw}$; (ii) the shocks with end state in the $\mathcal{S}_2^{ad}$ piece are faster than the shock (U_M, U_R); (iii) the shocks with end state in the $\mathcal{S}_2^{aw}$ piece are slower than the shock (U_M, U_R).*

Proof. There exists a local 2-shock part of $\mathcal{H}(U_M)$ having U_M as beginning point, so $\mathcal{S}_2$ exists, see [13]; locally, the speed decreases away from U_M. The Bethe-Wendroff theorem [15] ensures that the speed decreases (at least) until it equals a local characteristic speed, where the shock ceases to be a 2-shock, so Lemma 2.21 holds. □

Theorem 2.22. *Let U_L, U_M, U_R be points that do not lie on a straight line, $U_M \in \mathcal{H}(U_L)$, $U_R \in \mathcal{H}(U_L)$, $U_R \in \mathcal{H}(U_M)$ and U_M, U_R, $\mathcal{S}_2$, $\mathcal{S}_2^{ad}$ and $\mathcal{S}_2^{aw}$ as in Lemma 2.21. Then the following facts hold:*
i) $\mathcal{H}(U_L)$ and $\mathcal{H}(U_M)$ are transversal at U_R;
ii) If $U_R^* \in \mathcal{S}_2^{ad}$ the sequence of shocks (U_L, U_M, s), (U_M, U_R^*, s^*) is compatible; if $U_R^* \in \mathcal{S}_2^{aw}$ the sequence of shocks (U_L, U_M, s), (U_M, U_R^*, s^*) is incompatible (see Figure 5).

Proof. Since U_L, U_M, U_R do not lie on a straight line the triple shock rule [14] says that $s_0 = s(U_M, U_R) = s(U_L, U_M) = s(U_L, U_R)$.

Since (U_M, U_R, s_0) is a 2-shock, s_0 is not an eigenvalue of $DF(U_R)$, then the implicit function theorem says that the sets $\mathcal{H}(U_L)$ and $\mathcal{H}(U_M)$ are parametrized curves near U_R. Let $\dot{U}$ $(\dot{u})$ be a vector tangent to the curve $\mathcal{H}(U_L)$ $(\mathcal{H}(U_M))$ at U_R and $\dot{\Sigma}$ $(\dot{\sigma})$ the derivative of the shock speed $\Sigma = s(U_L, U_R)$ $(\sigma = s(U_M, U_R))$ at U_R, respectively. (The Bethe-Wendroff theorem [15] ensures that $\dot{\Sigma} \neq 0$, $\dot{\sigma} \neq 0$.)

We parameterize both curves and define the angles formed by the u-axis and the vectors $\dot{U}$ and $\dot{u}$ so that they lie in the interval $[0, \pi)$ and $|\dot{U}| = |\dot{u}| = 1$. We have: $(DF(U_R) - s_0 I)\dot{U} - (U_R - U_L)\dot{\Sigma} = 0$ and $(DF(U_R) - s_0 I)\dot{u} - (U_R - U_M)\dot{\sigma} = 0$ then $(DF(U_R) - s_0 I)(\dot{U} - \dot{u}) = (U_R - U_L)\dot{\Sigma} - (U_R - U_M)\dot{\sigma}$. Since neither $\dot{\Sigma}$ nor $\dot{\sigma}$ are zero and $U_L - U_M$, $U_M - U_R$ are not parallel, then $(U_R - U_L)\dot{\Sigma} - (U_R - U_M)\dot{\sigma} \neq 0$; thus $\dot{U} \neq \dot{u}$, i.e., $\mathcal{H}(U_L)$ and $\mathcal{H}(U_M)$ are transverse at U_R, (i) is proved.

FIGURE 4. The pieces $\mathcal{S}_2^{ad}$, $\mathcal{S}_2^{aw}$ of $\mathcal{S}_2$.

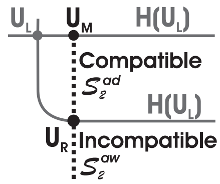

FIGURE 5. The shock sequence (U_L, U_M, s), (U_M, U_R^*, s^*) is compatible if $U_R^* \in \mathcal{S}_2^{ad}$ and it is incompatible if $U_R^* \in \mathcal{S}_2^{aw}$.

