COCYCLES OVER PARTIALLY HYPERBOLIC MAPS

ARTUR AVILA [1,2], JIMMY SANTAMARIA [2], MARCELO VIANA [2]

ABSTRACT. We give a general necessary condition for the extremal (largest and smallest) Lyapunov exponents of a Hölder continuous cocycle over a volume preserving partially hyperbolic diffeomorphism to coincide. This condition applies to smooth cocycles, with linear and projective cocycles as special cases. It is based on an abstract rigidity result for fiber bundle sections that are holonomy-invariant, or even just continuous, over the strong-stable leaves and the strong-unstable leaves of the diffeomorphism. As an application, we prove that the subset of Hölder continuous linear cocycles for which the extremal Lyapunov exponents do coincide is meager and even has infinite codimension.

CONTENTS

Date: December 5, 2008.

[1] CNRS UMR 7599, Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte Postale 188, 75252 Paris Cedex 05, France.

[2] IMPA – Estrada D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro – Brazil.

1. Introduction

Let $f : M \to M$ be a measurable transformation and $\pi : \mathcal{V} \to M$ be a finite-dimensional vector bundle over M. A *linear cocycle* over f is a transformation $F : \mathcal{V} \to \mathcal{V}$ satisfying $\pi \circ F = f \circ \pi$ and acting by linear isomorphisms $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ on the fibers. Let $\mathcal{V}$ be endowed with a measurable Riemannian metric, and let μ be an f-invariant probability measure on M. If

$$x \mapsto \max\{0, \log \|F_x\|\} \quad \text{is } \mu\text{-integrable.} \tag{1.1}$$

then, by the sub-additive ergodic theorem of Kingman [16],

$$\lambda_+(F, x) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\|$$

exists at μ-almost every point. Analogously, if

$$x \mapsto \max\{0, \log \|F_x^{-1}\|\} \quad \text{is } \mu\text{-integrable.} \tag{1.2}$$

then

$$\lambda_-(F, x) = \lim_{n\to\infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1}$$

exists at μ-almost every point. The numbers $\lambda_+(F, x)$ and $\lambda_-(F, x)$ are the *extremal Lyapunov exponents* of the cocycle. If (f, μ) is ergodic then they are constant on a full measure set. It is easy to see that $\lambda_+(F, x) \geq \lambda_-(F, x)$ whenever they are defined. We study conditions under which these two numbers coincide.

This problem has a long and rich history, initiated by the work of Furstenberg [11], which dealt with the case when the cocycle arises from independent choices of random matrices, that is, when (f, μ) is a Bernoulli shift and F_x depends only on the first coordinate of x. Furstenberg proved that the extremal Lyapunov exponents must be distinct except, possibly, in the very special event that there be some probability measure invariant under *all* the matrices F_x. Ledrappier [17] proposed an alternative approach to this and related results that is particularly well suited to our purposes here. A recent series of papers by Bonatti, Gomez-Mont, Viana [5, 6, 25] extended the conclusions to Hölder continuous cocycles over "chaotic" transformations: in the broadest set up, given in [25], one assumes that (f, μ) is hyperbolic, in the non-uniform sense of Pesin theory, and satisfies a local product condition.

Here we prove, for the first time, that a Furstenberg type criterion is compatible with the presence of neutral (neither expanding nor contracting) behavior of the base transformation, as long as this does not take place along *all* tangent directions. In precise terms, we take f to be a partially hyperbolic diffeomorphism preserving a Lebesgue measure μ on a manifold M, where partial hyperbolicity means that the derivative admits an invariant splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

where E^s is uniformly contracted and E^u is uniformly expanded by the derivative Df, but the behavior of Df along the central bundle E^c is only required to be in between (dominated by) the other two. The dynamics of volume preserving partially hyperbolic diffeomorphisms has been intensively studied in the last decade or so. We refer the reader to [8, 12, 13] and [4, Chapter 8], for updated surveys of progress in this field. We take advantage of some of this progress, specially of ideas introduced by Burns, Wilkinson [9] in their proof of the Pugh, Shub [20] ergodicity conjecture (under a mild center bunching property, that we also assume here).

1.1. **Generic linear cocycles.** Unless stated otherwise, we assume $f : M \to M$ to be a C^2 partially hyperbolic center bunched diffeomorphism (the precise definitions of these and other notions involved in the statements will be recalled later). The first main theorem states that every fiber bunched linear cocycle over such a partially hyperbolic diffeomorphism is approximated by another whose extremal Lyapunov exponents are stably distinct:

Theorem A. *Assume f is volume preserving and accessible, and the linear cocycle F is Hölder continuous and fiber bunched. Then F is approximated, in the $C^{r,\alpha}$ topology, by open sets of cocycles G such that $\lambda_-(G,x) < \lambda_+(G,x)$ at μ-almost every point. Even more, locally the subset of cocycles for which the Lyapunov exponents coincide has infinite codimension: it is contained in finite unions of closed submanifolds with arbitrarily high codimension.*

It is implicitly assumed that the vector bundle is sufficiently smooth for $C^{r,\alpha}$ regularity of the cocycle to be well defined. Notice that the exponents are constant on a full measure subset of M, because the hypothesis implies f is ergodic [9].

1.2. **Measurable rigidity – projective cocycles.** Theorem A will be deduced, in Section 9, from certain perturbation arguments together with the following rigidity result for cocycles whose extremal Lyapunov exponents coincide. A measurable set $S \subset M$ is *bi-saturated* if it consists of entire strong-stable leaves and entire strong-unstable leaves of the partially hyperbolic diffeomorphism f. Let $\mathbb{P}(F) : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$ be the projective cocycle induced by F in the projectivization of $\mathcal{V}$. By a slight abuse of language, we also denote by π the fiber bundle projection $\mathbb{P}(\mathcal{V}) \to M$.

Theorem B. *Assume f is volume preserving and the linear cocycle F is Hölder continuous and fiber bunched. Assume $\lambda_-(F,x) = \lambda_+(F,x)$ at μ-almost every point. Then every $\mathbb{P}(F)$-invariant probability m on $\mathbb{P}(\mathcal{V})$ such that $\pi_* m = \mu$ admits a disintegration $\{\tilde{m}_x : x \in M\}$ into conditional probabilities along the fibers such that*

- (a) *the disintegration is invariant under stable holonomy and under unstable holonomy of $\mathbb{P}(F)$ over a full measure bi-saturated set $M_F \subset M$;*
- (b) *if f is accessible then $M_F = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

Invariant probability measures m that project down to μ always exist in this setting, because the cocycle $\mathbb{P}(F)$ is continuous and the domain is compact. We postpone the definition of stable and unstable holonomies for a little while. Indeed, Theorem B is a special case of a much more general result, valid for smooth cocycles over partially hyperbolic diffeomorphisms, that we are going to state in the sequel.

1.3. **Smooth cocycles.** Let $\pi : \mathcal{E} \to M$ be a measurable fiber bundle whose leaves are manifolds endowed with a bounded Riemannian structure. By this we mean $\mathcal{E}$ comes with a system of local coordinates $\pi^{-1}(U) \to U \times N$ where N is a Riemannian manifold and the coordinate changes are measurable maps

$$(U \cap V) \times N \to (U \cap V) \times N, \qquad (x,\xi) \mapsto (x, g_x(\xi)) \tag{1.3}$$

such that every g_x is a diffeomorphism, depending measurably on the base point x relative to the C^1 topology, and both the derivative $Dg_x(\xi)$ and its inverse are uniformly bounded in norm. Then one may consider a Riemannian metric on the fibers, varying measurably with the base point, transported from N via these local coordinates. This metric depends on a choice of the coordinates, but only up to a uniformly bounded factor, which does not affect the notions that follow.

A *smooth cocycle* over f is a measurable transformation $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ \mathfrak{F} = f \circ \pi$, every $\mathfrak{F}_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a diffeomorphism depending measurably on x in the C^1 topology, and the norms of the derivative $D\mathfrak{F}_x(\xi)$ and its inverse are uniformly bounded. In particular, the functions

$$(x, \xi) \mapsto \log \|D\mathfrak{F}_x(\xi)\| \quad \text{and} \quad (x, \xi) \mapsto \log \|D\mathfrak{F}_x(\xi)^{-1}\|$$

are integrable, relative to any probability measure on $\mathcal{E}$. The *extremal Lyapunov exponents* of $\mathfrak{F}$ at a point $(x, \xi) \in \mathcal{E}$ are

$$\lambda_+(\mathfrak{F}, x, \xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}^n_x(\xi)\|$$
$$\lambda_-(\mathfrak{F}, x, \xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}^n_x(\xi)^{-1}\|^{-1} .$$

The limits exist m-almost everywhere, with respect to any $\mathfrak{F}$-invariant probability m on $\mathcal{E}$, by Kingman [16], and we have $\lambda_-(\mathfrak{F}, x, \xi) \le \lambda_+(\mathfrak{F}, x, \xi)$. We shall only be interested in measures m that project down to μ under π.

In our setting the base space M is a topological space (even a manifold). We will always *assume the fiber bundle is continuous and the smooth cocycle is continuous.* The first assumption means that local coordinates of $\mathcal{E}$ are defined on a neighborhood of every point and the coordinate changes (1.3) are homeomorphisms such that the diffeomorphisms g_x depend continuously on x in the C^1 topology (uniformly on compact parts of N). The second one means that $\mathfrak{F}$ is a continuous map such that the diffeomorphisms $\mathfrak{F}_x$ depend continuously on x in the C^1 topology.

Definition 1.1. We call *stable holonomy* for $\mathfrak{F}$ a family H^s of homeomorphisms $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ defined for all x and y in the same strong-stable leaf of the diffeomorphism f and satisfying

(a) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ and $H^s_{x,x} = \mathrm{id}$
(b) $\mathfrak{F}_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ \mathfrak{F}_x$
(c) $(x, y) \mapsto H^s_{x,y}(\xi)$, $\xi \in K$ is equi-continuous, for any compact $K \subset N$.

In the last condition, (x, y) varies in the set of pairs of points in the same local strong-stable leaf. The condition also ensures that $H^s_{x,y}(\xi)$ depends continuously on all three variables. *Unstable holonomy* is defined analogously, for pairs of points in the same strong-unstable leaf.

Example 1.2. Let $f : M \to M$ be a partially hyperbolic diffeomorphism for which there exists a central foliation $\mathcal{W}^c$ with compact leaves, that is, an invariant continuous foliation with compact smooth leaves tangent to the central subbundle E^c at every point. Let $\mathcal{E}$ be the disjoint union of the leaves of $\mathcal{W}^c$. The natural projection $P : \mathcal{E} \to M$ given by $P \mid \mathcal{W}^c(x) = x$ makes $\mathcal{E}$ a continuous fiber bundle, in the sense we have just given. Moreover, the map f induces a smooth cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$, mapping each $y \in \mathcal{W}^c(x)$ to $f(y) \in \mathcal{W}^c(f(x))$, and this cocycle is continuous, in the sense we have just given. Assume f is dynamically coherent, that is, there exist invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ with smooth leaves tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$, respectively. Then the cocycle $\mathfrak{F}$ admits stable and unstable holonomies: $H^s_{x,y}(z)$ is the point where the strong-stable leaf through $z \in \mathcal{W}^c(x)$ intersects the central leaf $\mathcal{W}^c(y)$, and analogously for unstable holonomy.

This construction, combined with Theorem 7.6 below, is used by Wilkinson [26] in a her very recent development of a Livsič theory for partially hyperbolic diffeomorphisms.

1.4. **Measurable rigidity – smooth cocycles.** Assume $\mathfrak{F}$ admits stable holonomy. Let m be any probability measure on $\mathcal{E}$ that projects down to μ, and let $\{m_x : x \in M\}$ be a disintegration into conditional probabilities along the fibers. The disintegration is *invariant under stable holonomy* (or *s-invariant*) if

$$(H^s_{x,y})_* m_x = m_y \tag{1.4}$$

for every x and y in the same strong-stable leaf. The definition of *invariance under unstable holonomy* (or *u-invariance*) is analogous. In either case, one speaks of *essential invariance* if the invariance relation (1.4) holds for x and y in some full measure subset of M.

Theorem C. *Assume f is volume preserving and the smooth cocycle $\mathfrak{F}$ admits stable and unstable holonomies. Let m be an $\mathfrak{F}$-invariant probability measure on $\mathcal{E}$ such that $\pi_* m = \mu$, and assume $\lambda_-(\mathfrak{F}, x, \xi) = 0 = \lambda_+(\mathfrak{F}, x, \xi)$ at m-almost every point. Then m admits a disintegration $\{\tilde{m}_x : x \in M\}$ into conditional probabilities along the fibers such that*

(a) *the disintegration is invariant under stable holonomy and under unstable holonomy of $\mathfrak{F}$ over a full measure bi-saturated set $M_{\mathfrak{F}} \subset M$;*
(b) *if f is accessible then $M_{\mathfrak{F}} = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

Theorem B corresponds to the special case of projective cocycles associated to linear cocycles: $\mathcal{E} = \mathbb{P}(\mathcal{V})$ and $\mathfrak{F}_x = \mathbb{P}(F_x)$. Indeed, it is not difficult to see that

$$\lambda_+(\mathbb{P}(F), x, \xi) = \lambda_+(F, x) - \lambda_-(F, x) = -\lambda_-(\mathbb{P}(F), x, \xi)$$

wherever these exponents are defined. Moreover, we are going to see that if F is fiber bunched then its projectivization admits stable and unstable holonomies. Therefore, Theorem B is indeed contained in Theorem C.

1.5. **Holonomy invariance.** The proof of Theorem C has two main stages. The first one, that we state as Theorem 5.1, is to show that every disintegration of m is essentially invariant under both stable holonomy and unstable holonomy. This is based on a non-linear extension of an abstract criterion of Ledrappier [17] for linear cocycles, proposed in Avila, Viana [3] and quoted here as Theorem 5.4. Center bunching and accessibility are not used at this point. On the other hand, they are very important for the second stage of the proof:

Theorem D. *Assume the smooth cocycle $\mathfrak{F}$ admits stable and unstable holonomies. Let Ψ be a measurable function assigning to each point x in M a probability measure on the fiber $\mathcal{E}_x$. Assume Ψ is essentially invariant under stable holonomy and essentially invariant under unstable holonomy. Then,*

(a) *Ψ coincides μ-almost everywhere with some function $\tilde{\Psi}$ defined on a full measure bi-saturated set $M_\Psi \subset M$ and invariant under stable holonomy and under unstable holonomy;*
(b) *if f is accessible then $M_\Psi = M$ and $\tilde{\Psi}$ is continuous, relative to the weak* topology.*

As before, μ denotes a Lebesgue measure. However, in this theorem we *do not* assume f to be volume preserving. The proof of part (a) is given in Section 7, and is based on ideas of Burns, Wilkinson [9] that we recall in Section 6. Part (b) of the theorem is proved in Section 8. It is worth pointing out that the cocycle $\mathfrak{F}$ itself plays no significant role here. Indeed, the results we actually prove, that contain Theorem D (Theorems 7.6 and 8.1), hold for sections of continuous fiber bundles invariant under stable and unstable holonomies, and make no mention to cocycles.

One can go one step further and dispose of the holonomies as well, as follows. The notions we are going to introduce and our next main result apply, in particular, to functions $\Psi : M \to P$ with values in some topological space P, viewed as sections of the trivial fiber bundle $\mathcal{X} = M \times P$.

