Conformal Killing graphs with prescribed mean curvature

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Abstract

We prove the existence and uniqueness of graphs with prescribed mean curvature function in a large class of Riemannian manifolds which comprises spaces endowed with a conformal Killing vector field.

1 Introduction

Our aim in this paper is to continue the research theme elaborated in [2] and [3] where the existence of a Killing vector field in a manifold or, more generally, a Riemannian submersion structure, permitted to formulate and solve Dirichlet problems associated to the mean curvature equation. For instance, given a function defined in a domain of the base space of the submersion, its Killing graph is a hypersurface in the total space transverse to the flow lines of the vector field. The function assigns to each point of the graph a value of the flow parameter calculated with respect to a fixed reference hypersurface. It is verified that the graph has prescribed mean curvature if the function satisfies a certain quasilinear elliptic PDE.

In this paper, we expand our scope by dealing with conformal Killing vector fields and the corresponding notion of graph. The result we obtain extends Theorem 1 in [2] and provides a significant improvement of the results in [1]. In fact, the method used here allows us to discard or weaken several assumptions in the latter paper, in particular, the conformal Killing field does not have to be closed.

To explain the framework of this paper we first fix some terminology. Let $\bar{M}^{n+1}$ denote a Riemannian manifold endowed with a conformal Killing vector field Y whose orthogonal distribution $\mathcal{D}$ we assume integrable. Thus, there exists a function $\rho \in C^\infty(\bar{M})$ such that $\pounds_Y \bar{g} = 2\rho\bar{g}$, where $\bar{g}$ is the metric in $\bar{M}$. It results that the integral leaves of $\mathcal{D}$ are totally umbilical hypersurfaces. If in addition Y is closed, then they are spherical, i.e., have constant mean curvature.

We denote by $\Phi : \mathbb{I} \times M^n \to \bar{M}^{n+1}$ the flow generated by Y, where $\mathbb{I} = (-\infty, a)$ is an interval with $a > 0$ and M^n is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as

*Partially supported by CNPq and Faperj.
†Partially supported by CNPq and Funcap.

$t = 0$. It may happen that $a = +\infty$, i.e., the vector field Y is complete. For instance, this occurs when the trajectories of Y are circles and we pass to the universal cover. Since $\Phi_t = \Phi(t, \cdot)$ is a conformal map for any fixed $t \in \mathbb{I}$, there exists a positive function $\lambda \in C^\infty(\mathbb{I})$ such that $\lambda(0) = 1$ and $\Phi_t^* \bar{g} = \lambda^2(t)\bar{g}$.

Given a bounded domain Ω in M, the conformal Killing graph Σ of a function z on $\bar{\Omega}$ is the hypersurface

$$\Sigma = \{\bar{u} = \Phi(z(u), u) : u \in \bar{\Omega}\}.$$

Proving the existence of a conformal Killing graph with prescribed mean curvature and boundary requires establishing apriori estimates. This is accomplished by the use of Killing cylinders as barriers. The Killing cylinder K over $\Gamma = \partial\Omega$ is the hypersurface ruled by the flow lines of Y through Γ, that is,

$$K = \{\bar{u} = \Phi(t, u) : u \in \Gamma\}.$$

Let Ω_0 denote the largest open subset of points of Ω that can be joined to Γ by a *unique* minimizing geodesic. At points of Ω_0, we denote

$$\mathrm{Ric}_{\bar{M}}^{rad}(x) = \mathrm{Ric}_{\bar{M}}(\eta, \eta)$$

where $\mathrm{Ric}_{\bar{M}}$ is the ambient Ricci tensor and $\eta \in T_x M$ is a unit vector tangent to the the unique minimizing geodesic from $x \in \Omega_0$ to Γ.

The following result assures the existence of conformal Killing graphs with prescribed mean curvature H and boundary data ϕ. Here, the functions H and ϕ are defined on $\bar{\Omega}$ and Γ, respectively. Moreover, H_K denotes the mean curvature of K when calculated pointing inwards.

Theorem 1. *Let $\Omega \subset M$ be a $C^{2,\alpha}$ bounded domain such that $\mathrm{Ric}_{\bar{M}}^{rad} \geq -n \inf_\Gamma H_K^2$. Assume $\lambda_t \geq 0$ and $(\lambda_t/\lambda)_t \geq 0$. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be such that $\inf_\Gamma H_K > H \geq 0$ and $\phi \leq 0$. Then, there exists a unique function $z \in C^{2,\alpha}(\bar{\Omega})$ whose conformal Killing graph has mean curvature function H and boundary data ϕ.*

Proposition 5 below implies that Theorem 1 holds under weaker but somewhat more technical assumptions on the ambient Ricci tensor. We also point out that we can prove with minor modifications an existence result for functions H depending also on t by imposing the condition $(\lambda H)_t \geq 0$ instead of $\lambda_t \geq 0$. In the case comprised in Theorem 1 the condition $\lambda_t \geq 0$ says that the mean curvatures of the leaves and of the graph have opposite signs.

It is worth to mention that $\bar{M}$ is conformal to a Riemannian product manifold $\mathbb{I} \times \tilde{M}$ where $\tilde{M}$ is conformal to M. A quite remarkable fact is that the mean curvature equation for a general conformal metric to a product metric like this does not satisfy, in general, the maximum principle. In fact, we will see that the class of metrics which we deal in this paper stands as a borderline for the validity of the elliptic techniques in the treatment of that equation.

The particular case of closed conformal Killing fields encompasses a broad range of examples, namely, product and warped ambient spaces, that have been extensively considered in the recent pertinent literature. In this case, we have the following consequence stated in terms of the Ricci tensor of M.

Corollary 2. *Theorem 1 holds when the conformal Killing field Y is closed if the assumption on the Ricci curvature is replaced by*

$$n\,\mathrm{Ric}_M^{rad} \geq -(n-1)^2 \inf_\Gamma H_\Gamma^2.$$

Moreover, we may assume just $\inf_\Gamma H_K \geq H$ *if Y is a Killing field.*

Theorem 1 generalizes under an unifying perspective several previous results, in particular, since conformal Killing graphs include the notion of radial graphs. Corollary 2 generalizes the main result in [1] for graphs of constant mean curvature and initial condition Γ. Our result removes the restriction on the Ricci curvature to be minimal in the direction of the conformal Killing field and weakens other requirements on that tensor. It also rules out some restrictions on λ, thus applying successfully to product ambient spaces.

