Local Solutions in Sobolev Spaces and Unconditional Well-Posedness for the Generalized Boussinesq Equation *

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Abstract

We study the local well-posedness of the initial-value problem for the nonlinear generalized Boussinesq equation with data in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, $s \geq 0$. Under some assumption on the nonlinearity f, local existence results are proved for $H^s(\mathbb{R}^n)$-solutions using an auxiliary space of Lebesgue type. Furthermore, under certain hypotheses on s, n and the growth rate of f these auxiliary conditions can be eliminated.

1 Introduction

In this paper we consider the generalized Boussinesq Equation

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u + \Delta f(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \end{cases} \tag{1}$$

where the nonlinearity f satisfies the following assumptions

$(f1)$ $f \in C^{[s]}(\mathbb{C}, \mathbb{C})$, where $s \geq 0$ and $[s]$ denotes the smallest positive integer greater than s;

$(f2)$ $|f^{(l)}(v)| \lesssim |v|^{k-l}$ for all integers l varying in the whole range $0 \leq l \leq [s] \leq k$ with $k > 1$;

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$(f3)$ If $s \leq \dfrac{n}{2}$ then $1 < k \leq 1 + \dfrac{4}{n-2s}$.

Equations of this type, in one dimension, but with the opposite sign in front of the fourth derivative term, were originally derived by Boussinesq [3] in his study of nonlinear, dispersive wave propagation. We should remark that it was the first equation proposed in the literature to describe this kind of physical phenomena. The equation (1) was also used by Zakharov [23] as a model of nonlinear string and by Falk *et al* [8] in their study of shape-memory alloys.

In one dimension, local and global results have been obtained for (1). Using Kato's abstract theory for quasilinear evolution equation, Bona and Sachs [2] showed local well-posedness (1), where $f \in C^\infty$ and initial data $u_0 \in H^{s+2}(\mathbb{R})$, $u_1 \in H^{s+1}(\mathbb{R})$ with $s > \frac{1}{2}$. Tsutsumi and Matahashi [20] established similar result when $f(u) = |u|^{k-1}u$, $k > 1$ and $u_0 \in H^1(\mathbb{R})$, $u_1 = \chi_{xx}$ with $\chi \in H^1(\mathbb{R})$. These results were improved by Linares [14]. He proved local well-posedness when $f(u) = |u|^{k-1}u$, $1 < k < 5$, $u_0 \in L^2(\mathbb{R})$, $u_1 = h_x$ with $h \in H^{-1}(\mathbb{R})$ and $f(u) = |u|^{k-1}u$, $k > 1$, $u_0 \in H^1(\mathbb{R})$, $u_1 = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, assuming smallness in the initial data, it was proved that these solutions can be extended globally in $H^1(\mathbb{R})$. The main tool used in [14] was the Strichartz estimates satisfied by solutions of the linear problem.

Another problem studied in the context of the Boussinesq equation is scattering of small amplitude solutions. This question was investigated by several authors, see for instance Linares and Scialom [16] and Liu [17] for results in one dimension and Cho and Ozawa [6] for arbitrary dimension.

Here we consider the local well-posedness and uniqueness problems, studying the integral equation associated to (1). To describe the integral formulation we first consider the following modified linear equation

$$\begin{cases} u_{tt} + \Delta^2 u = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \Delta\psi. \end{cases} \tag{2}$$

First, we recall that the solution to the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0 \end{cases}$$

is given by

$$S(t)u_0 \quad = \quad \left(e^{-it|\xi|^2}\widehat{u_0}\right)^\vee . \tag{3}$$

On the other hand, applying the Fourier transform to the equation (2), we obtain that the solution is given by

$$u(t) = B_c(t)\phi + B_s(t)\Delta\psi, \tag{4}$$

where the operators $B_c(t)$ and $B_s(t)\Delta$ are completely describe by the unitary group (3), that is

$$B_c(t) \quad = \quad \frac{1}{2}\left(S(t) + S(-t)\right) \tag{5}$$

and

$$B_s(t)\Delta \quad = \quad \frac{1}{2i}\left(S(t) - S(-t)\right). \tag{6}$$

Therefore, applying the Duhamel's principle, the integral formulation of the initial value problem (IVP)

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta(f(u) - u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \Delta\psi \end{cases} \tag{7}$$

is given by

$$u(t) = B_c(t)\phi + B_s(t)\Delta\psi + \int_0^t B_s(t-t')\Delta(f(u) - u)(t')dt' \tag{8}$$

Since the operators (5) and (6) are linear combinations of the Schrödinger unitary group and its adjoint, we have that the structure of (8) is very similar to the Schrödinger's integral equation. Therefore, applying well known results for this last equation we establish new results about local well-posedeness and uniqueness for the generalized Boussinesq equation (7).

The local well-posedness question was first raised by Hadamard [11] in the case of Laplace equation and refined by Kato in the case of an initial value problem (IVP) (see, for example, [12]). In the present work, we adopt the following definition

Definition 1.1 (Well-posedness) *We say that the IVP (7) is locally well-posed in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ if for any initial data $(\phi, \psi) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ there exists a time $T > 0$, a subset Ξ of $C([0,T]; H^s(\mathbb{R}^n))$ and a function $u \in \Xi$ such that*

(i) **Existence and Uniqueness** *The function u is the unique solution in Ξ of the integral equation (8) (interpreted in a distributional sense).*

(*ii*) **Continuous Dependence** *The flow map is (at least) continuous from a neighborhood of (ϕ, ψ) in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ into $C([0,T]; H^s(\mathbb{R}^n))$.*

Remark 1.1 *If T can be taken arbitrarily large, we say that the well-posedness is global.*

To obtain a solution of (7), we analyze the integral equation (8) applying a fixed point argument. That is, we find $T > 0$ and define a suitable complete subspace of $C([0,T]; H^s(\mathbb{R}^n))$, for instance Ξ_s, such that the integral equation is stable and contractive in this space. Then, by Banach's fixed point theorem, there exists a unique solution in Ξ_s.

However, to define the subset Ξ_s we will need some auxiliary conditions, which is based on the available Strichartz estimates for the Schrödinger equation

Definition 1.2 *We call (q, r) an admissible pair if they satisfy the condition:*

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right),$$

where

$$\begin{cases} 2 \le r \le \infty & , \textit{ if } n = 1, \\ 2 \le r < \infty & , \textit{ if } n = 2, \\ 2 \le r \le \dfrac{2n}{n-2} & , \textit{ if } n \ge 3. \end{cases}$$

Now, we can define the (auxiliary) space

$$\begin{aligned} \mathcal{Y}_s &= (1-\Delta)^{-\frac{s}{2}} \left(\bigcap \{L^q L^r : \text{ is an admissible pair}\} \right) \\ &= \bigcap \{L^q H^s_r : (q,r) \text{ is an admissible pair}\} \end{aligned}$$

where $H^s_r = (1-\Delta)^{-\frac{s}{2}} L^r$.

With these notations and definitions, we have the following result on the local well-posedness for the generalized Boussinesq equation (7).

Theorem 1.1 *Assume $(f1) - (f3)$ and $s \ge 0$. Then for any $(\phi, \psi) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, there are $T > 0$ and a unique solution u of (7) with the following properties*

(i) $u \in C([0,T]; H^s(\mathbb{R}^n))$;

(ii) $u \in \mathcal{Y}_s$.

Note that this theorem cover the cases analyzed by Linares [14]. In fact in one dimension and with nonlinearity $f(u) = |u|^{k-1}u$, assumptions $(f1)-(f3)$ impose the restrictions $k > 1$ and $1 < k < 5$ for $s = 0$ and $s = 1$, respectively.

On the other hand, the above theorem generalizes the local result of Linares [14] in two ways. Firstly, it provides local well-posedness in $H^s(\mathbb{R})$ with $s \neq \{0, 1\}$ and $s \leq 2 + 1/2$, which up to our knowledge are not present in the literature for the generalized Boussinesq equation in one dimension. Moreover, it includes the IVP associated to (7) in arbitrarily dimension.