Since $s(U_L, U_M) = s(U_M, U_R)$, Lemma 2.21 ensures that (ii) holds. □

The following sequences of shocks with equal speeds s_0 are fundamental in our work:

i) (U_L, U_M) is a L1-shock, (U_L, U_R) is a LO-shock and (U_M, U_R) is a 2-shock (see Figure 6, case U_{M_1});

ii) (U_L, U_M) is a 1-shock, (U_L, U_R) is a O-shock and (U_M, U_R) is a 2-shock (see Figure 6, case U_{M_2}).

iii) We need also the following consequence of the triple shock rule and Bethe-Wendroff theorem: assume that (U_L, U_M) is a 1-shock and (U_L, U_R) is a RO-shock; then, $\mathcal{H}(U_L)$ and $\mathcal{H}(U_M)$ are tangent at U_R and (U_M, U_R) is a R2-shock that may be followed by a 2-rarefaction (see Figure 6, case U_{M_3}).

3. The Local Riemann Problem with Non Local Solution

We study a model of type IV in Schaeffer and Shearer's [10] classification with the flux function

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3u^2 + v^2 \\ 2uv \end{pmatrix} + \begin{pmatrix} 2v \\ 0 \end{pmatrix}. \tag{3.1}$$

We have set $a = 3$ and $b = 0$ in the normal form given in [10]. We expect that other type IV models with nearby parameters lead to similar results.

The eigenvalues of the Jacobian of the flux are $\lambda_1 = 2u - \sqrt{u^2 + (v+1)^2 - 1}$ and $\lambda_2 = 2u + \sqrt{u^2 + (v+1)^2 - 1}$, so $\lambda_1 = \lambda_2$ along the circle $u^2 + (v+1)^2 = 1$, the coincidence locus. The interior of this circle is the elliptic region in this model.

We show that non local solutions arise from RPs with arbitrarily small data. (In Section 4 we show that this RP does not have local solutions.) This result is stated in the following theorems.

Theorem 3.1. *Let $\mathcal{O}$ be $(0,0)$. There exists an open set B in the strictly hyperbolic region (with $\mathcal{O}$ on ∂B) with the following property. Given a small $\beta > 0$, for any $U_R \in B$ with $|U_R - \mathcal{O}| < \beta$ the solution of the RP with data $(U_L = \mathcal{O}, U_R)$ has amplitude close to 4, consisting of two Lax shocks that are limits of traveling waves.*

This behavior can be extended for U_L, U_R in open sets near $\mathcal{O}$ in the hyperbolic region. For each β let $T(\beta)$ be the open triangle in the hyperbolic region

$$T(\beta) = \left\{(u,v) \in \mathbb{R}^2 : 0 < v < \beta^2/9 \text{ and } -v < u < v\right\}. \tag{3.2}$$

The choice $\beta^2/9$ is explained in the proof of Lemma 4.1.

Theorem 3.2. *Let be $\beta \gtrsim 0$. For every $U_L \in T(\beta)$ there is a non empty open set $A(U_L, \beta)$ with the following properties:*

i) $A(U_L, \beta)$ lies in the strictly hyperbolic region;

ii) The distance of all points of $A(U_L, \beta)$ to U_L is smaller than β;

iii) For all U_R in $A(U_L, \beta)$ the solution of the RP with data (U_L, U_R) has amplitude larger than 4: this solution consists of two Lax shocks;

iv) For all U_R in $A(U_L, \beta)$ there are no small solution of the RP with data (U_L, U_R).

We remark that both $T(\beta)$ and $A(U_L, \beta)$ approach $\mathcal{O}$ as β goes to zero.

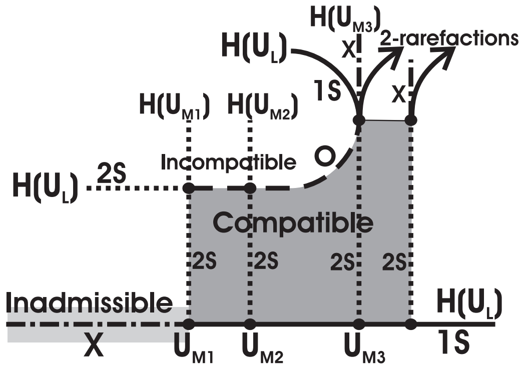

FIGURE 6. Typical configurations of global solutions of RP.