2 Preliminaries

Let $(\bar{M}^{n+1}, \bar{g})$ be a Riemannian manifold endowed with a conformal Killing vector field Y whose orthogonal distribution $\mathcal{D}$ is integrable. Let $\bar{\nabla}$ denote the Riemannian connection in $\bar{M}$ and

$$\langle X, Z\rangle = \bar{g}(X, Z).$$

From $\pounds_Y \bar{g} = 2\rho\bar{g}$ we deduce the conformal Killing equation

$$\langle \bar{\nabla}_X Y, Z\rangle + \langle \bar{\nabla}_Z Y, X\rangle = 2\rho\langle X, Z\rangle, \tag{1}$$

where $X, Z \in T\bar{M}$. It is a standard fact [8] that the conformal factor $\lambda \in C^\infty(\mathbb{I})$ and $\rho \in C^\infty(\mathbb{I})$ are related by

$$\rho = \lambda_t/\lambda. \tag{2}$$

Denote

$$|Y(t,u)|^2 = 1/\bar{\gamma}(t,u) \quad \text{and} \quad \gamma(u) = \bar{\gamma}(0,u).$$

It follows from (1) and (2) that

$$\bar{\gamma}(t,u) = \gamma(u)/\lambda^2(t).$$

We have from (1) and the integrability of $\mathcal{D}$ that

$$\langle \bar{\nabla}_X Y, Z\rangle = \rho\langle X, Z\rangle \tag{3}$$

for any $X, Z \in \mathcal{D}$. Thus, the leaves $M_t^n = \Phi_t(M)$ are totally umbilical and the mean curvature $k = k(t,u)$ of M_t with respect to the unit normal vector field $Y/|Y|$ is

$$k = -\frac{\rho}{|Y|} = -\frac{\lambda_t\sqrt{\gamma}}{\lambda^2}. \tag{4}$$

We assign coordinates $x^0 = t, x^1, \ldots, x^n$ to points in $\bar{M}$ of the form $\bar{u} = \Phi(t,u)$ where $x^1, \ldots, x^n$ are local coordinates in M. Then, the coordinate vector fields are

$$\partial_0|_{\bar{u}} = Y(\bar{u}) \quad \text{and} \quad \partial_i|_{\bar{u}} = \Phi_{t*}\partial_i|_u.$$

The components of the ambient line element $\mathrm{d}s^2$ in terms of these coordinates are

$$\bar{\sigma}_{00} = \langle \partial_0, \partial_0 \rangle = |Y|^2 = \lambda^2(t)/\gamma(u), \quad \bar{\sigma}_{0i} = \langle \partial_0, \partial_i \rangle = 0, \quad \bar{\sigma}_{ij}|_{\bar{u}} = \lambda^2(t)\sigma_{ij}|_u,$$

where σ_{ij} are the local components of the metric $\mathrm{d}\sigma^2$ in M. Therefore,

$$\mathrm{d}s^2 = \lambda^2(t)(\gamma^{-1}(u)\mathrm{d}t^2 + \mathrm{d}\sigma^2). \tag{5}$$

Thus $\bar{M}$ is conformal to the Riemannian warped product manifold

$$M^n \times_{1/\sqrt{\gamma}} \mathbb{I}$$

with conformal factor λ. Moreover, after the change of variable

$$r = r(t) = \int_0^t \lambda(\tau)d\tau,$$

we see that (5) takes the form of a Riemannian twisted product

$$\mathrm{d}s^2 = \gamma^{-1}(u)\mathrm{d}r^2 + \psi^2(r)\mathrm{d}\sigma^2$$

where $\psi(r) = \lambda(t(r))$.

Examples 3. Let $\phi \in C^\infty(M)$ be a positive function.
(a) By means of the change of variable $e^t = r$, we have that

$$\bar{M} = \mathbb{R}_+ \times M, \;\; \mathrm{d}\bar{s}^2 = \phi^2(u)\,\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2$$

is isometric to

$$\tilde{M} = \mathbb{R} \times M, \;\; \mathrm{d}\tilde{s}^2 = e^{2t}(\phi^2(u)\,\mathrm{d}t^2 + \mathrm{d}\sigma^2).$$

(b) By means of the change of variable $t = 1 - e^{-r}$, we have that

$$\bar{M} = \mathbb{R} \times M, \;\; \mathrm{d}\bar{s}^2 = \phi^2(u)\,\mathrm{d}r^2 + e^{2r}\mathrm{d}\sigma^2$$

is isometric to

$$\tilde{M} = (-\infty, 1) \times M, \;\; \mathrm{d}\tilde{s}^2 = \frac{1}{(1-t)^2}(\phi^2(u)\,\mathrm{d}t^2 + \mathrm{d}\sigma^2).$$

(c) By means of the change of variable $t = c + \log(b \tanh(t/2))$, where $c > 0$ and $b^{-1} = \tanh(c/2)$, we have that

$$\bar{M} = \mathbb{R}_+ \times M, \;\; \mathrm{d}\bar{s}^2 = \phi^2(u)\, \mathrm{d}r^2 + (\sinh t)^2 \mathrm{d}\sigma^2$$

is isometric to

$$\tilde{M} = (-\infty, c + \log b) \times M, \;\; \mathrm{d}\tilde{s}^2 = (\sinh(2 \operatorname{argtanh} b^{-1} e^{s-c}))^2 (\phi^2(u)\, \mathrm{d}t^2 + \mathrm{d}\sigma^2).$$

Suppose now that the conformal Killing field Y is closed, i.e.,

$$\langle \bar{\nabla}_X Y, Z \rangle = \rho \langle X, Z \rangle.$$

For simplicity, we take $\gamma(u) = 1$. Thus $\bar{M}^{n+1}$ has a warped product structure and is conformal with conformal factor λ to the Riemannian product manifold $\mathbb{I} \times M^n$. Observe that in this situation the leaves of $\mathcal{D}$ are spherical, that is, totally umbilical with constant mean curvature $k = k(t)$.

3 Killing cylinders

Let $\Omega \subset M^n$ be a bounded domain with regular boundary Γ. The *Killing cylinder* K over Γ determined by the conformal Killing field Y is the hypersurface defined by

$$K = \{\Phi(t, u) : t \in \mathbb{I}, \; u \in \Gamma\}.$$

Let $t^1, \ldots, t^{n-1}$ be local coordinates for Γ. We denote by (τ_{ij}) the components of the metric in Γ with respect to these coordinates. It results that $t, t^1, \ldots, t^{n-1}$ are local coordinates for K. Let η be the unit inward normal vector along Γ as a submanifold of M. Then

$$\bar{\eta} = \frac{1}{\lambda} \Phi_{t_*} \eta$$

is an unit normal vector field to K. Thus,

$$\langle \bar{\eta}, \partial_t \rangle = 0 = \langle \bar{\eta}, \partial_{t^i} \rangle$$

where $\partial_t = \partial/\partial t$ and $\partial_{t^j} = \partial/\partial t^j$. We deduce from (3) that

$$\langle \bar{\nabla}_{\partial_{t^i}} \partial_t, \bar{\eta} \rangle = \rho \langle \partial_{t^i}, \bar{\eta} \rangle = 0.$$

Hence ∂_t is a principal direction of K with corresponding principal curvature

$$\kappa = \bar{\gamma} \langle \bar{\nabla}_Y Y, \bar{\eta} \rangle.$$

It follows from (1) that

$$\kappa = -\frac{1}{2} \bar{\gamma} \bar{\eta}(\bar{\gamma}^{-1}) = -\frac{1}{2} \gamma \bar{\eta}(\gamma^{-1}) = \frac{1}{2\gamma} \bar{\eta}(\gamma) = \frac{1}{\lambda} \eta(\log \sqrt{\gamma}). \tag{6}$$

It was shown in [6] that the distance function $d(u) = \mathrm{dist}(u, \Gamma)$ in Ω_0 has the same regularity as Γ. Hence, local coordinates in $\bar{M}$ near K can be defined setting $t^0 = t$ and $t^n = d$. We denote by (t_{ij}) the components of the metric for these coordinates. Thus,

$$t_{ij}(t,u) = \lambda^2(t)\tau_{ij}(u) \ \text{ for } \ 1 \le i,j \le n-1.$$

Proposition 4. *The mean curvature of the Killing cylinder K is given by*

$$nH_K(t,u) = \kappa(t,u) + \frac{n-1}{\lambda(t)} H_\Gamma(u). \tag{7}$$

Proof: We have that

$$\begin{aligned} nH_K &= \kappa + t^{ij}\langle \bar{\nabla}_{\partial_{t^i}}\partial_{t^j}, \bar{\eta}\rangle|_{(t,u)} = \kappa + \lambda^{-2}\tau^{ij}\langle \Phi_{t*}\nabla_{\partial_{t^i}}\partial_{t^j}|_u, \lambda^{-1}\Phi_{t*}\eta|_u\rangle \\ &= \kappa + \lambda^{-1}\tau^{ij}\langle \nabla_{\partial_{t^i}}\partial_{t^j}|_u, \eta|_u\rangle, \end{aligned}$$

and the proof follows. ∎

We denote by Γ_ϵ and K_ϵ the level sets $d = \epsilon$ in M and $\bar{M}$, respectively. By H_{K_ϵ} we denote the mean curvature of the Killing cylinder K_ϵ over Γ_ϵ.