Based in the proof of Theorem 1.1, we can also obtain results in the life span and blow-up of the solutions given above. This is done in the following two theorems

Theorem 1.2 *Let $[0, T^*)$ be the maximal interval of existence for u in Theorem 1.1. Then T^* depends on ϕ, ψ in the following way*

(i) Let $s > \frac{n}{2}$ and $\sigma > 0$ such that $\frac{n}{2} < \sigma \leq s$. Then T^* can be estimate in terms of $\|\phi\|_{H^\sigma}$ and $\|\psi\|_{H^\sigma}$ only. Moreover,

$$T^* \to \infty \text{ when } \max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\} \to 0. \tag{9}$$

(ii) Let $s \leq \frac{n}{2}$ and $\sigma \geq 0$ such that

$$\sigma \in \left[0, \frac{n}{2}\right) \bigcap \left[\frac{n}{2} - \frac{2}{k-1}, s\right]; \tag{10}$$

(iia) If $\sigma > \frac{n}{2} - \frac{2}{k-1}$, Then T^* can be estimate in terms of $\|D^\sigma \phi\|_{L^2}$ and $\|D^\sigma \psi\|_{L^2}$ only. Moreover,

$$T^* \to \infty \text{ when } \max\{\|D^\sigma \phi\|_{L^2}, \|D^\sigma \psi\|_{L^2}\} \to 0.$$

(iib) If $\sigma = \frac{n}{2} - \frac{2}{k-1}$, the time T^* can be estimated in terms of $D^\sigma \phi, D^\sigma \psi \in L^2$, but not necessarily of their norms.

Theorem 1.3 *Suppose, in Theorem 1.2, that $T^* < \infty$. Then*

(a) In case (i), $\max\{\|u(t)\|_{H^\sigma}, \|\Delta^{-1} u_t(t)\|_{H^\sigma}\}$ blows up at $t = T^*$ for all σ such that $\frac{n}{2} < \sigma \leq s$;

(b) In case (iia), $\max\{\|D^\sigma u(t)\|_{L^2}, \|D^\sigma \Delta^{-1} u_t(t)\|_{L^2}\}$ *blows up at* $t = T^*$ *for all* $\sigma \neq \frac{n}{2} - \frac{2}{k-1}$ *and in (10).*

Note that part (ii) of Theorem 1.1 is an essential part; without such a condition, uniqueness might not hold. In this case, we say that (7) is conditionally well-posed in $H^s(\mathbb{R}^n)$, with the auxiliary space $\mathcal{Y}_s$.

A natural question arise in this context: Is it possible to remove the auxiliary condition? In other words, is it possible to prove that uniqueness of the solution for (7) holds in the whole space $C([0,T]; H^s(\mathbb{R}^n))$? If the answer for this last question is yes, then we say that (7) is unconditionally well-posed in $H^s(\mathbb{R}^n)$. In [12], Kato introduce this notion and extensively studies it for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + f(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0, \end{cases}$$

where f is a nonlinear function satisfying certain hypotheses.

Based on the integral formulation (8), we will use Kato's argument to prove the same kind of result for the generalized Boussinesq equation (7). The next theorem gives a precise statement of our uniqueness result.

Theorem 1.4 *Assume* $(f1) - (f3)$ *and let* $s \geq 0$*. Uniqueness for (7) holds in* $C([0,T]; H^s)$ *in each of the following cases*

(i) $s \geq \frac{n}{2}$;

(ii) $n = 1$, $0 \leq s < \frac{1}{2}$ *and* $k \leq \frac{2}{1-2s}$;

(iii) $n = 2$, $0 \leq s < 1$ *and* $k < \frac{s+1}{1-s}$;

(iv) $n \geq 3$, $0 \leq s < \frac{n}{2}$, $k \leq \min\left\{1 + \frac{4}{n-2s}, 1 + \frac{2s+2}{n-2s}\right\}$.

The fundamental tool to prove Theorem 1.4 are the classic Strichartz estimates satisfied by the solution of the Schrödinger equation. We remark that parts (i), (ii), and (iii) of the above theorem are identical, respectively, to (i), (iii), and (ii) for $n = 2$ of [12], Corollary 2.3. However, for $n \geq 3$, we include the high extreme point for the value of k, in the range of validity of the theorem. This is possible due to the improvement in the Strichartz estimates proved by Keel and Tao [13].

For the particular case where $f(u) = |u|^{k-1}u$, we can also improve Theorem 1.4 for a large range of values k. This is done in the following theorem.

Theorem 1.5 *Let $n \geq 3$, $0 < s < 1$ and $f(u) = |u|^{k-1}u$, with $k > 1$ satisfying $(f3)$. Uniqueness for (7) holds in $C([0,T]; H^s)$ if k verifies the following conditions*

(1) $k > 2$;

(2) $k > 1 + \dfrac{2s}{n-2s}$, $k < 1 + \min\left\{\dfrac{n+2s}{n-2s}, \dfrac{4s+2}{n-2s}\right\}$;

(3) $k < 1 + \dfrac{4}{n-2s}$;

(4) $k \leq 1 + \dfrac{n+2-2s}{n-2s}$.

Remark 1.2 *Note that the restriction $k \leq \dfrac{n+2s}{n-2s}$ seems natural. In fact, this assumption implies $|u|^{k-1}u \in L^1_{loc}(\mathbb{R}^n)$, which ensures that the equation*

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u + \Delta\left(|u|^{k-1}u\right) = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi, \quad u_t(x,0) = \Delta\psi \end{cases} \tag{11}$$

makes sense within the framework of the distribution.

Theorem 1.5 is inspired on the unconditional well-posed result proved by Furioli and Terraneo [9] for the case of nonlinear Schrödinger equation. As in [9], the proof of this theorem relies in the use of Besov space of negative indices.

The plan of this paper is as follows: in Section 2, we introduce some notation. Linear estimates and other preliminary results are proved in Section 3. The local existence theory is established in Section 4. Finally, the unconditional well-posedness problem is treated in Section 5.

2 Notations

In the sequel c denotes a positive constant which may differ at each appearance.

We use local in time versions of the space-time Lebesgue spaces $L^q_t L^r_x$, which we denote by $L^q_T L^r_x$, equipped with the norms

$$\|f\|_{L^q_T L^r_x} = \|\|f(t,\cdot)\|_{L^r(\mathbb{R}^n)}\|_{L^q([0,T])}.$$

Now we recall the definition of homogeneous Besov spaces. Let $\omega \in C_c^\infty(\mathbb{R}^n)$ be such that supp $\omega \subseteq \{\xi : 2^{-1} \leq \xi \leq 2\}$, $\omega(\xi) > 0$ for $2^{-1} < \xi < 2$

and $\sum_{j\in\mathbb{Z}}\omega(2^{-j}\xi)=1$ for $\xi\neq 0$. Denote $\widehat{\Delta_j f}=\omega(2^{-j}\xi)\widehat{f}(\xi)$. We have the following definition

Definition 2.1 *Let $s\in\mathbb{R}$, $1\leq p,q\leq\infty$. The homogeneous Besov space is defined as follows:*

$$\dot{B}^s_{p,q}(\mathbb{R}^n)=\{f\in S'(\mathbb{R}^n)/\mathcal{P}:\{2^{js}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}}\in l^q(\mathbb{Z})\}$$

where $\mathcal{P}$ is the space of polynomials in n variables.

It is well-known that $\dot{B}^s_{2,2}(\mathbb{R}^n)=\dot{H}^s_2(\mathbb{R}^n)$, where $\dot{H}^s_p(\mathbb{R}^n)$, with $1\leq p\leq\infty$, denote the homogeneous (generalized) Sobolev space defined as the completion of $S(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^n)}=\|(|\xi|^s\widehat{f}(\xi))^\vee\|_{L^p(\mathbb{R}^n)}.$$

For further details concerning the Besov and (generalized) Sobolev spaces we refer the reader to [1].

3 Preliminary results

To treat the integral equation (8), we need to obtain estimates for the operators $B_c(\cdot)$ and $B_s(\cdot)\Delta$. From the definition of these operator and the well-known Strichartz inequalities for solutions of Schrödinger Equation we can easily prove the following two lemmas

Lemma 3.1 *Let (q,r) and (γ,ρ) admissible pairs and $0<T\leq\infty$. Then*

(i) $\|B_c(\cdot)h\|_{L^q_TL^r}+\|B_s(\cdot)\Delta h\|_{L^q_TL^r}\leq c\|h\|_{L^2}$;

(ii) $\|B_I(g)\|_{L^q_TL^r}\leq c\|g\|_{L^{\gamma'}_TL^{\rho'}}$,
where $B_I(g)\equiv\int_0^t B_s(t-t')\Delta g(t')dt'$.

Proof Since the above estimates are valid for the Schrödinger group (see [15] Chapter 4), using (5) and (6) the lemma follows.