Definition 1.3. A measurable section $\Psi : M \to \mathcal{X}$ of a continuous fiber bundle $\pi : \mathcal{X} \to M$ is *s-continuous* if the map $(x, y, \Psi(x)) \mapsto \Psi(y)$ is continuous on the set of pairs of points (x, y) in the same local strong-stable leaf. The notion of *u-continuous* is analogous, considering strong-unstable leaves instead. Finally, Ψ is *bi-continuous* if it is both s-continuous and u-continuous.

Notice that s-continuity/u-continuity implies continuity on each local strong-stable/strong-unstable leaf.

We shall also consider essential versions of s-continuity and u-continuity, where the continuity condition is taken to hold on some full measure subset only, *but is required to be locally uniform* (see Remark 7.14). Then we say Ψ is *bi-essentially continuous* if it is both essentially s-continuous and essentially u-continuous.

Recall that a polish space is a complete metrizable space, and for metrizable spaces separability is the same as existence of a countable basis of open sets.

Theorem E. *Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle.*

(a) *Assume the fiber of $\mathcal{X}$ is a separable polish space. Then every bi-essentially continuous section $\Psi : M \to \mathcal{X}$ coincides μ-almost everywhere with some bi-continuous section $\tilde{\Psi} : M_\Psi \to \mathcal{X}$ defined on a full measure bi-saturated set $M_\Psi \subset M$.*
(b) *Assume f is accessible (not necessarily fiber bunched). Then every bi-continuous section $\tilde{\Psi} : M \to \mathcal{X}$ is continuous on the whole M.*

Acknowledgements. We are most grateful to Amie Wilkinson for explanations on the use of the methods of [9] in connection with the proof of Theorem D and for useful comments on earlier versions of this paper. This work was partly conducted during the period A. A. served as a Clay Research Fellow. J. S. was supported by a CNPq doctoral scholarship. M. V. was partially supported by CNPq, FAPERJ, and PRONEX-Dynamical Systems.

2. Partially hyperbolic diffeomorphisms

We say that a diffeomorphism $f : M \to M$ of a compact manifold is *partially hyperbolic* if there is a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

that is invariant under the derivative map Df, and there exists a Riemannian metric on M for which one can choose continuous positive functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with

ν, $\hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ such that, for any unit vector $v \in T_pM$,

$$\|Df(v)\| < \nu(p) \qquad \text{if } v \in E^s(p), \tag{2.1}$$

$$\gamma(p) < \|Df(v)\| < \hat{\gamma}(p)^{-1} \quad \text{if } v \in E^c(p), \tag{2.2}$$

$$\hat{\nu}(p)^{-1} < \|Df(v)\| \qquad \text{if } v \in E^u(p). \tag{2.3}$$

We say that f is *volume preserving* if it preserves a probability μ in the measure class of a volume induced by some Riemannian metric. This is not ambiguous since the volumes of any two Riemannian metrics lie in the same measure class.

We will often use the following notational convention: given any continuous function $\tau : M \to \mathbb{R}^+$, we denote

$$\tau^n(p) = \tau(p)\tau(f(p)) \cdots \tau(f^{n-1}(p)) \quad \text{for any } n \geq 1. \tag{2.4}$$

2.1. **Accessibility and center bunching.** The stable and unstable bundles E^s and E^u of f are uniquely integrable and their integral manifolds form two transverse (continuous) foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed submanifolds of the same class of differentiability as f. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under f, meaning that

$$f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^s(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through any $x \in M$. These foliations are, usually, *not* transversely smooth. But they are always absolutely continuous: the holonomy maps between any pair of cross-sections preserve the class of zero Lebesgue measure sets (see Section 2.2 for definitions and [1] for a proof of this fact in the present setting).

Definition 2.1. A measurable set is *s-saturated* (or *$\mathcal{W}^s$-saturated*) if it is a union of entire strong-stable leaves, *u-saturated* (or $\mathcal{W}^u$-saturated) if it is a union of entire strong-unstable leaves, and *bi-saturated* if it is both s-saturated and u-saturated. We say that f is *accessible* if $\emptyset$ and M are the only bi-saturated sets, and *essentially accessible* if every bi-saturated set has either zero measure or full measure.

Pugh, Shub [20] conjectured that essential accessibility implies ergodicity, for a C^2, partially hyperbolic, volume preserving diffeomorphism. In [21] they showed that this does hold under a few additional assumptions, called dynamical coherence and center bunching. To date, the best result in this direction is due to Burns, Wilkinson [9], who proved the Pugh-Shub conjecture assuming only a milder form of center bunching:

Definition 2.2. A C^2 partially hyperbolic diffeomorphism is *center bunched* if the functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ may be chosen to satisfy

$$\nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}. \tag{2.5}$$

When the diffeomorphism is just $C^{1+\alpha}$, for some $\alpha > 0$, the arguments of Burns, Wilkinson can still be carried out, as long as one assumes what they call strong center bunching (see [9, Theorem 0.3]). All our results extend to this setting.

2.2. **Adapted metric and local strong leaves.** Let $\mathcal{F}$ be a foliation of some n-dimensional manifold M, with d-dimensional smooth leaves. For $r > 0$, we denote by $\mathcal{F}(x, r)$ the connected component of x in the intersection of the leaf $\mathcal{F}(x)$ through x with the Riemannian ball $B(x, r)$. A *foliation box* for $\mathcal{F}$ is the image of $\mathbb{R}^d \times \mathbb{R}^{n-d}$ under a homeomorphism that maps each $\mathbb{R}^d \times \{y\}$ diffeomorphically to some subset of a leaf of $\mathcal{F}$; let us call the image a horizontal slice. A *cross-section* to $\mathcal{F}$ is a smooth codimension-d disk inside a foliation box that intersects each horizontal slice exactly once, transversely and with angle uniformly bounded from zero. Then, for any pair of cross-sections Σ and Σ' there is a well defined *holonomy map* $\Sigma \to \Sigma'$, assigning to each $x \in \Sigma$ the unique point of intersection of Σ' with the horizontal slice through x. The foliation is *absolutely continuous* if all these homeomorphisms map zero Lebesgue measure sets to zero Lebesgue measure sets. In all the cases we deal with in this work, we even have that the Jacobians (Radon-Nikodym derivatives) of all holonomy maps are bounded by a uniform constant. See [1] for the case of strong-stable and strong-unstable foliations of partially hyperbolic diffeomorphisms.

Let M be endowed with a Riemannian metric *adapted* to f, that is, such that properties (2.1)-(2.3) hold. Clearly, these properties are not affected by rescaling. At a few steps in the course of the argument we do allow for the Riemannian metric to be multiplied by some large constant. Let $R > 1$ be fixed, once and for all. Rescaling the metric, if necessary, we may assume that the Riemannian ball $B(x, R)$ is contained in foliation boxes for both $\mathcal{W}^s$ and $\mathcal{W}^u$, for every $x \in M$. For each symbol $* \in \{s, u\}$, define the *local leaf of $\mathcal{W}^*$ through* x to be $\mathcal{W}^*_{\text{loc}}(x) = \mathcal{W}^*(x, R)$. Rescaling the metric once more, if necessary, we may ensure that, given any $p \in M$ and x, $y \in B(p, R)$,

$$y \in \mathcal{W}^s_{\text{loc}}(x) \quad \text{implies} \quad \operatorname{dist}(f(x), f(y)) \leq \nu(p) \operatorname{dist}(x, y),$$

and, similarly,

$$y \in \mathcal{W}^u_{\text{loc}}(x) \quad \text{implies} \quad \operatorname{dist}(f^{-1}(x), f^{-1}(y)) \leq \hat{\nu}(f^{-1}(p)) \operatorname{dist}(x, y).$$

As a consequence, given any p, x, $y \in M$,

(a) $f(\mathcal{W}^s_{\text{loc}}(x)) \subset \mathcal{W}^s_{\text{loc}}(f(x))$ and $f^{-1}(\mathcal{W}^u_{\text{loc}}(x)) \subset \mathcal{W}^u_{\text{loc}}(f^{-1}(x))$.

(b) If $f^j(x) \in B(f^j(p), R)$ for $0 \leq j < n$, and $y \in \mathcal{W}^s_{\text{loc}}(x)$, then
$$\operatorname{dist}(f^n(x), f^n(y)) \leq \nu^n(p) \operatorname{dist}(x, y);$$

(c) If $f^{-j}(x) \in B(f^{-j}(p), R)$ for $0 \leq j < n$, and $y \in \mathcal{W}^u_{\text{loc}}(x)$, then
$$\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \leq \hat{\nu}^{-n}(p) \operatorname{dist}(x, y).$$

These properties of the strong-stable and strong-unstable foliations of f are useful guidelines to the notion of fake foliations introduced in [9], that we are going to recall in Section 6.2.

3. Linear cocycles: fiber bunching and holonomies

Let $F : \mathcal{V} \to \mathcal{V}$ be a linear cocycle over a volume preserving diffeomorphism. The theorem of Oseledets [18] states that, under the integrability assumptions (1.1) and (1.2), Lebesgue almost every point $x \in M$ admits a splitting of the corresponding fiber

$$\mathcal{V} = E^1_x \oplus \cdots \oplus E^k_x, \qquad k = k(x)$$

and real numbers $\lambda_1(F, x) > \cdots > \lambda_k(F, x)$ such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|F^n_x(v_i)\| = \lambda_i(F, x) \text{ for every non-zero } v_i \in E^i_x. \tag{3.1}$$

The *Lyapunov exponents* $\lambda_i(F,x)$ and the *Oseledets subspaces* E^i_x are uniquely defined μ-almost everywhere, and they vary measurably with the point x. Moreover $\lambda_1(F,x) = \lambda_+(F,x)$ and $\lambda_k(F,x) = \lambda_-(F,x)$. Finally, the exponents $\lambda_i(F,x)$ are constants on orbits, and so they are constant almost everywhere when f is ergodic.

Throughout, $\mathbb{K}$ will represent both $\mathbb{R}$ and $\mathbb{C}$. We focus on the case when the vector bundle is trivial: $\mathcal{V} = M \times \mathbb{K}^d$. Then the cocycle has the form

$$F(x,v) = (f(x), A(x)v) \quad \text{for some } A : M \to \mathrm{GL}(d,\mathbb{K}).$$

Conversely, any $A : M \to \mathrm{GL}(d,\mathbb{K})$ defines a cocycle over f, that we denote by F_A. Note that $F^n(x,v) = (f^n(x), A^n(x)v)$ for $n \in \mathbb{Z}$, where

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x) \quad \text{and} \quad A^{-n}(x) = \big(A^n(f^{-n}(x))\big)^{-1}$$

if $n \geq 1$, and $A^0(x) = \mathrm{id}$. We also write $\lambda_i(A,x)$ and $\lambda_\pm(A,x)$ to mean $\lambda_i(F_A,x)$ and $\lambda_\pm(F_A,x)$, respectively. The relation (3.1) translates to

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v_i\| = \lambda_i(A,x) \quad \text{for every non-zero } v_i \in E^i_x. \tag{3.2}$$

For each $r \in \{0,1,\dots\}$ and $0 \leq \alpha \leq 1$, let $\mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ be the space of $C^{r,\alpha}$ maps from M to $\mathrm{GL}(d,\mathbb{K})$, that is, maps whose derivative of order r exists and is α-Hölder continuous. The $C^{r,\alpha}$ topology is defined by the norm (for $\alpha = 0$ omit the last term)

$$\|A\|_{r,\alpha} = \max_{0\leq i\leq r} \sup_{x\in M} \|D^i A(x)\| + \sup_{x\neq y} \frac{\|D^r A(x) - D^r A(y)\|}{\mathrm{dist}(x,y)^\alpha}. \tag{3.3}$$

By continuity and compactness, every cocycle F_A with $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ satisfies the integrability conditions (1.1) and (1.2). We always assume $r + \alpha > 0$. Then every $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is Hölder continuous,

$$\|A(x) - A(y)\| \leq \|A\|_{0,\beta} \,\mathrm{dist}(x,y)^\beta, \quad \text{with} \quad \beta = \begin{cases} \alpha & \text{if } r = 0 \\ 1 & \text{if } r \geq 1. \end{cases}$$

3.1. **Fiber bunched linear cocycles.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism, and ν and $\hat\nu$ be the functions in (2.1) and (2.3).

Definition 3.1. We say that $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is *fiber bunched* if

$$\|A(x)\| \, \|A(x)^{-1}\| \nu(x)^\beta < 1 \ \text{ and } \ \|A(x)\| \, \|A(x)^{-1}\| \hat\nu(x)^\beta < 1, \tag{3.4}$$

for every $x \in M$ (interchangeably, we say that the cocycle F_A is fiber bunched).

Remark 3.2. This notion has appeared before in [5, 6, 25], where it was called *domination*. The present terminology seems preferable, for a number of reasons. One, is by analogy with the notion of center bunching in Definition 2.2. Perhaps more important, the natural notion of domination for smooth cocycles corresponds to a rather different condition, see Definition 4.1. The relation between the two is explained in Section 4.3: a linear cocycle is fiber bunched if and only if the associated projective cocycle is dominated. Moreover, a notion of fiber bunching can be defined for smooth cocycles as well (see [3]), similar to (3.4) and stronger than domination.

We are going to see that if A is fiber bunched then the linear cocycle F_A, and its projectivization $\mathbb{P}(F_A)$, admit stable and unstable holonomies, and these holonomies depend in a differentiable way on $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$. All our arguments hold, up to appropriate adjustments, under the weaker assumption that (3.4) holds for some power A^N, $N \geq 1$. Notice that fiber bunching is an open condition: if A is fiber bunched then so is every cocycle B in a C^0 neighborhood, just because M is compact. Even more, still by compactness, if A is fiber bunched then there exists $m < 1$ such that

$$\|B(x)\| \, \|B(x)^{-1}\| \nu(x)^{\beta m} < 1 \quad \text{and} \quad \|B(x)\| \, \|B(x)^{-1}\| \hat{\nu}(x)^{\beta m} < 1 \tag{3.5}$$

for every $x \in M$ and every B in a C^0 neighborhood of A. It is in this form that the definition will be used in the proofs.

Lemma 3.3. *Suppose $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there is $C > 0$ such that*

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \leq C \nu^n(x)^{-\beta m}$$

for all y, $z \in \mathcal{W}^s_{\text{loc}}(x)$, $x \in M$, and $n \geq 1$. Moreover, the constant C that may be taken uniform on a neighborhood of A.

