HELICOIDAL GRAPHS WITH PRESCRIBED MEAN CURVATURE

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ABSTRACT. We prove an existence result for helicoidal graphs with prescribed mean curvature in a large class of warped product spaces which comprises space forms.

1. HELICOIDAL GRAPHS

Motivated by a question posed to one of the authors by J. Ripoll, we address in this note the Dirichlet problem for helicoidal graphs with prescribed mean curvature. Although the question was originally raised for graphs in Euclidean three dimensional space, we solve it here in a very general context using our result in [1].

In order to made precise the notion of helicoidal graph, we first explain the geometric background we consider. Then, we present the proof of the general existence result. Finally, we discuss its particular version in the case of Riemannian product manifolds $M^2 \times \mathbb{R}$, that includes the Euclidean space.

Let $(M^n, d\sigma^2)$ be a Riemannian manifold endowed with a Killing vector field S_0. Given a positive function $\varrho \in C^\infty(M)$ so that $S_0(\varrho) = 0$, we consider the warped product manifold

$$\bar{M}^{n+1} = M^n \times_\varrho \mathbb{R}$$

with the warped metric

$$(1) \qquad d\bar{\sigma}^2 = \varrho^2 dt^2 + d\sigma^2.$$

We denote by S the lift of S_0 by the projection $(u, t) \in \bar{M} \mapsto u \in M$. Hence, it is easy to see that S is a Killing vector field in $(\bar{M}, d\bar{\sigma}^2)$. Thus, given constants $a, b \in \mathbb{R}$ with $b \neq 0$, it follows that

$$Y = aS + bT$$

is a Killing field in $\bar{M}$, where $T = \partial_t$. Identifying M^n with the immersed hypersurface $M \times \{0\} \subset \bar{M}$, we denote by $\Phi \colon \mathbb{R} \times M \to \bar{M}$ the flow generated by Y.

Given a bounded domain Ω in M^n with boundary Γ, the *helicoidal graph* of a function z defined on $\bar{\Omega}$ is the hypersurface

$$\Sigma^n = \{\Phi(z(u), u) : u \in \Omega\}.$$

The *cylinder* K over Γ is the hypersurface ruled by the flow lines of Y through Γ, i.e.,

$$K = \{\bar{u} = \Phi(s, u) : s \in \mathbb{R},\ u \in \Gamma\}$$

and H_K stands for its mean curvature when calculated inwards.

In the following statement $\mathrm{Ric}_{\bar{M}}$ denotes the ambient Ricci tensor.

Theorem 1. *Let Ω be a $C^{2,\alpha}$ bounded domain in M^n. Assume $H_K \geq 0$ and $\mathrm{Ric}_{\bar{M}}|_{T\Omega} \geq -n \inf_\Gamma H_K^2$. Let $H \in C^\alpha(\Omega)$ and $\varphi \in C^{2,\alpha}(\Gamma)$ be given such that*

$$|H| \leq \inf_\Gamma H_K.$$

Then, there exists a unique function $z \in C^{2,\alpha}(\bar{\Omega})$ satisfying $z|_\Gamma = \varphi$ whose helicoidal graph has mean curvature function H and boundary data φ.

2. Proof of theorem 1

Consider the submersion map $\pi \colon \bar{M} \to M$ defined by identifying points along flow lines of $\bar{Y}$, i.e.,

$$\pi(\Phi(s, u)) = u,$$

where $s \in \mathbb{R}$ and $u \in M^n$. Let $\mathsf{v}_1, \ldots, \mathsf{v}_n$ be a local frame defined in $\Omega \subset M^n$. For instance, we may take the coordinate frame relative to a choice of local coordinates in M^n. Then, let $D_1, \ldots, D_n$ be the basic vector fields π-related to $\mathsf{v}_1, \ldots, \mathsf{v}_n$. It follows that π is a Riemannian submersion if we consider M^n endowed with the metric defined by

$$\langle \mathsf{v}_i(u), \mathsf{v}_j(u) \rangle = \langle D_i(\Phi(s, u)), D_j(\Phi(s, u)) \rangle_\varrho,$$

where $\langle \cdot, \cdot \rangle_\varrho$ denotes the metric (1) in $\bar{M}^{n+1}$. Now, the proof follows directly from Theorem 1 in [1]. $\qquad\square$

3. Helicoidal graphs in $M^2 \times \mathbb{R}$

Now we consider the case when M^2 is endowed with a rotationally invariant metric. More precisely, we have polar coordinates r, θ such that the metric is written as

$$ds^2 = dr^2 + \psi^2(r)\, d\theta^2$$

for some positive smooth function ψ.