■

Lemma 3.2 *Let (q,r) and (γ,ρ) be admissible pairs. Then*

(i) $\|B_I(g)\|_{L^q_T\dot{B}^s_{r,2}}\leq c\|g\|_{L^{\gamma'}_T\dot{B}^s_{\rho',2}}$;

(ii) $\|B_c(\cdot)h\|_{L^q_T\dot{B}^s_{r,2}}+\|B_s(\cdot)\Delta h\|_{L^q_T\dot{B}^s_{r,2}}\leq c\|h\|_{\dot{H}^s}$.

Proof It follows from Theorem 2.2 in [5] and definitions (5) and (6). ■

Remark 3.1 *Lemma 3.2 is still valid if we replace* $\dot{B}^s_{q,2}(\mathbb{R}^n)$ *by the homogeneous Sobolev spaces* $\dot{H}^s_q(\mathbb{R}^n)$ *(see [5] page 814).*

Another important result are the estimates for nonlinear term that appear in equation (7). For the next two results see Lemmas A1-A4 in [12] and Lemma 2.3 in [9].

Lemma 3.3 *Assume (f1)-(f2) and for* $0 \leq s \leq k$*, define* $D^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$*, then*

(i) $\|D^s f(u)\|_{L^r} \leq c\|u\|^{k-1}_{L^{(k-1)r_1}}\|D^s u\|_{L^{r_2}}$

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$, $r_1 \in (1,\infty]$, $r_2 \in (1,\infty)$;

(ii) $\|D^s(uv)\|_{L^r} \leq c\,(\|D^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\|D^s v\|_{L^{r_2}})$

where $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{q_2} = \dfrac{1}{q_1} + \dfrac{1}{r_2}$, $r_i \in (1,\infty)$, $q_i \in (1,\infty]$, $i = 1, 2$.

Lemma 3.4 *Let* $k > 1$, $s \geq 0$, $p \in [1,\infty)$, $s < \min\left\{\dfrac{n}{p}, k\right\}$ *and* $\dfrac{1}{p} - \dfrac{s}{n} \leq \dfrac{1}{k}$*. Let* $\alpha = \dfrac{n}{s + k\left(\dfrac{n}{p} - s\right)}$*. Then there exists* $c > 0$ *such that for all* $g \in \dot{H}^s_p(\mathbb{R}^n)$*, we have*

(i) $\||g|^{k-1}g\|_{\dot{H}^s_\alpha} \leq c\|g\|^k_{\dot{H}^s_p}$;

(ii) $\||g|^k\|_{\dot{H}^s_\alpha} \leq c\|g\|^k_{\dot{H}^s_p}$.

4 Local well-posedness

First, we present some numerical facts that will be important in the proof of the local well-posedness result.

Lemma 4.1 *Let* $k > 1$*, there is* $q \geq 2$ *and an admissible pair* (γ, ρ)*, such that*

$$\frac{1}{\rho'} = \frac{1}{2} + \frac{k-1}{q}.$$

Proof In the case $n \geq 3$, we have to satisfy the following system

$$\begin{cases} \dfrac{1}{\rho} = \dfrac{1}{2} - \dfrac{(k-1)}{q} \\ 2 \leq \rho < \dfrac{2n}{n-2}, \end{cases}$$

thus, it is enough to choose $q > \max\{n(k-1), 2\}$.

In the case $n = 1, 2$ it is sufficient to satisfy the following system

$$\begin{cases} \dfrac{1}{\rho} = \dfrac{1}{2} - \dfrac{(k-1)}{q} \\ 2 \leq \rho < \infty, \end{cases}$$

which is clearly satisfied for every $q \geq \max\{2(k-1), 2\}$.

■

Now by $(f3)$ we have $\dfrac{n}{2} - \dfrac{2}{k-1} \leq s \leq \dfrac{n}{2}$, then it is always possible to choose $\sigma \geq 0$ satisfying (10).

Lemma 4.2 *Assume $(f3)$. Then, for all σ satisfying (10) there exist (p_1, p_2) and (q_1, q_2) such that*

(i) *(p_1, p_2) is an admissible pair;*

(ii) *There exists an admissible pair (q_1, β_2) such that:*

$$\frac{1}{q_2} = \frac{1}{\beta_2} - \frac{\sigma}{n};$$

(iii) *$p_1 < q_1$;*

(iv) *If $\dfrac{1}{r_i} \equiv \dfrac{1}{p_i} + \dfrac{k-1}{q_i}$, $i = 1, 2$, then there exists $s_1 \geq 1$ such that (s_1, r_2) is the dual of an admissible pair and*

$$\begin{aligned} &\frac{1}{r_1} < \frac{1}{s_1}, \quad \text{if} \quad \sigma \in \left[0, \frac{n}{2}\right) \bigcap \left(\frac{n}{2} - \frac{2}{k-1}, s\right]; \\ &\frac{1}{r_1} = \frac{1}{s_1}, \quad \text{if} \quad \sigma = s = \frac{n}{2} - \frac{2}{k-1} \geq 0. \end{aligned}$$

Proof To obtain the points $p_1, p_2, q_1, q_2, \beta_2, r_1, r_2$ and s_1, we need to solve the system of equations corresponding to conditions $(i) - (iv)$. We consider several cases.

(a) $\mathbf{n \geq 2}; \sigma \in \left[\mathbf{0}, \frac{\mathbf{n}}{\mathbf{2}}\right) \bigcap \left(\frac{\mathbf{n}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{k-1}}, \mathbf{s}\right]$

Set

$q_1 = \infty$, $\frac{1}{q_2} = \frac{1}{2} - \frac{\sigma}{n}$;

$\frac{1}{p_1} = \frac{k-1}{4} n \left(\frac{1}{2} - \frac{\sigma}{n}\right)$, $\frac{1}{p_2} = \frac{1}{2} - \frac{k-1}{2}\left(\frac{1}{2} - \frac{\sigma}{n}\right)$.

Then, for $\beta_2 = 2$, it is easy to verify properties $(i) - (iii)$. On the other hand, according to (vi), (r_1, r_2) are given by

$\frac{1}{r_1} = \frac{k-1}{4} n \left(\frac{1}{2} - \frac{\sigma}{n}\right)$, $\frac{1}{p_2} = \frac{1}{2} + \frac{k-1}{2}\left(\frac{1}{2} - \frac{\sigma}{n}\right)$.

Setting $\frac{1}{s_1} = 1 - \frac{k-1}{4} n \left(\frac{1}{2} - \frac{\sigma}{n}\right)$, we have that (s_1, r_2) is the dual of (p_1, p_2) and $\frac{1}{r_1} < \frac{1}{s_1}$, if and only if $\sigma > \frac{n}{2} - \frac{2}{k-1}$.

(b) $\mathbf{n \geq 3}; \sigma = \mathbf{s} = \frac{\mathbf{n}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{k-1}} \geq \mathbf{0}$

In this case we can easily verify properties $(i) - (iv)$ for the points

$q_1 = \infty$, $q_2 = \frac{n(k-1)}{2}$;

$p_1 = 2$, $\frac{1}{p_2} = \frac{1}{2} - \frac{1}{n}$;

$\beta_2 = 2$;

$r_1 = 2$, $\frac{1}{r_2} = \frac{1}{2} + \frac{1}{n}$.

Note that (r_1, r_2) is the dual of (p_1, p_2).

(c) $\mathbf{n = 2}; \sigma = \mathbf{s} = \mathbf{1} - \frac{\mathbf{2}}{\mathbf{k-1}} \geq \mathbf{0}$

For $n = 2$ the pair $(2, \infty)$ is not admissible. So in this case we choose

$q_1 = q_2 = 2(k-1)$;

$p_1 = 3$, $p_2 = 6$;

$r_1 = \frac{6}{5}$, $r_2 = \frac{3}{2}$.

Now it is easy to verify that properties $(i) - (iv)$ hold for $\frac{1}{\beta_2} = \frac{1}{2} - \frac{1}{2(k+1)}$. Note that $k \geq 3$ and thus (iii) holds. Moreover, (r_1, r_2) is the dual of the admissible pair $(6,3)$.

(d) $\mathbf{n=1}; \sigma \in \left[\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right) \bigcap \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{k-1}}, \mathbf{s}\right]$

In this case we consider two possibilities.