Proof. Since $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is β-Hölder, there exists $L_1 > 0$ such that

$$\begin{aligned} \|A(f^j(y))\| / \|A(f^j(x))\| &\leq \exp(L_1 \operatorname{dist}(f^j(x), f^j(y))^\beta) \\ &\leq \exp(L_1 \nu^j(x)^\beta \operatorname{dist}(x, y)^\beta) \end{aligned}$$

and similarly for $\|A(f^j(z))^{-1}\| / \|A(f^j(x))^{-1}\|$. By sub-multiplicativity of the norm

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \leq \prod_{j=0}^{n-1} \|A(f^j(y))\| \, \|A^n(f^j(z))^{-1}\|.$$

In view of the previous observations, the right hand side is bounded by

$$\exp \Big[L_1 \sum_{j=0}^{n-1} \nu^j(x) (\operatorname{dist}(x, y)^\beta + \operatorname{dist}(x, z)^\beta) \Big] \prod_{j=0}^{n-1} \|A(f^j(x))\| \, \|A^n(f^j(x))^{-1}\|$$

Since $\nu(\cdot)$ is strictly smaller than 1, the first factor is bounded by some $C > 0$. By fiber bunching (3.5), the second factor is bounded by $\nu^n(x)^{-\beta m}$. It is clear from the construction that L_1 and C may be chosen uniform on a neighborhood. □

Proposition 3.4. *Suppose $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there is $L > 0$ such that for every pair of points x, y in the same leaf of the strong-stable foliation $\mathcal{W}^s$,*

1. $H^s_{x,y} = \lim_{n\to\infty} A^n(y)^{-1} A^n(x)$ *exists (a linear isomorphism of $\mathbb{K}^d$)*
2. $H^s_{f^j(x), f^j(y)} = A^j(y) \circ H^s_{x,y} \circ A^j(x)^{-1}$ *for every $j \geq 1$*
3. $H^s_{x,x} = \operatorname{id}$ *and* $H^s_{x,y} = H^s_{z,y} \circ H^s_{x,z}$
4. $\|H^s_{x,y} - \operatorname{id}\| \leq L \operatorname{dist}(x, y)^\beta$ *whenever $y \in \mathcal{W}^s_{\text{loc}}(x)$.*
5. *Given $a > 0$ there is $\Gamma(a) > 0$ such that $\|H^s_{x,y}\| < \Gamma(a)$ for any x, $y \in M$ with $y \in \mathcal{W}^s(x)$ and $\operatorname{dist}_{\mathcal{W}^s}(x, y) < a$.*

Moreover, L and the function $\Gamma(\cdot)$ may be taken uniform on a neighborhood of A.

Proof. In order to prove claim (1), it is sufficient to consider the case $y \in \mathcal{W}^s_{\text{loc}}(x)$ because $A^{n+j}(y)^{-1} A^{n+j}(x) = A^j(y)^{-1} A^n(f^j(y))^{-1} A^n(f^j(x)) A^j(x)$. Furthermore, once this is done, claim (2) follows immediately from this same relation. Each difference $\|A^{n+1}(y)^{-1} A^{n+1}(x) - A^n(y)^{-1} A^n(x)\|$ is bounded by

$$\|A^n(y)^{-1}\| \, \|A(f^n(y))^{-1} A(f^n(x)) - \operatorname{id}\| \, \|A^n(x)\|.$$

Since A is β-Hölder, there is $L_2 > 0$ such that the middle factor is bounded by

$$L_2 \operatorname{dist}(f^n(x), f^n(y))^\beta \le L_2 \big[\nu^n(x) \operatorname{dist}(x, y)\big]^\beta.$$

Using Lemma 3.3 to bound the product of the other factors, we obtain

$$\|A^{n+1}(y)^{-1} A^{n+1}(x) - A^n(y)^{-1} A^n(x)\| \le C L_2 \big[\nu^n(x)^{(1-m)} \operatorname{dist}(x, y)\big]^\beta. \tag{3.6}$$

The sequence $\nu^n(x)^{\beta(1-m)}$ is uniformly summable, since $\nu(\cdot)$ is bounded away from 1. Let $K > 0$ be an upper bound for the sum. It follows that $A^n(y)^{-1} A^n(x)$ is a Cauchy sequence, and so it does converge. This finishes the proof of claim (1). Claim (3) is a direct consequence. Moreover, adding the last inequality over all n, we also get $\|H^s_{x,y} - \mathrm{id}\,\| \le L \operatorname{dist}(x, y)^\beta$ with $L = C L_2 K$. This proves claim (4). As a consequence, we also get that there exists $\gamma > 0$ such that $\|H^s_{x,y}\| < \gamma$ for any points x, y in the same local strong-stable leaf. To deduce claim (5), notice that for any x, y in the same (global) strong-stable leaf there exist points $z_0, \ldots, z_n$, where n depends only on an upper bound for the distance between x and y along the leaf, such that $z_0 = x$, $z_n = y$, and each z_i belongs to the local strong-stable leaf of z_{i-1} for every $i = 1, \ldots, n$. Together with (3), this implies $\|H^s_{x,y}\| < \gamma^n$. It is clear from the construction that L_2 and $\Gamma(\cdot)$ may be taken uniform on a neighborhood. The proof of the proposition is complete. □

The family of maps $H^s_{x,y}$ given by this proposition is a *stable holonomy* for A (or the cocycle F_A). The next proposition states that these maps vary continuously with the base point.

Proposition 3.5. *Suppose $A \in C^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then the map*

$$(x, y) \mapsto H^s_{x,y}$$

is continuous on $W^s_N = \{(x, y) \in M \times M : f^N(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^N(x))\}$, for every $N \ge 0$.

Proof. Notice that $\operatorname{dist}(x, y) \le 2R$ for all $(x, y) \in W^s_0$, by our definition of local strong-stable leaves in Section 2.2. So, the Cauchy estimate in (3.6)

$$\begin{aligned} \|A^{n+1}(y)^{-1} A^{n+1}(x) - A^n(y)^{-1} A^n(x)\| &\le C L_2 \big[\nu^n(x)^{(1-m)} \operatorname{dist}(x, y)\big]^\beta. \\ &\le C L_2 (2R)^\beta \nu^n(x)^{\beta(1-m)} \end{aligned} \tag{3.7}$$

is uniform on W^s_0. This implies that the limit in part (1) of Proposition 3.4 is uniform on W^s_0. That implies case $N = 0$ of the present proposition. The general case follows immediately, using property (2) in Proposition 3.4. □

Remark 3.6. Since the constants C and L_2 are uniform on some neighborhood of A, the Cauchy estimate (3.7) is also locally uniform on A. Thus, the limit in part (1) of Proposition 3.4 is locally uniform on A as well. Consequently, the stable holonomy also depends continuously on the cocycle, in the sense that

$$(A, x, y) \mapsto H^s_{A,x,y} \quad \text{is continuous on } \mathcal{G}^{r,\alpha}(M, d, \mathbb{K}) \times W^s_0.$$

Using property (2) in Proposition 3.4 we may even replace W^s_0 by any W^s_N.

Dually, one finds an *unstable holonomy* $(x, y) \mapsto H^u_{x,y}$ for A (or the cocycle F_A), given by

$$H^u_{x,y} = \lim_{n \to -\infty} A^n(y)^{-1} A^n(x)$$

whenever x and y are on the same strong-unstable leaf, and it is continuous on $W^u_N = \{(x,y) \in M \times M : f^{-N}(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^{-N}(x))\}$, for every $N \geq 0$. Even more,

$$(A, x, y) \mapsto H^u_{A,x,y} \quad \text{is continuous on every } \mathcal{G}^{r,\alpha}(M, d, \mathbb{K}) \times W^u_N.$$

3.2. **Differentiability of holonomies.** Now we study the differentiability of stable holonomies $H^s_{A,x,y}$ as functions of $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$. Notice that $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is an open subset of the Banach space of $C^{r,\alpha}$ maps from M to the space of all $d \times d$ matrices and so the tangent space at each point of $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is naturally identified with that Banach space. The next proposition is similar to Lemma 2.9 in [25], but our proof is neater: the previous argument used a stronger fiber bunching condition.

Proposition 3.7. *Suppose $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ of A such that, for any $x \in M$ and any y, $z \in \mathcal{W}^s(x)$, the map $B \to H^s_{B,y,z}$ is of class C^1 on $\mathcal{U}$, with derivative*

$$\partial_B H^s_{B,y,z} : \dot{B} \mapsto \sum_{i=0}^{\infty} B^i(z)^{-1} \Big[H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \Big] B^i(y). \tag{3.8}$$

Proof. Recall fiber bunching is an open condition and the constants in Lemma 3.3 and Proposition 3.4 may be taken uniform on some neighborhood $\mathcal{U}$ of A. There are three main steps. First, we suppose that y, z are in the local strong-stable leaf of x, and prove that the expression $\partial_B H^s_{B,y,z} \dot{B}$ is well defined for every $B \in \mathcal{U}$ and every $\dot{B}$ in $T_B \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$. Next, still in the local case, we show that this expression indeed gives the derivative of our map with respect to the cocycle. Finally, we extend the conclusion to arbitrary points on the global strong-stable leaf of x.

Step 1. For each $i \geq 0$, write

$$H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \tag{3.9}$$

as the following sum

$$(H^s_{B,f^i(y),f^i(z)} - \mathrm{id}) B(f^i(y))^{-1} \dot{B}(f^i(y)) + B(f^i(z))^{-1} \dot{B}(f^i(z)) (\mathrm{id} - H^s_{B,f^i(y),f^i(z)}) + [B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z))].$$

By property (4) in Proposition 3.4, the first term is bounded by

$$L \, \|B(f^i(y))^{-1}\| \, \|\dot{B}(f^i(y))\| \; \mathrm{dist}(f^i(y), f^i(z))^\beta \leq L \, \|B^{-1}\|_{0,0} \, \|\dot{B}\|_{0,0} \Big[\nu^i(x) \, \mathrm{dist}(y,z)\Big]^\beta \tag{3.10}$$

and analogously for the second one. The third term may be written as

$$\|B(f^i(y))^{-1} [\dot{B}(f^i(y)) - \dot{B}(f^i(z))] + [B(f^i(y))^{-1} - B(f^i(z))^{-1}] \dot{B}(f^i(z))\|.$$

Using the triangle inequality, we conclude that this is bounded by

$$\Big(\|B(f^i(y))^{-1}\| \, H_\beta(\dot{B}) + H_\beta(B^{-1}) \, \|\dot{B}(f^i(z))\| \Big) \, \mathrm{dist}(f^i(y), f^i(z))^\beta. \leq \; \|B^{-1}\|_{0,\beta} \, \|\dot{B}\|_{0,\beta} \Big[\nu^i(x) \, \mathrm{dist}(y,z)\Big]^\beta, \tag{3.11}$$

where $H_\beta(\phi)$ means the smallest $C \geq 0$ such that $\|\phi(z) - \phi(w)\| \leq C \operatorname{dist}(z,w)^\beta$ for all $z, w \in M$. Notice, from the definition (3.3), that

$$\|\phi\|_{0,0} + H_\beta(\phi) = \|\phi\|_{0,\beta} \leq \|\phi\|_{r,\alpha} \quad \text{for any function } \phi. \tag{3.12}$$

Let $C_1 = \sup\left\{\|B^{-1}\|_{0,\beta} : B \in \mathcal{U}\right\}$. Replacing (3.10) and (3.11) in the expression preceding them, we find that the norm of (3.9) is bounded by

$$(2L+1)C_1\, \nu^i(x)^\beta \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta}$$

Hence, the norm of the ith term in the expression of $\partial_B H^s_{B,y,z}\, \dot{B}$ is bounded by

$$\begin{aligned}(3.13)\quad 2(L+1)C_1\; \nu^i(x)^\beta \|B^i(z)^{-1}\|\, \|B^i(y)\|\; \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta}& \\ \leq C_2\, \nu^i(x)^{\beta(1-m)}\; \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta}&\end{aligned}$$

where $C_2 = 2C(L+1)C_1$ and C is the constant in Lemma 3.3. In this way we find,

$$\|\partial_B H^s_{B,y,z}(\dot{B})\| \leq C_2 \sum_{i=0}^{\infty} \nu^i(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \tag{3.14}$$

for any $x \in M$ and $y, z \in \mathcal{W}^s_{\mathrm{loc}}(x)$. This shows that the series defining $\partial_B H^s_{B,y,z}(\dot{B})$ does converge at such points.

Step 2. By part (1) of Proposition 3.4 together with Remark 3.6, the map $H^s_{B,y,z}$ is the uniform limit $H^n_{B,y,z} = B^n(z)^{-1}B^n(y)$ when $n \to \infty$. Clearly, every $H^n_{B,y,z}$ is a differentiable function of B, with derivative

$$\begin{aligned}\partial_B H^n_{B,y,z}(\dot{B}) = \sum_{i=0}^{n-1} B^i(z)^{-1}\big[&H^{n-i}_{B,f^i(y),f^i(z)} B(f^i(y))^{-1}\dot{B}(f^i(y)) \\ &- B(f^i(z))^{-1}\dot{B}(f^i(z)) H^{n-i}_{B,f^i(y),f^i(z)}\big] B^i(y).\end{aligned}$$

So, to prove that $\partial_B H^s_{B,y,z}$ is indeed the derivative of the holonomy with respect to B, it suffices to show that $\partial H^n_{B,y,z}$ converges uniformly to $\partial H^s_{B,y,z}$ when $n \to \infty$.

Write $1 - m = 2\tau$. From (3.6) and the fact that $\nu(\cdot)$ is strictly smaller than 1,

$$\begin{aligned}\|H^n_{B,y,z} - H^s_{B,y,z}\| &\leq C L_2 \sum_{j=n}^{\infty} \nu^j(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \\ &\leq C_3 \nu^n(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta \leq C_3 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta\end{aligned}$$

for some uniform constant C_3 (the last inequality is trivial, but it will allow us to come out with a positive exponent for $\nu^i(x)$ in (3.15) below). More generally, and for the same reasons,

$$\begin{aligned}\|H^{n-i}_{B,f^i(y),f^i(z)} - H^s_{B,f^i(y),f^i(z)}\| &\leq C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \operatorname{dist}(f^i(y), f^i(z))^\beta \\ &\leq C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \nu^i(x)^\beta \operatorname{dist}(y,z)^\beta \\ &= C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta\end{aligned}$$

for all $0 \leq i \leq n$, and all y, z in the same local strong-stable leaf. It follows, using also Lemma 3.3, that the norm of the difference between the ith terms in the expressions of $\partial_B H^n_{B,y,z}$ and $\partial_B H^s_{B,y,z}$ is bounded by

$$\begin{aligned}(3.15)\quad C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta \|B^i(z)^{-1}\|\, \|B^i(y)\|& \\ \leq C C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta.&\end{aligned}$$

Combining this with (3.13), we find that $\|\partial_B H^n_{B,y,z} - \partial_B H^s_{B,y,z}\|$ is bounded by

$$CC_3 \sum_{i=0}^{n-1} \nu^i(x)^{\beta\tau} \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta + C_2 \sum_{i=n}^{\infty} \nu^i(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta.$$

Since $\nu^i(x)$ is bounded away from 1, the sum is bounded by $C_4 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta$, for some uniform constant C_4. This latter expression tends to zero uniformly when $n \to \infty$, and so the argument is complete.

Step 3. From the property $B(z)\, H^s_{B,y,z} = H^s_{B,f(y),f(z)}\, B(y)$ in Proposition 3.4, we find that if $H^s_{B,f(y),f(z)}$ is differentiable on B then so is $H^s_{B,y,z}$ and the derivative is determined by

$$\dot{B}(z)\, H^s_{B,y,z} + B(z) \cdot \partial_B H^s_{B,y,z}(\dot{B}) = H^s_{B,y,z} \cdot \dot{B}(y) + \partial_B H^s_{B,y,z}(\dot{B}) \cdot B(y). \tag{3.16}$$

Combining this observation with the previous two steps, we conclude that $H^s_{B,y,z}$ is differentiable on B for any pair of points y, z in the same (global) strong-stable leaf: just note that $f^n(y)$, $f^n(z)$ are in the same local strong-stable leaf for large n. Moreover, a straightforward calculation shows that the expression in (3.8) satisfies the relation (3.16). Therefore, (3.8) is the expression of the derivative for all points y, z in the same strong-stable leaf. The proof of the proposition is now complete. $\square$

Corollary 3.8. *Suppose $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists $\theta < 1$ and a neighborhood $\mathcal{U}$ of A and, for each $a > 0$, there exists $C_5(a) > 0$ such that*

$$\begin{aligned} \Big\| \sum_{i=k}^{\infty} B^i(z)^{-1} \Big[& H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) \\ & - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \Big] B^i(y) \Big\| \le C_5(a)\, \theta^k\, \|\dot{B}\|_{0,\beta} \end{aligned} \tag{3.17}$$

for any $B \in \mathcal{U}$, $k \ge 0$, $x \in M$, and y, $z \in \mathcal{W}^s(x)$ with $\operatorname{dist}_{\mathcal{W}^s}(y,z) < a$.