Proposition 5. *Assume that the Ricci curvature tensor of $\bar{M}$ satisfies*

$$\inf_{\Omega_0}\left\{\mathrm{Ric}_{\bar{M}}^{rad} + \left(nk^2 - \sqrt{\gamma}\, k_t\right)|_{t=0}\right\} \ge -n \inf_{\Gamma} H_K^2. \tag{8}$$

Then, $H_{K_\epsilon}|_{x_0} \ge H_K|_{y_0}$ if $y_0 \in \Gamma$ is the closest point to a given point $x_0 \in \Gamma_\epsilon \subset \Omega_0$.

Proof: The coordinate d-curve in $\Phi(t,u)$ is the image by Φ_t of the coordinate d-curve passing through $u \in M$. Thus,

$$\bar{\eta}|_{\Phi(t,u)} = \frac{1}{\lambda}\Phi_{t\,*}(u)\partial_d|_u = \frac{1}{\lambda}\partial_d|_{\Phi(t,u)} = \frac{1}{\lambda}\partial_{t^n}|_{\Phi(t,u)}.$$

Extend $\bar{\eta}$ near K as the velocity vector field of the geodesics in M_t departing orthogonally from $K \cap M_t$. We obtain for $1 \le i,j \le n-1$ that

$$\begin{aligned} \lambda^2\bar{\eta}\langle \bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \partial_{t^j}\rangle &= \partial_{t^n}\langle \bar{\nabla}_{\partial_{t^i}}\partial_{t^n}, \partial_{t^j}\rangle \\ &= \langle \bar{\nabla}_{\partial_{t^i}}\bar{\nabla}_{\partial_{t^n}}\partial_{t^n}, \partial_{t^j}\rangle + \langle \bar{R}(\partial_{t^n}, \partial_{t^i})\partial_{t^n}, \partial_{t^j}\rangle + \langle \bar{\nabla}_{\partial_{t^i}}\partial_{t^n}, \bar{\nabla}_{\partial_{t^j}}\partial_{t^n}\rangle \\ &= -\lambda^2(\bar{\gamma}\langle \bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle\langle \bar{\nabla}_{\partial_{t^i}}\partial_{t^j}, Y\rangle + \langle \bar{R}(\partial_{t^i}, \bar{\eta})\bar{\eta}, \partial_{t^j}\rangle - \langle \bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \bar{\nabla}_{\partial_{t^j}}\bar{\eta}\rangle). \end{aligned}$$

Using (3) and (4), we have

$$\bar{\eta}\langle \bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \partial_{t^j}\rangle = -k^2 t_{ij} - \langle \bar{R}(\partial_{t^i}, \bar{\eta})\bar{\eta}, \partial_{t^j}\rangle + \langle A_\epsilon^2\partial_{t^i}, \partial_{t^j}\rangle \tag{9}$$

where A_ϵ denotes the Weingarten map of K_ϵ relative to $\bar{\eta}$.

For the remaining case $i = j = 0$, we have

$$\bar{\eta}\langle\bar{\nabla}_{\partial_{t^0}}\bar{\eta}, \partial_{t^0}\rangle = \langle\bar{\nabla}_Y\bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle + \langle\bar{R}(\bar{\eta}, Y)\bar{\eta}, Y\rangle + \langle\bar{\nabla}_{[\bar{\eta},Y]}\bar{\eta}, Y\rangle + \langle\bar{\nabla}_Y\bar{\eta}, \bar{\nabla}_{\bar{\eta}}Y\rangle.$$

However,

$$[\bar{\eta}, Y] = -[Y, \bar{\eta}] = -[\partial_{t^0}, \lambda^{-1}\partial_{t^n}] = \frac{\lambda_t}{\lambda^2}\partial_{t^n} = \rho\bar{\eta}. \tag{10}$$

Thus,

$$\langle\bar{\nabla}_{[\bar{\eta},Y]}\bar{\eta}, Y\rangle = \rho\langle\bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle = -\rho^2.$$

Using (10) we have

$$\langle\bar{\nabla}_Y\bar{\eta}, \bar{\nabla}_{\bar{\eta}}Y\rangle = \langle\bar{\nabla}_Y\bar{\eta}, \bar{\nabla}_Y\bar{\eta}\rangle + \langle\bar{\nabla}_Y\bar{\eta}, [\bar{\eta}, Y]\rangle = \langle A_\epsilon^2 Y, Y\rangle$$

and

$$\langle\bar{\nabla}_Y\bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle = Y\langle\bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle - \bar{\gamma}\langle\bar{\nabla}_{\bar{\eta}}\bar{\eta}, Y\rangle\langle\bar{\nabla}_Y Y, Y\rangle = -Y(\rho) + \rho^2.$$

It follows that

$$\bar{\eta}\langle\bar{\nabla}_{\partial_{t^0}}\bar{\eta}, \partial_{t^0}\rangle = -Y(\rho) - \langle\bar{R}(Y, \bar{\eta})\bar{\eta}, Y\rangle + \langle A_\epsilon^2 Y, Y\rangle. \tag{11}$$

We also have

$$\bar{\eta}\langle\bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \partial_{t^j}\rangle = -\langle\bar{\nabla}_{\bar{\eta}}A_\epsilon\partial_{t^i}, \partial_{t^j}\rangle - \langle A_\epsilon\partial_{t^i}, \bar{\nabla}_{\bar{\eta}}\partial_{t^j}\rangle = -\langle(\bar{\nabla}_{\bar{\eta}}A_\epsilon)\partial_{t^i}, \partial_{t^j}\rangle + 2\langle A_\epsilon^2\partial_{t^i}, \partial_{t^j}\rangle$$

for $0 \le i, j \le n-1$. Taking traces with respect to the metric (t_{ij}) in K with $t^{00} = \bar{\gamma}$ and $t^{0i} = 0$, and using (9) and (11) gives

$$\begin{aligned}
\mathrm{tr}\bar{\nabla}_{\bar{\eta}}A_\epsilon &= t^{ij}\langle(\bar{\nabla}_{\bar{\eta}}A_\epsilon)\partial_{t^i}, \partial_{t^j}\rangle = -t^{ij}\bar{\eta}\langle\bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \partial_{t^j}\rangle + 2t^{ij}\langle A_\epsilon^2\partial_{t^i}, \partial_{t^j}\rangle \\
&= -\bar{\gamma}\bar{\eta}\langle\bar{\nabla}_{\partial_{t^0}}\bar{\eta}, \partial_{t^0}\rangle + 2\bar{\gamma}\langle A_\epsilon^2\partial_{t^0}, \partial_{t^0}\rangle - \sum_{i,j=1}^{n-1} t^{ij}(\bar{\eta}\langle\bar{\nabla}_{\partial_{t^i}}\bar{\eta}, \partial_{t^j}\rangle - 2\langle A_\epsilon^2\partial_{t^i}, \partial_{t^j}\rangle) \\
&= \bar{\gamma}Y(\rho) + (n-1)k^2 + \mathrm{Ric}_{\bar{M}}(\eta, \eta) + \mathrm{tr}A_\epsilon^2.
\end{aligned}$$