We define cylindrical coordinates r, θ, z in the Riemannian product manifold $\bar{M}^3 = M^2 \times \mathbb{R}$. We identify M^2 with the slice $z = 0$. Given the Killing vector field ∂_θ in M^2 and $a, b \in \mathbb{R}$ with $b \neq 0$, then

$$Y = a\partial_\theta + b\partial_z$$

is a Killing vector field in $\bar{M}^3$. In terms of the cylindrical coordinates the flow generated by Y is described by

$$\Phi_s(r, \theta, z) = (r, \theta + as, z + bs), \quad s \in \mathbb{R}.$$

As in the general case, we define the submersion $\pi \colon \bar{M}^3 \to M^2$ by identifying points in the same orbit through a point of M^2. Hence,

$$\pi(r, \theta, z) = (r, \theta - \frac{a}{b}z).$$

Notice that the vector fields ∂_r and $-b\partial_\theta + a\psi^2\partial_z$ span the horizontal subspace of π with respect to the metric in $\bar{M}^3$. Moreover,

(2) $$\pi_*\partial_r = \partial_r, \quad \pi_*\partial_\theta = \partial_\theta, \quad \pi_*\partial_z = -\frac{a}{b}\partial_\theta,$$

what implies that the horizontal vector fields

$$D_1 = \partial_r, \quad D_2 = \frac{b}{a^2\psi^2 + b^2}(b\partial_\theta - a\psi^2\partial_z)$$

are π-related to the vector fields ∂_r and ∂_θ in $\mathbb{R}^2$, respectively.

Therefore, the metric in $\bar{M}^3$ restricted to horizontal subspaces has components

$$\langle D_1, D_1 \rangle = 1, \quad \langle D_1, D_2 \rangle = 0, \quad \langle D_2, D_2 \rangle = \frac{b^2\psi^2}{a^2\psi^2 + b^2}.$$

Thus, we conclude that $\pi \colon \bar{M}^3 \to M^2$ is a Riemannian submersion if we consider in M^2 the metric

$$ds^2 = dr^2 + \frac{b^2\psi^2}{a^2\psi^2 + b^2}\, d\theta^2.$$
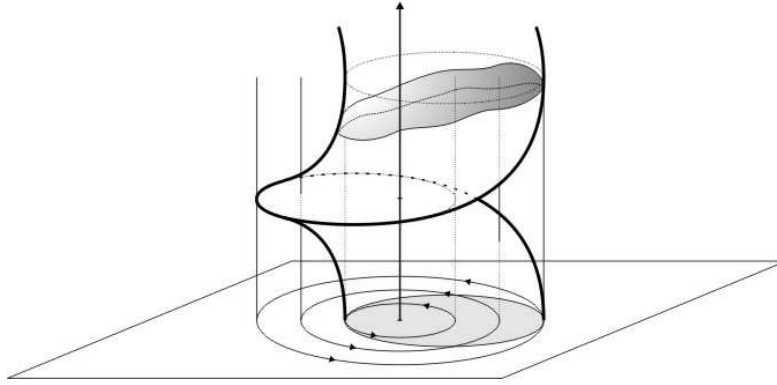
which coincides with the Euclidean metric when $a = 0$ and $\psi(r) = r$.

Theorem 1 in the particular case of $\mathbb{R}^3$ gives the following result.

Corollary 1. *Let $\Omega \subset \mathbb{R}^2$ be a $C^{2,\alpha}$ bounded domain with boundary Γ so that $H_K \geq 0$. Let $H \in C^\alpha(\Omega)$ and $\varphi \in C^{2,\alpha}(\Gamma)$ be given such that*

$$|H| \leq \inf_\Gamma H_K.$$

Then, there exists a unique function $z \in C^{2,\alpha}(\bar{\Omega})$ satisfying $z|_\Gamma = \varphi$ whose helicoidal graph in $\mathbb{R}^3$ has mean curvature H and boundary data φ.

It is noteworthy to observe that this corollary reduces to Serrin's classical existence theorem [2] if we take $a = 0$. We also point out that similar results may be stated for helicoidal graphs in hyperbolic spaces and spheres with respect to linear combinations of Killing vector fields generating translations along a geodesic and rotations.

References

[1] M. Dajczer and J. H. de Lira. *Killing graphs with prescribed mean curvature and Riemannian submersions.* To appear in Annales de l'Institut Henri Poincaré - Analyse non linéaire.

[2] J. Serrin. *The problem of Dirichlet for quasilinear elliptic equations with many independent variables.* Philos. Trans. Roy. Soc. London Ser. A **264**, (1969) 413–496.

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