Proof. Let $\theta < 1$ be an upper bound for $\nu(\cdot)^{\beta(1-m)}$. Suppose first $\operatorname{dist}_{\mathcal{W}^s}(y,z) < R$. Then y, z are in the same local strong-stable leaf, and we may use (3.13) to get that the expression in (3.17) is bounded above by

$$C_2 \sum_{i=k}^{\infty} \nu^i(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \le C_5'\, \theta^k\, \|\dot{B}\|_{0,\beta}$$

for some uniform constant C_5'. This settles the case $a \le R$, with $C_5(a) = C_5'$. In general, there is $l \ge 0$ such that $\operatorname{dist}_{\mathcal{W}^s}(y,z) < a$ implies $\operatorname{dist}_{\mathcal{W}^s}(f^l(y), f^l(z)) < R$. Suppose first that $k \ge l$. Clearly, the expression in (3.17) does not change if we replace y, z by $f^l(y)$, $f^l(z)$ and replace k by $k - l$. Then, by the previous special case, (3.17) is bounded above by

$$C_5'\, \theta^{k-l}\, \|\dot{B}\|_{0,\beta}$$

and so it suffices to choose $C_5(a) \ge C_5' \theta^{-l}$. If $k < l$ then begin by splitting (3.17) into two sums, respectively, over $k \le i < l$ and over $i \ge l$. The first sum is bounded by $C_5''(a) \|\dot{B}\|_{0,\beta}$ for some constant $C_5''(a) > 0$ that depends only on a (and l, which is itself a function of a). The second one is bounded by $C_5'\, \|\dot{B}\|_{0,\beta}$, as we have just seen. The conclusion follows, assuming we choose $C_5(a) \ge C_5' \theta^{-l} + C_5''(a) \theta^{-l}$. $\square$

For future reference, let us also state the dual analogues of Proposition 3.7 and Corollary 3.8 for unstable holonomies:

Proposition 3.9. *Suppose $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that, for any $x \in M$ and any $y, z \in \mathcal{W}^u(x)$, the map $B \to H^u_{B,y,z}$ is of class C^1 on $\mathcal{U}$ with derivative*

$$\begin{aligned}(3.18)\quad \partial_B H^u_{B,y,z} : \dot{B} \mapsto -\sum_{i=1}^{\infty} B^{-i}(z)^{-1}\Big[&H^u_{B,f^{-i}(y),f^{-i}(z)} B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y)) \\ &- B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z)) H^u_{B,f^{-i}(y),f^{-i}(z)}\Big] B^{-i}(y).\end{aligned}$$

Corollary 3.10. *In the same setting as Proposition 3.9, if $\operatorname{dist}_{\mathcal{W}^u}(y,z) < a$ then,*

$$\begin{aligned}(3.19)\quad \Big\| \sum_{i=k}^{\infty} B^{-i}(z)^{-1}\Big[&H^u_{B,f^{-i}(y),f^{-i}(z)} B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y)) \\ &- B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z)) H^u_{B,f^{-i}(y),f^{-i}(z)}\Big] B^{-i}(y)\Big\| \le C_5(a)\,\theta^k\,\|\dot{B}\|_{0,\beta}.\end{aligned}$$

for every $k \ge 0$.

4. Smooth cocycles: domination and holonomies

We are going to introduce a concept of domination for smooth cocycles, related to the notion of fiber bunching in the linear setting, and observe that dominated smooth cocycles admit stable and unstable holonomies (Proposition 4.2) and these holonomies vary continuously with the cocycle (Proposition 4.3). We also include some comments on the special case of projective cocycles. These facts are mentioned to make the analogy to the linear case more apparent, but they will otherwise not be used in the present paper: whenever we deal with smooth cocycles we simply assume stable and unstable holonomies exist.

4.1. **Dominated smooth cocycles.** For each $\beta > 0$ let $\mathcal{C}^\beta(f,\mathcal{E})$ be the space of cocycles $\mathfrak{F}$ that are β-Hölder continuous, meaning $\operatorname{dist}_{C^1}(\mathfrak{F}_x,\mathfrak{F}_y) \le C \operatorname{dist}(x,y)^\beta$ for some $C > 0$ and every x, $y \in M$.

Definition 4.1. A cocycle $\mathfrak{F} \in \mathcal{C}^\beta(f,\mathcal{E})$ is *dominated* if there is $\theta < 1$ such that

$$(4.1)\qquad \|D\mathfrak{F}_x(\xi)^{-1}\|\,\nu(x)^\beta \le \theta \quad\text{and}\quad \|D\mathfrak{F}_x(\xi)\|\,\hat{\nu}(x)^\beta \le \theta \quad \text{for all } (x,\xi) \in \mathcal{E}$$

In other words, all the contractions of $\mathfrak{F}$ along the fibers are strictly weaker than the contractions of f along strong-stable leaves, and analogously for the expansions. The observations that follow extend, after straightforward adjustments, to the case when (4.1) holds instead for some iterate $\mathfrak{F}^\ell$, $\ell \ge 1$.

This condition is designed so that the usual graph transform argument yields a "strong-stable" lamination and a "strong-unstable" lamination for the map $\mathfrak{F}$:

Proposition 4.2. *Assume $\mathfrak{F} \in \mathcal{C}^\beta(f,\mathcal{E})$ is dominated. Then there exists a unique partition $\mathcal{W}^s = \{\mathcal{W}^s(x,\xi) : (x,\xi) \in \mathcal{E}\}$ of the fiber bundle $\mathcal{E}$ such that*

1. *every $\mathcal{W}^s(x,\xi)$ is a β-Hölder graph over $\mathcal{W}^s(x)$, with Hölder constant C uniform on x*
2. *$\mathfrak{F}(\mathcal{W}^s(x,\xi)) \subset \mathcal{W}^s(\mathfrak{F}(x,\xi))$ for all $(x,\xi) \in \mathcal{E}$*

(3) *the family of maps* $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ *defined by* $(y, H^s_{x,y}(\xi)) \in \mathcal{W}^s(x,\xi)$*, when* $y \in \mathcal{W}^s(x)$*, is a stable holonomy for* $\mathfrak{F}$

(4) *each map* $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$*, with* $y \in \mathcal{W}^s(x)$*, coincides with the uniform limit of* $(\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x$ *as* $n \to \infty$*.*

Moreover, there is a dual statement for strong-unstable leaves.

Outline of the proof. This follows from the same partial hyperbolicity methods (see Hirsch, Pugh, Shub [14]) used in the previous section for linear cocycles. Existence (1) and invariance (2) of the family $\mathcal{W}^s$ follow from a standard application of the graph transform argument [14]. Property (3) is a consequence, in view of the definition of $H^s_{x,y}$. To prove (4), notice that

$$H^s_{x,y} = (\mathfrak{F}^n_y)^{-1} \circ H^s_{f^n(x),f^n(y)} \circ \mathfrak{F}^n_x,$$

because the lamination $\mathcal{W}^s$ is invariant under $\mathfrak{F}$. Also, by part (1), the uniform C^0 distance from $H^s_{f^n(x),f^n(y)}$ to the identity is bounded by

$$C \operatorname{dist}(f^n(x), f^n(y))^\beta \le C\big[\nu^n(x) \operatorname{dist}(x,y)\big]^\beta.$$

Putting these two observations together, we find that

$$\begin{aligned}\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) &\le \operatorname{Lip}\big((\mathfrak{F}^n_y)^{-1}\big) \operatorname{dist}_{C^0}(H^s_{f^n(x),f^n(y)}, \mathrm{id}) \\ &\le C \sup_\xi \|D\mathfrak{F}^n_y(\xi)^{-1}\|\, \nu^n(x)^\beta \operatorname{dist}(x,y)^\beta.\end{aligned}$$

So, by the domination condition (4.1),

$$\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) \le C\theta^n \operatorname{dist}(x,y)^\beta.$$

Thus we obtain (4), and this closes our outline of the proof. □

4.2. **Continuous dependence of holonomies.** Let $\mathcal{D}^\beta(f,\mathcal{E}) \subset \mathcal{C}^\beta(f,\mathcal{E})$ denote the subset of dominated cocycles. It is clear from the definition that this is an open subset, relative to the uniform C^1 metric

$$\operatorname{dist}_{C^1}(\mathfrak{F}, \mathfrak{G}) = \sup_{x\in M} \operatorname{dist}_{C^1}(\mathfrak{F}_x, \mathfrak{G}_x). \tag{4.2}$$

We are going to see that stable holonomies vary continuously with the cocycle inside $\mathcal{D}^\beta(f,\mathcal{E})$, relative to this metric.

Let $\mathcal{W}^s(\mathfrak{G}) = \{\mathcal{W}^s(\mathfrak{G},x,\xi) : (x,\xi) \in \mathcal{E}\}$ denote the strong-stable lamination of a dominated cocycle $\mathfrak{G}$, as in Proposition 4.2, and $H^s_{\mathfrak{G}} = H^s_{\mathfrak{G},x,y}$ be the corresponding stable holonomy:

$$(y, H^s_{\mathfrak{G},x,y}(\xi)) \in \mathcal{W}^s(\mathfrak{G},x,\xi). \tag{4.3}$$

Recall $\mathcal{W}^s(\mathfrak{G},x,\xi)$ is a graph over $\mathcal{W}^s(x)$. We also denote by $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{G},x,\xi)$ the subset of points $(y, H^s_{\mathfrak{G},x,y}(\xi))$ with $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$.

Proposition 4.3. *Let* $(\mathfrak{F}_k)_k$ *be a sequence of cocycles converging to* $\mathfrak{F}$ *in* $\mathcal{D}^\beta(f,\mathcal{E})$*. Then*

(1) *every* $\mathcal{W}^s(\mathfrak{F}_k,x,\xi)$ *is a Lipschitz graph, with Lipschitz constant uniform on* x*,* ξ*, and* k

(2) $\mathcal{W}^s_\varepsilon(\mathfrak{F}_k,x,\xi)$ *converges to* $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{F},x,\xi)$*, as graphs over the same domain, uniformly on* $(x,\xi) \in \mathcal{E}$

(3) $H^s_{\mathfrak{F}_k,x,y}(\xi)$ *converges to* $H^s_{\mathfrak{F},x,y}(\xi)$ *for every* $x \in M$*,* $y \in \mathcal{W}^s(x)$*, and* $\xi \in \mathcal{E}_x$*, and the convergence is uniform over all* $y \in \mathcal{W}_{\mathrm{loc}}(x)$*.*

Outline of the proof. This is another standard consequence of the classical graph transform argument [14]. Indeed, the assumptions imply that the graph transform of $\mathfrak{F}_k$ converges to the graph transform of $\mathfrak{F}$ in an appropriate sense, so that the corresponding fixed points converge as well. This yields (1) and (2). Part (3) is a direct consequence of (2) and the definition (4.3), in the case $y \in \mathcal{W}^s_\varepsilon(x)$. The general statement follows, using the invariance property (h2):

$$H^s_{\mathfrak{F}_k,x,y} = (\mathfrak{F}^n_{k,y})^{-1} \circ H_{\mathfrak{F}_k,f^n(x),f^n(y)} \circ \mathfrak{F}^n_{k,x}.$$

Related facts have been proved in [25, Section 4] for linear cocycles, along these lines. □

4.3. **Projective cocycles.** The *projective cocycle* defined by $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is the smooth cocycle

$$\mathfrak{F}_A = \mathbb{P}(F_A) : M \times \mathbb{P}(\mathbb{K}^d) \to M \times \mathbb{P}(\mathbb{K}^d)$$

given by $\mathfrak{F}_A(x,[v]) = (f(x),[A(x)v])$, where $[w]$ denotes the projective class of a non-zero vector $w \in \mathbb{K}^d$. Then, for every x, n, and $\xi \in \mathbb{P}(\mathbb{K}^d)$,

$$\mathfrak{F}^n_{A,x}(\xi) = \frac{A^n(x)\xi}{\|A^n(x)\xi\|}$$

(on the right hand side, think of ξ as a unit vector in $\mathbb{K}^d$). It follows that,

$$D\mathfrak{F}^n_{A,x}(\xi)\dot{\xi} = \frac{\operatorname{proj}_{A^n(x)\xi}(A^n(x)\dot{\xi})}{\|A^n(x)\xi\|},$$

where $\operatorname{proj}_w v = v - w(w\cdot v)/(w\cdot w)$ is the projection of v to the orthogonal complement of w. This implies that

$$\|D\mathfrak{F}^n_{A,x}(\xi)\| \le \|A^n(x)\|/\|A^n(x)\xi\| \le \|A^n(x)\|\|A^n(x)^{-1}\| \tag{4.4}$$

for every x, ξ, and n. Analogously, replacing A^n by its inverse,

$$\|D\mathfrak{F}^n_{A,x}(\xi)^{-1}\| \le \|A^n(x)^{-1}\|\|A^n(x)\| \tag{4.5}$$

for every x, ξ, and n. These two inequalities imply

$$\lambda_+(\mathfrak{F}_A,x,\xi) \le \lambda_+(A,x) - \lambda_-(A,x) \quad \text{and} \quad \lambda_-(\mathfrak{F}_A,x,\xi) \ge \lambda_-(A,x) - \lambda_+(A,x)$$

It also follows from (4.4)-(4.5) that A is fiber bunched (Definition 3.1) if and only if $\mathfrak{F}_A$ is dominated (Definition 4.1). Then we could use Proposition 4.2 to conclude that $\mathfrak{F}_A$ admits stable and unstable holonomies. However, it is also possible to exhibit these holonomies explicitly: if $H^s_{x,y}$ and $H^u_{x,y}$ are holonomies of F_A then $\mathbb{P}(H^s_{x,y})$ and $\mathbb{P}(H^u_{x,y})$ are holonomies of $\mathfrak{F}_A = \mathbb{P}(F_A)$.

5. Invariant measures of smooth cocycles

In this section we prove the following result, and use it to reduce the proof of Theorem C to proving Theorem D:

Theorem 5.1. *Let f be a C^2 partially hyperbolic, volume preserving diffeomorphism, $\mathfrak{F}$ be a smooth cocycle over f admitting stable and unstable holonomies, and m be an $\mathfrak{F}$-invariant probability on $\mathcal{E}$ such that $\pi_* m = \mu$ and $\lambda_-(\mathfrak{F},x,\xi) = 0 = \lambda_+(\mathfrak{F},x,\xi)$ at m-almost every point. Then, for any disintegration $\{m_x : x \in M\}$ of m into conditional probabilities along the fibers, there exists a full μ-measure subset M^s such that $m_z = (H^s_{y,z})_* m_y$ for every y, $z \in M^s$ in the same strong-stable leaf.*

Remark 5.2. The hypotheses of the theorem are invariant under time reversion. So, replacing f and $\mathfrak{F}$ by their inverses, we get that the disintegration is also invariant under strong-unstable holonomy over some full μ-measure subset M^u.

