

POLYNOMIAL ODES WITH MANY REAL OVALS IN THE SAME COMPLEX SOLUTION

BY A. LINS NETO ¹

Abstract. Let $Fol_{\mathbb{R}}(2, d)$ be the space of real algebraic foliations of degree d in $\mathbb{RP}(2)$. For fixed d , let $Int_{\mathbb{R}}(2, d) = \{\mathcal{F} \in Fol_{\mathbb{R}}(2, d) \mid \mathcal{F} \text{ has a non-constant rational first integral}\}$. Given $\mathcal{F} \in Int_{\mathbb{R}}(2, d)$, with primitive first integral G , set $O(\mathcal{F}) = \text{number of real ovals of the generic level } (G = c)$. Let $O(d) = \sup\{O(\mathcal{F}) \mid \mathcal{F} \in Int_{\mathbb{R}}(2, d)\}$. The main purpose of this paper is to prove that $O(d) = +\infty$ for all $d \geq 5$.

CONTENTS

1.	Introduction	1
2.	Some properties of the family $\mathbb{F}_4$	4
3.	Pulling-back the family $\mathbb{F}_4$	10
4.	Other families	13

1. INTRODUCTION

It is well known (Harnack's theorem) that a real algebraic curve of degree d in $\mathbb{RP}(2)$ has at most $\frac{(d-1)(d-2)}{2} + 1$ connected components (ovals). A similar question in the context of real algebraic foliations can be posed. Let $Fol_{\mathbb{R}}(2, d)$ be the set of algebraic foliations in $\mathbb{RP}(2)$ of degree d . A foliation $\mathcal{F} \in Fol_{\mathbb{R}}(2, d)$ can be complexified to one in $\mathbb{CP}(2)$ that we denote $\mathcal{F}_{\mathbb{C}}$. We will study the following problem :

Problem 1. Given $d \geq 2$ does there exists $N(d) \in \mathbb{N}$ such that for any complex algebraic leaf L of a foliation $\mathcal{F}_{\mathbb{C}}$, $\mathcal{F} \in Fol_{\mathbb{R}}(2, d)$, then the number of ovals of $L \cap \mathbb{RP}(2)$ is $\leq N(d)$?

We would like to remark that for $d = 1$ there exists such a bound : $N(1) = 1$.

The main purpose of this paper is to prove that there is no such a bound for all $d \geq 5$. In order to pose the main result, let us recall some facts and fix some notations concerning the subject.

Let $Fol(2, d)$ be the space of holomorphic foliations of degree d in $\mathbb{CP}(2)$. Any foliation $\mathcal{F} \in Fol(2, d)$ can be represented in homogeneous coordinates by an integrable 1-form

$$(1) \quad \Omega_{\mathcal{F}} = P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

where P , Q and R are homogeneous polynomials of degree $d + 1$ and $x.P + y.Q + z.R \equiv 0$ (cf. [LN]). Geometrically the degree of $\mathcal{F}$ is the number of tangencies of $\mathcal{F}$

¹This research was partially supported by Pronex.

with a generic straight line of $\mathbb{CP}(2)$. For instance, in the affine coordinate system ($z = 1$) the foliation is defined by the differential equation $\omega = 0$, where

$$\omega = P(x, y, 1) dx + Q(x, y, 1) dy ,$$

and the number of tangencies of $\mathcal{F}$ in $\mathbb{C}^2 = (z = 1) \subset \mathbb{CP}(2)$ with the line ($y = ax + b$) is the number of complex solutions of the equation $f(x) = 0$, where $f(x) = P(x, ax + b) - aQ(x, ax + b)$. With the condition $x.P + y.Q + z.R = 0$ and if $a, b \in \mathbb{C}$ are generic then the polynomial $f(x)$ has degree d .

We will denote by $\text{sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. If $\Omega_{\mathcal{F}}$ is as in (1) then in homogeneous coordinates we have $\text{sing}(\mathcal{F}) = (P = Q = R = 0)$.

The set of real algebraic foliations of degree d , in these coordinates, is

$$\text{Fol}_{\mathbb{R}}(2, d) = \{\mathcal{F} \mid \Omega_{\mathcal{F}} = P dx + Q dy + R dz, \ x.P + y.Q + z.R = 0, \ P, Q, R \in \mathbb{R}[x, y, z]\}$$

and P, Q and R are homogeneous of degree $d + 1$.

From now on, we will suppose the homogeneous coordinate fixed.

Let $\text{Int}(2, d) = \{\mathcal{F} \in \text{Fol}(2, d) \mid \mathcal{F} \text{ has a non-constant rational first integral}\}$ and $\text{Int}_{\mathbb{R}}(2, d) = \text{Int}(2, d) \cap \text{Fol}_{\mathbb{R}}(2, d)$. It is well known that if $\mathcal{F} \in \text{Fol}_{\mathbb{R}}(2, d)$ has a non-constant rational first integral then it has one, say F/G , where $F, G \in \mathbb{R}[x, y, z]$, that is with real coefficients. For a fixed $\mathcal{F} \in \text{Int}_{\mathbb{R}}(2, d)$ we will denote by $O(\mathcal{F})$ the number of ovals of the generic level ($G_{\mathcal{F}} = c$), $c \in \mathbb{R}$, where $G_{\mathcal{F}}$ is a real primitive rational first integral of $\mathcal{F}$.

The main goal of this paper is to prove the following result :

Theorem 1. *For all $d \geq 5$ there are families $(\mathcal{F}_{\alpha})_{\alpha \in J}$, $J = (a < t < b) \subset \mathbb{R}$, in $\text{Fol}_{\mathbb{R}}(2, d)$ with the following properties :*

- (P.1). $\mathcal{F}_{\alpha} \in \text{Int}_{\mathbb{R}}(2, d)$ if, and only if, $\alpha \in \mathbb{Q} \cap J$.
- (P.2). The set $\{O(\mathcal{F}_{\alpha}) \mid \alpha \in \mathbb{Q} \cap J\}$ is unbounded.
- (P.3). If $\alpha \notin \mathbb{Q} \cap J$ then for almost all complex leaves L of the complexification of $\mathcal{F}_{\alpha}$ such that $L \cap \mathbb{RP}(2) \neq \emptyset$ then $L \cap \mathbb{RP}(2)$ has an infinite number of connected components.

In particular, (P.2) implies that for all $d \geq 5$ we have

$$\sup\{O(\mathcal{F}) \mid \mathcal{F} \in \text{Int}_{\mathbb{R}}(2, d)\} = +\infty .$$

The proof of theorem 1 will be based in [LN] and in some results of [LN-1]. In [LN], for any degree $d \geq 2$, we give examples of 1-parameter families of foliations $\mathbb{F}_d = (\mathcal{F}_{\alpha}^d)_{\alpha \in \overline{\mathbb{C}}}$ in $\text{Fol}(2, d)$, with the following properties :

- (I). The set $E_d = \{\alpha \in \overline{\mathbb{C}} \mid \mathcal{F}_{\alpha}^d \text{ has a non-constant rational first integral}\}$ is countable and dense in $\overline{\mathbb{C}}$. Denote by G_{α}^d a primitive rational first integral of $\mathcal{F}_{\alpha}^d$.
- (II). The set $\{dg(G_{\alpha}^d) \mid \alpha \in E_d\}$ is unbounded ($dg = \text{degree}$).
- (III). The family is *generically equisingular*, in the following sense :
 - (a). There exists a finite subset F of $\overline{\mathbb{C}}$ such that for any $\alpha \in \overline{\mathbb{C}} \setminus F$ the singularities of $\mathcal{F}_{\alpha}^d$ are non-degenerate.
 - (b). For any fixed $\alpha_o \notin F$ and any singularity p_o of $\mathcal{F}_{\alpha_o}^d$ there exists a holomorphic function $p(\alpha)$ defined in a neighborhood of α_o such that $p(\alpha_o) = p_o$, $p(\alpha) \in \text{sing}(\mathcal{F}_{\alpha}^d)$ and the germs of $\mathcal{F}_{\alpha_o}^d$ and $\mathcal{F}_{\alpha}^d$ at p_o and $p(\alpha)$, respectively, are holomorphically equivalent.

The families of degrees $d = 2, 3, 4$ are exhibited explicitly in [LN]. For instance, $\mathbb{F}_4$ is defined in affine coordinates by the family of polynomial 1-forms on $\mathbb{C}^2 \subset \mathbb{CP}(2)$, $(\omega_\alpha := \omega - \alpha \cdot \omega_\infty)_{\alpha \in \overline{\mathbb{C}}}$, where

$$(2) \quad \begin{cases} \omega = (x^3 - 1)x dy - (y^3 - 1)y dx \\ \omega_\infty = (x^3 - 1)y^2 dy - (y^3 - 1)x^2 dx \end{cases}$$

It is shown in [LN] that for $d \in \{2, 3, 4\}$ and $\alpha \in E_d$ the normalization of a generic level $(G_\alpha^d = c)$ is an elliptic curve biholomorphic to $\mathbb{C}/\langle 1, e^{2\pi i/3} \rangle$, where $\langle 1, a \rangle$ denotes the lattice $\{m + n.a \mid m, n \in \mathbb{Z}\}$. In particular, these families cannot satisfy condition (P.2) of theorem 1. However, for $d \geq 5$ the family $\mathbb{F}_d$ is obtained by pulling-back one of the families $\mathbb{F}_2, \mathbb{F}_3$ or $\mathbb{F}_4$ by some fixed rational map $\Phi: \mathbb{CP}(2) \rightarrow \mathbb{CP}(2)$, that is $\mathbb{F}_d = \Phi^*(\mathbb{F}_j)$, for some $j \in \{2, 3, 4\}$. In this way, for $d \geq 5$, it is shown in [LN] that $\mathbb{F}_d$ satisfies :

(IV). For a fixed $\alpha \in E_d$ denote by $g(\alpha)$ the genus of the generic level $(G_\alpha^d = c)$.