However,

$$\sqrt{\bar{\gamma}}\,k_t = \frac{Y(k)}{|Y|} = -\frac{1}{|Y|^2}Y(\rho) + kY\left(\frac{1}{|Y|}\right) = -\bar{\gamma}Y(\rho) + k^2. \tag{12}$$

We conclude that

$$\mathrm{tr}\bar{\nabla}_{\bar{\eta}}A_\epsilon = \mathrm{tr}A_\epsilon^2 + \mathrm{Ric}_{\bar{M}}(\bar{\eta}, \bar{\eta}) + nk^2 - \sqrt{\bar{\gamma}}\,k_t.$$

Since $n\dot{H}_{K_\epsilon} = \bar{\nabla}_{\bar{\eta}}\mathrm{tr}A_\epsilon = \mathrm{tr}\bar{\nabla}_{\bar{\eta}}A_\epsilon$, at $d = \epsilon$, it follows that

$$n\dot{H}_{K_\epsilon} \ge nH_{K_\epsilon}^2 + \mathrm{Ric}_{\bar{M}}(\bar{\eta}, \bar{\eta}) + nk^2 - \sqrt{\bar{\gamma}}\,k_t.$$

Therefore, at $t = 0$ and by the assumption on $\mathrm{Ric}_{\bar{M}}^{rad}$ there exist constants $c > 0$ and $d_0 > 0$ such that

$$\dot{H}_{K_d} \ge c(H_{K_d} - \inf_\Gamma H_K)$$

for $d \in [0, d_0]$. Hence, H_{K_d} does not decrease with increasing d. ∎

4 Conformal Killing graphs

The *conformal Killing graph* Σ^n of a function $z\colon \bar\Omega \subset M^n \to \mathbb{I}$ is the hypersurface in $\bar M^{n+1}$ given by

$$\Sigma^n = \{X(u) = \Phi(z(u), u) : \ u \in \bar\Omega\}.$$

We show next that the partial differential equation for a prescribed mean curvature function H in $\bar\Omega$ is the quasilinear elliptic equation of divergence form

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{\gamma + |\nabla z|^2}}\right) - \frac{1}{\sqrt{\gamma + |\nabla z|^2}}\left(\frac{\langle \nabla\gamma, \nabla z\rangle}{2\gamma} + \frac{n\gamma\lambda_t}{\lambda}\right) - n\lambda H = 0. \tag{13}$$

Here, the gradient ∇ and the divergence div are differential operators in the leaf M^n endowed with the metric $d\sigma^2$.

A sufficient condition to have a maximum principle for (13) (see Theorem 10.1 in [5]) is

$$(\lambda_t/\lambda)_t = \rho_t \geq 0 \quad \text{and} \quad \lambda_t H \geq 0. \tag{14}$$

The graph Σ is parametrized in terms of local coordinates by

$$X(u) \in \Sigma \mapsto \big(z(x^1, \ldots, x^n), x^1, \ldots, x^n\big).$$

Thus, the tangent space to Σ at $X(u)$ is spanned by the vectors

$$X_i = z_i \partial_0|_{X(u)} + \partial_i|_{X(u)}, \tag{15}$$

where $z_i = \partial z/\partial x^i$. Hence, the metric induced on Σ is

$$g_{ij}|_{X(u)} = \lambda^2(z(u))\left(\sigma_{ij}(u) + \frac{z_i z_j}{\gamma(u)}\right).$$

The inverse is

$$g^{ij}|_{X(u)} = \frac{1}{\lambda^2(z(u))}\left(\sigma^{ij}(u) - \frac{z^i z^j}{\gamma(u) + |\nabla z|^2}\right)$$

where $z^i = \sigma^{ik} z_k$ and $|\nabla z|^2 = z^j z_j$ as usual.

Fix the orientation on Σ given by the unit normal vector field

$$N|_{X(u)} = \frac{1}{\lambda^2 W}\big(\gamma(u)\partial_0|_{X(u)} - \Phi_{z(u)*}\nabla z(u)\big), \tag{16}$$

where

$$\lambda^2 W^2 = \gamma + |\nabla z|^2.$$

Notice that $\langle N, Y\rangle > 0$. We compute the second fundamental form

$$a_{ij} = \langle \bar\nabla_{X_i} X_j, N\rangle$$

of Σ. From (15) and (16) we obtain

$$\begin{aligned}\lambda^2 W a_{ij} &= \gamma z_{ij}\langle\partial_0,\partial_0\rangle + \gamma z_i z_j\langle\bar\nabla_{\partial_0}\partial_0,\partial_0\rangle + \gamma z_j\langle\bar\nabla_{\partial_i}\partial_0,\partial_0\rangle + \gamma z_i\langle\bar\nabla_{\partial_j}\partial_0,\partial_0\rangle\\ &\quad + \gamma\langle\bar\nabla_{\partial_i}\partial_j,\partial_0\rangle - z_i z_j\langle\bar\nabla_{\partial_0}\partial_0,\Phi_{z(u)*}\nabla z\rangle - z_j\langle\bar\nabla_{\partial_i}\partial_0,\Phi_{z(u)*}\nabla z\rangle\\ &\quad - z_i\langle\bar\nabla_{\partial_j}\partial_0,\Phi_{z(u)*}\nabla z\rangle - \langle\bar\nabla_{\partial_i}\partial_j,\Phi_{z(u)*}\nabla z\rangle.\end{aligned}$$

The Levi-Civita connections in M and M_t are determined by the same Christoffel symbols since $\bar\sigma_{ij}|_{\bar u} = \lambda^2\sigma_{ij}|_u$ if $u\in M$ and $\bar u = \Phi_t(u)\in M_t$. We have from (2) and (3) that

$$\langle\bar\nabla_{\partial_i}\partial_j,\partial_0\rangle = -\rho(z(u))\langle\partial_i|_{X(u)},\partial_j|_{X(u)}\rangle = -\lambda\lambda_t(z(u))\sigma_{ij}(u).$$

The terms involving the flow lines acceleration are

$$\langle\bar\nabla_{\partial_0}\partial_0,\partial_0\rangle|_{X(u)} = \frac{1}{2}\partial_t|_{t=z(u)}(\lambda^2/\gamma) = \frac{\lambda\lambda_t(z(u))}{\gamma(u)}$$

and

$$\langle\bar\nabla_{\partial_i}\partial_0,\partial_0\rangle|_{X(u)} = \frac{1}{2}\partial_i|_{X(u)}\big(\lambda^2/\gamma\big) = -\frac{\lambda^2(z(u))\gamma_i(u)}{2\gamma^2(u)}.$$