Let us recall that a *disintegration* of m is a family of probability measures $\{m_x : x \in M\}$ on the fibers $\mathcal{E}_z$, such that

$$m(E) = \int m_x(\mathcal{E}_x \cap X)\, d\mu(x)$$

for every measurable subset X. Such a family exists and is essentially unique, meaning that any two coincide on a full measure subset. See Rokhlin [22].

Before proving Theorem 5.1, let us deduce Theorem C. Given any disintegration $\{m_x : x \in M\}$ of the probability m, define $\Psi(x) = m_x$ at every point. According to Theorem 5.1 and Remark 5.2, Ψ is essentially s-invariant and essentially u-invariant. By Theorem D, there exists a bi-invariant function $\tilde{\Psi}$ defined on some bi-saturated full measure set $\tilde{M}$ and coinciding with Ψ almost everywhere. Then we get a new disintegration $\{\tilde{m}_x : x \in M\}$ by setting $\tilde{m}_x = \tilde{\Psi}(x)$ when $x \in \tilde{M}$ and extending the definition arbitrarily to the complement. The conclusion of Theorem D means that this new disintegration is both s-invariant and u-invariant on $\tilde{M}$. Moreover, it is continuous if f is accessible.

5.1. **Abstract invariance criterion.** Let $(M_*, \mathcal{M}_*, \mu_*)$ be a Lebesgue space, that is, a complete separable probability space. Every Lebesgue space is isomorphic $\bmod 0$ to the union of an interval, endowed with Lebesgue measure, and a finite or countable set of atoms. See Rokhlin [22, § 2]. Let $T : M_* \to M_*$ be an invertible measurable transformation. A σ-algebra $\mathcal{B}_* \subset \mathcal{M}_*$ is *generating* if its iterates $T^n(\mathcal{B}_*)$, $n \in \mathbb{Z}$ generate the whole $\mathcal{M}_* \bmod 0$: for every $E \in \mathcal{M}_*$ there exists E' in the smallest σ-algebra that contains all the $T^n(\mathcal{B}_*)$ such that $\mu_*(E \Delta E') = 0$.

Theorem 5.3 (Ledrappier [17])**.** *Let $B : M_* \to \mathrm{GL}(d, \mathbb{K})$ be a measurable map such that the functions $x \mapsto \log \|B(x)^{\pm 1}\|$ are μ_*-integrable. Let $\mathcal{B} \subset \mathcal{M}_*$ be a generating σ-algebra such that both T and B are $\mathcal{B}$-measurable* $\bmod 0$*. If $\lambda_-(B, x) = \lambda_+(B, x)$ at μ_*-almost every $x \in M_*$ then, for any $\mathbb{P}(F_B)$-invariant probability m that projects down to μ_*, any disintegration $x \mapsto m_x$ of m along the fibers is $\mathcal{B}$-measurable* $\bmod 0$.

The proof of Theorem 5.1 is based on an extension of this result for smooth cocycles that was recently proved by Avila, Viana [3]. For the statement one needs to introduce the following notion. A *deformation of a smooth cocycle* $\mathfrak{F}$ is a continuous transformation $\tilde{\mathfrak{F}} : \mathcal{E} \to \mathcal{E}$ of the form

$$\tilde{\mathfrak{F}} = \mathcal{H} \circ \mathfrak{F} \circ \mathcal{H}^{-1}$$

where $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ is a homeomorphism of the form $\mathcal{H}(x, \xi) = (x, \mathcal{H}_x(\xi))$ such that the $\mathcal{H}_x : \mathcal{E}_x \to \mathcal{E}_x$ are Hölder continuous, with uniform Hölder constants. To each $\mathfrak{F}$-invariant probability m corresponds an $\tilde{\mathfrak{F}}$-invariant probability $\tilde{m} = \mathcal{H}_* m$.

Theorem 5.4 (Avila, Viana [3])**.** *Let $\tilde{\mathfrak{F}}$ be a deformation of a smooth cocycle $\mathfrak{F}$. Let $\mathcal{B} \subset \mathcal{B}_*$ be a generating σ-algebra such that both T and $x \mapsto \tilde{\mathfrak{F}}_x$ are $\mathcal{B}$-measurable* $\bmod 0$*. Let $\tilde{m}$ be an $\tilde{\mathfrak{F}}$-invariant probability that projects down to μ_*. If $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ for m-almost every $(x, \xi) \in \mathcal{E}$ then any disintegration $x \mapsto \tilde{m}_x$ of $\tilde{m}$ along the fibers is $\mathcal{B}$-measurable* $\bmod 0$.

5.2. **Global essential invariance.** Here we prove Theorem 5.1 from the following local version, whose proof is postponed until Section 5.4. Recall that, for each symbol $* \in \{s,u\}$, we denote by $\mathcal{W}^*(x,r)$ the intersection of the leaf $\mathcal{W}^*(x)$ with the Riemannian ball of radius r around x, and we write $\mathcal{W}^*_{\text{loc}}(x) = \mathcal{W}^*(x,R)$.

Proposition 5.5. *Assume the setting of Theorem 5.1. Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ of f and let $\delta \in (0,R)$. Then there exists a full μ-measure subset $\mathcal{N}^s(\Sigma,\delta)$ of*

$$\mathcal{N}(\Sigma,\delta) = \bigcup_{z\in\Sigma} \mathcal{W}^s(z,\delta)$$

such that $m_z = (H^s_{y,z})_* m_y$ for every y, $z \in \mathcal{N}^s(\Sigma,\delta)$ in the same strong-stable leaf. Then there exists a full Lebesgue measure subset Σ_0 of the cross-section such that $\mathcal{N}^s(\Sigma,\delta)$ intersects every $\mathcal{W}^s(z,\delta)$, $z \in \Sigma_0$ on a full Lebesgue measure subset.

Fix any $\delta < R$. For each $x \in M$, consider a cross-section $\Sigma(x)$ such that $\mathcal{N}(\Sigma(x),\delta)$ contains x in its interior, and let $\mathcal{N}^s(x) \subset \mathcal{N}(\Sigma(x),\delta)$ and $\Sigma_0(x) \subset \Sigma(x)$ be full Lebesgue measure subsets as in Proposition 5.5. By compactness, we may find $\varepsilon > 0$ and points $x_1,\dots,x_N$ such that $\mathcal{N}(\Sigma(x_j),\delta)$, $j = 1,\dots,N$ cover M and, even more, the Riemannian ball of radius ε around every point of M is contained in some $\mathcal{N}(\Sigma(x_j),\delta)$. Define

$$M^s = \Big(\bigcup_{j=1}^{n} \mathcal{N}^s(x_j)\Big) \setminus \Big(\bigcup_{j=1}^{n} \bigcup_{z\in\Sigma(x_j)\setminus\Sigma_0(x_j)} \mathcal{W}^s(z)\Big). \tag{5.1}$$

The union of all strong-stable leaves through the $\Sigma(x_j) \setminus \Sigma_0(x_j)$, $j = 1,\dots,N$ has zero μ-measure, because these sets have zero Lebesgue measure inside the corresponding cross-sections, and the strong-stable foliation is absolutely continuous; see [2, 7, 19]. Thus, M^s has full μ-measure. Given any y, $z \in M^s$ inside the same strong-stable leaf $\mathcal{W}^s(x)$, we may find $y = s_0, s_1, \dots, s_{k-1}, s_k = z$ inside $\mathcal{W}^s(x)$ and such that $\operatorname{dist}(s_{i-1},s_i) < \varepsilon$ for every $1 \le i \le k$. Then, for each i we may find j such that s_{i-1} and s_i are both contained in $\mathcal{N}(\Sigma(x_j),\delta)$. By construction, the subset $\mathcal{N}(\Sigma(x_j),\delta)$ has full Lebesgue measure inside $\mathcal{W}^s(z,\delta)$ for every z in $\Sigma_0(x_j)$. So, up to replacing the s_i by appropriate nearby points inside the same local leaf, we also have that s_{i-1} and s_i are both contained in $\mathcal{N}^s(\Sigma(x_j),\delta)$. Then

$$(H^s_{s_{i-1},s_i})_*(m_{s_{i-1}}) = m_{s_i} \quad \text{for every } 1 \le i \le k, \quad \text{and so } (H^s_{y,z})_* m_y = m_z.$$

This reduces the proof of Theorem 5.1 to proving Proposition 5.5.

5.3. **A local Markov construction.** The proof of Proposition 5.5 can be outlined as follows. The assumption that the cocycle admits stable holonomy allows us to construct a special deformation $\tilde{\mathfrak{F}}$ of the smooth cocycle $\mathfrak{F}$ which is measurable $\operatorname{mod} 0$ with respect to a certain σ-algebra $\mathcal{B}$. Applying Theorem 5.4 we get that the disintegration of $\tilde{m}$ is also $\mathcal{B}$-measurable $\operatorname{mod} 0$, where $\tilde{m}$ is the $\tilde{\mathfrak{F}}$-invariant measure corresponding to m. When translated back to the original setting, this $\mathcal{B}$-measurability property means that the disintegration of m is essentially invariant on the domain $\mathcal{N}(\Sigma,\delta)$, as stated in Proposition 5.5.

In this section we construct $\tilde{\mathfrak{F}}$ and $\mathcal{B}$. The next proposition is the main tool. It is essentially taken from Proposition 3.3 in [25], so here we just outline the construction.

Proposition 5.6. *Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ and $\delta \in (0, R/2)$. Then there exists $N \geq 1$ and a family of sets $\{S(z) : z \in \Sigma\}$ such that*

(1) $\mathcal{W}^s(z,\delta) \subset S(z) \subset \mathcal{W}^s_{\mathrm{loc}}(z)$ *for all* $z \in \Sigma$*;*
(2) *for all* $l \geq 1$ *and* $z, \zeta \in \Sigma$*, if* $f^{lN}(S(\zeta)) \cap S(z) \neq \emptyset$ *then* $f^{lN}(S(\zeta)) \subset S(z)$*.*

Outline of the proof. Fix N big enough so that $\nu^N(x) < 1/4$ for all $x \in M$, and denote $g = f^N$. For each $z \in \Sigma$ define $S_0 = \mathcal{W}^s(z, \delta)$ and

$$S_{n+1}(z) = S_0(z) \cup \bigcup_{(j,w)\in Z_n(z)} g^j(S_n(w)) \tag{5.2}$$

where $Z_n(z) = \big\{(j,w) \in \mathbb{N} \times \Sigma : g^j(S_n(w)) \cap S_0(z) \neq \emptyset\big\}$. Clearly, $S_0(z) \subset S_1(z)$ and $Z_0(z) \subset Z_1(z)$. Notice that if $S_{n-1}(z) \subset S_n(z)$ and $Z_{n-1}(z) \subset Z_n(z)$ for every $z \in \Sigma$, then,

$$\bigcup_{(j,w)\in Z_{n-1}(z)} g^j(S_{n-1}(w)) \subset \bigcup_{(j,w)\in Z_n(z)} g^j(S_n(w)).$$

Therefore, by induction, $S_n(z) \subset S_{n+1}(z)$ and $Z_n(z) \subset Z_{n+1}(z)$ for every $n \geq 0$. Define

$$S_\infty(z) = \bigcup_{n=0}^{\infty} S_n(z) \quad \text{and} \quad Z_\infty(z) = \bigcup_{n=0}^{\infty} Z_n(z).$$

Then $Z_\infty(z)$ is the set of $(j,w) \in \mathbb{N} \times \Sigma$ such that $g^j(S_\infty(w))$ intersects $S_0(z)$, and

$$S_\infty(z) = S_0(z) \cup \bigcup_{(j,w)\in Z_\infty(z)} g^j(S_\infty(w)).$$

The choice of N ensures that $S_\infty(z) \subset \mathcal{W}^s(z, 2\delta)$. Finally, define

$$S(z) = S_\infty(z) \setminus \bigcup_{(k,\xi)\in V(z)} g^k(S_\infty(\xi))$$

where $V(z) = \big\{(k,\xi) \in \mathbb{N} \times \Sigma : g^k(S_\infty(\xi)) \not\subset S_\infty(z)\big\}$. This family of sets satisfies the conclusion of the proposition. □

Since the conclusion of Proposition 5.5 is not affected when f and $\mathfrak{F}$ are replaced by its iterates f^N and $\mathfrak{F}^N$, we may assume the integer N in Proposition 5.6 to be equal to 1. Let $M_* = M$ and $T = f$. Let $\mathcal{M}_*$ be the μ-completion of the Borel σ-algebra of M and μ_* be the canonical extension of μ to $\mathcal{M}_*$. Then $(M_*, \mathcal{M}_*, \mu_*)$ is a Lebesgue space and T is an automorphism in it.

For each $z \in \Sigma$ let $r(z) \geq 0$ be the largest integer (possibly infinite) such that $f^j(S(z))$ does not intersect the union of $S(w)$, $w \in \Sigma$ for all $0 \leq j \leq r(z)$. Let $\mathcal{B}$ be the σ-algebra of sets $E \in \mathcal{M}_*$ such that, for every z and j, either E contains $f^j(S(z))$ or is disjoint from it. Notice that an $\mathcal{M}$-measurable function on M is $\mathcal{B}$-measurable precisely if it is constant on every $f^j(S(z))$. Define $\tilde{\mathfrak{F}} : \mathcal{E} \to \mathcal{E}$ to be $\tilde{\mathfrak{F}} = \mathcal{H} \circ \mathfrak{F} \circ \mathcal{H}^{-1}$, where

$$\mathcal{H}_x = \begin{cases} H^s_{x,f^j(z)} & \text{if } x \in f^j(S(z)) \text{ for some } z \in \Sigma \text{ and } 0 \leq j \leq r(z) \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

It is easy to check that the family $\{\mathcal{H}_x : x \in M\}$ is uniformly Hölder continuous. The definition implies that

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),f^{j+1}(z)} \circ \mathfrak{F}_x \circ H^s_{f^j(z),x} = \mathfrak{F}_{f^j(z)} \tag{5.3}$$

if $x \in f^j(S(z))$ for some $z \in \Sigma$ and $0 \le j < r(z)$. Moreover,

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),w} \circ \mathfrak{F}_x \circ H^s_{f^{r(z)}(z),x} \tag{5.4}$$

if $x \in f^{r(z)}(S(z))$ for some $z \in \Sigma$, where $w \in \Sigma$ is given by $f^{r(z)+1}(S(z)) \subset S(w)$. In all other cases, $\tilde{\mathfrak{F}}_x = \mathfrak{F}_x$.

Lemma 5.7. *The following properties hold*

1. $T = f$ *and* $x \mapsto \tilde{\mathfrak{F}}_x$ *are* $\mathcal{B}$*-measurable*
2. $\operatorname{dist}_{C^0}(\mathcal{H}_x, \mathrm{id})$ *is uniformly bounded*
3. $\{T^n(\mathcal{B}) : n \in \mathbb{N}\}$ *generates* $\mathcal{M}_*$ $\bmod 0$.