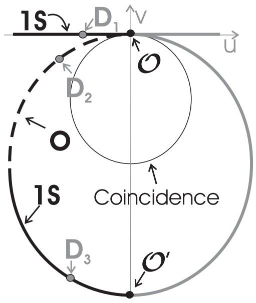

FIGURE 7. The curve $\mathcal{H}(\mathcal{O})$. The 1-shocks: solid curve; O-shocks: dashed.

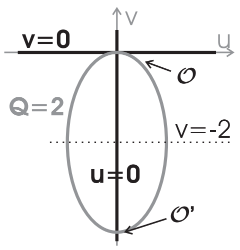

FIGURE 8. Quadratic curves for $U_- = \mathcal{O}$ and $s = 0$.

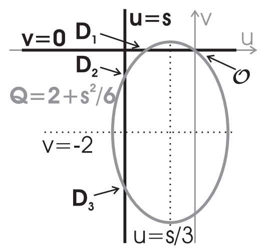

FIGURE 9. Quadratic curves for $U_- = \mathcal{O}$ and $s \lesssim 0$.

4. PROOF OF THE THEOREMS

The proof of the Theorems 3.1 and 3.2 is divided in three parts. In subsections 4.1 and 4.2 we analyze the solutions formed by a sequence of two shocks, under the conditions of the Theorem 3.1 and 3.2, respectively. We prove that only high amplitude shocks satisfy the compatibility condition.

In the subsection 4.3 we study solutions formed by small 1-rarefactions followed by 2-shocks and we prove that they do not provide solutions for our RP.

The RPs in the Theorems 3.1 and 3.2 do not include 2-rarefactions because in a neighborhood of U_R the speed along 2-rarefactions starting on a 1-wave curve decreases towards U_R.

4.1. **Proof of Theorem 3.1 for sequences of shocks.** Substituting (3.1) in the RH relation (2.5) yields

$$-s(u_+ - u_-) + 3(u_+^2 - u_-^2)/2 + (v_+^2 - v_-^2)/2 + 2(v_+ - v_-) = 0, \tag{4.1a}$$

$$-s(v_+ - v_-) + u_+v_+ - u_-v_- = 0. \tag{4.1b}$$

Fixing (u_-, v_-), these curves are conic sections in the variables (u_+, v_+), so there are 0, 2 or 4 intersections or zeros counting multiplicity. Since $U_+ = U_-$ is always a solution of (4.1), there are 2 or 4 zeros.

For $U_- = \mathcal{O} = (0, 0)$, Eqs. (4.1) reduce to:

$$Q \equiv \tfrac{3}{2}(u_+ - \tfrac{s}{3})^2 + \tfrac{1}{2}(v_+ + 2)^2 = 2 + \tfrac{s^2}{6} \quad \text{and} \quad (u_+ - s)v_+ = 0. \tag{4.2}$$

The RH locus $\mathcal{H}(\mathcal{O})$ defined in (2.6) consists of the horizontal axis $v_+ = 0$ together of the circle $u_+^2 + (v_+ + 2)^2 = 4$. On the horizontal axis the shock velocity is given by $s = \frac{3}{2}u_+$. On the circle, $s = u_+$, so we see that $s < \lambda_1(U_+)$ if and only if $u_+ > 0$ and $-2 < v_+ < 0$; also $s > \lambda_2(U_+)$ if and only if $u_+ < 0$ and $-2 < v_+ < 0$.

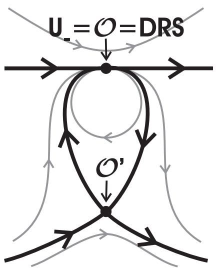

FIGURE 10. Phase portrait, $U_{-} = \mathcal{O}$, $s = 0$. The coincidence curve contains an orbit

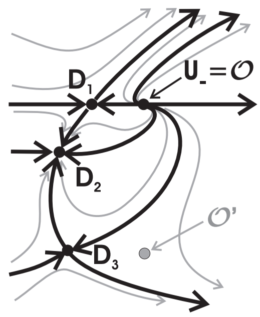

FIGURE 11. Phase portrait for $U_{-} = \mathcal{O}$, $s \lesssim 0$.