Similarly,

$$\langle\bar\nabla_{\partial_i}\partial_j|_{X(u)},\Phi_{z(u)\,*}\nabla z(u)\rangle = \langle\Phi_{z(u)\,*}\bar\nabla_{\partial_i}\partial_j|_u,\Phi_{z(u)\,*}\nabla z(u)\rangle = \lambda^2(z(u))\langle\nabla_{\partial_i}\partial_j|_u,\nabla z|_u\rangle$$

and

$$\langle\bar\nabla_{\partial_i}\partial_0,\Phi_{z(u)*}\nabla z\rangle = \langle\bar\nabla_{\partial_i}\partial_0|_{X(u)},z^j\partial_j|_{X(u)}\rangle = z^j\rho\langle\partial_i|_{X(u)},\partial_j|_{X(u)}\rangle = z_i\lambda\lambda_t(z(u)).$$

Since $z_{i;j} = z_{ij} - \langle\nabla_{\partial_i}\partial_j,\nabla z\rangle$ are the Hessian components, we have

$$W a_{ij} = z_{i;j} - \frac{\lambda_t}{\lambda}z_i z_j - \frac{\lambda_t}{\lambda}\gamma\sigma_{ij} - \frac{\gamma_i}{2\gamma}z_j - \frac{\gamma_j}{2\gamma}z_i - \frac{\gamma_k}{2\gamma^2}z_i z_j z^k.$$

We easily obtain

$$\begin{aligned}\lambda^4 W^3 a^i_k &= \lambda^4 W^3 g^{ij}a_{jk}\\ &= \big(\lambda^2 W^2\sigma^{ij} - z^i z^j\big)z_{j;k} - \frac{1}{2}z^i\gamma_k - \lambda^2 W^2\left(\frac{\gamma^i}{2\gamma}z_k + \frac{\gamma\lambda_t}{\lambda}\delta^i_k\right).\end{aligned}\tag{17}$$

Taking traces after dividing both sides by $\lambda^3 W^3$ yields

$$n\lambda H = \frac{1}{\lambda W}\left(\sigma^{ij} - \frac{z^i z^j}{\lambda^2 W^2}\right)z_{j;i} - \frac{\gamma_i z^i}{2\lambda^3 W^3} - \frac{1}{\lambda W}\left(\frac{\gamma_i z^i}{2\gamma} + \frac{n\gamma\lambda_t}{\lambda}\right).\tag{18}$$

Equivalently,

$$\mathcal{Q}_H[z] := \left(\frac{z^i}{\sqrt{\gamma + z^k z_k}}\right)_{;i} - \frac{1}{\sqrt{\gamma + z^k z_k}}\left(\frac{\gamma_i z^i}{2\gamma} + \frac{n\gamma\lambda_t}{\lambda}\right) - n\lambda H = 0.\tag{19}$$

In what follows, we will be concerned with finding a conformal Killing graph Σ with prescribed mean curvature function H and boundary $\bar{\Gamma}$. This amounts to be equivalent to solving the Dirichlet problem

$$\begin{cases} \mathcal{Q}_H[z] = 0 \\ z|_\Gamma = \phi. \end{cases} \tag{20}$$

In the next sections, our goal is to establish a priori estimates for solutions of the above Dirichlet problem under the hypothesis of Theorem 1.

5 Height estimates

In this section, we obtain an a priori height estimate.

Lemma 6. *Under the assumptions of Theorem 1 there exists a positive constant $C = C(\Omega, H)$ such that*

$$|z|_0 \leq C + |\phi|_0$$

for any solution z of the Dirichlet problem (20).

Proof: The second condition in (14) implies that $k \leq 0 \leq H$. Thus, it follows from the tangency principle (see [4]) that any solution z of (20) satisfies $z \leq \sup_\Gamma \phi \leq 0$.

We construct barriers on Ω_0 which are subsolutions to (20) of the form

$$\varphi(u) = \inf_\Gamma \phi + f(d(u)) \tag{21}$$

where $d(u) = \mathrm{dist}(u, \Gamma)$ and the real function f will be chosen later. Hence,

$$\varphi_i = f' d_i \quad \text{and} \quad \varphi_{i;j} = f'' d_i d_j + f' d_{i;j}. \tag{22}$$

At points in Ω_0, we have

$$|\nabla d| = 1. \tag{23}$$

It follows that

$$d^i d_{i;j} = 0 \tag{24}$$

and

$$2\langle \nabla_{\partial_d} \nabla d, \partial_d \rangle = \partial_d |\nabla d|^2 = 0.$$

Moreover,

$$d^i{}_{;i} = \sigma^{ij} d_{i;j} = \sigma^{ij} \langle \nabla_{\partial_i} \nabla d, \partial_j \rangle = -(n-1) H_{\Gamma_\epsilon}, \tag{25}$$

where H_{Γ_ϵ} denotes the mean curvature of $\Gamma_\epsilon \subset \Omega_0$ with respect to η.

Combining (18) and

$$\langle \nabla \gamma, \nabla z \rangle = -\frac{2}{|Y|^4} \langle \bar{\nabla}_{\nabla z} Y, Y \rangle = 2\gamma^2 \langle \bar{\nabla}_Y Y, \nabla z \rangle.$$

yields

$$\mathcal{Q}_H[\varphi] = \frac{1}{U}\left(\varphi^i{}_{;i} - \frac{\varphi^i\varphi^j\varphi_{i;j}}{U^2}\right) - \frac{\gamma}{U^3}(\gamma + U^2)\langle\bar{\nabla}_Y Y, \nabla\varphi\rangle - \frac{n\gamma\lambda_t}{\lambda U} - n\lambda H.$$

where

$$U = \lambda W = \sqrt{\gamma + f'^2}.$$

Using (22) and (25) we obtain

$$\mathcal{Q}_H[\varphi] = \frac{\gamma}{U^3}(f'' - \gamma\langle\bar{\nabla}_Y Y, \eta\rangle f') - \frac{f'}{U}((n-1)H_{\Gamma_\epsilon} + \gamma\langle\bar{\nabla}_Y Y, \eta\rangle) - \frac{n\gamma\rho}{U} - n\lambda H.$$

We take for (21) the test function

$$f = \frac{e^{DB}}{D}\left(e^{-Dd} - 1\right)$$

where $B > \mathrm{diam}(\Omega)$ and $D > 0$ is a constant to be chosen later. Then,

$$f' = -e^{D(B-d)} \quad \text{and} \quad f'' = -Df'.$$

By assumption and using (12) it follows that

$$nk^2 - \sqrt{\bar{\gamma}}k_t = (n-1)k^2 + \bar{\gamma}\rho_t \geq 0.$$

We obtain from Proposition 5 that

$$H_{K_\epsilon} \geq H^* > H \geq 0$$

where $H^* = \inf_\Gamma H_K > 0$. Since $\lambda(\varphi) \leq 1$, it follows that $\mathcal{Q}_H[\varphi] > 0$ if

$$\mathcal{Q}_H[\varphi] > -\frac{\gamma f'}{U^3}(D + \kappa_\epsilon) - \frac{f'}{U}nH^* - \frac{n\gamma\rho}{U} - nH.$$

We require $D > \sup_{\Omega_0}|\kappa_\epsilon|$ and denote $nD_0 = D + \kappa_\epsilon$. Thus $\mathcal{Q}_H[\varphi] > 0$ if

$$HU^3 < -H^* f' U^2 - \gamma\rho U^2 - \gamma D_0 f'. \tag{26}$$

Since $f'^2 \to +\infty$ as $D \to +\infty$, we conclude that for D sufficiently large the inequality holds. If Y is a Killing field ($\rho = 0$) and we only assume that $H^* \geq H$, then (26) is equivalent to

$$(H^*)^2(\gamma + f'^2)^2 f'^2 - H^2(\gamma + f'^2)^3 + 2H^*\gamma D_0 f'^2(\gamma + f'^2) + \gamma^2 D_0^2 f'^2 > 0.$$

Clearly, the last inequality holds for D sufficiently large. Hence, we have shown that

$$\mathcal{Q}_H[\varphi] > 0 = \mathcal{Q}_H[z], \quad \varphi|_\Gamma \leq z|_\Gamma$$

To prove that $\varphi \leq z$ on $\bar{\Omega}$ we just follow the reasoning in the proof of Lemma 6 in [2] (see [5], p. 171). We conclude that φ is a continuous subsolution for the Dirichlet problem (20).