If $d \geq 5$ then the set $\{g(\alpha) \mid \alpha \in E_d\}$ is unbounded.

As we will see in §3, when we pull-back the family $\mathbb{F}_4$ by an appropriate rational map Φ with real coefficients then we get a real family of degree $d = 8$ satisfying (P.1), (P.2), and (P.3) of theorem 1. In §4 we will sketch how to obtain families of any degree $d \geq 5$ satisfying (P.1), (P.2) and (P.3).

We would like to observe that property (P.1) will be a consequence of the following result of [LN-1] :

(V). For all $d \geq 2$ we have

$$E_d = \{a + b.e^{2\pi i/3} \mid a, b \in \mathbb{Q}\} \cup \{\infty\}$$

In particular, $E_d \cap \mathbb{R} = \mathbb{Q}$.

Remark 1.1. In [LN-1] it is exhibited another family of degree three such that the set of parameters for which the correspondent foliation has a first integral is $\{a + b.i \mid a, b \in \mathbb{Q}\} \cup \{\infty\}$, $i = \sqrt{-1}$. Families satisfying properties (P.1), (P.2) and (P.3) of theorem 1 can be also constructed by pulling-back this particular one.

Theorem 1 motivates the following problems :

Problem 2. Is the statement of theorem 1 true for degrees $d = 2, d = 3$ or $d = 4$?

Problem 3. For an algebraic curve $L \subset \mathbb{CP}(2)$ denote by $O(L)$ be the number of connected components of $L \cap \mathbb{RP}(2)$. Given $\mathcal{F} \in \text{Fol}_{\mathbb{R}}(2, d)$ set

$$O(\mathcal{F}) = \max\{O(L) \mid L \text{ is an algebraic leaf of } \mathcal{F}\}.$$

We would like to observe that an algebraic leaf is automatically irreducible (by the definition of leaf).

A natural question is the following : does there exists $d \geq 2$ such that

$$\sup\{O(\mathcal{F}) \mid \mathcal{F} \in \text{Fol}_{\mathbb{R}}(2, d) \setminus \text{Int}_{\mathbb{R}}(2, d)\} < +\infty ?$$

Remark 1.2. Concerning problem 3 the following result was proved in [C] by M. Carnicer : let $\mathcal{F}$ be a foliation of degree d in $\mathbb{CP}(2)$ without dicritical singularities. If L is an algebraic leaf of $\mathcal{F}$ then $dg(L) \leq d + 2$. In particular, $L \cap \mathbb{RP}(2)$ has at most $\frac{d(d+1)}{2} + 1$ ovals.

Remark 1.3. In this remark we consider families of logarithmic foliations in $Fol_{\mathbb{R}}(2, d)$ from the point of view problem 1. Let $f_1, \dots, f_r$ be real irreducible polynomials in two variables, two by two relatively primes, and consider the family of foliations in $\mathbb{RP}(2)$ defined in an affine coordinate system by the (r-1)-parametric differential equation $\omega_{\lambda} = 0$, where

$$(3) \quad \omega_{\lambda} = f_1 \dots f_r \sum_{j=1}^r \lambda_j \frac{df_j}{f_j}, \quad \sum_{j=1}^r \lambda_j \cdot dg(f_j) = 0.$$

Denote by $\mathcal{G}_{\lambda}$ the foliation in $\mathbb{RP}(2)$ defined by $\omega_{\lambda} = 0$. If $\lambda \neq (0, \dots, 0)$, it can be shown that $\mathcal{G}_{\lambda}$ has degree d , where $d = dg(f_1) + \dots + dg(f_r) - 2$. Moreover, $\mathcal{G}_{\lambda} \in Int_{\mathbb{R}}(2, d)$ if, and only if $\lambda = c \cdot \mu$, where $\mu \in \mathbb{Z}^r$. If $\lambda = (n_1, \dots, n_r) \in \mathbb{Z}^r$ and $\gcd(n_1, \dots, n_r) = 1$ then a primitive first integral of $\mathcal{G}_{\lambda}$ is $F_{\lambda} = f_1^{n_1} \dots f_r^{n_r}$. In particular, the set

$$\{dg(F_{\lambda}) \mid \lambda \in \mathbb{Z}^r\}$$

is unbounded. On the other hand, this family doesn't satisfy property (P.2) of theorem 1. This is a consequence of the results of Khovanskii in [Kh] which assert that the maximal number of ovals of the level $(F_{\lambda} = c)$ has a bound $N(f_1, \dots, f_r)$, which does not depend on λ , but only on the number of non-zero monomials of $f_1, \dots, f_r$. This fact, of course, implies that the maximal number of ovals that can be obtained from a solution of an equation like in (3) inducing a foliation of degree d has a bound which depends only of d .

I would like to thank Jean Jacques Risler who convinced me of the importance of the subject and strongly stimulated me to write this paper.

2. SOME PROPERTIES OF THE FAMILY $\mathbb{F}_4$

We begin this section by describing some properties of the family $\mathbb{F}_4$ that will be needed (see [LN] and [LN-1]). We will use the notation $j = e^{2\pi i/3}$.

Let $\mathcal{F}_{\alpha}^4$ be the foliation defined in the affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{CP}(2)$ by $\omega_{\alpha} = \omega - \alpha \cdot \omega_{\infty}$, where ω and ω_{∞} are as in (2). We will consider also $\mathcal{F}_{\alpha}^4$ defined by the dual vector field X_{α} of ω_{α} : $X_{\alpha} = X - \alpha \cdot X_{\infty}$, where $X = (x^3 - 1)x \partial_x + (y^3 - 1)y \partial_y$ and $X_{\infty} = (x^3 - 1)y^2 \partial_x + (y^3 - 1)x^2 \partial_y$.

The tangency divisor of $\mathcal{F}_0^4$ and $\mathcal{F}_{\infty}^4$, denoted by $\mathcal{L}$, is the union of 9 lines in $\mathbb{CP}(2)$, say $\ell_1, \dots, \ell_9$, defined in $\mathbb{C}^2$ by

$$(4) \quad (x^3 - 1)(y^3 - 1)(y^3 - x^3) = 0$$

These lines are invariant by all foliations in the family. They intersect two by two in the 12 points of the set

$$\mathcal{R} = \{(a, b) \in \mathbb{C}^2 \mid a, b \in \{1, j, j^2\}\} \cup \{[0 : 1 : 0], [1 : 0 : 0]\}.$$

The points $[0 : 1 : 0]$ and $[1 : 0 : 0]$ are contained in L_{∞} , the line at infinity of $\mathbb{C}^2 \subset \mathbb{CP}(2)$. For all $\alpha \in \overline{\mathbb{C}}$ we have $\mathcal{R} \subset \text{sing}(\mathcal{F}_{\alpha}^4)$.

Let us describe $\mathcal{F}_{\alpha}^4$ for $\alpha \notin \{1, j, j^2, \infty\}$. In this case, the foliation $\mathcal{F}_{\alpha}^4$ has 21 singularities, the 12 in the set $\mathcal{R}$ and 9 more in the set

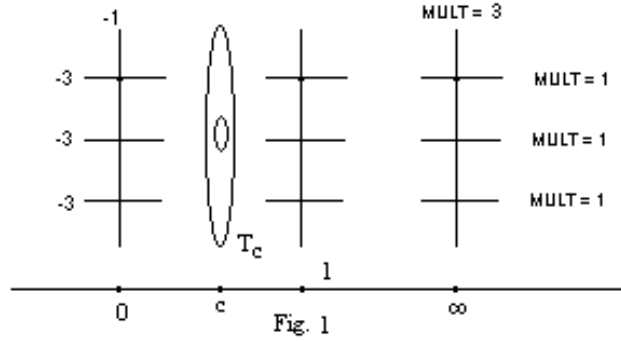
$$\mathcal{S}_{\alpha} = (x^3 - 1 = y - \alpha x^2 = 0) \cup (y^3 - 1 = x - \alpha y^2 = 0) \cup (y^3 - x^3 = x - \alpha y^2 = 0).$$

In particular, each line of $\mathcal{L}$ contains 4 singularities of $\mathcal{F}_\alpha^4$. For instance, the line $(y = x)$ contains the singularities $(1, 1)$, (j, j) , (j^2, j^2) and $(1/\alpha, 1/\alpha)$. The 12 points in $\mathcal{R}$ are of radial type for $\mathcal{F}_\alpha^4$: if $p \in \mathcal{R}$ then there exists a local chart (u, v) around p with $u(p) = v(p) = 0$ such that $X_\alpha = a(u\partial_u + v\partial_v)$, $a \neq 0$. In particular, v/u is a local meromorphic first integral of $\mathcal{F}_\alpha^4$. The 9 points in $\mathcal{S}_\alpha$ are of saddle type: if $q \in \mathcal{S}_\alpha \cap \ell_k$, $k \in \{1, \dots, 9\}$, then there exists a local chart $(W, (u, v))$ around q such that $u(q) = v(q) = 0$, $\ell_k \cap W = (v = 0)$ and $X_\alpha = a(u\partial_u - 3v\partial_v)$, $a \neq 0$. In particular, u^3v is a local holomorphic first integral of $\mathcal{F}_\alpha^4$.