If $k > 3$ set

$\frac{1}{q_1} = \frac{1}{4}\left(\frac{1}{2} - \sigma\right), \frac{1}{q_2} = \frac{1}{2}\left(\frac{1}{2} - \sigma\right)$

If $k \leq 3$ then there exists $m \in \mathbb{N} - \{1,2\}$ such that $1 + \frac{8}{2^{m-1}} \geq k > 1 + \frac{8}{2^m}$. Then, set

$\frac{1}{q_1} = \frac{1}{2^m}\left(\frac{1}{2} - \sigma\right), \frac{1}{q_2} = \left(1 - \frac{1}{2^{m-1}}\right)\left(\frac{1}{2} - \sigma\right)$

For (p_1, p_2) set, in both cases

$\frac{1}{p_1} = \frac{k-1}{8}\left(\frac{1}{2} - \sigma\right), \frac{1}{p_2} = \frac{1}{2} - \frac{k-1}{4}\left(\frac{1}{2} - \sigma\right).$

A simple calculation shows that $(i) - (iv)$ hold for

$$\frac{1}{\beta_2} = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \sigma\right) + \sigma & , k > 3, \\ \left(1 - \frac{1}{2^{m-1}}\right)\left(\frac{1}{2} - \sigma\right) + \sigma & , \text{ otherwise} \end{cases}$$

and

$$\frac{1}{s_1} = \begin{cases} 1 - \frac{k-1}{8}\left(\frac{1}{2} - \sigma\right) & , k > 3, \\ 1 - (k-1)\left(\frac{1}{2} - \sigma\right)\left(\frac{3}{8} - \frac{1}{2^m}\right) & , \text{ otherwise.} \end{cases}$$

(e) $\mathbf{n=1}; \sigma = \mathbf{s} = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{k-1}} \geq \mathbf{0}$

Set

$q_1 = \frac{4}{3}(k-1)$, $q_2 = 2(k-1)$;

$p_1 = 5$, $p_2 = 10$.

Therefore

$\beta_2 = \frac{1}{2} - \frac{3}{2(k-1)}$, $r_1 = \frac{20}{19}$ and $r_2 = \frac{5}{3}$.

We have that (r_1, r_2) is the dual of the admissible pair $(20, \frac{5}{2})$. Moreover, (iii) is verified since $k \geq 5$.

∎

Now we have all tools to prove our first main result.

Proof of Theorem 1.1

We consider three cases.

Case (i) $\mathbf{s > \frac{n}{2}}$

Choose $\sigma \in \left(\frac{n}{2}, s\right]$ and define

$$X^s = \{u \in L^\infty_T H^s : \|u\|_{L^\infty_T H^\sigma} \leq N \text{ and } \|D^s u\|_{L^\infty_T L^2} \leq K\}.$$

Remark 4.1 *Note that X^s with the $L^\infty_T L^2$-metric is a complete metric space.*

By the Sobolev Embedding we have for all $q \geq 2$ and $\gamma = \frac{n}{2} - \frac{n}{q}$ (note that $\gamma < \frac{n}{2} < \sigma$)

$$\|u(t)\|_{L^q} \leq c\|D^\gamma u(t)\|_{L^2} \leq c\|u(t)\|_{H^\sigma}.$$

Then, we obtain

$$\|u\|_{L^\infty_T L^q} \leq cN.$$

We need to show that N, K and T can be chosen so that the integral operator

$$\Phi(u)(t) = B_c(t)\phi + B_s(t)\Delta\psi + \int_0^t B_s(t-t')\Delta(f(u) - u)(t')dt' \qquad (12)$$

maps X^s into X^s and becomes a contraction map in the $L^\infty_T L^2$-metric.

Since D^σ commute with B_c, B_s and B_I (see (8)), we have

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_T H^\sigma} &\leq \|\Phi(u)\|_{L^\infty_T L^2} + \|D^\sigma \Phi(u)\|_{L^\infty_T L^2} \\
&\leq c\Big(\|\phi\|_{H^\sigma} + \|\psi\|_{H^\sigma} + \|B_I(f(u) - u)\|_{L^\infty_T L^2} + \\
&\quad + \|B_I(D^\sigma(f(u) - u))\|_{L^\infty_T L^2}\Big) \\
&\leq c\Big(\|\phi\|_{H^\sigma} + \|\psi\|_{H^\sigma} + \|B_I(u)\|_{L^\infty_T L^2} + \|B_I(D^\sigma u)\|_{L^\infty_T L^2} \\
&\quad + \|B_I(f(u))\|_{L^\infty_T L^2} + \|B_I(D^\sigma f(u))\|_{L^\infty_T L^2}\Big).
\end{aligned}$$

So using Lemma 3.1 (i), we have for all (γ,ρ) admissible pair

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_T H^\sigma} &\le c\Big(\|\phi\|_{H^\sigma}+\|\psi\|_{H^\sigma}+\|u\|_{L^1_T L^2}+\|D^\sigma u\|_{L^1_T L^2}\\
&\quad +\|f(u)\|_{L^{\gamma'}_T L^{\rho'}}+\|D^\sigma f(u)\|_{L^{\gamma'}_T L^{\rho'}}\Big)\\
&\le c\left(\|\phi\|_{H^\sigma}+\|\psi\|_{H^\sigma}+T\|u\|_{L^\infty_T H^\sigma}\right)+\\
&\quad +cT^{1/\gamma'}\Big(\|f(u)\|_{L^\infty_T L^{\rho'}}+\|D^\sigma f(u)\|_{L^\infty_T L^{\rho'}}\Big).
\end{aligned}$$

Let q,γ and ρ be given by Lemma 4.1. Then, using $(f2)$, Hölder's inequality $(\dfrac{1}{\rho'}=\dfrac{1}{2}+\dfrac{k-1}{q})$ and Lemma 3.3 we obtain

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_T H^\sigma} \le& c\left(\|\phi\|_{H^\sigma}+\|\psi\|_{H^\sigma}+T\|u\|_{L^\infty_T H^\sigma}\right)+\\
&+cT^{1/\gamma'}\Big(\|u\|_{L^\infty_T L^2}\|u\|^{k-1}_{L^\infty_T L^q}+\|D^\sigma u\|_{L^\infty_T L^2}\|u\|^{k-1}_{L^\infty_T L^q}\Big)\\
\le& c\left(\|\phi\|_{H^\sigma}+\|\psi\|_{H^\sigma}+T\|u\|_{L^\infty_T H^\sigma}\right)+\\
&+cT^{1/\gamma'}\Big(\|u\|_{L^\infty_T H^\sigma}\|u\|^{k-1}_{L^\infty_T L^q}\Big)\\
\le& c\left(\|\phi\|_{H^\sigma}+\|\psi\|_{H^\sigma}\right)+cN\Big(T+T^{1/\gamma'}N^{k-1}\Big).
\end{aligned} \tag{13}$$

By an analogous argument, we obtain

$$\|D^s\Phi(u)\|_{L^\infty_T L^2} \le c\left(\|D^s\phi\|_{L^2}+\|D^s\psi\|_{L^2}\right)+cK\Big(T+T^{1/\gamma'}N^{k-1}\Big).$$

Since $\gamma\neq 1$, it is clear that we can choose N,K and T such that Φ maps X^s into X^s.

Now we have to prove that Φ is a contraction in the $L_T^\infty L^2$-metric. Indeed, using Lemma 3.3 (i) and Hölder's inequality we have

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L_T^\infty L^2} &\leq \|B_I(f(u)-f(v))\|_{L_T^\infty L^2}+\|B_I(u-v)\|_{L_T^\infty L^2}\\
&\leq c\left(\|f(u)-f(v)\|_{L_T^{\gamma'}L^{\rho'}}+\|u-v\|_{L_T^1L^2}\right)\\
&\leq c\left(T^{1/\gamma'}\left\|\int_0^1 f'(\lambda u+(1-\lambda)v)(u-v)d\lambda\right\|_{L_T^\infty L^{\rho'}}+\right.\\
&\quad \left.+T\|u-v\|_{L_T^\infty L^2}\right)\\
&\leq c\left(T^{1/\gamma'}\int_0^1\|f'(\lambda u+(1-\lambda)v)\|_{L_T^\infty L^{\frac{q}{k-1}}}d\lambda\right)\\
&\quad \cdot\|u-v\|_{L_T^\infty L^2}+cT\|u-v\|_{L_T^\infty L^2}\\
&\leq c\left(T^{1/\gamma'}\left(\|u\|^{k-1}_{L_T^\infty L^q}+\|v\|^{k-1}_{L_T^\infty L^q}\right)+T\right)\|u-v\|_{L_T^\infty L^2}\\
&\leq c\left(T^{1/\gamma'}N^{k-1}+T\right)\|u-v\|_{L_T^\infty L^2}.
\end{aligned}$$

Then Φ is a contraction in the $L_T^\infty L^2$-metric for suitable N and $T>0$ and by standard arguments there a unique solution $u\in C([0,T];H^s(\mathbb{R}^n))\cap\mathcal{Y}_s$ to (7).