6 Boundary gradient estimates

In this section, we establish an a priori gradient estimate along the boundary of the domain.

Lemma 7. *Under the assumptions of Theorem 1 there exists a positive constant* $C = C(\Omega, H, \phi, |z|_0)$ *such that*

$$\sup_{\Gamma} |\nabla z| \leq C$$

for any solution z *of the Dirichlet problem (20).*

Proof: We use barriers of the form $w + \phi$ along a tubular neighborhood Ω_ϵ of Γ. We set $w = f(d)$ and, for simplicity, we extend the boundary data ϕ to Ω_ϵ by taking $\phi(t^i, d) = \phi(t^i)$. We choose

$$f(d) = -\mu \ln(1 + \tilde{\mu} d)$$

where μ and $\tilde{\mu}$ are positive constants. Hence,

$$f' = -\frac{\mu\tilde{\mu}}{1 + \tilde{\mu} d} \quad \text{and} \quad f'' = \frac{1}{\mu} f'^2.$$

Choosing

$$\tilde{\mu} = \frac{1}{\ln(1+\mu)},$$

we have that

$$f'(0) \to -\infty \quad \text{as} \quad \mu \to \infty. \tag{27}$$

A simple estimate gives

$$\begin{aligned} \mathcal{Q}_H[w + \phi] &= a^{ij}(x, \nabla w + \nabla\phi)(w_{i;j} + \phi_{i;j}) + b(x, \nabla w + \nabla\phi) - n\lambda H \\ &\geq a^{ij} w_{i;j} + \Lambda |\phi|_{2,\alpha} + b - n\lambda H. \end{aligned}$$

Here, $\Lambda = \gamma/U^3$ is the lowest eigenvalue of the matrix

$$a^{ij} = \frac{\delta^{ij}}{U} - \frac{1}{U^3}(w^i + \phi^i)(w^j + \phi^j)$$

and

$$b = -\frac{\gamma}{U^3}\big(\gamma + U^2\big)\langle \bar{\nabla}_Y Y, \nabla w + \nabla\phi\rangle - \frac{n\gamma\rho}{U},$$

where from (23) we have

$$U^2 = \theta + f'^2 \;\text{ and }\; \theta = \gamma + |\nabla\phi|^2.$$

It follows from (23) and (24) that

$$w^i w^j w_{i;j} = f'^2 d^i d^j (f'' d_i d_j + f' d_{i;j}) = f'^2 f'',$$

$$w^i\phi^j w_{i;j} = f'd^i\phi^j(f''d_id_j + f'd_{i;j}) = f'f''\langle\nabla d, \nabla\phi\rangle = 0$$

and

$$\phi^i\phi^j w_{i;j} = \phi^i\phi^j(f''d_id_j + f'd_{i;j}) = f'\phi^i\phi^j d_{i;j}.$$

Since

$$\Delta w = f'' + f'\Delta d = f'' - (n-1)f'H_{\Gamma_d},$$

we obtain

$$a^{ij}w_{i;j} = -(n-1)\frac{f'}{U}H_{\Gamma_d} + \frac{f''}{U^3}(\gamma + |\nabla\phi|^2) - \frac{f'}{U^3}\phi^i\phi^j d_{i;j}.$$

Since $\nabla w = f'\eta$ and $\kappa = \gamma\langle\bar{\nabla}_Y Y, \eta\rangle$, a suitable expression for b is

$$b \;=\; -\frac{f'}{U}\Big(\frac{\gamma}{U^2} + 1\Big)\kappa - \frac{\gamma}{U}\Big(\frac{\gamma}{U^2} + 1\Big)\langle\bar{\nabla}_Y Y, \nabla\phi\rangle - \frac{n\gamma\rho}{U}$$

Using Proposition 4, we conclude that

$$\begin{aligned}\mathcal{Q}_H[w+\phi]U^3 \geq& -n(f'H_{K_d} + \lambda HU)U^2 + \gamma|\phi|_{2,\alpha} - n\gamma\rho U^2 - \gamma\langle\bar{\nabla}_Y Y, \nabla\phi\rangle U^2\\ &+ f''(\gamma + |\nabla\phi|^2) - f'\gamma\kappa + f'\phi^i\phi^j d_{i;j} - \gamma^2\langle\bar{\nabla}_Y Y, \nabla\phi\rangle.\end{aligned}$$

Since $\phi \leq 0$, we have $\lambda(\phi) \leq 1$ and $\rho(\varphi) \leq \rho_0$. At points of Γ, we obtain

$$\begin{aligned}\mathcal{Q}_H[w+\phi]U^3 \geq& -n(f'H_K + H\sqrt{\theta + f'^2})(\theta + f'^2) + \ln(1+\mu)(\gamma + |\nabla\phi|^2)f'^2\\ &-\gamma(n\rho_0 + \langle\bar{\nabla}_Y Y, \nabla\phi\rangle)(\theta + f'^2) - (\gamma\kappa + \phi^i\phi^j d_{i;j})f' + \gamma|\phi|_{2,\alpha} - \gamma^2\langle\bar{\nabla}_Y Y, \nabla\phi\rangle\end{aligned}$$

where $f' = f'(0)$ satisfies (27). It is easy to see using $\inf_\Gamma H_K \geq H \geq 0$ and choosing μ large enough assures that $\mathcal{Q}_H[w+\phi] > 0$ on a small tubular neighborhood Ω_ϵ of Γ and that $w + \phi \leq z$ on both boundary components. Therefore, $w+\phi$ is a locally defined lower barrier for the Dirichlet problem (20). ∎

7 Interior gradient estimates

In this section, we establish an a priori global estimate for the gradient.

Lemma 8. *Under the assumptions of Theorem 1 there exists a positive constant $\hat{C} = \hat{C}(\Omega, H, \phi, |\nabla z|_\Gamma|_0)$ such that*

$$\sup_\Omega |\nabla z| \leq \hat{C}$$

for any solution z of the Dirichlet problem (20).

Proof: The proof follows the same guidelines as in [2]. Consider on Σ the function

$$\chi = ve^{2Cz}$$

where $v = |\nabla z|^2 = z^i z_i$ and $C > 0$ is a constant. If χ achieves its maximum on Γ then we already have the desired bound. Thus, we may assume that χ attains maximum value at an interior point $\bar{u} \in \Omega$ where $|\nabla z| \neq 0$. This assumption enables us to choose a local normal coordinate system $x^1, \ldots, x^n$ so that $\partial_1|_{\bar{u}} = \nabla z/|\nabla z|$ and $\sigma_{ij}(\bar{u}) = \delta_{ij}$. Therefore, at $\bar{u}$ we have

$$z_1 = |\nabla z| > 0 \ \text{ and } \ z_j = 0 \ \text{ if } \ j \geq 2.$$

Since $v_j = 2z^l z_{l;j}$, we also obtain at $\bar{u}$ from

$$0 = \chi_j = 2e^{2Cz}\big(Cvz_j + z^l z_{l;j}\big)$$

that $z^l z_{l;j} = -Cvz_j$. Thus, at $\bar{u}$ we conclude that

$$v_1 = -2Cv^{3/2}, \quad z_{1;1} = -Cv \quad \text{and} \quad v_j = 0 = z_{1;j} \ \text{ if } \ j \geq 2. \tag{28}$$

Rotating the vector fields ∂_j we may assume that $(z_{i;j})$, $2 \leq i, j \leq n$, is diagonal.