The resolution of $\mathcal{F}_\alpha^4$, in the sense of Seidenberg (cf. [Se] or [Br]), is done by blowing-up in the 12 points of $\mathcal{R}$. Denote by $\pi: M \rightarrow \mathbb{CP}(2)$ this blowing-up procedure and set $\tilde{\mathcal{F}}_\alpha := \pi^*(\mathcal{F}_\alpha^4)$. Let $\tilde{\ell}_k$ denote the strict transform of ℓ_k by π , $k = 1, \dots, 9$, and set $\tilde{\mathcal{L}} = \bigcup_{k=1}^9 \tilde{\ell}_k$. The following properties are proved in [LN]:

- (I). If $\alpha \neq \beta$ then $\tilde{\mathcal{F}}_\alpha$ and $\tilde{\mathcal{F}}_\beta$ are transversal outside $\tilde{\mathcal{L}}$.
- (II). If $\beta \in \{1, j, j^2, \infty\}$ then the foliation $\tilde{\mathcal{F}}_\beta$ has a holomorphic first integral $\tilde{F}_\beta: M \rightarrow \mathbb{CP}(1)$. Moreover, $\tilde{F}_\beta$ is an elliptic fibration with three singular fibers. For instance, $G(x, y) := (y^3 - 1)/(x^3 - 1)$ is a rational first integral of $\mathcal{F}_\infty^4$ and $\tilde{F}_\infty = G \circ \pi: M \rightarrow \mathbb{CP}(1)$. The critical levels of $\tilde{F}_\infty$ are $0, 1, \infty \in \mathbb{CP}(1)$. The fibration is sketched in figure 1. After blowing-down the -1 -component of one of the critical fibers, it becomes of type *IV* in Kodaira's classification of critical fibers elliptic fibrations (cf. [K]).

Remark 2.1. Set $T_c := \tilde{F}_\infty^{-1}(c)$, where $c \neq 0, 1, \infty$. We would like to observe that T_c is an elliptic curve biholomorphic to $\mathbb{C}/\Gamma$, where $\Gamma = \{m + n.j \mid m, n \in \mathbb{Z}\}$, $j = e^{2\pi i/3}$.



This is a consequence of the fact that $G^{-1}(c) = (y^3 - c.x^3 + c - 1 = 0)$ admits the automorphism $\varphi(x, y) = (j.x, j.y)$ which has period three and fixed points at $G^{-1}(c) \cap L_\infty$, where L_∞ is the line at infinity of $\mathbb{C}^2$.

We will denote by $\Pi_c: \mathbb{C} \rightarrow T_c$ a holomorphic universal covering of T_c , so that $T_c \simeq \mathbb{C}/\Gamma$, for all $c \notin \{0, 1, \infty\}$. For $z \in \mathbb{C}$ we will denote by $z \text{ md}(\Gamma)$ its equivalence class in $\mathbb{C}/\Gamma$.

Remark 2.2. Since $\tilde{\mathcal{F}}_\alpha$ is transversal to $\tilde{\mathcal{F}}_\infty$ outside $\tilde{\mathcal{L}}$, we can define a global holonomy representation $\Phi_\alpha: \Pi_1(\mathbb{CP}(1) \setminus \{0, 1, \infty\}, c) \rightarrow \text{Aut}(T_c)$, $c \notin \{0, 1, \infty\}$. If we consider the set of generators of $\Pi_1(\mathbb{CP}(1) \setminus \{0, 1, \infty\}, c)$ sketched in figure 2, then we can write $\Phi_\alpha(\gamma_k) := f_\alpha^k$, $k = 1, 2$, as $f_\alpha^k(z) = (j.z + A_k(\alpha)) \text{ md}(\Gamma)$, where $A_k: \mathbb{C} \rightarrow \mathbb{C}/\Gamma$ is holomorphic, $k = 1, 2$ (cf. [LN]). In particular, the global holonomy is the sub-group $\langle f_\alpha^1, f_\alpha^2 \rangle$ of $\text{Aut}(T_c)$, generated by f_α^1 and f_α^2 .

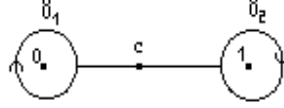


Fig.2

Let $a(\alpha) := (1 - j)^{-1} A_1(\alpha) \, md(\Gamma)$ and $h: \mathbb{C}/\Gamma \rightarrow \mathbb{C}/\Gamma$ be defined by $h(z) = (z + a(\alpha)) \, md(\Gamma)$. Since f_α^1 has a fixed point at $a(\alpha)$ we have $h^{-1} \circ f_\alpha^1 \circ h(z) = j \cdot z \, md(\Gamma)$ and $h^1 \circ f_\alpha^2 \circ h(z) = (j \cdot z + A(\alpha)) \, md(\Gamma)$, where $A(\alpha) = A_2(\alpha) - A_1(\alpha)$. In particular, the global holonomy group of $\tilde{\mathcal{F}}_\alpha$ is conjugated to the group G_α generated by $f(z) := j \cdot z \, md(\Gamma)$ and $f_\alpha(z) = (j \cdot z + A(\alpha)) \, md(\Gamma)$. In [LN] it is proved that $\alpha \mapsto A(\alpha)$ is non-constant.

Another important fact, proved in [LN-1], is that the functions $A_k(\alpha)$, $k = 1, 2$, are affine, that is $A_k(\alpha) = B_k \cdot \alpha + C_k \, md(\Gamma)$, where $B_k, C_k \in \mathbb{C}$. Hence, $A(\alpha)$ is also affine and we can write $A(\alpha) = (B \cdot \alpha + C) \, md(\Gamma)$, where $B \in \mathbb{C}^*$ and $C \in \mathbb{C}$.

We are mainly interested in the real foliations $\tilde{\mathcal{F}}_\alpha$, $\alpha \in \mathbb{R}$, and how their real leaves cut the real levels of $\tilde{F}_\infty$. Let us denote by $M_\mathbb{R}$ the strict transform of $\mathbb{RP}(2)$ by $\pi: M \rightarrow \mathbb{CP}(2)$. Remark that $M_\mathbb{R}$ is $\mathbb{RP}(2)$ blown-up in four points : the four real points of $\mathcal{R}$, $q_1 := (0, 0)$, $q_2 := (1, 1)$, $q_3 := [1 : 0 : 0]$ and $q_4 := [0 : 1 : 0]$. In particular, $M_\mathbb{R}$ is diffeomorphic to the non-orientable surface of Euler characteristic -3 . Let us denote by S_c the real trace of T_c , $S_c = M_\mathbb{R} \cap T_c$, $c \in \mathbb{R} \setminus \{0, 1\}$.

Lemma 2.1. *If $c \in \mathbb{R} \setminus \{0, 1\}$ then S_c is connected and homeomorphic to the circle $\mathbb{S}^1$. Moreover, there exists an universal covering $\Pi_c: \mathbb{C} \rightarrow T_c$ such that $\Pi_c(\mathbb{R}) = \Pi_c(0 \leq t < 1) = S_c$.*

Proof. If $c \in \mathbb{R} \setminus \{0, 1\}$ then S_c is the strict transform of $G^{-1}(c) \cap \mathbb{RP}(2)$ by π . On the other hand, $G^{-1}(c) = (y^3 - c \cdot x^3 + c - 1 = 0)$. For each $x \in \mathbb{R}$ there is only one $y \in \mathbb{R}$ such that $y = [c \cdot x^3 + 1 - c]^{1/3}$. Therefore, $G^{-1}(c) \cap \mathbb{R}^2$ is a graph, and so $G^{-1}(c) \cap \mathbb{RP}(2)$ is connected and homeomorphic to $\mathbb{S}^1$. Since S_c is the strict transform of $G^{-1}(c) \cap \mathbb{RP}(2)$, the same is true for it.

Let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma^*} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right), \quad \Gamma^* = \Gamma \setminus \{0\},$$

be the Weierstrass $\mathcal{P}$ -function associated to the lattice Γ . The cubic $G^{-1}(c)$ in the Weierstrass normal form can be written as $v^2 = 4(u^3 - C)$, where $C = 35 \sum_{\omega \in \Gamma^*} \frac{1}{\omega^6} \in \mathbb{R}_+$. It is known that $z \in \mathbb{C} \mapsto (\mathcal{P}(z), \mathcal{P}'(z)) = (u, v)$ parametrizes $G^{-1}(c)$ in the normal form, that is $(\mathcal{P}')^2 = 4(\mathcal{P}^3 - C)$. On the other hand, the projective automorphism $\Psi(u, v) = (x, y)$ given by

$$\begin{cases} x = \frac{b \cdot B \cdot u}{A \cdot v - 1/2} \\ y = \frac{b(A \cdot v + 1/2)}{A \cdot v - 1/2} \end{cases}, \quad \text{where } b = (1 - c)^{1/3}, \quad A = 1/\sqrt{48 \cdot C} \text{ and } B = 1/(4 \cdot c \cdot C)^{1/3}$$

sends $G^{-1}(c)$ to the normal form : $v^2 = 4(u^3 - C)$. By writing explicitly the inverse Ψ^{-1} we get that $G^{-1}(c)$ in the original affine coordinate system is parametrized by

$$z \in \mathbb{C} \mapsto \left(\frac{b.B.\mathcal{P}(z)}{A.\mathcal{P}'(z) - 1/2}, \frac{b(A.\mathcal{P}'(z) + 1/2)}{A.\mathcal{P}'(z) - 1/2} \right) := \Phi_c(z)$$

and $\Phi_c: \mathbb{C} \rightarrow G^{-1}(c)$ is an universal covering. If $c \in \mathbb{R}$ then we can take $b, A, B \in \mathbb{R}$, because $C > 0$. Since $\mathcal{P}(\mathbb{R}), \mathcal{P}'(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$ we get $\Phi_c(\mathbb{R}) \subset G^{-1}(c) \cap \mathbb{RP}(2)$. Therefore, $\Phi_c(\mathbb{R}) = G^{-1}(c) \cap \mathbb{RP}(2)$, because $G^{-1}(c) \cap \mathbb{RP}(2)$ is connected. On the other hand, $\pi_c := \pi|_{T_c}: T_c \rightarrow G^{-1}(c)$ is a biholomorphism. Hence, $\Pi_c := \pi_c^{-1} \circ \Phi_c: \mathbb{C} \rightarrow T_c$ is an universal covering and $\Pi_c(\mathbb{R}) = S_c$. Since 1 is a period of Π_c , we get $\Pi_c(0 \leq t < 1) = S_c$. $\square$

From now on we will fix the fiber $T_c := T$, $c \in \mathbb{R} \setminus \{0, 1\}$, where the global holonomy group is calculated. We will set $S := S_c = T \cap M_{\mathbb{R}}$.