Remark 4.2 *Note that, if $\Phi(u)=u\in X^s$, then by the proof of (13), we have*

$$\|u\|_{L_T^qH_r^s}\leq c\left(\|\phi\|_{H^s}+\|\psi\|_{H^s}\right)+c\left(T(N+K)+T^{1/\gamma'}N^{k-1}(N+K)\right) \quad (14)$$

for all (q,r) admissible pair. Therefore $u\in\mathcal{Y}_s$.

Case (ii) $\mathbf{s\leq\frac{n}{2}}$, $\sigma\in\left[\mathbf{0},\frac{\mathbf{n}}{\mathbf{2}}\right)\cap\left(\frac{\mathbf{n}}{\mathbf{2}}-\frac{\mathbf{2}}{\mathbf{k-1}},\mathbf{s}\right]$

Consider (p_1,p_2) and (q_1,q_2) given by Lemma 4.2 and define the following complete metric space

$$Y^s=\left\{u\in(1-\Delta)^{-\frac{s}{2}}\left(L_T^\infty L^2\cap L_T^{p_1}L^{p_2}\right):\begin{array}{l}\|u\|_{L_T^\infty L^2},\|u\|_{L_T^{p_1}L^{p_2}}\leq L;\\ \|D^su\|_{L_T^\infty L^2},\|D^su\|_{L_T^{p_1}L^{p_2}}\leq K;\\ \|D^\sigma u\|_{L_T^\infty L^2},\|D^\sigma u\|_{L_T^{p_1}L^{p_2}}\leq N\end{array}\right\}$$

$d(u,v)=\|u\|_{L_T^\infty L^2}+\|u\|_{L_T^{p_1}L^{p_2}}$.

By Sobolev embedding, we have

$$\|u\|_{L_T^{q_1} L^{q_2}} \leq c\|D^\sigma u\|_{L_T^{q_1} L^{\beta_2}} \text{ ; where } \frac{1}{\beta_2} = \frac{1}{q_2} + \frac{\sigma}{n}.$$

Recall that (q_1, β_2) is an admissible pair. Therefore, in view of (iii) in Lemma 4.2, we can interpolate between $L_T^\infty L^2$ and $L_T^{p_1} L^{p_2}$ and find $0 < \alpha < 1$ such that

$$\|u\|_{L_T^{q_1} L^{q_2}} \leq c\|D^\sigma u\|_{L_T^\infty L^2}^{1-\alpha} \|D^\sigma u\|_{L_T^{p_1} L^{p_2}}^{\alpha} \leq cN. \tag{15}$$

Moreover, by (iv) in Lemma 4.2 together with (i) in Lemma 3.1 there exists $\theta > 0$ such that

$$\begin{aligned} \|\Phi(u)\|_{L_T^a L^b} &\leq \|B_c(t)\phi\|_{L_T^a L^b} + \|B_s(t)\Delta\psi\|_{L_T^a L^b} + \|B_I(f(u) - u)\|_{L_T^a L^b} \\ &\leq c\left(\|\phi\|_{L^2} + \|\psi\|_{L^2} + \|B_I(f(u))\|_{L_T^a L^b} + \|B_I(u)\|_{L_T^a L^b}\right) \\ &\leq c\left(\|\phi\|_{L^2} + \|\psi\|_{L^2} + T^\theta \|f(u)\|_{L_T^{r_1} L^{r_2}} + \|u\|_{L_T^1 L^2}\right) \end{aligned}$$

where $(a, b) \in \{(\infty, 2), (p_1, p_2)\}$.

Now using $(f2)$, the definition of (r_1, r_2) in Lemma 4.2 and Hölder's inequality, we obtain

$$\begin{aligned} \|\Phi(u)\|_{L_T^a L^b} \leq & c\left(\|\phi\|_{L^2} + \|\psi\|_{L^2} + T^\theta \|u\|_{L_T^{p_1} L^{p_2}} \|u\|_{L_T^{q_1} L^{q_2}}^{k-1} + \right. \\ & \left. + T\|u\|_{L_T^\infty L^2}\right) \\ \leq & c\left(\|\phi\|_{L^2} + \|\psi\|_{L^2} + T^\theta N^{k-1} L + TL\right). \end{aligned} \tag{16}$$

Following the same arguments, using the estimates for fractional derivatives (remember that $p_2 \neq \infty$) and the fact that D^s and D^σ commute with B_I, B_c and $B_s\Delta$, we have

$$\|D^s\Phi(u)\|_{L_T^a L^b} \leq c\left(\|D^s\phi\|_{L^2} + \|D^s\psi\|_{L^2} + T^\theta N^{k-1} K + TK\right) \tag{17}$$

and

$$\|D^\sigma\Phi(u)\|_{L_T^a L^b} \leq c\left(\|D^\sigma\phi\|_{L^2} + \|D^\sigma\psi\|_{L^2} + T^\theta N^{k-1} N + TN\right). \tag{18}$$

On the other hand, from an argument analogous to the one used in case (i), we have for $(a,b)\in\{(\infty,2),(p_1,p_2)\}$

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L^a_TL^b} &\leq \|B_I(f(u)-f(v))\|_{L^a_TL^b}+\|B_I(u-v)\|_{L^a_TL^b}\\
&\leq c\left(T^\theta\|f(u)-f(v)\|_{L^{r_1}_TL^{r_2}}+\|u-v\|_{L^1_TL^2}\right)\\
&\leq c\left(T^\theta\left\|\int_0^1 f'(\lambda u+(1-\lambda)v)(u-v)d\lambda\right\|_{L^{r_1}_TL^{r_2}}+\right.\\
&\quad\left.+T\|u-v\|_{L^\infty_TL^2}\right)\\
&\leq cT^\theta\left(\|u\|^{k-1}_{L^{q_1}_TL^{q_2}}+\|v\|^{k-1}_{L^{q_1}_TL^{q_2}}\right)\|u-v\|_{L^{p_1}_TL^{p_2}}+\\
&\quad+cT\|u-v\|_{L^\infty_TL^2}\\
&\leq c\left(T^\theta N^{k-1}+T\right)d(u,v).
\end{aligned}$$

The proof follows by choosing suitable L,N,K and T.

Case (iii) $\mathbf{s}\leq\dfrac{\mathbf{n}}{\mathbf{2}},\sigma=\dfrac{\mathbf{n}}{\mathbf{2}}-\dfrac{\mathbf{2}}{\mathbf{k-1}}$

Let $\tau<1$ and (p_1,p_2), (p_1,p_2) be given by Lemma 4.2. Define the following complete metric space

$$Y^s_\tau=\left\{u\in\left(L^\infty_TH^s\cap L^{p_1}_TH^s_{p_2}\right):\begin{array}{l}\|u\|_{L^\infty_TL^2},\|u\|_{L^{p_1}_TL^{p_2}}\leq L;\\ \|D^su\|_{L^\infty_TL^2},\|D^su\|_{L^{p_1}_TL^{p_2}}\leq K;\\ \|D^\sigma u\|_{L^\infty_TL^2}\leq N;\ \|D^\sigma u\|_{L^{p_1}_TL^{p_2}}\leq\tau N<N\end{array}\right\}$$

$d(u,v)=\|u\|_{L^\infty_TL^2}+\|u\|_{L^{p_1}_TL^{p_2}}$.

Then, following the same arguments of (15), there exists $0<\alpha<1$, such that

$$\|u\|_{L^{q_1}_TL^{q_2}}\leq c\|D^\sigma u\|^{1-\alpha}_{L^\infty_TL^2}\|D^\sigma u\|^\alpha_{L^{p_1}_TL^{p_2}}\leq c\tau^\alpha N.$$

As in the inequalities (16) and (17), we have for $(a,b)\in\{(\infty,2),(p_1,p_2)\}$

$$\|\Phi(u)\|_{L^a_TL^b}\leq c\left(\|\phi\|_{L^2}+\|\psi\|_{L^2}+(\tau^\alpha N)^{k-1}L+TL\right), \tag{19}$$

and

$$\|D^s\Phi(u)\|_{L^a_TL^b}\leq c\left(\|D^s\phi\|_{L^2}+\|D^s\psi\|_{L^2}+(\tau^\alpha N)^{k-1}K+TK\right). \tag{20}$$

On the other hand, the inequality (18) should be replaced by the following two estimates

$$\|D^\sigma \Phi(u)\|_{L^\infty_T L^2} \leq c\left(\|D^\sigma\phi\|_{L^2} + \|D^\sigma\psi\|_{L^2} + \tau^{1+\alpha(k-1)}N^k + TN\right) \quad (21)$$

and

$$\begin{aligned}\|D^\sigma \Phi(u)\|_{L^{p_1}_T L^{p_2}} \leq & c\left(\|B_c(\cdot)D^\sigma\phi\|_{L^{p_1}_T L^{p_2}} + \|B_s\Delta(\cdot)D^\sigma\psi\|_{L^{p_1}_T L^{p_2}}\right) \\ & + c\left(\tau^{1+\alpha(k-1)}N^k + TN\right).\end{aligned} \quad (22)$$

Taking T small the terms $\|B_c(\cdot)D^\sigma\phi\|_{L^{p_1}_T L^{p_2}}$ and $\|B_s\Delta(\cdot)D^\sigma\psi\|_{L^{p_1}_T L^{p_2}}$ can be made small enough (note that $p_1 \neq \infty$). So it is clear that the operator Φ maps Y^s_τ into Y^s_τ (choosing suitable L, N, K, T, τ). The reminder of the proof follows from a similar argument as the one previously used and it will be omitted.