Covariantly differentiating $\mathcal{Q}_H[z] = 0$ in the tensorial form $a^i_i = \psi = n\lambda H$ gives

$$a^i_{i;l} = \psi_z z_l + \psi_l,$$

where the subscript in ψ indicates taking derivative. Contraction with z^l yields

$$z^l a^i_{i;l} = \psi_z |\nabla z|^2 + \psi_l z^l.$$

Thus,

$$z^l (W^3\lambda^4 a^i_i)_{;l} = z^l W^3_{;l}\lambda^4\psi + W^3\lambda^4(\psi_z|\nabla z|^2 + \psi_l z^l) + 4W^3\lambda^3\lambda_t|\nabla z|^2 a^i_i.$$

At $\bar{u}$ and since $\lambda^2 W^2 = \gamma + v$, we obtain

$$z^l W^3_{;l} = \frac{3}{2\lambda^2} W(\gamma_1 v^{1/2} - 2Cv^2).$$

Therefore,

$$z^l (W^3\lambda^4 a^i_i)_{;l} = \frac{3}{2} W\lambda^2\psi(\gamma_1 v^{1/2} - 2Cv^2) + W^3\lambda^4(\psi_z|\nabla z|^2 + \psi_l z^l) + 4(\gamma + v)^{3/2} v\lambda_t\psi.$$

is a polynomial in C of first order. We have from (17) that

$$W^3\lambda^4 a^i_k = \big((\gamma + v)\sigma^{ij} - z^i z^j\big) z_{j;k} - \frac{1}{2} z^i\gamma_k - (\gamma + v)\left(\frac{\gamma^i}{2\gamma} z_k + \frac{\lambda_t\gamma}{\lambda}\delta^i_k\right).$$

Thus,

$$z^l(W^3\lambda^4 a^i_i)_{;l} = C_1 + C_2 \tag{29}$$

where

$$C_1 = z^l((\gamma+v)\sigma^{ij} - z^i z^j)z_{j;i})_{;l} \quad \text{and} \quad C_2 = -z^l\Big(\frac{1}{2}z^i\gamma_i + (\gamma+v)\Big(\frac{\gamma_i}{2\gamma}z^i + \frac{n\lambda_t\gamma}{\lambda}\Big)\Big)_{;l}.$$

At $\bar{u}$, we obtain

$$C_2 = -\frac{z_1}{2}\Big(z_{1;1}\gamma_1 + z_1\gamma_{1;1} + (\gamma_1+v_1)\Big(\frac{\gamma_1}{\gamma}z_1 + \frac{2n\lambda_t\gamma}{\lambda}\Big) + (\gamma+v)\Big(\Big(\frac{\gamma_1}{\gamma}\Big)_1 z_1 + \frac{\gamma_1}{\gamma}z_{1;1} + \frac{2n\lambda_t\gamma_1}{\lambda}\Big)\Big).$$

It follows from (28) that C_2 is a polynomial in C of first order.

Computing C_1 at $\bar{u}$, we have

$$\begin{aligned} C_1 &= z^1\big(((\gamma+v)\sigma^{ij} - z^i z^j)z_{j;i}\big)_{;1} \\ &= (\gamma_1 v^{1/2} - 2Cv^2)z^i_{\ ;i} - 2C^2v^3 + v^{1/2}D^{ji}z_{j;i1} \end{aligned} \tag{30}$$

where

$$D^{ij} = (\gamma+v)\sigma^{ij} - z^i z^j.$$

For the first term in (30), it follows from (18) that

$$(\gamma+v)z^i_{\ ;i} = n\lambda H(\gamma+v)^{\frac{3}{2}} - Cv^2 + \frac{\gamma_1}{2\gamma}(2\gamma+v)v^{\frac{1}{2}} + \frac{n\gamma\lambda_t}{\lambda}(\gamma+v) = -Cv^2 + B$$

where B is independent of C. We use the Hessian matrix of χ for computing the last term in (30). We have,

$$\chi_{j;k} = 2e^{2Cz}(2C^2z_jz_kv + 2Cz_kz^lz_{l;j} + Cz_{j;k}v + 2Cz_jz^lz_{l;k} + z^l_{\ ;k}z_{l;j} + z^lz_{l;jk}),$$

which is nonpositive at $\bar{u}$. Thus $D^{ji}\chi_{j;i} \leq 0$. Therefore,

$$\begin{aligned} 0 &\geq 2C^2D^{ji}z_jz_iv + 2CD^{ji}z_iz^lz_{l;j} + CD^{ji}z_{j;i}v + 2CD^{ji}z_jz^l + D^{ji}z^l_{;i}z_{l;j} + D^{ji}z^lz_{l;ji}z_{l;i} \\ &= (\gamma+v)(Cvz^i_{\ ;i} + z^l_{\ ;i}z^i_{\ ;l}) - 2C^2\gamma v^2 + D^{ji}z^lz_{l;ji}. \end{aligned}$$

Since $z_{i;k}$ is diagonal, it follows that

$$z^l_{\ ;i}z^i_{\ ;l} = (z^l_{\ ;l})^2 \geq (z^1_{\ ;1})^2 = C^2v^2.$$

Hence,

$$D^{ji}z^lz_{l;ji} \leq 2\gamma C^2v^2 - (\gamma+v)Cvz^i_{;i} - (\gamma+v)C^2v^2 = \gamma C^2v^2 + CBv.$$

Next we use the Ricci identity in the form

$$z_{i;jk} - z_{k;ij} = R_{ijkm}z^m.$$

Since $R_{jklm}z^l z^m = 0$, we obtain

$$\begin{aligned} D^{ji}z^l z_{j;il} &= \big(\gamma+v\big)\sigma^{ij}z^l z_{j;il} - z^i z^j z^l z_{j;il} \\ &= \big(\gamma+v\big)\sigma^{ij}z^l\big(z_{l;ji} + R_{jilm}z^m\big) - z^i z^j z^l\big(z_{l;ji} + R_{jilm}z^m\big) \\ &= ((\gamma+v)\sigma^{ij} - z^i z^j)z^l z_{l;ji} \\ &= D^{ji}z^l z_{l;ji}. \end{aligned}$$

Therefore,

$$v^{1/2}D^{ji}z_{j;i1} \le \gamma C^2 v^2 + CBv.$$

It follows that

$$C_1 \le (\gamma_1 v^{1/2} - 2Cv^2)\frac{B - Cv^2}{\gamma+v} - 2C^2v^3 + \gamma C^2 v^2 + CBv.$$

As a polynomial in C we obtain an inequality of the form

$$(\gamma+v)C_1 \le \gamma(\gamma-v)v^2C^2 + mC + n.$$

We now use (29) where we have seen that only the term C_1 is a polynomial in C of second order. We obtain a polynomial inequality in C of the form

$$\gamma(\gamma-v)v^2C^2 + bC + c \ge 0.$$

Since $C > 0$ is arbitrary it follows that $v \le \gamma$, and this concludes the proof. ∎

8 Proof of the theorem

We apply the well-known continuity method to the family of Dirichlet problems

$$\begin{cases} \mathcal{Q}_{\tau H}[z] = 0, \\ z|_\Gamma = \tau\phi \end{cases}$$

where $\tau \in [0,1]$. The subset I of $[0,1]$ consisting of values of τ for which the above Dirichlet problem has a $C^{2,\alpha}$ solution is open in view of (14). That I is closed follows from standard theory of quasilinear elliptic equations [5] and the a priori estimates we had proved in Lemmas 6, 7 and 8 above. Thus, it remains to prove that I is non-empty. In fact, we prove that $0 \in I$. This corresponds to guarantee the existence of a minimal graph with boundary Γ.