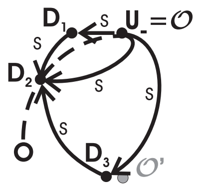

FIGURE 12. The five shocks defined by $U_{-} = \mathcal{O}$, $s \lesssim 0$.

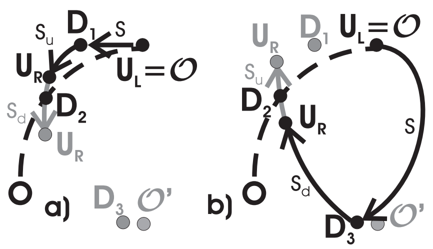

FIGURE 13. Solutions of the Riemann problem with $U_L = \mathcal{O}$ and U_R out the compressive sector but yet near D_2.

The classification of the points in $\mathcal{H}(\mathcal{O})$ according to the Definitions 2.5 and 2.7 is shown in Fig. 7. No bifurcation exists that interrupts the viscous profiles, which were checked numerically anyway. The 1-shock $(\mathcal{O}, \mathcal{O}', 0)$ is left-characteristic, i.e., $s(\mathcal{O}, \mathcal{O}') = \lambda_1(\mathcal{O}) = 0$. The points D_1, D_2 and D_3 will be used later.

The intersections of the two curves in (4.2) are the equilibria of the associated ODE (2.4). With $s = 0$ there are just two equilibria, $\mathcal{O}$ and $\mathcal{O}' = (0, -4)$, see Fig. 8. The equilibrium $\mathcal{O}'$ plays an important role.

The phase portrait for the ODE (2.4) associated to the shock $(\mathcal{O}, \mathcal{O}', s = 0)$ is shown in Fig. 10. For this EDO the nilpotent singularity $\mathcal{O}$ is a possibly degenerate elliptic equilibrium in the classification given by Dumortier, Roussarie and Sotomaior, see [4] and [1]. Thus $\mathcal{O}$ is called the DRS point in this phase space. One can verify that the coincidence curve contains an homoclinic orbit of $\mathcal{O}$, thus the orbits that connect the equilibrium $\mathcal{O}$ to the saddle $\mathcal{O}'$ lie in the hyperbolic region.

The phase portrait for $U_{-} = \mathcal{O}$ with shock speed $s \lesssim \lambda_1(\mathcal{O})$ has four equilibria, as it can be easily seen using 4.2, see Figs. 9 and 11. We see that $\mathcal{O}$ splits into three equilibria, $\mathcal{O}$, D_1 and D_2, while $\mathcal{O}'$ moves to D_3; D_1 and D_3 lie on the 1-shock parts of $\mathcal{H}(\mathcal{O})$, or $1S$, while D_2 lies on the over-compressive part O near $\mathcal{O}$, see again Fig. 7. The Jacobian DF at the equilibrium $\mathcal{O}$ has only one eigenvector, with double eigenvalue $-s$. It is easy to check that D_1 is a saddle and D_2 is an attractor. Since $\mathcal{O}'$ was a saddle for $s = 0$, D_3 is also a saddle. Thus, the four equilibria define five shocks with the same speed s (see Figs. 11 and 12): the 1-shocks $(\mathcal{O}, D_1)$ and $(\mathcal{O}, D_3)$, the O-shock $(\mathcal{O}, D_2)$ and the 2-shocks (D_1, D_2) and (D_3, D_2). Therefore the RP with $U_L = \mathcal{O}$ and $U_R = D_2$ has three solutions in phase space that coincide in physical space.

Now we remove the degeneracy of the Riemann solution. The states $\mathcal{O}$, D_1 and D_2 form a triple shock then we may apply Theorem 2.22. Let U_R be a point on the 2-shock part of $\mathcal{H}(D_1)$ near D_2 that not belong to O (see Fig. 13.a). If U_R lies above D_2 then the sequence of shocks $(\mathcal{O}, D_1, s)$ followed by (D_1, U_R, s_u) is compatible $(s < s_u)$. On the other hand, if U_R lies below D_2 then the sequence of shocks $(\mathcal{O}, D_1, s)$ followed by (D_1, U_R, s_d) is incompatible $(s_d < s)$.