Corollary 2.1. *For any $\alpha \in \mathbb{R}$ there exists an universal covering $\Pi_\alpha: \mathbb{C} \rightarrow T$ with the following properties :*

- (a). *The global holonomy group in the fiber T is $G_\alpha = \langle f, f_\alpha \rangle$, where $f(z) = j.z \text{ md}(\Gamma)$ and $f_\alpha(z) = (j.z + B.\alpha + C) \text{ md}(\Gamma)$, with $B \in \mathbb{C}^*$ and $C \in \mathbb{C}$.*
- (b). *$\Pi_\alpha(\mathbb{R} - a(\alpha)) = S$, where $\mathbb{R} - a(\alpha) = \{t - a(\alpha) | t \in \mathbb{R}\}$ and $a(\alpha) = (1 - j)^{-1} A_1(\alpha) \text{ md}(\Gamma)$ is as before.*

In particular, $\Pi_\alpha^{-1}(S) = \cup_{n \in \mathbb{Z}} (\mathbb{R} - a(\alpha) + n.j)$.

Proof. Consider the universal covering $\Pi_c: \mathbb{C} \rightarrow T$ as in lemma 2.1, for which $\Pi_c(\mathbb{R}) = S$. The global holonomy group in the fiber T is generated by f_α^1 and f_α^2 , which are covered by the maps $\hat{f}_\alpha^k(z) := j.z + A_k(\alpha)$, that is $\Pi_c \circ \hat{f}_\alpha^k(z) = f_\alpha^k(z) \circ \Pi_c$, $k = 1, 2$. On the other hand, if $\hat{h}(z) := z + a(\alpha)$ then $\hat{f} := \hat{h}^{-1} \circ \hat{f}_\alpha^1 \circ \hat{h}$ and $\hat{f}_\alpha := \hat{h}^{-1} \circ \hat{f}_\alpha^2 \circ \hat{h}$ are of the form $\hat{f}(z) = j.z$ and $\hat{f}_\alpha(z) = j.z + A(\alpha)$, where $A(\alpha) = A_2(\alpha) - A_1(\alpha) = B.\alpha + C$. Therefore, the universal covering $\Pi_\alpha := \Pi_c \circ \hat{h}$ satisfies $\Pi_\alpha(\mathbb{R} - a(\alpha)) = S$, $\Pi_\alpha \circ \hat{f} = f_\alpha^1 \circ \Pi_\alpha$ and $\Pi_\alpha \circ \hat{f}_\alpha = f_\alpha^2 \circ \Pi_\alpha$. $\square$

Recall that $E = \{\alpha \in \mathbb{CP}(1) | \tilde{\mathcal{F}}_\alpha \text{ has a first integral}\}$. In [LN] it is proved that the following condition are equivalent :

- (I). $\alpha \in E$.
- (II). G_α is finite.
- (III). $A(\alpha) \in \mathbb{Q}.\Gamma := \{a + b.j | a, b \in \mathbb{Q}\}$.

Notations :

- (a). For $\alpha \in \mathbb{C}$ fixed, set $\Gamma(\alpha) = \{c.A(\alpha) \text{ md}(\Gamma) | c \in \Gamma\}$ and for $\alpha \in \mathbb{R}$, set $\Gamma_{\mathbb{R}}(\alpha) = \Gamma(\alpha) \cap \mathbb{R} \text{ md}(\Gamma)$.
- (b). We have seen in lemma 2.1 and corollary 2.1 that $\Pi_c(\mathbb{R}) = \Pi_\alpha(\mathbb{R} - a(\alpha)) = S$. We define a segment I on S as the image of an open interval $I := \Pi_c((a, b)) \subset S$, $(a, b) \subset \mathbb{R}$, $a < b$.
- (c). Given $\alpha \in \mathbb{C} \cup \{\infty\}$ and $q \in M \setminus \text{sing}(\tilde{\mathcal{F}}_\alpha)$ denote by $\ell(\alpha, q)$ the complex leaf of $\tilde{\mathcal{F}}_\alpha$ through q . If $\alpha \in \mathbb{R} \cup \{\infty\}$ and $q \in M_{\mathbb{R}} \setminus \text{sing}(\tilde{\mathcal{F}}_\alpha)$ then set $\ell_{\mathbb{R}}(\alpha, q) := \ell(\alpha, q) \cap M_{\mathbb{R}}$.
- (d). Given $\alpha \in E \cap \mathbb{R}$, a point $q \in M_{\mathbb{R}} \setminus \text{sing}(\tilde{\mathcal{F}}_\alpha)$ and a segment I of S define

$$N(\alpha, q, I) = \#(\ell_{\mathbb{R}}(\alpha, q) \cap I)$$

Note that $N(\alpha, q, I)$ is always finite, because $\tilde{\mathcal{F}}_\alpha$ is transverse to S and has rational first integral for any $\alpha \in E \cap \mathbb{R}$.

Lemma 2.2. *For any fixed $\alpha \in \mathbb{R}$, $q_o \in S$ and $z_o \in \mathbb{C}$ such that $\Pi_\alpha(z_o) = q_o$ we have :*

$$(5) \quad \ell_{\mathbb{R}}(\alpha, q) \cap S = \Pi_\alpha(z_o + \Gamma_{\mathbb{R}}(\alpha)) ,$$

where $z_o + \Gamma_{\mathbb{R}}(\alpha) = \{z_o + t \mid t \in \Gamma_{\mathbb{R}}(\alpha)\}$. Moreover,

- (a). $\ell_{\mathbb{R}}(\alpha, q_o) \cap S$ is dense in $S \iff \alpha \in \mathbb{R} \setminus \mathbb{Q}$.
- (b). Let $\alpha_o \in \mathbb{R} \setminus \mathbb{Q}$ and $(\alpha_n)_{n \geq 1}$ be a sequence in $\mathbb{Q}$ such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha_o$.

Then for any segment I of S and any sequence $(q_n)_{n \geq 1}$ in S we have

$$\lim_{n \rightarrow \infty} N(\alpha_n, q_n, I) = +\infty$$

Proof. Since $S = \Pi_\alpha(\mathbb{R} - a(\alpha))$, we can suppose that $z_o \in \mathbb{R} - a(\alpha)$.

The global holonomy group G_α is constructed in such a way that for any $q \in T$ we have $\ell(\alpha, q) \cap T = \{g(q) \mid g \in G_\alpha\}$. Let $\hat{G}_\alpha = \{\hat{g} \in \text{Aut}(\mathbb{C}) \mid \Pi_\alpha \circ \hat{g} = g \circ \Pi_\alpha, g \in G_\alpha\}$. If $\hat{G}_\alpha(z_o)$ denotes the orbit of z_o by $\hat{G}_\alpha$ then

$$\ell(\alpha, q_o) \cap T = \Pi_\alpha(\hat{G}_\alpha(z_o)) .$$

Since $S = T \cap M_{\mathbb{R}} = \Pi_\alpha(\mathbb{R} - a(\alpha))$ we get

$$\ell_{\mathbb{R}}(\alpha, q_o) \cap S = \Pi_\alpha(\hat{G}_\alpha(z_o) \cap [\mathbb{R} - a(\alpha)]) .$$

The transformations $\hat{g} \in \hat{G}_\alpha$ for which $\hat{g}(\mathbb{R} - a(\alpha)) = \mathbb{R} - a(\alpha)$ are in the group $\hat{H}_\alpha = \{\hat{g} \in \hat{G}_\alpha \mid \hat{g}(z) = z + t, t \in \Gamma_{\mathbb{R}}(\alpha)\}$. Therefore,

$$\ell_{\mathbb{R}}(\alpha, q_o) \cap S = \Pi_\alpha(z_o + \Gamma_{\mathbb{R}}(\alpha)) .$$

It follows from (5) that $\ell_{\mathbb{R}}(\alpha, q_o) \cap S$ is dense in S if, and only if, $\Gamma_{\mathbb{R}}(\alpha)$ is dense in $\mathbb{R} \, md(\Gamma) = \mathbb{R} \, md(1)$. The next claim implies assertion (a) of lemma 2.2.