Finally, we remark that once we established that Φ is a contraction in appropriate spaces the proof of continuous dependence is straightforward.

■

Proof of Theorem 1.2

(i) By (13) we have to choose N, T such that

$$c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right) + c_0 N\left(T + T^{1/\gamma'}N^{k-1}\right) \leq N. \quad (23)$$

Setting $N = 2c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right)$ this inequality becomes

$$T + T^{1/\gamma'}\left(2c_0\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right)^{k-1} \leq 1/2c_0.$$

This inequality is clearly satisfied for

$$T = \frac{1}{4c_0}\min\left\{1, 2\left(\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right)^{\gamma'(1-k)}\right\}.$$

Now setting $c = 1/4c_0$ and $\theta = 1/\gamma'$ we have

$$T^* \geq c\left(\min\left\{1, 2\left(\max\left\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\right\}\right)^{\frac{1-k}{\theta}}\right\}\right).$$

Note that (9) does not follow direct from the inequality above. To prove (9) we will use an iterative argument. Set $T = \bar{T} = 1/2c_0$. Thus, inequality (23) becomes

$$c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right) + c_1 N^{k-1} \leq N/2 \quad (24)$$

for some $c_1 > 0$.

It is clear that (24) has a solution N if $\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}$ is sufficiently small. In fact, we have more than that. An application of the implicit function theorem tell us that there are $\bar{\delta} > 0$ and $\lambda > 1$ such that if $\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\} \leq \delta \leq \bar{\delta}$ then $N \leq \lambda\delta$, where N is the solution of (24).

It follows that if $\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\} \leq \lambda^{-n}\bar{\delta}$ then we can find $N_1 \leq \lambda^{-n+1}\bar{\delta}$ such that the solution exists in the interval $[0, \bar{T}]$. Moreover by construction

$$\|u(\bar{T})\|_{H^\sigma} \leq N_1 \leq \lambda^{-n+1}\bar{\delta}.$$

We want to repeat this argument. Therefore, we first need to control the growth of $\|\Delta^{-1}u_t(t)\|_{H^\sigma}$. Since $u(t)$ is given by (8) we have that

$$\Delta^{-1}u_t(t) = B_s(t)\Delta\phi - B_c(t)\psi - \int_0^t B_c(t-t')(f(u)-u)(t')dt'.$$

Thus, applying the same argument used to prove (13), we obtain

$$\begin{aligned}
\|\Delta^{-1}u_t(\bar{T})\|_{H^\sigma} &\leq \|\Delta^{-1}u_t(\bar{T})\|_{L^\infty_{\bar{T}}H^\sigma} \\
&\leq c_0\left(\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}\right) + N/2 + c_1 N^{k-1}.
\end{aligned}$$

Since N_1 is the solution of (24) we also have

$$\|\Delta^{-1}u_t(\bar{T})\|_{H^\sigma} \leq N_1 \leq \lambda^{-n+1}\bar{\delta}.$$

Now, solving equation (7) with initial data $u(\bar{T})$ and $\Delta^{-1}u_t(\bar{T})$, we can find $N_2 \leq \lambda^{-n+2}\bar{\delta}$ such that the solution exists in the interval $[\bar{T}, 2\bar{T}]$. Moreover,

$$\max\left\{\|u(2\bar{T})\|_{H^\sigma}, \|\Delta^{-1}u_t(2\bar{T})\|_{H^\sigma}\right\} \leq \lambda^{-n+2}\bar{\delta}.$$

Repeating this process we can find N_i, $i = 1, \ldots, n$, such that the solution exists on the intervals $[0, \bar{T}], \ldots, [(n-1)\bar{T}, n\bar{T}]$, so that $T^* \geq \bar{T}$. Thus T^* is arbitrarily large if $\max\{\|\phi\|_{H^\sigma}, \|\psi\|_{H^\sigma}\}$ is sufficiently small.

(iia) The proof is essentially the same as (i) using inequality (18) instead of (13).

(iib) In view of (21) and (22), we have to choose N, T, τ such that

$$c\left(\max\{\|D^\sigma\phi\|_{L^2}, \|D^\sigma\psi\|_{L^2}\} + \tau^{1+\alpha(k-1)}N^k + TN\right) \leq N$$

and

$$c\left(\max\{B_1, B_2\} + \tau^{1+\alpha(k-1)}N^k + TN\right) \leq \tau N.$$

where $B_1 \equiv \|B_c(\cdot)D^\sigma\phi\|_{L_T^{p_1}L^{p_2}}$ and $B_2 \equiv \|B_s\Delta(\cdot)D^\sigma\psi\|_{L_T^{p_1}L^{p_2}}$.

But the sizes of B_1 and B_2 depend on T and $D^\sigma\phi$, $D^\sigma\psi$ (but not necessarily on their norms). That is why T^* cannot be estimated only in terms of $\|D^\sigma\phi\|_{L^2}$ and $\|D^\sigma\psi\|_{L^2}$.

■

Proof of Theorem 1.3 We use an argument first used by [21] (see also [5] page 826).

(a) Let T^* given by Theorem 1.2 and $t < T^*$. If we consider $u(t)$ and $\Delta^{-1}u_t(t)$ as the initial data, the solution cannot be extended to a time $\geq T^*$. Setting $D(t) = \max\{\|u(t)\|_{H^\sigma}, \|\Delta^{-1}u_t(t)\|_{H^\sigma}\}$, it follows from (13) and the fixed point argument that if for some $N > 0$,

$$cD(t) + cN\left((T-t) + (T-t)^{1/\gamma'}N^{k-1}\right) \leq N$$

then $T < T^*$.

Thus for all $N > 0$, we have

$$cD(t) + cN\left((T^*-t) + (T^*-t)^{1/\gamma'}N^{k-1}\right) \geq N.$$

Now, choosing $N = 2cD(t)$ and letting $t \to T^*$ we have the blow up result.

(b) The proof is similar to part (a).

■

5 Unconditional well-posedness

The aim of this section is to prove Theorems 1.4 and 1.5. We start with the following uniqueness result.

Lemma 5.1 *Let* (p_1, p_2) *and* (q_1, q_2) *such that*

(*i*) (p_1, p_2) *is an admissible pair;*

(*ii*) *There exists* $\delta \in [0,1]$ *such that*
$$\frac{1}{p_1} \geq \frac{1-\delta}{q_1} \text{ and } \frac{1}{p_2} = \frac{1-\delta}{q_2} + \frac{\delta}{2};$$

(*iii*) *If* $\frac{1}{r_i} \equiv \frac{1}{p_i} + \frac{k-1}{q_i}$, $i = 1, 2$, *then there exists* $s_1 \geq 1$ *such that* (s_1, r_2) *is the dual of an admissible pair and* $s_1 \leq r_1$.

Then uniqueness holds in $\mathcal{X} \equiv L^\infty_T L^2 \cap L^{q_1}_T L^{q_2}$.

Proof The proof follows the same ideas of Lemma 3.1 in [12].

Using Hölder's inequality and interpolation we have, in view of (*ii*), that $\mathcal{X} \subset L^{p_1}_T L^{p_2}$.