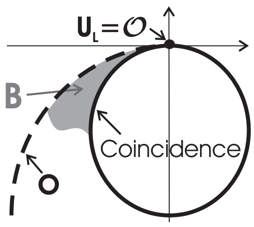

FIGURE 14. The open set B in Theorem 3.1.

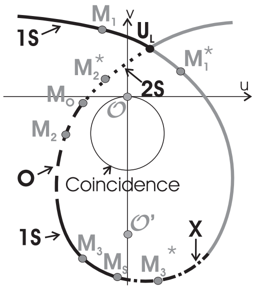

FIGURE 15. $\mathcal{H}(U_L)$ for $U_L \in T(\beta)$. The 1-shocks: solid curve; O-shocks: dashed; 2-shocks: dotted curve; dot-dashed curve: X-disc.

The states $\mathcal{O}$, D_3 and D_2 form a triple shock then we may apply Theorem 2.22. Let U_R be a point on the 2-shock part of $\mathcal{H}(D_3)$ near D_2 that not belong to O (see Fig. 13.b). If U_R lies below D_2 then the sequence of shocks $(\mathcal{O}, D_3, s)$ followed by (D_3, U_R, s_d) is compatible $(s < s_d)$. On the other hand, if U_R lies above D_2 then the sequence of shocks $(\mathcal{O}, D_3, s)$ followed by (D_3, U_R, s_u) is incompatible $(s_u < s)$. We remark that U_R is an attractor and D_1 and D_3 are a saddles for every s_u (or s_d) close to s.

In summary, the solution of the RP with data $\mathcal{O}$, U_R has D_1 as middle state if U_R lies above C; and D_3 as middle state if U_R lies below C: in the latter case, the RP does not have a local solution formed by two small shocks.

Because O is adjacent to $\mathcal{O}$ we can choose U_R as close to $\mathcal{O}$ as we wish, so there are RPs with data $U_L = \mathcal{O}$, U_R with non local solutions formed by shocks. The open set B (see Fig. 14) lies in the gap between O and the coincidence, so B lies in the hyperbolic region. The proof of Theorem 3.1 is complete for sequence of shocks.

4.2. **Proof of Theorem 3.2 for sequences of shocks.** We now show that the behavior shown in Subsection 4.1 actually occurs also for U_L lying in open triangles above $\mathcal{O}$. Let $T(\beta)$ be the family of triangles defined in (3.2). For U_L in $T(\beta)$ the RH curve is shown in Fig. 15; the points M_i will be defined later. Because U_L now lies in the hyperbolic region it has two characteristic speeds, and we set $s_0 = \lambda_1(U_L)$. The Lax theorem guarantees that the O-shock part of the RH is not adjacent to U_L. Moreover, in Lemma 4.1 we calculate the two points on $\mathcal{H}(U_L)$ where $s = s_0$. It is also possible to calculate the two points where $s = \lambda_2(U_L)$ and the other two points where s equals one of the right-characteristic speeds. Knowing these points it is possible to classify $\mathcal{H}(U_L)$ based on the nature of the equilibria, as shown in Fig. 15.

The phase portrait for $U_L \in T(\beta)$ with $s_- \lesssim s_0$ has four equilibria, U_L, M_1, M_2 and M_3, see Figure 16.a. The equilibria define the following shocks with the same speed s_-: the 1-shocks (U_L, M_1) and (U_L, M_3), the O-shock (U_L, M_2), and the 2-shocks (M_1, M_2) and (M_3, M_2). By increasing the speed back to s_0 the equilibria M_1 and U_L collapse into each other (U_L is a repeller-saddle) but M_2 stays away. In this case we rename M_2 and M_3 as, respectively, M_O and M_S, see Figures 15 and 16.b. The equilibria define the following shocks with the same speed s_0: the L1-shock (U_L, M_S), the LO-shock (U_L, M_C) and the the 2-shock (M_S, M_C). For $s_+ \gtrsim s_0$ there is just one shock starting at U_L, namely the 2-shock (U_L, M_2^*, s_+), see Figures 15 and 16.c.