Claim 2.1. The set $\Gamma(\alpha)$ is an additive sub-group of $\mathbb{C}/\Gamma$, whereas $\Gamma_{\mathbb{R}}(\alpha)$ is an additive sub-group of $\mathbb{R} \, md(1) \simeq \mathbb{S}^1$, if $\alpha \in \mathbb{R}$. Moreover, $\Gamma_R(\alpha)$ is dense in $\mathbb{R} \, md(1)$ if, and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. The first two assertions are consequence of the definitions. Recall that $A(\alpha) = B.\alpha + C \, md(\Gamma)$ where $B \in \mathbb{C}^*$ and $C \in C$. We will use the following facts proved in [LN] :

- (i). If $\alpha \in \{1, j, j^2\}$ then $\#(G_\alpha) = 3$. In particular, $G_1 = G_j = G_{j^2} = \{z \, md(\Gamma), j.z \, md(\Gamma), j^2.z \, md(\Gamma)\}$.
- (ii). $\#(G_0) = 9$.
- (iii). The group G_α can be explicitly written as :

$$G_\alpha = \{(\lambda.z + b.A(\alpha) \, md(\Gamma)) \mid \lambda \in \{1, j, j^2\} \text{ and } b \in \Gamma\}$$

We can deduce from (i) and (iii) that $A(1) = A(j) = A(j^2) = 0 \, md(\Gamma)$, and so $B + C = B.j + C = B.j^2 + C \in \Gamma$. Therefore,

$$A(\alpha) = B.z - B \, md(\Gamma) = B(z - 1) \, md(\Gamma)$$

and

$$A(j) = A(j^2) = 0 \, md(\Gamma) \implies B(j - 1), B(j^2 - 1) \in \Gamma \implies B \in \frac{1}{3}\Gamma$$

It follows from (ii) that :

$$A(0) = -B \notin \Gamma .$$

Therefore, we can write $B = \frac{k}{3}(m + n.j)$, where, $k, m, n \in \mathbb{Z}$, $3 \nmid k$, and either $(m, n) = (1, 0)$, or $(m, n) = (0, 1)$, or $m, n \neq 0$ and $\gcd(m, n) = 1$.

Let $\lambda := \frac{1}{3}(m^2 + n^2 - m.n) = (m + n.j^2).B \in \mathbb{Q} \setminus \{0\}$. We assert that if $\alpha \in \mathbb{R}$ then

$$(6) \quad \Gamma_{\mathbb{R}}(\alpha) = \{\ell.\lambda(\alpha - 1)md(1) \mid \ell \in \mathbb{Z}\} ,$$

In fact, recall that

$$\Gamma_{\mathbb{R}}(\alpha) = \{tmd(1) \mid t = c.B(\alpha - 1) , c \in \Gamma , c.B \in \mathbb{R}\} .$$

In particular, $\Gamma_{\mathbb{R}}(1) = \{0md(1)\}$. If $\alpha \neq 1$, set $X := \{\ell.\lambda(\alpha - 1)md(1) \mid \ell \in \mathbb{Z}\}$. Since $\lambda(\alpha - 1) \in \mathbb{R}$, $\lambda = (m + n.j^2).B$ and $m + n.j^2 \in \Gamma$ we get $\lambda(\alpha - 1) \in \Gamma_{\mathbb{R}}(\alpha)$. Hence, $X \subset \Gamma_{\mathbb{R}}(\alpha)$, because $\Gamma_{\mathbb{R}}(\alpha)$ is an additive sub-group of $\mathbb{R}md(1)$. On the other hand, if $c.B.(\alpha - 1) \in \mathbb{R}$ for some $c \in \Gamma$, then

$$c.B \in \mathbb{R} \implies c.(m + n.j) \in \mathbb{R} \cap \Gamma = \mathbb{Z} \implies c = \ell.(m + n.j^2), \ell \in \mathbb{Z} \implies$$

$$c.B = \ell.\lambda, \ell \in \mathbb{Z} \implies \Gamma_{\mathbb{R}}(\alpha) \subset X ,$$

which proves (6). Finally, we have

$$\alpha \in \mathbb{R} \setminus \mathbb{Q} \iff \lambda.(\alpha - 1) \in \mathbb{R} \setminus \mathbb{Q} \iff \Gamma_{\mathbb{R}}(\alpha) \text{ is dense in } \mathbb{R}md(1) ,$$

which proves the claim. $\square$

Proof of (b) of lemma 2.2. If $\alpha \in \mathbb{Q} \setminus \{1\}$ then $\lambda.(\alpha - 1) \in \mathbb{Q} \setminus \{0\}$. Set $\lambda.(\alpha - 1) = r/s$, where $r, s \in \mathbb{Z}$, $\gcd(r, s) = 1$ and $s > 0$. Let $p, q \in \mathbb{Z}$ be such that $p.r - s.q = 1$. Hence, $p.\lambda(\alpha - 1) = p.r/s = 1/smd(1)$, which implies that

$$\Gamma_{\mathbb{R}}(\alpha) = \left\{ \frac{n}{s}md(1) \mid n \in \mathbb{Z} \right\} .$$

Let $(a, b) \subset \mathbb{R}$ be an open interval with $b - a > 1/s$. Then there exist $u \leq v \in \mathbb{Z}$ such that

$$\frac{u-1}{s} \leq a < \frac{u}{s} \leq \frac{v}{s} < b \leq \frac{v+1}{s} \implies$$

$$\# \left\{ \frac{t}{s} \mid t \in \mathbb{Z}, \frac{t}{s} \in (a, b) \right\} = v - u + 1 \geq (b - a)s - 1 .$$

In particular, if $q_o \in S$, $\Pi_{\alpha}(z_o) = q_o$ and $I = \Pi_{\alpha}(z_o + (a, b))$, then

$$(7) \quad N(\alpha, q_o, I) = \# [(z_o + \Gamma_{\mathbb{R}}(\alpha)md(1)) \cap (z_o + (a, b)md(1))] \geq (b - a)s - 1 .$$

Fix $\alpha_o \in \mathbb{R} \setminus \mathbb{Q}$ and sequences $(\alpha_n)_{n \geq 1}$ in $\mathbb{Q}$ and $(q_n)_{n \geq 1}$ in S , where $\lim_{n \rightarrow \infty} \alpha_n = \alpha_o$. Since $\alpha_o \notin \mathbb{Q}$, we can write $\lambda(\alpha_n - 1) = r_n/s_n$, where $r_n, s_n \in \mathbb{Z}$, $\gcd(r_n, s_n) = 1$ and $\lim_{n \rightarrow \infty} s_n = +\infty$. It follows from (7) that

$$N(\alpha_n, q_n, I) \geq (b - a)s_n - 1 \implies \lim_{n \rightarrow \infty} N(\alpha_n, q_n, I) = +\infty .$$

$\square$

3. PULLING-BACK THE FAMILY $\mathbb{F}_4$

In the process of pulling-back the pencil $\mathbb{F}_4$ we will consider a polynomial map, depending on $\alpha \in \mathbb{R}$, $\Phi_\alpha: \mathbb{RP}(2) \rightarrow \mathbb{RP}(2)$ with a fold. The map will satisfy the following properties :

- (i). There are algebraic curves $F, F_\alpha \subset \mathbb{RP}(2)$ and $p_o \in F$, $q_o \in F_\alpha$ such that $\Phi_\alpha(F) = F_\alpha$ and $\Phi_\alpha(p_o) = q_o$.
- (ii). There are local coordinates $(U, (u, v) \in \mathbb{R}^2)$ and $(U_\alpha, (x, y) \in \mathbb{R}^2)$ such that $u(p_o) = v(p_o) = 0$, $x(q_o) = y(q_o) = 0$, $F \cap U = (v = 0)$, $F_\alpha \cap U_\alpha = (y = 0)$ and $\Phi_\alpha(u, v) = (u, v^2) = (x, y)$. In other words, Φ_α folds around F in the sense of Whitney.

Let us suppose that the point q_o is not contained in the set $\mathcal{L}$. Since $\mathcal{F}_0^4$ and $\mathcal{F}_\infty^4$ are transversal outside $\mathcal{L}$, there is an unique $\alpha \in \mathbb{R} \cup \{\infty\}$ such that the leaf of $\mathcal{F}_\alpha^4$ through q_o is tangent to F_1 at q_o . This condition implies that :

- (iii). The foliation $\mathcal{F}_\alpha^4$ can be defined in $(U_\alpha, (x, y))$ by a differential equation of the form $dy - Q(x, y) dx = 0$, where $Q(0, 0) = 0$.

Let us assume further that $Q_x(0, 0) = -a < 0$. The pull-back foliation $\mathcal{F}_\alpha := \Phi^*(\mathcal{F}_\alpha^4)$ is defined in the chart $(U, (u, v))$ by $\Phi_\alpha^*(dy - Q(x, y) dx) = 2v dv - Q(u, v^2) du = 0$, or by the vector field $X = 2v \partial_u + Q(u, v^2) \partial_v$. The eigenvalues of $DX(0, 0)$ are $\pm i \sqrt{2a}$ and the singularity $p_o = (0, 0)$ is a center for $\mathcal{F}_\alpha$.

In fact, with the condition $Q_x(0, 0) < 0$, the real leaf of $\mathcal{F}_\alpha^4$ through q_o has a tangency of order one with the line $(y = 0)$, as shown in figure 3. A nearby leaf of $\mathcal{F}_\alpha^4|_U$ in the set $H_+ := (y \geq 0)$, say γ , cuts the line $(y = 0)$ in two points and $\gamma \cap H_+$ is a segment. In this case, $\Phi^{-1}(\gamma \cap H_+)$ is a closed curve in the plane (u, v) . Hence, p_o is a center for $\mathcal{F}_\alpha$.