Proof. The relations (5.3) and (5.4) show that $\tilde{\mathfrak{F}}_x$ is constant on $f^j(S(z))$ for every $z \in \Sigma$ and $0 \le j \le r(z)$. Thus, $x \mapsto \tilde{\mathfrak{F}}_x$ is $\mathcal{B}$-measurable. $\mathcal{B}$-measurability of f is a simple consequence of the Markov property in Proposition 5.6. Indeed, let $E \in \mathcal{B}$ and let $z \in \Sigma$ and $0 \le j \le r(z)$ be such that $f^{-1}(E)$ intersects $f^j(S(z))$. Then E intersects $f^{j+1}(S(z))$. We claim that E contains $f^{j+1}(S(z))$. When $j+1 \le r(z)$ this follows immediately from $E \in \mathcal{B}$. When$j = r(z)$, notice that $f^{j+1}(S(z)) \subset S(w)$ for some $w \in S(z)$, and $E \in \mathcal{B}$ must contain $S(w)$. So the claim holds in all cases. It follows that $f^{-1}(E)$ contains $f^j(S(z))$. This proves that $f^{-1}(E) \in \mathcal{B}$, and so the proof of claim (1) is complete. To prove claim (2), observe that

$$\operatorname{diam} f^j(S(z)) \le \operatorname{diam}_{\mathcal{W}^s} S(z) \le R,$$

for all $z \in \Sigma$ and $j \ge 0$, and so

$$\sup_{x \in M} \operatorname{dist}_{C^0}(\mathcal{H}_x, \mathrm{id}) \le \sup_{\operatorname{dist}(a,b) \le R} \operatorname{dist}_{C^0}(H^s_{a,b}, \mathrm{id}).$$

The right hand side is uniformly bounded, since the stable holonomy depends continuously on the base points, and the space of $(a, b) \in M \times M$ with $\operatorname{dist}(a, b) \le R$ is compact. This proves claim (2). To prove the last claim, observe that $f^n(\mathcal{B})$ is the σ-algebra of sets $E \in \mathcal{M}_*$ such that every $f^{j+n}(S(z))$ either is contained in E or is disjoint from E. Observe that the diameter of $f^{j+n}(S(z))$ goes to zero, uniformly, when n goes to ∞. It follows that every open set can be written as a union of sets $E_n \in f^n(\mathcal{B})$ and, hence, belongs to the σ-algebra generated by $\{f^n(\mathcal{B}) : n \in \mathbb{N}\}$. This proves that the latter σ-algebra coincides $\bmod 0$ with the completion $\mathcal{M}_*$ of the Borel σ-algebra. $\square$

5.4. **Local essential invariance.** Now we deduce Proposition 5.5. By assumption, $\lambda_-(\mathfrak{F}, x, \xi) = \lambda_+(\mathfrak{F}, x, \xi)$ at m-almost every point. Lemma 5.7 ensures that all the other assumptions of Theorem 5.4 are fulfilled as well. We conclude from the theorem that the disintegration $\{\tilde{m}_x : x \in M\}$ of the measure $\tilde{m} = \mathcal{H}_* m$ is measurable $\bmod 0$ with respect to the σ-algebra $\mathcal{B}$. Then, there exists a full μ-measure set $X^s \subset M$ such that the restriction of the disintegration to X^s is constant on every $f^j(S(z))$ with $z \in \Sigma$ and $0 \le j \le r(z)$. The disintegrations of m and $\tilde{m}$ are related to one another by

$$\tilde{m}_x = (\mathcal{H}_x)_* m_x = \begin{cases} (H^s_{x,f^j(z)})_* m_x & \text{if } x \in f^j(S(z)) \text{ for } z \in \Sigma \text{ and } 0 \le j \le r(z) \\ m_x & \text{otherwise.} \end{cases}$$

Define $\mathcal{N}^s(\Sigma(x_j), \delta) = X^s \cap \mathcal{N}(\Sigma, \delta)$. Recall that $\mathcal{W}(z, \delta) \subset S(z)$ for all $z \in \Sigma$. Then, for every z_1, $z_2 \in \mathcal{N}^s(\Sigma(x_j), \delta)$ in the same $\mathcal{W}(z, \delta)$,

$$(H_{z,z_1})_* m_{z_1} = \tilde{m}_{z_1} = \tilde{m}_{z_2} = (H_{z,z_2})_* m_{z_2} \quad \text{and so} \quad m_{z_2} = (H_{z_1,z_2})_* m_{z_1}.$$

This proves the first claim in the proposition. The second one is an immediate consequence, since the strong-stable foliation is absolutely continuous (see [1, 7, 19]). The proofs of Proposition 5.5 and Theorem 5.1 are now complete.

6. Density points

Here we recall ideas of Burns, Wilkinson [9], that will have an important role in Section 7. Let λ be the Riemannian volume associated to the metric adapted to f described in Section 2.2. We denote by λ_S the volume of the Riemannian metric induced on an immersed submanifold S. Given a foliation $\mathcal{F}$ with smooth leaves, we denote by $\lambda_{\mathcal{F}}(A)$ the volume of a measurable subset A of some leaf F, relative to the Riemannian metric λ_F induced on that leaf.

6.1. **Density sequences.** It is clear that λ and the invariant volume μ have the same zero measure sets. More important for our proposes, they have the same Lebesgue density points. Recall that $x \in M$ is a *Lebesgue density point* of a set $X \subset M$ if

$$\lim_{\delta \to 0} \lambda(X : B(x, \delta)) = 1$$

where $\lambda(A : B) = \lambda(A \cap B)/\lambda(B)$ is defined for general subsets A, B with $\lambda(B) > 0$. The Lebesgue Density Theorem asserts that $\lambda(X \,\Delta\, \mathrm{DP}(X)) = 0$ for any measurable set X, where $\mathrm{DP}(X)$ is the set of Lebesgue density points of X.

Balls may be replaced in the definition by other, but not arbitrary, families of neighborhoods of the point. We say that a sequence of measurable sets $(Y_n)_n$ is a *Lebesgue density sequence at $x \in M$* if

(a) $(Y_n)_n$ *nests* at a point x: $Y_n \supset Y_{n+1}$ for every n and $\cap_n Y_n = \{x\}$
(b) $(Y_n)_n$ is *regular*: there is $\delta > 0$ such that $\lambda(Y_{n+1}) \geq \delta \lambda(Y_n)$ for every n
(c) x is a density point of a set X if and only if:

$$\lim_{n \to \infty} \lambda(X : Y_n) = 1.$$

Some of the sequences we are going to mention satisfy these conditions for special classes of sets only. In particular, we say $(Y_n)_n$ is a *Lebesgue density sequence at x for bi-essentially saturated sets* if (a), (b), (c) hold for every bi-essentially saturated set. Let us recall the definition of this last notion.

Definition 6.1. A measurable set $X \subset M$ is *essentially s-saturated* if there exists a set $X^s \subset M$ consisting of entire strong-stable leaves (i.e. an s-saturated set) such that $\mu(X \Delta X^s) = 0$. Analogously, $X \subset M$ is *essentially u-saturated* if there exists a set $X^u \subset M$ consisting of entire strong-stable leaves (i.e. a u-saturated set) such that $\mu(X \Delta X^u) = 0$. Moreover, X is *bi-essentially saturated* if it is both essentially s-saturated and essentially u-saturated.

Burns, Wilkinson [9] propose two main techniques for constructing new Lebesgue density sequences: internested sequences and the Cavalieri's principle. The first one is quite simple and applies to general measurable sets. Two sequences $(Y_n)_n$ and $(Z_n)_n$ that nest at x are said to be *internested* if there is $k \geq 1$ such that

$$Y_{n+k} \subseteq Z_n \quad \text{and} \quad Z_{n+k} \subseteq Y_n \quad \text{for all } n \geq 0.$$

Lemma 6.2 (Lemma 2.1 in [9])**.** *If $(Y_n)_n$ and $(Z_n)_n$ are internested then one sequence is regular if and only if the other one is. Moreover,*

$$\lim_{n \to \infty} \lambda(X : Y_n) = 1 \quad \iff \quad \lim_{n \to \infty} \lambda(X : Z_n) = 1,$$

for any measurable set $X \subset M$.

Consequently, if $(Y_n)_n$ and $(Z_n)_n$ are internested then one is a Lebesgue density sequence (for bi-essentially saturated sets) if and only if the other one is.

6.1.1. *Cavalieri's principle.* The second technique is a lot more subtle and is specific to subsets essentially saturated by some absolutely continuous foliation $\mathcal{F}$ (with bounded Jacobians). Let U be a foliation box for $\mathcal{F}$ and Σ be a cross-section to $\mathcal{F}$ in U. The *fiber* of a set $Y \subset U$ over a point $q \in \Sigma$ is the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base* of $Y \subset U$ is the set Σ_Y of points $q \in \Sigma$ whose fiber $Y(q)$ is a measurable set and has positive $\lambda_{\mathcal{F}}$-measure. The absolute continuity of $\mathcal{F}$ ensures the base is a measurable set. We say Y *fibers* over some set $Z \subset \Sigma$ if the basis $\Sigma_Y = Z$. Given $c \geq 1$, a sequence of sets Y_n contained in U has *c-uniform fibers* if

$$c^{-1} \leq \frac{\lambda_{\mathcal{F}}(Y_n(q_1))}{\lambda_{\mathcal{F}}(Y_n(q_2))} \leq c \quad \text{for all } q_1,\, q_2 \in \Sigma_{Y_n} \text{ and every } n \geq 0. \tag{6.1}$$

Proposition 6.3 (Proposition 2.7 in [9])**.** *Let $(Y_n)_n$ be a sequence of measurable sets in U with c-uniform fibers, for some c. Then, for any locally $\mathcal{F}$-saturated measurable set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda_\Sigma(\Sigma_X : \Sigma_{Y_n}) = 1.$$

By locally $\mathcal{F}$-saturated we mean that the set is a union of local leaves of $\mathcal{F}$ in the foliation box U. Sets that differ from a locally $\mathcal{F}$-saturated one by zero Lebesgue measure subsets are called essentially locally $\mathcal{F}$-saturated.

Proposition 6.4 (Proposition 2.5 in [9])**.** *Let $(Y_n)_n$ and $(Z_n)_n$ be two sequences of measurable subsets of U with c-uniform fibers, for some c, and $\Sigma_{Y_n} = \Sigma_{Z_n}$ for all n. Then, for any essentially locally $\mathcal{F}$-saturated set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda(X : Z_n) = 1.$$

6.2. **Fake foliations and juliennes.** Juliennes were proposed by Pugh, Shub [20] as density sequences particularly suited for partially hyperbolic dynamical systems. These are sets constructed by means of invariant foliations that are assumed to exist (dynamical coherence) tangent to the invariant subbundles E^s, E^u, $E^{cs} = E^c \oplus E^s$, $E^{cu} = E^c \oplus E^u$, and E^c, and they have nice properties of invariance under iteration and under the holonomy maps of the strong-stable and strong-unstable foliations. As mentioned before, strong-stable and strong-unstable foliations (tangent to the subbundles E^s and E^u, respectively) always exist in the partially hyperbolic setting. However, that is not always true about the center, center-stable, center-unstable subbundles E^c, E^{cs}, E^{cu}.

One main novelty in Burns, Wilkinson [9] was that, for the first time, they avoided the dynamical coherence assumption. A version of the julienne construction is still important in their approach, but now the definition involves, instead, certain "approximations" to the, possibly nonexistent, invariant foliations, that they call fake foliations. We will not need to use fake foliations nor juliennes directly in this paper but, for the reader's convenience, we briefly describe their main features.

6.2.1. *Fake foliations.* The central result about fake foliations is Proposition 3.1 in [9]: for any $\varepsilon > 0$ there exist constants $0 < \rho < r < R$ such that the ball of radius r around every point admits foliations

$$\widehat{\mathcal{W}}^u_p, \quad \widehat{\mathcal{W}}^s_p, \quad \widehat{\mathcal{W}}^c_p, \quad \widehat{\mathcal{W}}^{cu}_p, \quad \widehat{\mathcal{W}}^{cs}_p.$$

with the following properties, for any $* \in \{u, s, c, cs, cu\}$:

(1) For every $x \in B(p, \rho)$, the leaf $\widehat{\mathcal{W}}^*_p(x)$ is C^1 and the tangent space $T_x\widehat{\mathcal{W}}^*_p(x)$ is contained in the cone of radius ε around E^*_x.

(2) For every $x \in B(p, \rho)$,
$$f(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f(p)}(f(x)) \quad \text{and} \quad f^{-1}(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f^{-1}(p)}(f^{-1}(x)).$$

(3) Given x and $n \geq 1$ such that $f^j(x) \in B(f^j(p), r)$ for $0 \leq j < n$,
   - if $y \in \widehat{\mathcal{W}}^s_p(x, \rho)$ then $f^n(y) \in \widehat{\mathcal{W}}^s_p(f^n(x), \rho)$ and
   $$\operatorname{dist}(f^n(x), f^n(y)) \leq \nu^n(p) \operatorname{dist}(x, y)$$
   - if $f^j(y) \in \widehat{\mathcal{W}}^{cs}_p(f^j(q), \rho)$ for $0 \leq j < n$ then $f^n(y) \in \widehat{\mathcal{W}}^{cs}_p(f^n(x))$ and
   $$\operatorname{dist}(f^n(x), f^n(y)) \leq \hat{\gamma}^n(p)^{-1} \operatorname{dist}(x, y).$$

   There is a similar statement with f, $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^{cs}$ replaced by f^{-1}, $\widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^{cu}$.

(4) $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cu}_p$, and $\widehat{\mathcal{W}}^s_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cs}_p$.

(5) $\widehat{\mathcal{W}}^s_p(p) = \mathcal{W}^s(p, r)$ and $\widehat{\mathcal{W}}^u_p(p) = \mathcal{W}^u(p, r)$.

(6) All the fake foliations $\widehat{\mathcal{W}}^*$, $* \in \{u, s, c, cs, cu\}$ are Hölder continuous, and so are their tangent distributions.

(7) Assuming f is center bunched, every leaf of $\widehat{\mathcal{W}}^{cs}_p$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}^s_p$ and every leaf of $\widehat{\mathcal{W}}^{cu}_p$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}^u_p$.

The local invariance property (2) and the exponential bounds (3) should be compared to the corresponding facts (a), (b), (c) for genuine foliations in Section 2.2. Concerning the uniqueness property (5), notice that the fake strong-stable and strong-unstable foliations need not coincide with the genuine ones, $\mathcal{W}^s$ and $\mathcal{W}^u$, at points other than p. The regularity properties (6) and (7) hold uniformly in $p \in M$.

6.2.2. *Juliennes.* Another direct use of the center bunching condition, besides the smoothness property (7) above, is in the definition of juliennes. In view of the first center bunching condition, $\nu < \gamma\hat{\gamma}$ (there is a dual construction starting from $\hat{\nu} < \gamma\hat{\gamma}$ instead), we may find continuous functions τ and σ such that

$$\nu < \tau < \sigma\gamma \quad \text{and} \quad \sigma < \min\{\hat{\gamma}, 1\}.$$

Let $p \in M$ be fixed. For any $x \in \mathcal{W}^s(p, 1)$ and $n \geq 0$, define

$$\widehat{B}^c_n(x) = \widehat{\mathcal{W}}^c_p(x, \sigma^n(p)) \quad \text{and} \quad S_n(p) = \bigcup_{x \in \mathcal{W}^s(p,1)} \widehat{B}^c_n(x).$$

The (fake) *center-unstable julienne* of order $n \geq 0$ centered at $x \in \mathcal{W}^s(p, 1)$ is defined by

$$\widehat{J}^{cu}_n(x) = \bigcup_{y \in \widehat{B}^c_n(x)} \widehat{J}^u_n(y), \quad \text{where} \quad \widehat{J}^u_n(y) = f^{-n}(\widehat{\mathcal{W}}^u_{f^n(p)}(f^n(y), \tau^n(p))).$$

The latter is the (fake) *unstable julienne* of order $n \geq 0$ centered at y, and is defined for every $y \in S_n(p)$. See Figure 1.