For right states near the over-compressive part O of $\mathcal{H}(U_L)$ there are two kinds of solutions, see Fig. 17. If U_R lies above C, the solution is a 1-shock from U_L to M_1 followed by a faster 2-shock from M_1 to U_R (the equilibrium U_R is an attractor). We remark that the sequence of a 1-shock from U_L to M_3 followed by a 2-shock from M_3 to U_R has incompatible shock speeds, as stated by Theorem 2.22. Therefore, the RP with data U_L and U_R above O has a a solution formed by local shocks with middle state M_1 as in Lax Theorem.

On other hand if U_R lies below C, see again Fig. 17, the solution is a 1-shock from U_L to M_3 followed by a faster 2-shock from M_3 to U_R. We will show that M_3 stays away from U_L, therefore, the RP with data U_L, U_R below O has a large amplitude solution and for such Riemann

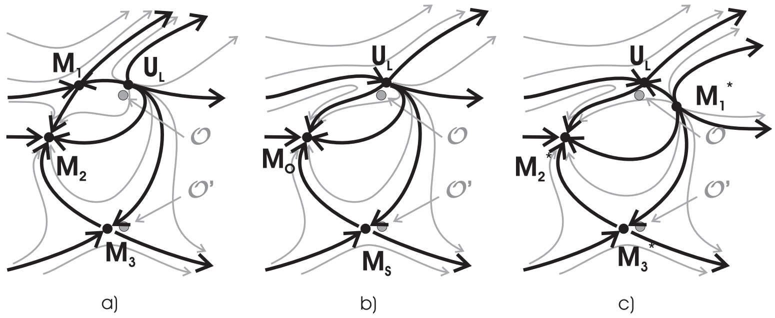

FIGURE 16. Phase portraits for $U_L \in T(\beta)$ with different speeds: a) $s_- \lesssim s_0$; b) s_0 (the equilibrium $M_1 = U_L$ is a repeller-saddle); c) $s_+ \gtrsim s_0$.

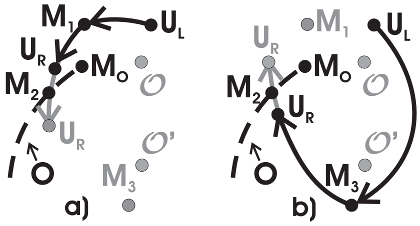

FIGURE 17. Solutions of the Riemann problem with $U_L \in T(\beta)$ and U_R out of compressive sector but near M_2.

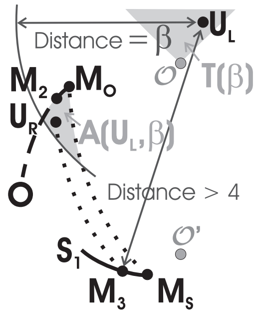

FIGURE 18. Solution for $U_L \in T(\beta)$, $U_R \in A(U_L,\beta)$.

data there is no solution formed by small amplitude shocks. We remark that the sequence of a 1-shock from U_L to M_1 followed by a 2-shock from M_1 to U_R below O has incompatible shock speeds, as stated by Theorem 2.22. We need to locate the points M_O, separating the 2-shock and O-shock parts, and M_S, separating the 1-shock and X-disc. parts of $\mathcal{H}(U_L)$.

Lemma 4.1. *For $U_L \in T(\beta)$ with small β we have $|U_L - M_C| < \beta$ and $|U_L - M_S| > 4$.*

Proof. Let us find the location of $M_C \equiv (u_C, v_C)$ and $M_S \equiv (u_S, v_S)$. If $U_L = (\alpha v_L, v_L) \in T(\beta)$, with $-1 < \alpha < 1$ and $0 < v_L < \frac{\beta^2}{9}$, straightforward calculations using (4.1) with $s = \lambda_1(U_L)$ lead to $v_C = -v_L - 2 + b$, $v_S = -v_L - 2 - b$ and $u_i = 2\alpha v_L - a - (\alpha v_L^2 + a v_L)/v_i$ for $i = C, S$, with $a = \sqrt{2v_L + (1+\alpha^2)v_L^2}$ and $b = \sqrt{4 + (6\alpha a - 2)v_L - 6(\alpha^2 - 2)v_L^2}$. The quantity a is real in the hyperbolic region; b is real in part of the hyperbolic region, e.g. where $v_L < \sqrt{3}u_L + 1$ and $v_L > -\frac{1}{2}$, or where $v_L > -\sqrt{3}u_L + 1$ and $v_L < -\frac{1}{2}$. For small positive β both M_O, M_S lie in the strictly hyperbolic region.