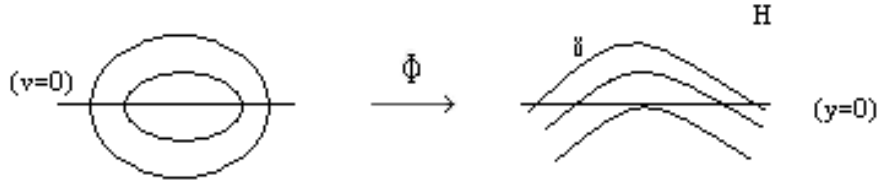


Fig. 3

Let us specify the map Φ_α . Choose the point $q_o \in \mathbb{RP}(2)$ as $q_o = \pi(q_1)$, where $q_1 \in S$. Note that $q_o \notin \mathcal{L}$. Let $(x, y) \in \mathbb{R}^2 \subset \mathbb{C}^2$ be the affine coordinate system fixed in section 2, and $q_o = (a_o, b_o)$ in this coordinate system. Denote by ℓ_α the leaf of $\mathcal{F}_\alpha^4$ through q_o and by L_α the straight line tangent to ℓ_α at q_o . Since $\omega_\alpha = (x^3 - 1)(x - \alpha y^2) dy - (y^3 - 1)(y - \alpha x^2) dx$, the slope of L_α at q_o with

respect to the x -axis is

$$\phi(\alpha) := \frac{(b_o^3 - 1)(b_o - \alpha a_o^2)}{(a_o^3 - 1)(a_o - \alpha b_o^2)} .$$

If we choose $\alpha \neq a_o/b_o^2$ then L_α is not vertical and can be parametrized as $s \mapsto (s + a_o, \phi(\alpha).s + b_o)$.

Set

$$(8) \quad \Phi_\alpha(s, t) = (s + a_o, \pm t^2 + \phi(\alpha).s + b_o) = (x, y) ,$$

where the sign $+$ or $-$ will be choosed next. Note that $\Phi_\alpha(0, 0) = q_o$ and $\Phi_\alpha(t = 0) = L_\alpha$. From now on, we will denote by $\mathbb{R}_{(s,t)}^2$ the domain of Φ_α .

If the sign in the second component of (8) is $+$ then

$$\Phi_\alpha(\mathbb{R}_{(s,t)}^2) = \{(x, y) \mid y \geq \phi(\alpha).(x - a_o) + b_o\} := H_+(\alpha) ,$$

whereas if the sign is $-$ then

$$\Phi_\alpha(\mathbb{R}_{(s,t)}^2) = \{(x, y) \mid y \leq \phi(\alpha).(x - a_o) + b_o\} := H_-(\alpha) .$$

In particular, Φ_α folds $\mathbb{R}_{(s,t)}^2$ around the line $(t = 0)$.

We have to impose an additional condition on α to guarantee that the tangency of ℓ_α with L_α at q_o is of order one. The set of points $p \in \mathbb{CP}(2)$ where the tangency of the leaf of some foliation $\mathcal{G}$ through p with its tangent line at p is of order greater than one, called the inflection divisor of $\mathcal{G}$, was computed in [JP]. Applying this computation in the case of $\mathcal{F}_\alpha^4$ we find the following equation for its inflection divisor:

$$P(x, y, \alpha) = (y^3 - 1)(x^3 - 1)(x^3 - y^3)([2 + \alpha^3]xy - \alpha x^3 - \alpha y^3 - \alpha) .$$

Since $q_o = (a_o, b_o) \notin \mathcal{L} = ((x^3 - 1)(y^3 - 1)(y^3 - x^3) = 0)$ we have to choose α in such a way that $[2 + \alpha^3]a_o b_o - \alpha a_o^3 - \alpha b_o^3 - \alpha \neq 0$.

The sign of $\pm t^2$ in (8) is choosen to be $+$ if the leaf ℓ_α (near q_o) is contained in the region $H_-(\alpha)$ and $-$ otherwise. Note that in the first case $\Phi_\alpha(\mathbb{R}_{(s,t)}^2) \subset H_+(\alpha)$ and in the second $\Phi_\alpha(\mathbb{R}_{(s,t)}^2) \subset H_-(\alpha)$. With these conditions, $p_o = (0, 0)$ is a center for the real pull-back foliation

$$\mathcal{F}_\alpha := \Phi_\alpha^*(\mathcal{F}_\alpha^4) .$$

Fix $\alpha_o \in \mathbb{R}$ satisfying the above conditions and assume that ℓ_{α_o} near q_o is contained in the region $H_-(\alpha_o)$, so that we choose the sign $+$ in (8) for Φ_{α_o} .

From continuity and the arguments already exposed there exist $\epsilon > 0$ and $\delta, \delta_1 > 0$ such that if $\alpha \in J := (\alpha_o - \epsilon, \alpha_o + \epsilon)$ then :

- (iv). If $p_o = (0, 0)$ then $\Phi_\alpha(p_o) = q_o$ and p_o is a center for $\mathcal{F}_\alpha$ (see figure 3).
- (v). If $K := (-\delta, +\delta) \times \{0\} \subset \mathbb{R}_{(s,t)}^2$ then the segment $\Phi_\alpha(K) \subset L_\alpha$ contains a segment I_α of the line L_α of euclidean length $2\delta_1$ and centered at q_o .
- (vi). $\mathcal{F}_\alpha^4$ is transverse to L_α in all points of $I_\alpha \setminus \{q_o\}$.

Let $D \subset \mathbb{R}^2$ be the disk of radius δ_1 centered at q_o . Denote $D_+^\alpha := H_+(\alpha) \cap D$. Recall that $\pi(S)$ is the real leaf of $\mathcal{F}_\infty^4$ through q_o . Since $\mathcal{F}_\infty^4$ and $\mathcal{F}_\alpha^4$ are transversal at q_o we can choose a segment I contained in S with the following properties :

- (vii). q_o is in one of the extremities of $\pi(I)$ and $\pi(I) \subset D_+(\alpha)$.
- (viii). For all $q \in \pi(I)$ and $\alpha \in J$ the leaf ℓ_q of $\mathcal{F}_\alpha^4|_D$ cuts I_α in two points, say q_+ and q_- , one in each side of q_o in I_α .



(ix). Given $p \in I_1(\alpha) \setminus \{p_o\}$ the real leaf of $\mathcal{F}_\alpha$ through p is a closed curve (diffeomorphic to $\mathbb{S}^1$).

Let us prove that the family $(\mathcal{F}_\alpha)_{\alpha \in J}$ satisfies properties (P.1), (P.2) and (P.3) in the statement of theorem 1. Property (P.1) follows from the fact that $E_4 \cap \mathbb{R} = \mathbb{Q}$, where $E_4 = \{\alpha \in \mathbb{C} \mid \mathcal{F}_\alpha^4 \text{ has a non-constant rational first integral}\}$.

Lemma 3.1. *In the above situation, given a leaf L of $\mathcal{G}$ not contained in $CV(\Psi)$, define L^* as the saturated set of $\Psi^{-1}(L \setminus CV(\Psi))$ by the foliation $\mathcal{G}^*$. Then L^* is an union at most k leaves of $\mathcal{G}^*$.*

Proof. In fact, since Ψ is non-degenerate, $\Psi^{-1}(CV(\Psi))$ is a holomorphic curve in M . Let $X := M \setminus \Psi^{-1}(CV(\Psi))$ and $Y := N \setminus CV(\Psi)$. Then X and Y are open and dense in M and N , respectively, and $\psi := \Psi|_X: X \rightarrow Y$ is a holomorphic covering map with k -sheets. Let $\mathcal{G}_Y := \mathcal{G}|_Y$ and $\mathcal{G}_X^* := \mathcal{G}^*|_X$. Since L is a leaf of $\mathcal{G}$ not contained in $CV(\Psi)$, which is a curve, $L \cap CV(\Psi)$ is a discrete subset of L in its intrinsic topology (cf. [C-LN]). Therefore, $L_Y := L \cap Y$ is connected, which implies that it is a leaf of $\mathcal{G}_Y$. Set $L_X := \psi^{-1}(L_Y)$ and $\psi_L := \psi|_{L_X}: L_X \rightarrow L_Y$. Note that L_X is a union of leaves of $\mathcal{G}_X^*$, because $\psi|_X$ is a local biholomorphism. If we consider these leaves with the intrinsic topology, the map ψ_L is a covering map with k -sheets. This implies that L_X has at most k connected components, so that

it is an union of at most k leaves of $\mathcal{G}_X^*$. This implies that L^* is an union of at most k leaves of $\mathcal{G}^*$. $\square$

Denote by $\Phi_{\mathbb{C},\alpha}$ and $\mathcal{F}_{\mathbb{C},\alpha}$ the complexifications of Φ_α and $\mathcal{F}_\alpha$, respectively. With these notations we have $\mathcal{F}_{\mathbb{C},\alpha} = \Phi_{\mathbb{C},\alpha}^*(\mathcal{F}_\alpha^4)$. Note that $\Phi_{\mathbb{C},\alpha}$ can be considered as a rational map $\mathbb{CP}(2) \dashrightarrow \mathbb{CP}(2)$ of topological degree two. Moreover, $CP(\Phi_{\mathbb{C},\alpha}) \cap \mathbb{C}^2 = (t = 0)$ and $CV(\Phi_{\mathbb{C},\alpha}) \cap \mathbb{C}^2 = L_\alpha$. Given $q \in \mathbb{R}^2 \setminus \text{sing}(\mathcal{F}_\alpha^4)$ denote the complex leaf of $\mathcal{F}_\alpha^4$ through q by $\ell(q, \alpha)$. Following the notation of lemma 3.1, let $\ell^*(q, \alpha)$ be the saturated set of $\Phi_{\mathbb{C},\alpha}^{-1}(\ell(q, \alpha) \setminus CV(\Phi_{\mathbb{C},\alpha}))$ by the foliation $\mathcal{F}_{\mathbb{C},\alpha}$. According to lemma 3.1, $\ell^*(q, \alpha)$ contains at most two leaves of $\mathcal{F}_{\mathbb{C},\alpha}$.