Returning to the uniqueness question, suppose there are two fixed points $u, v \in \mathcal{X}$ of the integral equation (12). Then $w \equiv u - v$ may be written as
$$w = B_I(f(u) - f(v)) - B_I(u - v).$$

But for $(a, b) \in \{(\infty, 2), (p_1, p_2)\}$, we have by Lemma 3.1 (*ii*) that
$$\begin{aligned} \|B_I(u-v)\|_{L^a_T L^b} &\leq c\|u-v\|_{L^1_T L^2} &(25)\\ &\leq cT\|u-v\|_{L^\infty_T L^2}. &(26) \end{aligned}$$

It remains to estimate the term $B_I(f(u) - f(v))$. Suppose first that $s_1 < r_1$. In this case, using (*iii*), Lemma 3.1 (*ii*), the Mean Value Theorem and Hölder's inequality, we obtain for $\theta \equiv \frac{1}{s_1} - \frac{1}{r_1} > 0$
$$\begin{aligned} \|B_I(f(u)-f(v))\|_{L^a_T L^b} &\leq c\|f(u)-f(v)\|_{L^{s_1}_T L^{r_2}}\\ &\leq cT^\theta \|f(u)-f(v)\|_{L^{r_1}_T L^{r_2}}\\ &\leq cT^\theta \left\| \left(|u|^{k-1} + |v|^{k-1}\right)(u-v)\right\|_{L^{r_1}_T L^{r_2}}\\ &\leq cT^\theta \left(\|u\|^{k-1}_{L^{q_1}_T L^{q_2}} + \|u\|^{k-1}_{L^{q_1}_T L^{q_2}}\right)\|u-v\|_{L^{p_1}_T L^{p_2}}. \end{aligned}$$

When $s_1 = r_1$ we have $\theta = 0$ in the above inequality. To overcome this difficulty we use an argument introduced by Cazenave (see [4] Proposition 4.2.5.). Define
$$\begin{aligned} f^N &= 1_{\{|u|+|v|>N\}}(f(u) - f(v)),\\ f_N &= 1_{\{|u|+|v|\leq N\}}(f(u) - f(v)). \end{aligned}$$

Therefore by Lemma 3.1 (ii) we have for $(a,b)\in\{(\infty,2),(p_1,p_2)\}$ that

$$\begin{aligned}\|B_I f_N\|_{L^a_T L^b} \leq & \quad cN^{k-1}\|u-v\|_{L^1_T L^2}\\ \leq & \quad cN^{k-1}T\|u-v\|_{L^\infty_T L^2}.\end{aligned}$$

On the other hand, using (iii), Lemma 3.1 (ii), the Mean Value Theorem and Hölder's inequality, we obtain

$$\|B_I f^N\|_{L^a_T L^b} \leq \quad c\left(\left\|1_{\{|u|+|v|>N\}}(|u|+|v|)\right\|_{L^{q_1}_T L^{q_2}}\right)^{k-1}\|u-v\|_{L^{p_1}_T L^{p_2}}.$$

Since $|u|+|v|\in L^{q_1}_T L^{q_2}$, it follows by dominated convergence that

$$\left\|1_{\{|u|+|v|>N\}}(|u|+|v|)\right\|_{L^{q_1}_T L^{q_2}} \to 0, \text{ when } N\to\infty.$$

By choosing N large enough, we can find $\bar{c}>0$ such that

$$\|u-v\|_{L^{p_1}_T L^{p_2}} + \|u-v\|_{L^\infty_T L^2} \leq \bar{c}TN^{k-1}\|u-v\|_{L^\infty_T L^2}.$$

Set $d(w)=\|w\|_{L^{p_1}_T L^{p_2}}+\|w\|_{L^\infty_T L^2}$. Therefore, in both cases we can find a function $H(T)$ such that $H(T)\to 0$ when $T\to 0$ and

$$d(w)\leq H(T)d(w).$$

Taking $T_0>0$ small enough such that $H(T_0)<1$, we conclude that $d(w)$ must be zero in $[0,T_0]$. Now, since the argument does not depend on the initial data, we can reapply this process a finite number of times to extend the uniqueness result to the whole existence interval $[0,T]$.

■

By Sobolev embedding, we know that for $s\geq 0$

$$C([0,T];H^s)\subset L^\infty_T(L^2\cap L^q), \text{ for some } q>2.$$

Therefore, in view of Lemma 5.1, the unconditional well-posed problem can be reduced to find pairs (p_1,p_2) and (q_1,q_2) satisfying the hypotheses (i), (ii) and (iii). We remark that is at this point that the restrictions on k and s appear (see Theorem 1.4). In the next lemma, we treat this geometric problem.

Lemma 5.2 *We have three cases:*

(i) If $n=1$, uniqueness holds in $L^\infty_T(L^2\cap L^q)$ for all

$$q\geq\max\{k,2\};$$

(ii) If $n = 2$, uniqueness holds in $L^\infty_T(L^2 \cap L^q)$ for all

$$\frac{1}{q} < \frac{1}{k} \quad \textit{and} \quad \frac{1}{q} \le \min\left\{\frac{1}{2}, \frac{1}{k-1}\right\};$$

(iii) If $n \ge 3$, uniqueness holds in $L^\infty_T(L^2 \cap L^q)$ for all

$$\frac{1}{q} \le \min\left\{\left(\frac{1}{2} + \frac{1}{n}\right)\frac{1}{k}, \frac{1}{2}, \frac{2}{n(k-1)}\right\}.$$

Proof Affirmations (i) and (ii) follow from Corollary 2.2 (see also Theorem 2.1) in [12]. On the other hand, the proof of (iii) is a little bit different from Kato's proof since we have one more admissible pair, namely $\left(2, \dfrac{2n}{n-2}\right)$. So we will give a detailed proof of this item. We consider several cases separately

(a) $\mathbf{1 < k \le 1 + \dfrac{2}{n}}$

Set $(p_1, p_2) = (q_1, q_2) = (\infty, 2)$. It is easy to see that there exists $s_1 \ge 1$ satisfying $(i) - (iii)$ of Lemma 5.1 (with $\delta = 0$). Then uniqueness holds in $L^\infty_T L^2$ and therefore in $L^\infty_T L^2 \cap L^{q_1}_T L^{q_2}$ fol all (q_1, q_2). Note that if $k = 1 + \dfrac{2}{n}$, we have that (r_1, r_2) must be given by $r_1 = \infty$ and $\dfrac{1}{r_2} = \dfrac{1}{2} + \dfrac{1}{n}$. Therefore, $(2, r_2)$ is the dual of the admissible pair $\left(2, \dfrac{2n}{n-2}\right)$.

(b) $\mathbf{1 + \dfrac{2}{n} < k < 1 + \dfrac{4}{n-2}}$

Let $b_k \equiv \left(\dfrac{1}{2} + \dfrac{1}{n}\right)\dfrac{1}{k}$. By the restriction on k we have $\dfrac{1}{2} - \dfrac{1}{n} < b_k < \dfrac{1}{2}$. Therefore there exists an admissible pair (α_k, β_k) such that $\beta_k = \dfrac{1}{b_k}$. Let (∞, q) such that $\dfrac{1}{q} \le b_k$. By interpolation we obtain

$$L^\infty_T L^2 \cap L^\infty_T L^q \subseteq L^\infty_T L^2 \cap L^\infty_T L^{\beta_k}.$$

If uniqueness holds on $L^\infty_T L^2 \cap L^\infty_T L^{\beta_k}$, then it holds, a fortiori, in $L^\infty_T L^2 \cap L^\infty_T L^q$. Therefore, we just need to verify that $(p_1, p_2) =$

(α_k,β_k), $(q_1,q_2)=(\infty,\beta_k)$ satisfy the hypotheses of Lemma 5.1. Indeed, in this case $(i)-(ii)$ can be easily verified (for $\delta=0$). On the other hand, (r_1,r_2) must be given by $\dfrac{1}{r_1}=\dfrac{1}{\alpha_k}$ and $\dfrac{1}{r_2}=\dfrac{k}{\beta_k}$.

Thus, (s_1,r_2), with $s_1=2$ is the dual of the admissible pair $\left(2,\dfrac{2n}{n-2}\right)$. Moreover

$$s_1<r_1 \Longleftrightarrow \frac{1}{2}>\frac{n}{2}\left(\frac{1}{2}-\frac{1}{\beta_k}\right)\Longleftrightarrow k<1+\frac{4}{n-2}.$$

(c) $\mathbf{k\geq 1+\dfrac{4}{n-2}}$

In this case

$$\frac{2}{n(k-1)}\leq\frac{1}{2}-\frac{1}{n}<\frac{1}{2}. \tag{27}$$

Let (∞,q) such that $\dfrac{1}{q}\leq\dfrac{2}{n(k-1)}$. By the same argument used in item (b) it is sufficient to prove that uniqueness holds in $L^\infty_T L^2\cap L^\infty_T L^{\widetilde{q}}$, where $\dfrac{1}{\widetilde{q}}=\dfrac{2}{n(k-1)}$. Therefore, we need to verify that $(p_1,p_2)=(2,\dfrac{2n}{n-2})$ and $(q_1,q_2)=(\infty,\widetilde{q})$ satisfy the hypotheses of Lemma 5.1. It is clear that (i) holds. On the other hand, in view of (27) we can find $\delta\in[0,1]$ such that (ii) holds. Now, we turn to property (iii). The pair (r_1,r_2) must be given by $r_1=2$ and $\dfrac{1}{r_2}=\dfrac{1}{2}+\dfrac{1}{n}$, which is the dual of the admissible pair $\left(2,\dfrac{2n}{n-2}\right)$.