Expanding the Euclidean distances from U_L to M_S and M_O in power series in v_L near the origin we have:

$$|U_L, M_C| \simeq (5\sqrt{2v_L} - \alpha v_L)/3 \quad \text{and} \quad |U_L, M_S| \simeq 4 + \tfrac{7}{4}v_L, \tag{4.3}$$

with error $O(v_L^{3/2})$, so for small positive v_L we have

$$|U_L, M_C| < 3\sqrt{v_L} < \beta \quad \text{and} \quad |U_L, M_S| > 4. \tag{4.4}$$

The proof is complete. □

Lets us examine the Riemann solution for U_R lying in the region below the part of O to the left of M_O (see Fig. 18). The 1-shock from U_L to M_3 near M_S has speed s slightly lower than s_0; the 2-shock from M_3 to U_R near M_2 and M_O has speed higher than s. By continuity we have $|U_L, M_3| > 4$ and $|U_L, U_R| < \beta$.

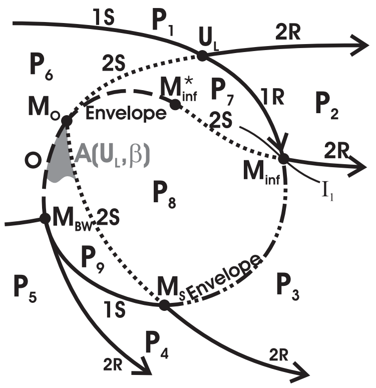

FIGURE 19. Solution of the RP for $U_L \in T(\beta)$.

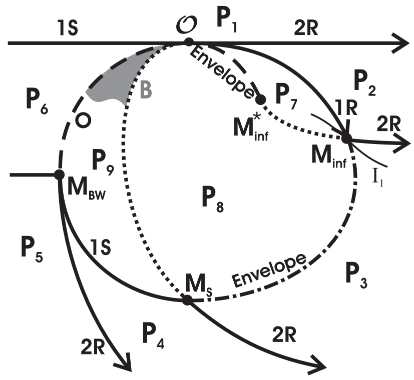

FIGURE 20. Solution of the RP for $U_L = \mathcal{O}$.

The curves $\mathcal{H}(U_L)$ and $\mathcal{H}(M_S)$ are transversal at M_O, as stated by Theorem 2.22. So we can define an open set $A(U_L, \beta)$, see Fig. 18, with corner on M_O and angle given by the tangents of $\mathcal{H}(U_L)$ and $\mathcal{H}(M_S)$ at M_O. Imposing that the distance of all points of $A(U_L, \beta)$ to U_L should be is smaller than β the proof of theorem 3.2 is complete for sequences of shocks.

4.3. **Absence of small rarefactions: proof.** For a $U_L \in T(\beta)$ there exists a LO-shock (U_L, M_C, s_0). The solutions using a small 1-rarefaction form an envelope that is a compatibility boundary, see Corollary 2.18. Because the envelope and the O-shock part of $\mathcal{H}(U_L)$ join differentiably the set $A(U_L, \beta)$ is not covered by 2-shocks starting on the 1-rarefaction. We also note that the set $A(U_L, \beta)$ is not reached by 2-rarefactions starting at 1-waves from U_L (see the global solution in Section 5). Therefore, there are no small amplitude solutions for a RP with $U_L \in T(\beta)$ and $U_R \in A(U_L, \beta)$. The proof of the Theorem 3.2 for small waves is complete.

Theorem 3.1 is a limit case of Theorem 3.2, where the LO-shock is degenerate. Despite this degeneracy, there exists an envelope that joins differentiably with the O-shock part. Therefore, the 2-shocks that start on the 1-rarefaction do not cover the open set B. We also note that the set B is not reached by 2-rarefactions starting at 1-waves from U_L. The proof of Theorem 3.1 is complete.