Remark 3.1. If $\ell(q, \alpha)$ is transverse to L_α at some point of $\ell(q, \alpha) \cap L_\alpha \cap \mathbb{C}^2$ then $\ell^*(q, \alpha)$ is a leaf of $\mathcal{F}_{\mathbb{C},\alpha}$.

Proof. Assume that $\ell^*(q, \alpha)$ contains two different leaves of $\mathcal{F}_{\mathbb{C},\alpha}$, say ℓ_1 and ℓ_2 . In this case, if $m \in \ell(q, \alpha) \setminus L_\alpha$ then $\Phi_{\mathbb{C},\alpha}^{-1}(m)$ contains two points, one in ℓ_1 and the other in ℓ_2 . Let $m_o \in L_\alpha \cap \ell(q, \alpha) \cap \mathbb{C}^2$ and suppose by contradiction that $\ell(q, \alpha)$ is transverse to L_α at m_o . Note that $\Phi_{\mathbb{C},\alpha}^{-1}(m_o) = \{m_1\}$. It follows from (8) that there are germs of coordinates systems (u, v) and (z, w) such that $u(m_1) = v(m_1) = 0$, $z(m_o) = w(m_o) = 0$, $L_\alpha = (w = 0)$, $\Phi_{\mathbb{C},\alpha}(u, v) = (u, v^2) = (z, w)$ and $\ell(q, \alpha)$ near m_o can be parametrized by $w \mapsto (\psi(w), w)$, $\psi(0) = 0$. In this case, the curve C parametrized by $v \mapsto (\psi(v^2), v)$ satisfies $\Phi_{\mathbb{C},\alpha}(C) \subset \ell(q, \alpha)$, so that $C \subset \ell^*(q, \alpha)$. On the other hand, if $w \neq 0$ and $m = (\psi(w), w)$ then $\Phi_{\mathbb{C},\alpha}^{-1}(m)$ contains two points in C which implies that $\ell_1 = \ell_2$. Hence, $\ell^*(q, \alpha)$ is a leaf of $\mathcal{F}_{\mathbb{C},\alpha}$. $\square$

As a consequence of (ix) and of remark 3.1 we obtain the following :

- (x). With the notations of remark 3.1, assume that $\ell(q, \alpha)$ is transverse to L_α at some point of $L_\alpha \cap \ell(q, \alpha)$. Then for each point $q_1 \in \pi(I) \cap \ell(q, \alpha)$ the real foliation $\mathcal{F}_\alpha$ has a closed leaf contained in $\ell^*(q, \alpha) \cap \mathbb{RP}(2)$.

For fixed $\lambda \in J \cap \mathbb{Q}$ and $p \in I_1(\lambda) \setminus \{p_o\}$, denote by F_λ a primitive real rational first integral of $\mathcal{F}_\lambda$ and by $O(\lambda, p)$ the number of real connected components of $F_\lambda^{-1}(F_\lambda(p))$. Let $F_{\mathbb{C},\lambda}$ be the complexification of F_λ . Note that the leaf of $\mathcal{F}_{\mathbb{C},\lambda}$ through p is

$$\tilde{\ell}(p) := F_{\mathbb{C},\lambda}^{-1}(F_\lambda(p)) \setminus \text{sing}(\mathcal{F}_{\mathbb{C},\alpha}) .$$

It follows from remark 3.1 that $\tilde{\ell}(p) = \ell^*(q, \lambda)$, where $q = \Phi_\alpha(p)$, because $\ell(q, \lambda)$ cuts transversely L_α at q_+ and q_- (see figure 4). Let $\hat{q} \in I \subset S$ be such that $\pi(\hat{q}) = q$. If $N(\lambda, \hat{q}, I)$ is like in lemma 2.2 then we get from (x) that

$$O(\lambda, p) \geq N(\lambda, \hat{q}, I) .$$

Therefore, (b) of lemma 2.2 implies that the family $(\mathcal{F}_\alpha)_{\alpha \in J}$ satisfies property (P.2). With the same type of argument, it is possible to prove that (a) of lemma 2.2 implies that it satisfies property (P.3). We leave the details for the reader.

Finally, the family $(\mathcal{F}_\alpha)_{\alpha \in J}$ is in $Fol_{\mathbb{R}}(2, 8)$ because $\Phi_\alpha^*(\omega - \alpha.\omega_\infty) = P_\alpha(s, t)dt - Q_\alpha(s, t)ds$, where P_α and Q_α have degree 8 and the homogeneous term of degree eight of $t.P_\alpha(s, t) - s.Q_\alpha(s, t)$ is not identically zero, as the reader can check.

4. OTHER FAMILIES

In this section we will describe briefly how to obtain families of foliations of any degree ≥ 5 satisfying properties (P.1), (P.2) and (P.3) of theorem 1. These families

will be obtained by pulling-back the family $\mathbb{F}_2 := (\mathcal{F}_\alpha^2)_{\alpha \in \mathbb{R}}$ already mentined in §1. The foliation $\mathcal{F}_\alpha^2$ is defined by the differential equation $\omega_\alpha := \omega - \alpha \omega_\infty = 0$, where

$$\begin{cases} \omega = (4x - 9x^2 + y^2)dy - (6y - 12xy)dx \\ \omega_\infty = (2y - 4xy)dy - 3(x^2 - y^2)dx \end{cases}$$

We would like to remark that the set $Tang(\mathcal{F}_0^2, \mathcal{F}_\infty^2)$ consists of two irreducible curves : the line at infinity of $\mathbb{C}^2$, L_∞ , and the quartic $Q = (P = 0)$, where $P(x, y) = 4y^2(1 - 3x) - 4x^3 + (3x^2 + y^2)^2$. These two curves are invariant for all foliations in the pencil. Moreover, $sing(\mathcal{F}_\alpha^2) \subset Q \cup L_\infty$ for any $\alpha \in \overline{\mathbb{C}}$.

The connection between the familes $\mathbb{F}_2$ and $\mathbb{F}_4$ is that there exists a rational map of topological degree two $\Psi: \mathbb{CP}(2) \rightarrow \mathbb{CP}(2)$ such that $\Psi^*(\mathcal{F}_\alpha^2) = \mathcal{F}_\alpha^4$ for all $\alpha \in \overline{\mathbb{C}}$. This map satisfies $\Psi(\mathbb{RP}(2)) = \mathbb{RP}(2)$ (cf. [LN]). As a consequence, the set $E_2 = \{\alpha \in \overline{\mathbb{C}} \mid \mathcal{F}_\alpha^2 \text{ has a non-constant rational first integral}\}$ coincides with $E_4 = \mathbb{Q} \cup \{\infty\}$. Moreover, a statement analogous to lemma 2.2 is true.

Let us precise the last assertion, but before that, we will fix some notations. In order to avoid confusion, when $\alpha \in \mathbb{R} \cup \{\infty\}$ we will denote by $\mathcal{F}_{\mathbb{R}, \alpha}^2$ the real foliation induced by ω_α in $\mathbb{RP}(2)$. Given $q \in \mathbb{RP}(2)$ denote by $\ell(q, \alpha)$ the (complex) leaf of $\mathcal{F}_\alpha^2$ through q . Set

$$\ell_{\mathbb{R}}(q, \alpha) = \mathbb{RP}(2) \cap \ell(q, \alpha) .$$

We would like to remark that $\ell_{\mathbb{R}}(q, \alpha)$ is an union of leaves of $\mathcal{F}_{\mathbb{R}, \alpha}^2$. By Poincaré-Bendixson theorem, each leaf of $\mathcal{F}_{\mathbb{R}, \alpha}^2$ is homeomorphic to $\mathbb{R}$ and accumulates in at most two points of $sing(\mathcal{F}_{\mathbb{R}, \alpha}^2)$: the foliation has no closed leaf (homeomorphic to $\mathbb{S}^1$). When $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $q \in \mathbb{RP}(2) \setminus (Q \cup L_\infty)$ then $\ell_{\mathbb{R}}(q, \alpha)$ has an infinite number of connected components, whereas if $\alpha \in \mathbb{Q} \cup \{\infty\}$ then it has a finite number.

In the case of $\mathcal{F}_\infty^2$, we have the first integral $G_\infty(x, y) := P(x, y)/(2x - 1)^3$ (cf. [LN-1]). In particular, given $q_o \in \mathbb{RP}(2) \setminus (Q \cup L_\infty)$ then the closure of $\ell_{\mathbb{R}}(q_o, \infty)$ is precisely $G_\infty^{-1}(G_\infty(q_o)) \cap \mathbb{RP}(2)$. Moreover, if $q_1 \in \mathbb{RP}(2)$ is such that $\Psi(q_1) = q_o$ then $G_\infty^{-1}(G_\infty(q_o)) = \Psi(\pi(S_c))$, for some $c \in \mathbb{R}$, where $q_1 \in S_c$ and S_c is like in section 2. From now on, we will fix $q_o \in \mathbb{RP}(2) \setminus (Q \cup L_\infty)$ and set $S_1 := \ell_{\mathbb{R}}(q_o, \infty)$.