We first show that an extreme of the functional

$$\mathcal{I}[z] = \int_\Omega F(u, z(u), \nabla z(u))\mathrm{d}x,$$

where

$$F(u, z, \nabla z) = \frac{\lambda^n}{\sqrt{\gamma}}\sqrt{\gamma + |\nabla z|^2}\sqrt{\det \sigma_{ij}}$$

defined in

$$\mathcal{C} = \{z \in C^{0,1}(\bar{\Omega}) : z|_\Gamma = 0\}$$

provides a weak solution of the mean curvature equation (19) for minimal graphs ($H = 0$) with boundary Γ. The corresponding Euler-Lagrange equation is

$$\frac{\partial F}{\partial z} - \left(\frac{F z^i}{\sqrt{\gamma + |\nabla z|^2}} \right)_{,i} = 0. \tag{31}$$

However, we have

$$\frac{\partial F}{\partial z} = \frac{n\lambda_t}{\lambda} F$$

and

$$\frac{\partial F}{\partial z^i} = \frac{\lambda^n}{2U\sqrt{\gamma}} \frac{\partial}{\partial z^i}(\sigma_{kl} z^k z^l) \sqrt{\det \sigma_{kl}} = \frac{G}{U} \sigma_{ik} z^k = \frac{G}{U} z_i,$$

where

$$U = \sqrt{\gamma + |\nabla z|^2} \quad \text{and} \quad G = \frac{F}{U} = \frac{\lambda^n}{\sqrt{\gamma}} \sqrt{\det \sigma_{kl}}.$$

Since

$$\left(\frac{G z^i}{U} \right)_{,i} = G \left(\frac{z^i}{U} \right)_{,i} + \frac{z^i G_{,i}}{U} = G \left(\frac{z^i}{U} \right)_{,i} + G \frac{z^i}{U} \left(\frac{n\lambda_t z_i}{\lambda} - \frac{\gamma_i}{2\gamma} + \frac{\partial_i \sqrt{\det \sigma_{kl}}}{\sqrt{\det \sigma_{kl}}} \right),$$

we obtain from (31) that

$$-\left(\frac{z^i}{U} \right)_{,i} + \frac{1}{U} \Big(\frac{z^i \gamma_i}{2\gamma} + \frac{n\gamma\lambda_t}{\lambda} \Big) + \frac{z^i}{U} \frac{\partial_i \sqrt{\det \sigma_{kl}}}{\sqrt{\det \sigma_{kl}}} = 0.$$

Since

$$\sqrt{\det \sigma_{kl}} = |\partial_1 \wedge \ldots \wedge \partial_n| \quad \text{and} \quad \nabla_{\partial_i} (\partial_1 \wedge \ldots \wedge \partial_n) = \sum_{j=1}^n \Gamma^j_{ij} \, \partial_1 \wedge \ldots \wedge \partial_n$$

we have

$$\frac{\partial_i \sqrt{\det \sigma_{kl}}}{\sqrt{\det \sigma_{kl}}} = \Gamma^j_{ij}.$$

We conclude that (19) for $H = 0$ is the Euler-Lagrange equation for $\mathcal{I}$ as claimed.

Lipschitz extrema for $\mathcal{I}$ in $\mathcal{C}$ are C^2 functions by standard regularity results (see Theorem 1.10.4 (i) in [7]). Thus, these extrema are classical solutions for the mean curvature equation for $H = 0$. The a priori estimates combined with the fact that $C^1(\bar{\Omega})$ is continuously immersed in $C^{0,1}(\bar{\Omega})$ assure that there exists a constant $L > 0$ so that Lipschitz extrema of $\mathcal{I}$ in $\mathcal{C}$ satisfy $|u|_{0,1} \leq L$. Thus, we must seek for extrema in the subset

$$\mathcal{C}_L = \{z \in \mathcal{C} : |z|_{0,1} \leq L\}.$$

Since $0 \in \mathcal{C}_L$ this is a non-empty subset of $\mathcal{C}$. Moreover, it results that F is convex since this condition is equivalent to the ellipticity of the PDE. Thus, we conclude from Theorem 11.10 in [5] that there exists a solution for the problem of extremizing $\mathcal{I}$ in $\mathcal{C}_L$. This establishes the existence of a minimal graph with boundary Γ, and concludes the proof of the theorem. ∎

Proof of Corollary 2: Being Y closed we may assume $\gamma = 1$. Thus (6) and (7) yield

$$n^2 H_K^2 = (n-1)^2 H_\Gamma^2.$$

On the other hand, the relation between the Ricci tensors of $\bar{M}^{n+1}$ and M^n is

$$\mathrm{Ric}_{\bar{M}}(X,Z) = \mathrm{Ric}_M(X,Z) - (nk^2 - k_t)\langle X, Z\rangle$$

for any $X, Z \in TM$. Thus (8) is equivalent to

$$\inf_{\Omega_0} \mathrm{Ric}_M^{rad} \geq -\frac{(n-1)^2}{n} \inf_{\Gamma} H_\Gamma^2,$$

and the proof follows. ∎

Finally, we point out that our existence results still hold if ϕ is only assumed continuous. We may approximate ϕ uniformly by smooth boundary data and use the interior gradient estimate to obtain strong convergence on compact subsets of Ω. A local barrier argument shows that the limiting solutions achieves the given boundary data.

9 Final Remark

We have from (5) that the ambient space $\bar{M}$ is the product manifold $\mathbb{I} \times M$ endowed with the metric

$$ds^2 = \lambda^2(t)\gamma^{-1}(u)(dt^2 + \gamma(u)d\sigma^2).$$

It is thus natural to consider the general situation of an ambient space $\mathbb{I} \times M$ endowed with the conformal metric

$$\tilde{g} = \lambda^2(t,u) g = e^{2\varphi(t,u)} g$$

where g is the product metric in $\mathbb{I} \times M$. In this case, the mean curvature equation for the graph $X = (z(u), u)$ is

$$\mathrm{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) - \frac{n}{\sqrt{1+|\nabla z|^2}}\left(\langle \bar{\nabla} z, \bar{\nabla}\varphi\rangle - \varphi_t\right) - n\lambda H = 0,$$

where $z(t,u) = z(u)$ and we compute $\langle \bar{\nabla} z, \bar{\nabla}\varphi\rangle$ in the ambient space.

We conclude that in order to have a maximum principle for the above equation we have to ask φ_t to be independent of t, that is, the function λ has to separate variables. But this is precisely the case we studied in this paper.

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