5. SOLUTION FOR ANY U_R

First we present the solution of the RP with $U_L \in T(\beta)$ and any U_R. There are six important points (see Fig. 19): (i) U_L; (ii) M_O; (iii) M_S; (iv) M_{BW}, the point between the O-shock and 1-shock parts of $\mathcal{H}(U_L)$, where the Bethe–Wendroff ensures the tangency of $\mathcal{H}(U_L)$ with an $\mathcal{H}(U_M)$ for some U_M; (v) M_{Inf}, the point on the 1-rarefaction where the genuine non-linearity is lost; (vi) M^*_{Inf}, the point such that (M_{Inf}, M^*_{Inf}, s) is a LO-shock.

The state plane is divided in nine parts, P_1 to P_9, by the following fourteen curves:

- the 1-shock part of $\mathcal{H}(U_L)$ containing U_L;
- the 1-rarefaction that starts at U_L and stops at M_{Inf};
- the envelope that starts at M_S and glues with M_{Inf};
- the 1-shock part of $\mathcal{H}(U_L)$ between M_S and M_{BW};
- the O-shock part of $\mathcal{H}(U_L)$ between M_{BW} and M_O;
- the 2-shock part of $\mathcal{H}(U_L)$ between M_O and U_L;
- the envelope that starts at M_O and stops at M^*_{Inf};
- the 2-shock part of $\mathcal{H}(M_{Inf})$ between M_{Inf} and M^*_{Inf};
- the 2-rarefaction starting at U_L;
- the 2-rarefaction starting at M_{Inf};
- the 2-rarefaction starting at M_S;
- the 2-rarefaction starting at M_{BW};
- the 2-shock part of $\mathcal{H}(M_S)$ between M_S and M_O;

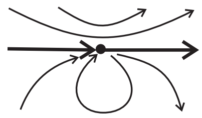

FIGURE 21. Phase portrait for the elliptic singularity DRS.

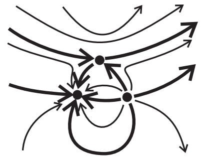

FIGURE 22. One of the possible perturbations of the phase portrait for the elliptic singularity DRS.

- the line of critical speeds that starts at M_{BW}.

The solution of the RP for U_R in:

- P_1 is a 1-shock followed by a 2-rarefaction;
- P_2 is a 1-rarefaction followed by a 2-rarefaction;
- P_3 is a 1-RS group followed by a 2-rarefaction;
- P_4 is a 1-shock followed by a 2-rarefaction;
- P_5 is a 1-shock followed by a 2-SR group;
- P_6 is a 1-shock followed by a 2-shock;
- P_7 is a 1-rarefaction followed by a 2-shock;
- P_8 is a 1-RS group followed by a 2-shock;
- P_9 is a 1-shock followed by a 2-shock.

The solution of the RP with $U_L = \mathcal{O}$ is the limit of a solution for $U_L \in T(\beta)$ when M_O collapses with U_L, see Fig. 20. The state space is also divided in nine parts, P_1 to P_9, with the same kinds of solution.

6. Remarks

Dumortier, Roussarie and Sotomaior studied the versal bifurcation for a nilpotent singularity for a planar vector field with three parameters (see [4]). They classify the codimension-3 bifurcation types as saddle, focus and elliptic. In [1] it is proved that saddle and elliptic bifurcations occur in quadratic models; moreover for a type IV flux with identity viscosity matrix the singularity is elliptic. The phase portrait for this kind of nilpotent singularity is sketched in Fig. 21, while one of the sixteen stable deformation is shown on Fig. 22. No high amplitude solutions arise directly from the local bifurcation. In fact, looking only at local solutions would lead to nonexistence of Riemann solution. However the phase portraits of the solution for $U_L \in T(\beta)$ contain an extra equilibrium M_3 near $\mathcal{O}'$ which is fundamental for defining the non local solution, see again Fig. 16.a.

For Riemann Problems with a type IV umbilic point, which arise for homogeneous quadratic polynomials, it is shown in [7] that high amplitude solutions do not appear. The singularity ceases to be nilpotent, since when the umbilic point is taken as U_L the latter contains all four equilibria points. In other words, the phase portrait for U_L on the umbilic point with speed lower than characteristic is topologically equivalent to the phase portrait for $\mathcal{O}$ with $s \lesssim 0$, see again Fig. 11. However, if s equals the left-characteristic speed the phase portraits are not topologically equivalent any more: there is just one equilibrium in the umbilic case and two equilibria $\mathcal{O}$, $\mathcal{O}'$ in our case.

7. Acknowledgments

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