An interval $I \subset S_1$ will be by convention the image of some interval $I_1 \subset S_c$: $I = \Psi(\pi(I_1))$. Given $q \in S_1 \setminus (Q \cup L_\infty)$ set

$$N(\alpha, q, I) = \# [I \cap \ell_{\mathbb{R}}(\alpha, q)] .$$

If G_α is a real primitive first integral of $\mathcal{F}_{\mathbb{R}, \alpha}^2$ then

$$N(\alpha, q, I) = \# [I \cap G_\alpha^{-1}(G_\alpha(q))] .$$

We have the following consequence of lemma 2.2 :

Corollary 4.1. *With the above notations, we have :*

- (a). *If $q \in S_1$ then $\ell_{\mathbb{R}}(\alpha, q) \cap S_1$ is dense in $S_1 \iff \alpha \in \mathbb{R} \setminus \mathbb{Q}$.*
- (b). *Let $\alpha_o \in \mathbb{R} \setminus \mathbb{Q}$ and $(\alpha_n)_{n \geq 1}$ be a sequence in $\mathbb{Q}$ such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha_o$.*

Then for any non-trivial segment I of S_1 and any sequence $(q_n)_{n \geq 1}$ in S_1 we have

$$\lim_{n \rightarrow \infty} N(\alpha_n, q_n, I) = +\infty$$

It follows that we can apply to the family $\mathbb{F}_2$ the same method of section 3 to obtain families satisfying properties (P.1), (P.2) and (P.3) of theorem 1. The inflexion divisor of $\mathcal{F}_\alpha^2$ is a curve of degree six in $\mathbb{CP}(2)$ which contains the line L_∞ .

Its intersection with $\mathbb{R}^2 = \mathbb{RP}(2) \setminus L_\infty$ is an irreducible curve C_α of degree five. Given $q = (a, b) \in \mathbb{R}^2$ denote by $L_\alpha(q)$ the line tangent to $\mathcal{F}_\alpha$ at q . The slope of $L_\alpha(q)$ is

$$\phi(q, \alpha) = \frac{6b - 12ab - 3\alpha(a^2 - b^2)}{4a - 9a^2 + b^2 - \alpha(2b - 4ab)}.$$

It is clear that we can choose $q_o = (a_o, b_o) \in \mathbb{R}^2$ and α_o in such a way that $\phi(q_o, \alpha_o) \neq \infty$ and $q_o \notin C_{\alpha_o}$. Let $\epsilon > 0$ be such that $\phi(\alpha) := \phi(q_o, \alpha) \neq \infty$ and $q_o \notin C_\alpha$ for all $\alpha \in (\alpha_o - \epsilon, \alpha_o + \epsilon) := J$. The line L_α is not vertical and can be parametrized by $s \mapsto (s + a_o, \phi(\alpha).s + b_o)$.

When we pull-back $(\mathbb{F}_\alpha^2)_{\alpha \in J}$ by the family of maps $(\Phi_\alpha)_{\alpha \in J}$ given in (8) we obtain a family of foliations $(\mathcal{F}_\alpha)_{\alpha \in J}$ of degree five. In fact, if we set $\omega_\alpha = P_\alpha(x, y) dy - Q_\alpha(x, y) dx$, where $P_\alpha(x, y) = 4x - 9x^2 + y^2 - \alpha(2y - 4xy)$ and $Q_\alpha(x, y) = 6y - 12xy - 3\alpha(x^2 - y^2)$, then

$$\Phi_\alpha^*(\omega_\alpha) = P_\alpha \circ \Phi_\alpha(s, t) (\pm t dt + \phi(\alpha).ds) - Q_\alpha \circ \Phi_\alpha(s, t) ds := \hat{P}_\alpha(s, t) dt - \hat{Q}_\alpha(s, t) ds.$$

As the reader can check, $\hat{P}_\alpha(s, t)$ has degree 5 and $\hat{Q}_\alpha(s, t)$ degree ≤ 4 , and so the foliation $\mathcal{G}_\alpha := \Phi_\alpha^*(\mathcal{F}_\alpha^2)$ has degree five. We then choose the sign $+$ or $-$, as indicated in section 3, in such a way that $\mathcal{G}_\alpha$ has a real center at the point $p_o = (0, 0)$ ($\Phi_\alpha(p_o) = q_o$). In this way, we get a family of degree 5 satisfying (P.1), (P.2) and (P.3) of theorem 1.

Based in the same idea, we can obtain families of any degree $k \geq 5$ as follows. Consider $\Phi_\alpha: \mathbb{RP}(2) \rightarrow \mathbb{RP}(2)$ defined in $\mathbb{R}^2 \subset \mathbb{RP}(2)$ by

$$\Phi_\alpha(s, t) := (s + a_o, q(s).p(t) + \phi(\alpha).s + b_o),$$

where $p(t) = \sum_{j=2}^d t^j$ ($dg(p) = d$) and $q \in \mathbb{R}[s]$ is a polynomial of degree $k \geq 0$ such that $q(0) \neq 0$.

Let $\mathcal{G}_\alpha = \Phi_\alpha^*(\mathcal{F}_\alpha^2)$, $\alpha \in J = (\alpha_o - \epsilon, \alpha_o + \epsilon)$. The reader can check that $\Phi_\alpha^*(\omega_\alpha) = \hat{P}_\alpha(s, t) dt - \hat{Q}_\alpha(s, t) ds$, where

$$\begin{cases} \hat{P}_\alpha(s, t) = q(s).p'(t).P_\alpha \circ \Phi_\alpha(s, t) \\ \hat{Q}_\alpha(s, t) = Q_\alpha \circ \Phi_\alpha(s, t) - (q'(s).p(t) + \phi(\alpha)).P_\alpha \circ \Phi_\alpha(s, t) \end{cases}.$$

Note that $dg(\hat{P}_\alpha(s, t)) = 3d + k - 1$ and $dg(\hat{Q}_\alpha(s, t)) \leq 3d + k - 1$. The line at infinity L of the plane (s, t) is invariant for $\mathcal{G}_\alpha$. This can be proved by observing that $\Phi_\alpha(L) = [0 : 1 : 0] \in L_\infty$, where L_∞ is the line at infinity of the plane (x, y) , which is invariant for $\mathcal{F}_\alpha^2$. This implies that the degree of the foliation $\mathcal{G}_\alpha$ is $3d + k - 1$. By letting $k \in \{0, 1, 2\}$ we obtain in this way families of foliations of degrees $3d - 1, 3d, 3d + 1$, $d \geq 2$, and so families of any degree ≥ 5 .

Now, the critical set of Φ_α in $\mathbb{R}^2$ is given by $CV(\Phi_\alpha) = (q(s).p'(t) = 0)$ and so it contains the line $(t = 0)$. Moreover, $\Phi_\alpha(s, 0) = (s + a_o, \phi(\alpha).s + b_o)$ and $\Phi_\alpha(t = 0) = L_\alpha$, the line tangent to the leaf of $\mathcal{F}_\alpha^2$ through q_o . On the other hand, if we fix $s = s_o \in \mathbb{R}$ such that $q(s_o) \neq 0$, we get

$$\Phi_\alpha(s_o, t) = (s_o + a_o, q(s_o).p(t) + \phi(\alpha).s_o + b_o),$$

which implies that Φ_α sends the line $(s = s_o)$ into the line $(x = s_o + a_o)$ folding near $t = 0$ because $p(t) = t^2 + h.o.t.$. Since $q(0) \neq 0$, it follows that there exists a disk D around $0 \in \mathbb{R}^2$, which does not depends on $\alpha \in J$, if ϵ is small enough, such that $\Phi_\alpha|_D$ has a fold line at $(t = 0) \cap D$ and $\Phi_\alpha(D) \subset H_+(\alpha)$ if $q(0) > 0$, whereas $\Phi_\alpha(D) \subset H_-(\alpha)$ if $q(0) < 0$. Let ℓ_{α_o} be the germ at q_o of the leaf of $\mathcal{F}_{\alpha_o}^2$.

through q_o . If $\ell_{\alpha_o} \subset H_-(\alpha_o)$, as in figure 4, we choose q so that $q(0) = 1$, and if $\ell_{\alpha_o} \subset H_+(\alpha_o)$ we choose $q(0) = -1$. With this condition, the foliation $\mathcal{G}_\alpha$ has a real center at $p_o = (0, 0)$, as in figure 3. By applying corollary 4.1 we obtain families of foliations of any degree ≥ 5 satisfying properties (P.1), (P.2) and (P.3) of theorem 1.

References

- [Br] M. Brunella : "Birational geometry of foliations" ; text book for a course in the First Latin American Congress of Mathematics, IMPA (2000).
- [C] M. Carnicer: "The Poincaré problem in the non-dicritical case"; Ann. de Math. 140 (1994) pg. 289-294.
- [C-LN] C. Camacho e A. Lins Neto : "Geometric theory of foliations"; Birkhauser 1985.
- [JP] Jorge Vitório Pereira: "Vector fields, invariant varieties and linear systems"; Ann. Inst. Fourier (Grenoble) 51 (2001), no. 5, pg. 1385-1405.
- [K] K. Kodaira : "On Compact Analytic Surfaces: II" ; Annals of Mathematics, vol. 77, n. 3 (1963).
- [Kh] Khovanskii : "Fewnomials"; Translations of the AMS monographs, # 88 (1991).
- [LN] A. Lins Neto: "Some examples for the Poincaré and Painlevé problems"; Ann. Scientifiques de l'École Norm. Sup., 4^{eme} série, t. 35, pg. 231-266 (2002).
- [LN-1] A. Lins Neto: "Curvature of pencils of foliations"; Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I. Astérisque No. 296 (2004), 167–190.
- [Se] A. Seidenberg: "Reduction of singularities of the differential equation $Ady = Bdx$ "; Amer. J. de Math. 90 (1968), pg. 248-269.

A. LINS NETO

Instituto de Matemtica Pura e Aplicada

Estrada Dona Castorina, 110

Horto, Rio de Janeiro, Brasil

E-Mail: `alcides@impa.br`