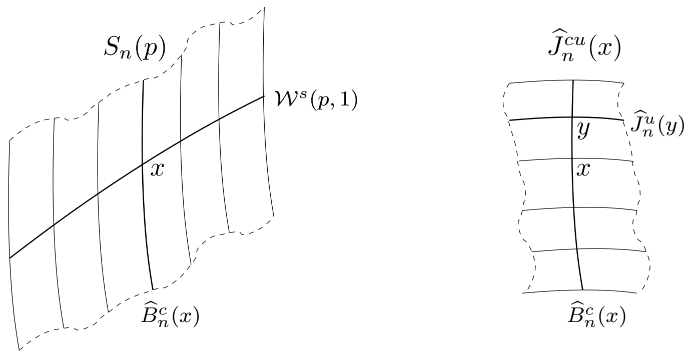

FIGURE 1.

Observe that $\widehat{J}^{cu}_n(x)$ is contained in the smooth submanifold $\widehat{\mathcal{W}}^{cu}_p(x)$, and it has positive measure relative to the Riemannian volume $\lambda_{\widehat{cu}}$ defined by the restriction of the Riemannian metric to $\widehat{\mathcal{W}}^{cu}_p(x)$. Notice also that fake center-unstable leaves are transverse to the strong-stable foliation, as a consequence of property (1) in Section 6.2.1. One key feature of center-unstable juliennes is that, unlike balls for instance, they are approximately preserved by the holonomy maps of the strong-stable foliation:

Proposition 6.5 (Proposition 5.3 in [9]). *For any x, $x' \in \mathcal{W}^s(p,1)$, the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested, where $h^s : \widehat{\mathcal{W}}^{cu}_p(x) \to \widehat{\mathcal{W}}^{cu}_p(x')$ is the holonomy map induced by the strong-stable foliation $\mathcal{W}^s$.*

6.3. **Lebesgue and julienne density points.** Let S be a locally s-saturated set in a neighborhood of p. For notational simplicity, we write

$$\lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x)) = \lambda_{\widehat{cu}}(S \cap \widehat{\mathcal{W}}^{cu}_p(x) : \widehat{J}^{cu}_n(x)).$$

Notice that $S \cap \widehat{\mathcal{W}}^{cu}_p(x)$ coincides with the base of S over $\widehat{\mathcal{W}}^{cu}_p(x)$.

Definition 6.6. We call $x \in \mathcal{W}^s(p,1)$ a *cu-julienne density point* of S if

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x)) = 1.$$

Observe that if $x \in \mathcal{W}^s(p,1)$ is a cu-julienne density point of S then so is every $x' \in \mathcal{W}^s(p,1)$. Indeed, absolute continuity (with bounded Jacobians) gives that

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x)) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

By Proposition 6.5 $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested. Hence, by Lemma 6.2,

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x')) = 1.$$

Another crucial property of center-unstable juliennes is

Proposition 6.7 (Proposition 5.5 in [9]). *Let X be a measurable set that is both s-saturated and essentially u-saturated. Then $x \in \mathcal{W}^s(p)$ is a Lebesgue density point of X if and only if x is a cu-julienne density point of X.*

We are not in a position to use this proposition directly, because the saturation hypotheses are not fully satisfied by the sets we deal with. On the other hand, the proof of this proposition has several steps, involving various nesting sequences, and each step uses only part of the conditions in the hypothesis. We are going to detail the main steps, and recall the definitions of the relevant nesting sequences $B_n(x)$, $C_n(x)$, $D_n(x)$, $G_n(x)$, in order to be able to use them individually in our context.

By definition, $B_n(x)$ is just the Riemannian ball of radius $\sigma^n(p)$ centered at x:

$$B_n(x) = B(x, \sigma^n(p)).$$

Lemma 6.8. *Let $S \subset M$ be any measurable set. Then, x is a Lebesgue density point of S if and only if $\lim_{n\to\infty} \lambda(S : B_n(x)) = 1$.*

Proof. This follows from the fact that the ratio $\sigma^{n+1}(p)/\sigma^n(p) = \sigma(f^n(p))$ of successive radii is less than 1, and is uniformly bounded away from both 0 and 1. $\square$

Next, let us introduce nesting sequences

$$C_n(x) = \bigcup_{q\in D_n^{cs}(x)} \mathcal{W}^u(q, \sigma^n(p)) \quad \text{and} \quad D_n(x) = \bigcup_{q\in D_n^{cs}(x)} f^{-n}(\mathcal{W}^u(f^n(q), \tau^n(p))),$$

fibering over the same sequence of bases

$$D_n^{cs}(x) = \bigcup_{y\in\widehat{\mathcal{W}}_p^s(x,\sigma^n(p))} \widehat{B}_n^c(y) = \bigcup_{y\in\widehat{\mathcal{W}}_p^s(x,\sigma^n(p))} \widehat{\mathcal{W}}_p^c(y, \sigma^n(p)).$$

By property (2) in Section 6.2.1, $D_n^{cs}(x)$ is contained in the submanifold $\widehat{\mathcal{W}}^{cs}(x)$.

Lemma 6.9. *Let $S \subset M$ be any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : B_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : C_n(x)) = 1.$$

Proof. Continuity and transversality of the fake foliations $\widehat{\mathcal{W}}_p^c$ and $\widehat{\mathcal{W}}_p^s$ imply that the sequences $D_n^{cs}(x)$ and $\widehat{\mathcal{W}}^{cs}(x, \sigma^n(p))$ are internested. Then, similarly, continuity and transversality of the foliations $\mathcal{W}^u$ and $\widehat{\mathcal{W}}_p^{cs}$ imply that the sequences $C_n(x)$ and $B_n(x)$ are internested. Then the claim follows from Lemma 6.2. $\square$

Lemma 6.10. *Let $S \subset M$ be locally essentially u-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : C_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : D_n(x)) = 1.$$

Proof. By definition, $C_n(x)$ and $D_n(x)$ both fiber over $D_n^{cs}(x)$, with fibers contained in strong-unstable leaves. The fibers of $C_n(x)$ are uniform, in the sense of (6.1), because they are all comparable to balls of fixed radius $\sigma^n(p)$ inside strong-unstable leaves. Proposition 5.4 in [9] gives that the fibers of $D_n(x)$ are uniform as well. Then the claim follows from Proposition 6.4. $\square$

Finally, we also define

$$G_n(x) = \bigcup_{q\in\widehat{J}_n^{cu}(x)} \mathcal{W}^s(q, \sigma^n(p)).$$

Lemma 6.11. *Let $S \subset M$ any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : D_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : G_n(x)) = 1.$$

Proof. The sequences $D_n(x)$ and $G_n(x)$ are internested, according to Lemma 8.1 and Lemma 8.2 in [9]. Then the claim follows from Lemma 6.2. □

Lemma 6.12. *Let $S \subset M$ be locally s-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : G_n(x)) = 1 \iff \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}_n^{cu}(x)) = 1.$$

Proof. By definition, $G_n(x)$ fibers over $\widehat{J}_n^{cu}(x)$. By Proposition 5.4 of [9], the fibers are uniform. Then the claim follows from Proposition 6.3. □

Proposition 6.7 is obtained in [9] by concatenating Lemmas 6.8 through 6.12.

7. Bi-essential invariance implies essential bi-invariance

Let $f : M \to M$ be a partially hyperbolic diffeomorphism and $\pi : \mathcal{X} \to M$ be a continuous fiber bundle with fibers modelled on some topological space P: by this we mean $\mathcal{X}$ is equipped with local coordinates $\pi^{-1}(U) \to U \times P$ over the neighborhood $U \subset M$ of any point, such that all coordinate changes $(U \cap V) \times P \to (U \cap V) \times P$ are homeomorphisms. Thus, every fiber $\mathcal{X}_x$, $x \in M$ is a topological space homeomorphic to P.

Definition 7.1. A *stable holonomy* on $\mathcal{X}$ is a family $h^s_{x,y} : \mathcal{X}_x \to \mathcal{X}_y$ of homeomorphisms defined for all x, y in the same strong-stable leaf of f and satisfying

(a) $h^s_{y,z} \circ h^s_{x,y} = h^s_{x,z}$ and $h^s_{x,x} = \mathrm{id}$
(b) the map $(x, y, \eta) \mapsto h^s_{x,y}(\eta)$ is continuous.

Unstable holonomy is defined analogously, for pairs of points in the same strong-unstable leaf.

In what follows we assume stable and unstable holonomies exist on $\mathcal{X}$ and have been chosen once and for all.

Definition 7.2. A measurable section $\Psi : M \to \mathcal{X}$ of the fiber bundle $\mathcal{X}$ is called *s-invariant* if

$$h^s_{x,y}(\Psi(x)) = \Psi(y) \quad \text{for every } x,\, y \text{ in the same strong-stable leaf}$$

and *essentially s-invariant* if this relation holds restricted to some full measure subset. The definition of *u-invariant* and *essentially u-invariant* functions is analogous, considering unstable holonomies and strong-unstable leaves instead. Finally, Ψ is *bi-invariant* if it is both s-invariant and u-invariant, and it is *bi-essentially invariant* if it is both essentially s-invariant and essentially u-invariant.

Remark 7.3. It is clear from the definitions that if a section $\Psi : M \to \mathcal{X}$ is s-invariant then it is s-continuous, in the sense introduced in Section 1:

$$(x, y, \Psi(x)) \mapsto \Psi(y) = h^s_{x,y}(\Psi(x))$$

is continuous on the set of pairs of points in the same strong-stable leaf. Analogously, u-invariant sections are u-continuous. In Remark 7.14 we make a similar observation for essential $*$-invariance and essential $*$-continuity.

These notions extend immediately to sections defined over bi-saturated subsets of M. We also need the following mild condition on the topological space:

Definition 7.4. A (Hausdorff) topological space P is *refinable* if there exists an increasing sequence of finite or countable partitions $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ into measurable subsets such that any sequence $(Q_n)_n$ with $Q_n \in \mathcal{Q}_n$ for every n and $\cap_n Q_n \neq \emptyset$ converges to some point $\eta \in P$, in the sense that every neighborhood of η contains Q_n for all large n. (Then, clearly, η is unique and $\cap_n Q_n = \{\eta\}$.)

Remark 7.5. Every topological space with a countable basis $\{U_n : n \in \mathbb{N}\}$ of open sets is refinable: take $\mathcal{Q}_n$ to be the finite partition of M generated by $\{U_1, \ldots, U_n\}$.

We call a continuous fiber bundle $\mathcal{X}$ *refinable* if the fibers $\mathcal{X}_x$, $x \in M$ are refinable.

Theorem 7.6. *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. Then, given any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, there exists a bi-saturated set M_Ψ with full measure, and a bi-invariant section $\tilde{\Psi} : M_\Psi \to \mathcal{X}$ that coincides with Ψ at almost every point.*

Before proving this theorem, let us note that Theorem D(a) is a particular case. Take P to be the space of probability measures on N, endowed with the weak* topology, that is, the smallest topology for which the integration operator

$$P \to \mathbb{R}, \quad \eta \mapsto \int \varphi \, d\eta$$

is continuous, for every continuous function $\varphi : N \to \mathbb{R}$ with compact support.

Lemma 7.7. *P is a separable polish space and, in particular, it is refinable.*

Proof. It is well known that the space of continuous functions on a compact topological space X, endowed with the uniform norm, admits a countable dense subset. Then the same is true for the space $C^0_c(X, \mathbb{R})$ of continuous functions with compact support on a σ-compact space X. Let $\{\varphi_k : N \to \mathbb{R} : k \in \mathbb{N}\}$ be a countable dense subset of the unit ball in $C^0_c(N, \mathbb{R})$. Then

$$\operatorname{dist}(\eta_1, \eta_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \Big| \int \varphi_k \, d\eta_1 - \int \varphi_2 \, d\eta_2 \Big|$$

defines a metric on $\mathcal{X}$ that induces the weak* topology. This metric is complete. Indeed, if $(\eta_n)_n$ is a Cauchy sequence in P then every $(\int \varphi_k \, d\eta_n)_n$ is a Cauchy sequence in $\mathbb{R}$. It follows that

$$\lim_n \int \varphi \, d\eta_n \tag{7.1}$$

exists for every $\varphi = \varphi_k$, $k \in \mathbb{N}$. Then it actually exists for every $\varphi \in C^0_c(N, \mathbb{R})$, because any continuous function with compact support is uniformly approximated by linear combinations of the φ_k. The operator defined by (7.1) on $C^0_c(N, \mathbb{R})$ is linear and positive. Hence, by the Riesz representation theorem ([23, Theorem 2.14]), it represents some positive measure η on N:

$$\int \varphi \, d\eta = \lim_n \int \varphi \, d\eta_n \quad \text{for all } \varphi \in C^0_c(N, \mathbb{R}). \tag{7.2}$$

It is easy to check η is a probability. The definition (7.2) means that the Cauchy sequence $(\eta_k)_n$ converges to η in the weak* topology. Finally, the family

$$\left\{ \eta \in \mathcal{X} : \alpha_k < \int \varphi_k \, d\eta < \beta_k \text{ for } 1 \leq k \leq m \right\},$$

indexed by $m \in \mathbb{N}$ and $\alpha_k, \beta_k \in \mathbb{Q}$, is a countable basis of open sets for P. By Remark 7.5, it follows that P is refinable. $\square$

Associated to $\pi : \mathcal{E} \to M$, we have a new fiber bundle $\Pi : \mathcal{X} \to M$, whose fiber over a point $x \in M$ is the space of probability measures on the corresponding $\mathcal{E}_x$. It is easy to see that this is a continuous fiber bundle with leaves modelled on the space P we have just introduced: if $\pi^{-1}(U) \to U \times N$, $v \mapsto (\pi(v), \psi_{\pi(v)}(v))$ is a continuous local chart for $\mathcal{E}$ then

$$\Pi^{-1}(U) \to U \times P, \quad \eta \mapsto (\Pi(\eta), (\psi_{\Pi(\eta)})_*(\eta))$$

is a continuous local chart for $\mathcal{X}$. The cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ induces a cocycle on $\mathcal{X}$, by push-forward, but this will not be needed here. More important for our purposes, the stable and unstable holonomies of $\mathfrak{F}$ induce homeomorphisms

$$h^s_{x,y} = (H^s_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y \quad \text{and} \quad h^u_{x,y} = (H^u_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y$$

for points x, y in the same strong-stable leaf or the same strong-unstable leaf, respectively. It is easy to see that these homeomorphisms form stable and unstable holonomies on $\mathcal{X}$. Indeed, the group property (a) and the continuity property (b) in Definition 7.1 follow easily from the corresponding properties for H^s and H^u in Definition 1.1: for (a) this is obvious, and for (b) it is checked in the next lemma. Since the statement is local, we may pretend the fiber bundle is trivial ($\mathcal{X} = M \times P$) and so the holonomies are homeomorphisms of P.