■

Now we can prove our first uniqueness result.

Proof of Theorem 1.4 This is an immediate consequence of the last lemma. Using Sobolev embedding and decreasing T if necessary we have

$$C([0,T];H^s)\subset L^\infty_T(L^2\cap L^{\bar{q}})$$

where

$$\bar{q}=\begin{cases}2n/(n-2s) & \text{if } s<n/2;\\ \text{any } \bar{q}<\infty & \text{if } s=n/2;\\ \infty & \text{if } s>n/2.\end{cases}$$

So we have only to verify that uniqueness holds in $L^\infty_T(L^2 \cap L^{\bar q})$, but Lemma 5.2 tell us when it happens.

■

Now, we turn to the proof of Theorem 1.5. First of all, define $H(u,v)$ by

$$H(u,v) \equiv \int_0^1 |\lambda u + (1-\lambda)v|^{k-1} d\lambda. \tag{28}$$

We will need some preliminary lemmas. The following result can be found in [9] Lemma 3.8.

Lemma 5.3 *Let $n \geq 3$, $0 < s < 1$, $k > 2$ and $k \leq 1 + \dfrac{2n-2s}{n-2s}$. Let $h \in \dot H^s_\tau(\mathbb{R}^n)$ with $\tau = \dfrac{n}{s + (k-1)\left(\dfrac{n}{2} - s\right)}$. If k also verifies the following conditions:*

(i) $k > 1 + \dfrac{2s}{n-2s}$;

(ii) $k < 1 + \min\left\{\dfrac{4s+2}{n-2s}, \dfrac{4}{n-2s}, \dfrac{n+2s}{n-2s}\right\}$;

(iii) $k \leq 1 + \dfrac{n+2-2s}{n-2s}$.

Then there exist σ, p verifying $\sigma - \dfrac{n}{p} = s - \dfrac{n}{2}$ and

(1) $s - 1 \leq \sigma \leq s$;

(2) $-s < \sigma < 0$;

(3) $s - (k-1)\left(\dfrac{n}{2} - s\right) \leq \sigma \leq \min\left\{s+1, \dfrac{n}{2}\right\} - (k-1)\left(\dfrac{n}{2} - s\right)$.

Such that if $g \in \dot B^\sigma_{p,2}(\mathbb{R}^n)$, then $gh \in \dot B^\sigma_{r',2}(\mathbb{R}^n)$ with

$$\|gh\|_{\dot B^\sigma_{r',2}} \leq c \|g\|_{\dot B^\sigma_{p,2}} \|h\|_{\dot H^s_\tau}$$

where $\dfrac{1}{r'} = \dfrac{1}{p} + \dfrac{(k-1)\left(\dfrac{n}{2} - s\right)}{n}$ *and* $\dfrac{2n}{n+2} \leq r' \leq 2$.

To estimate (28) we use the following lemma.

Lemma 5.4 *Let $n \geq 3$, $k > 2$, $0 \leq s < \dfrac{n}{2}$ and $s < k-1$. Suppose also that $(k-1)\left(\dfrac{1}{2} - \dfrac{s}{n}\right) \leq 1$ and define $\tau = \dfrac{n}{s + (k-1)(\frac{n}{2} - s)}$. If $u, v \in L_T^\infty \dot{H}^s$, then $H(u,v) \in L_T^\infty \dot{H}_\tau^s$. Moreover, $\tau \geq 1$ if and only if $k \leq 1 + \dfrac{2n-2s}{n-2s}$.*

Proof By definition of $H(u,v)$, we have

$$\|H(u,v)\|_{\dot{H}_\tau^s} = \int_0^1 \||\lambda u + (1-\lambda)v)|^{k-1}\|_{\dot{H}_\tau^s} \leq c\left(\||u|^{k-1}\|_{\dot{H}_\tau^s} + \||v|^{k-1}\|_{\dot{H}_\tau^s}\right)$$

and using Lemma 3.4 (ii) we have the desire estimate.

■

Furthermore, we have the following embedding lemma which proof can be found in [18].

Lemma 5.5 *$\dot{H}^s \hookrightarrow \dot{B}_{p,2}^\sigma$ for all $\sigma \leq s$ and $\sigma - \dfrac{n}{p} = s - \dfrac{n}{2}$. Moreover, there exists $\gamma \geq 1$ such that (γ, p) is an admissible pair if and only if $s-1 \leq \sigma \leq s$.*

Now, we can proof our last uniqueness result.

Proof of Theorem 1.5 First, we recall that $\dot{B}_{2,2}^\sigma = \dot{H}^\sigma$, $H^s \subseteq \dot{H}^\sigma$ for all $\sigma, s \in \mathbb{R}$ and $\sigma \leq s$. Then, using Lemma 5.5, we conclude that $(u-v) \in L_T^\infty \dot{B}_{p,2}^\sigma \cap L_T^\infty \dot{B}_{2,2}^\sigma$, where σ and p satisfy conditions $(1)-(3)$ of Lemma 5.3. Moreover, in view of Lemma 5.5 and condition (1) of Lemma 5.3, there exists $\gamma \geq 1$ such that (γ, p) is an admissible pair.

Thus, by Lemma 3.2 (i), we have for $(a,b) \in \{(\infty, 2), (\gamma, p)\}$

$$\begin{aligned}
\|u-v\|_{L_T^a \dot{B}_{b,2}^\sigma} &\leq \|B_I(f(u)-f(v))\|_{L_T^a \dot{B}_{b,2}^\sigma} + \|B_I(u-v)\|_{L_T^a \dot{B}_{b,2}^\sigma} \\
&\leq c\|f(u)-f(v)\|_{L_T^{q'} \dot{B}_{r',2}^\sigma} + c\|u-v\|_{L_T^1 \dot{B}_{2,2}^\sigma} \\
&\leq c\|(u-v)H(u,v)\|_{L_T^{q'} \dot{B}_{r',2}^\sigma} + cT\|u-v\|_{L_T^\infty \dot{B}_{2,2}^\sigma}
\end{aligned}$$

where $\dfrac{1}{r'} = \dfrac{1}{p} + \dfrac{(k-1)\left(\frac{n}{2} - s\right)}{n}$ and $\dfrac{2n}{n+2} \leq r' \leq 2$. Recall that this last condition implies that (q', r') is the dual of an admissible pair.

Then by Lemma 5.3, we obtain:

$$\|u-v\|_{L_T^a \dot{B}_{b,2}^\sigma} \leq c\|\|u-v\|_{\dot{B}_{p,2}^\sigma} \|H(u,v)\|_{\dot{H}_\tau^s}\|_{L_T^{q'}} + cT\|u-v\|_{L_T^\infty \dot{B}_{2,2}^\sigma}.$$

But $\frac{1}{q'} - \frac{1}{\gamma} = 1 - \frac{(k-1)}{2}\left(\frac{n}{2} - s\right) \equiv \theta > 0$ since $k < 1 + \frac{4}{n-2s}$. Thus

$$\|u-v\|_{L^a_T \dot{B}^\sigma_{b,2}} \leq cT^\theta \|u-v\|_{L^\gamma_T \dot{B}^\sigma_{p,2}} \|H(u,v)\|_{L^\infty_T \dot{H}^s_\tau} + cT\|u-v\|_{L^\infty_T \dot{B}^\sigma_{2,2}}.$$

Set $\omega(u,v) \equiv \|u-v\|_{L^\infty_T \dot{B}^\sigma_{2,2}} + \|u-v\|_{L^\gamma_T \dot{B}^\sigma_{p,2}}$, therefore we conclude that

$$\omega(u,v) \leq c\left(T^\theta \|H(u,v)\|_{L^\infty_T \dot{H}^s_\tau} + T\right)\omega(u,v).$$

Hence, for $T_0 > 0$ small enough, $u(t) = v(t)$ on $[0, T_0]$ and to recover the whole interval we apply the same argument as the one used in the proof of Lemma 5.1.

■

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