Lemma 7.8. *Let x, $y \in M$ and $\eta \in P$. For any neighborhood $V \subset P$ of $h_{x,y}(\eta)$ there exists $\delta > 0$ and a neighborhood $U \subset P$ of η, such that $h_{z,w}(U) \subset V$ for every (z,w) with $\operatorname{dist}(x,z) \le \delta$ and $\operatorname{dist}(y,w) \le \delta$.*

Proof. Consider any x, $y \in M$ and $\eta \in P$. Let $(x_n)_n \to x$ and $(y_n)_n \to y$, and let $(\eta_n)_n \to \eta$ in P. We want to prove that

$$(H^s_{x_n,y_n})_*\eta_n \to (H^s_{x,y})_*\eta\,. \tag{7.3}$$

To this end, let $\varphi : N \to \mathbb{R}$ be any continuous function with compact support. Given $\varepsilon > 0$, fix $\delta > 0$ such that $|\varphi(z) - \varphi(w)| \le \varepsilon$ whenever $\operatorname{dist}(z,w) \le \delta$. By the continuity condition (c) in Definition 1.1, given any compact set $K \subset N$, there exists $\rho_1 > 0$ such that

$$\operatorname{dist}(H^s_{x,y}(\xi), H^s_{x',y'}(\xi)) \le \delta \quad \text{for all } x' \in B(x,\rho_1),\ y' \in B(y,\rho_1), \text{ and } \xi \in K.$$

Choose K large enough so that it contains some neighborhood of $H^s_{y,x}(\operatorname{supp}\varphi)$. Then we may find $\rho_2 > 0$ such that

$$\operatorname{supp}\varphi \subset H^s_{x',y'}(K) \quad \text{for every } x' \in B(x,\rho_2) \text{ and } y' \in B(y,\rho_2).$$

It follows that $|\varphi \circ H^s_{x,y}(\xi) - \varphi \circ H^s_{x_n,y_n}(\xi)| \le \varepsilon$ for every large n and every $\xi \in N$ (consider the cases $\xi \in K$ and $\xi \notin K$ separately). As a consequence,

$$\Big| \int \varphi \circ H^s_{x,y}\, d\eta_n - \int \varphi \circ H^s_{x_n,y_n}\, d\eta_n \Big| \le \varepsilon$$

for every large n. Moreover, the hypothesis $(\eta_n)_n \to \eta$ implies

$$\Big| \int \varphi \circ H^s_{x,y} d\eta - \int \varphi \circ H^s_{x,y} d\eta_n \Big| \le \varepsilon,$$

since $\varphi \circ H^s_{x,y}$ is a continuous function with compact support. Adding the last two inequalities we find that $\int \varphi \circ H^s_{x_n,y_n}\, d\eta_n$ converges to $\int \varphi \circ H^s_{x,y} d\eta$ when $n \to \infty$.

Since φ is an arbitrary function with compact support, this implies (7.3), and so the lemma is proved. $\square$

Now it is clear that Theorem D(a) corresponds to the statement of Theorem 7.6 in the special case of the section $\Psi(x) = m_x$ of the fiber bundle $\mathcal{X}$ wee have defined.

7.1. **Lebesgue densities.** Let $\Psi : M \to P$ be a measurable function with values in a refinable space.

Definition 7.9. A point $x \in P$ is a *point of measurable continuity* of Ψ if there is $\upsilon \in P$ such that x is a Lebesgue density point of $\Psi^{-1}(V)$ for every neighborhood $V \subset P$ of υ. Then υ is called a *density value* of Ψ.

Let $\mathrm{MC}(\Psi)$ denote the set of measurable continuity points of Ψ. It is easy to see that the density value is unique, when it exists. Thus, we have a well defined function $\tilde{\Psi} : \mathrm{MC}(\Psi) \to P$ assigning to each point x of measurable continuity its density value $\tilde{\Psi}(x)$. We call $\tilde{\Psi}$ the *Lebesgue density* of Ψ.

Lemma 7.10. *For any measurable function $\Psi : M \to P$, the set $\mathrm{MC}(\Psi)$ has full Lebesgue measure and $\Psi = \tilde{\Psi}$ almost everywhere.*

Proof. Let $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ be a sequence of partitions of the space P as in Definition 7.4. Let

$$\tilde{M} = \bigcap_{n \geq 1} \bigcup_{Q \in \mathcal{Q}_n} \Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q)).$$

Since $\Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q))$ has full measure in $\Psi^{-1}(Q)$, and $\{\Psi^{-1}(Q) : Q \in \mathcal{Q}_n\}$ is a partition of M for every n, the set on the right hand side has full measure in M for every n. This proves that $\tilde{M}$ is a full measure subset of M. Next, we check that $\tilde{M}$ is contained in the set of points of measurable continuity of Ψ. Indeed, given any point $x \in \tilde{M}$, let $Q_n \in \mathcal{Q}_n$ be the sequence of atoms such that $x \in \Psi^{-1}(Q_n)$. Then x is a density point of $\Psi^{-1}(Q_n)$ for every $n \geq 1$, in view of the definition of $\tilde{M}$. Notice that $\cap_n Q_n$ is non-empty, since it contains $\Psi(x)$. Then, according to Definition 7.4, there exists $\upsilon \in \mathcal{X}$ such that every neighborhood V contains some Q_n. It follows that x is a density point of $\Psi^{-1}(V)$ for any neighborhood $V \subset \mathcal{X}$ of υ, that is, υ is a density value for Ψ at x. This shows that $x \in \mathrm{MC}(\Psi)$ with $\tilde{\Psi}(x) = \upsilon$. Moreover, υ must coincide with $\Psi(x)$, since the intersection of all Q_n contains exactly one point. In other words, $\tilde{\Psi}(x) = \Psi(x)$ for every $x \in \tilde{M}$. $\square$

More generally, let $\Psi : M \to \mathcal{X}$ be a measurable section of a refinable fiber bundle $\mathcal{X}$. Let $x \in M$ be fixed. Using a local chart, one may identify the fiber $\mathcal{X}_y$ over every point y in an neighborhood U of x with the fiber $\mathcal{X}_x$ over x and, thus, view $\Psi \mid U$ as a function with values in $\mathcal{X}_x$. Two such local expressions $\Psi_1 : U \to \mathcal{X}_x$ and $\Psi_2 : U \to \mathcal{X}_x$ of the section Ψ are related by

$$\Psi_1(y) = h_y(\Psi_2(y)),$$

where $(y, \xi) \mapsto (y, h_y(\xi))$ is a homeomorphism with $h_x = \mathrm{id}$. So, a point $\upsilon \in \mathcal{X}_x$ is a density value of Ψ_1 at x if and only if it is a density value of Ψ_2 at x. Moreover, any local expression $\Psi_3 : V \to \mathcal{X}_z$ of the section Ψ near any other point $z \in U$ is related to $\Psi_1 : U \to \mathcal{X}_x$ by

$$\Psi_1(y) = g_y(\Psi_3(y)),$$

where $(y, \xi) \mapsto (y, g_y(\xi))$ is a homeomorphism. So, z is a point of measurable continuity for Ψ_3 if and only if it is a point of measurable continuity for Ψ_1.

These observations allow us to extend Definition 7.9 to sections of refinable fiber bundles, as follows. We call $\upsilon \in \mathcal{X}_x$ a *density value* of the section $\Psi : M \to \mathcal{X}$ at the point x if it is a density value for some (and, hence, any) local expression $U \mapsto \mathcal{X}_x$ as before. We call x a *point of mesurable density* of the section Ψ if it admits some density value or, equivalently, if it is a point of measurable density for some (and, hence, any) local expression of Ψ. The subset $\mathrm{MC}(\Psi)$ of points of measurable continuity has full Lebesgue measure in M, since it intersects every domain U of local chart on a full Lebesgue measure subset. Recall Lemma 7.10. Finally, the *Lebesgue density* of Ψ is the section $\mathrm{MC}(\Psi) \to \mathcal{X}$ assigning to each point x of measurable continuity its (unique) density value.

7.2. **Proof of bi-invariance.** Now Theorem 7.6 is a direct consequence of the next proposition: it suffices to take $M_\Psi = \mathrm{MC}(\Psi)$ and $\tilde{\Psi}$ = the Lebesgue density of Ψ, and apply the proposition together with Lemma 7.10.

Proposition 7.11. *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. For any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, the set $\mathrm{MC}(\Psi)$ is bi-saturated and the Lebesgue density $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-invariant on $\mathrm{MC}(\Psi)$.*

Proof. For any $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^s(x, 1)$, we are going to prove $h^s_{x,y}(\tilde{\Psi}(x))$ is a density value of Ψ at y. It will follow that $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^s_{x,y}(\tilde{\Psi}(x))$. Analogously, one gets that if $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^u(x, 1)$ then $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^u_{x,y}(\tilde{\Psi}(x))$. The proposition is an immediate consequence of these facts. It is convenient to think of $\pi : \mathcal{X} \to M$ as a trivial bundle on neighborhoods U_x of x and U_y of y, identifying $\pi^{-1}(U_x) \approx U_x \times P$ and $\pi^{-1}(U_y) \approx U_y \times P$ via local coordinates, and we do so in what follows.

Let $V \subset P$ be a neighborhood of $h^s_{x,y}(\tilde{\Psi}(x))$. We are going to show that y is a density point of $\Psi^{-1}(V)$. By the continuity of unstable holonomies (property (b) in Definition 7.1), there exists a neighborhood $W \subset V$ of $h^s_{x,y}(\tilde{\Psi}(x))$ and a number $\varepsilon > 0$ such that

$$h^u_{w_1,w_2}(W) \subset V \quad \text{for all } w_1,\, w_2 \in B(y, \varepsilon) \text{ with } w_1 \in \mathcal{W}^u_{\mathrm{loc}}(w_2). \tag{7.4}$$

By the same continuity property for stable holonomies, there exists a neighborhood $U \subset P$ of $\tilde{\Psi}(x)$ and a number $\delta_0 > 0$ such that

$$h^s_{z,w}(U) \subset W \quad \text{for every } z \in B(x, \delta_0) \text{ and } w \in B(y, \varepsilon). \tag{7.5}$$

The assumption that Ψ is bi-essentially invariant (Definition 7.2) implies that there exists a full measure set S^{su} such that

$$\begin{aligned} &h^s_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad \text{for any } \xi,\, \eta \in S^{su} \text{ in the same strong-stable leaf} \\ &h^u_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad \text{for any } \xi,\, \eta \in S^{su} \text{ in the same strong-unstable leaf.} \end{aligned} \tag{7.6}$$

We also need the following lemma, whose proof we postpone for a while:

Lemma 7.12. *Let x be a point measurable continuity of Ψ. Then for any open neighborhood U of the point $\tilde{\Psi}(x) \in P$ there exists $\delta > 0$ and $L \subset B(x, \delta)$ such that*

1. *$\Psi(L \cap S^{su}) \subset U$.*
2. *L is a union of local leaves of $\mathcal{W}^s$ inside $B(x, \delta)$.*
3. *Each of these local leaves contains some point of S^{su}.*

(4) *x is a cu-julienne density point of L:* $\lim_{n\to\infty}\lambda_{\widehat{cu}}(L:\widehat{J}^{cu}_n(x))=1$.

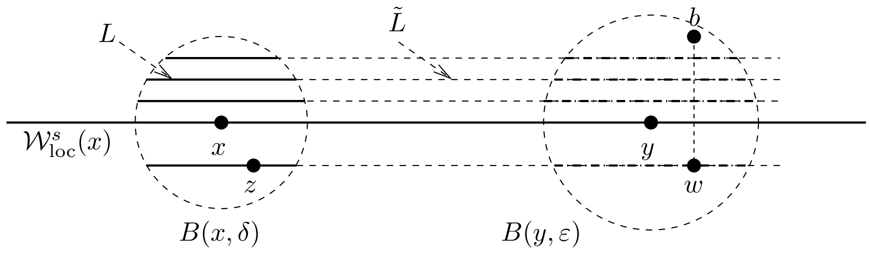

FIGURE 2.

Let L and δ be as given by this lemma. Of course, we may suppose $\delta\le\delta_0$. We extend the local leaves in L along $\mathcal{W}^s_{\mathrm{loc}}(x)$, long enough so as to cross $B(y,\varepsilon)$. Let $\tilde{L}$ denote this extended set. See Figure 2. Since cu-julienne density points of locally s-saturated sets are preserved by stable holonomy, as we have seen in Section 6.3, property (4) in Lemma 7.12 yields

$$\lim_{n\to\infty}\lambda_{\widehat{cu}}(\tilde{L}:\widehat{J}^{cu}_n(y))=1.$$

Applying Lemmas 6.12 and 6.11, we deduce that $\lim_{n\to\infty}\lambda(\tilde{L}:D_n(y))=1$. Now define $A=\tilde{L}\cap S^{su}\cap B(y,\varepsilon)$. Since S^{su} has full measure, we have

$$\lim_{n\to\infty}\lambda(A:D_n(y))=1.$$

Next, let A^u be the u-saturate A inside $B(y,\varepsilon)$ and define $B=A^u\cap S^{us}$. Since $A^u\supset A$ and S^{su} has full measure, we have

$$\lim_{n\to\infty}\lambda(B:D_n(y))=1.$$

Now, by construction, the set B is locally essentially u-saturated. So, we may use Lemmas 6.10, 6.9, and 6.8 to conclude that y is a Lebesgue density point of B.

So, to prove y is a Lebesgue density point of $\Psi^{-1}(V)$, it suffices to show that $\Psi(B)\subset V$. Consider any point $b\in B$. By definition $b\in S^{su}\cap B(y,\varepsilon)$ and there exists $w\in\tilde{L}\cap S^{su}\cap B(y,\varepsilon)$ such that b and w are on the same local strong-unstable leaf. By part (3) of Lemma 7.12, there exists $z\in L\cap S^{su}$ in the same local strong-stable leaf as w. By part (1) of Lemma 7.12, we have $\Psi(z)\in U$. Then, by (7.6) and (7.5),

$$\Psi(w)=h^s_{z,w}(\Psi(z))\in W.$$

Finally, by (7.6) and (7.4),

$$\Psi(b)=h^u_{w,b}(\Psi(w))\in V,$$

as we claimed. This reduces the proof of the proposition to the

Proof of Lemma 7.12. By the continuity of stable holonomies (Definition 7.1), there exists $\delta_2>0$ and a neighborhood $U_2\subset U$ of $\tilde{\Psi}(x)$ such that

$$(h^s_{z_1,z_2})(U_2)\subset U \quad \text{if } z_1,\, z_2\in B(x,\delta_2) \text{ are on the same local strong-stable leaf.}$$

and there exists $\delta_1>0$ and a neighborhood $U_1\subset U_2$ of $\tilde{\Psi}(x)$ such that

$$(h^u_{z_1,z_2})(U_1)\subset U_2 \quad \text{if } z_1,\, z_2\in B(x,\delta_1) \text{ are on the same local strong-unstable leaf.}$$

Let $\delta = \min\{1, \delta_1, \delta_2\}$. Since x is a point of measurable continuity of Ψ, it is a Lebesgue density point of $\Psi^{-1}(U_1)$. Then, since S^{su} has full measure, x is also a density point of $L_1 = \Psi^{-1}(U_1) \cap S^{su}$. Let L_1^u be the local u-saturate of S_1 inside $B(x,\delta)$ and let $L_2 = L_1^u \cap S^{su}$. It is follows that x is a Lebesgue density point of L_1^u, because $L_1^u \supset L_1$, and then it is also a density point of L_2, because S^{su} has full measure. Then, using Lemmas 6.8, 6.9, 6.10, and 6.11 we conclude that

$$\lim_{n\to\infty} \lambda(L_2 : G_n(x)) = 1.$$

Notice that in Lemma 6.10 we used the fact that L_2 is essentially u-saturated.

Take L to be the local s-saturate of L_2 inside $B(x,\delta)$. Consider any point $z \in L \cap S^{su}$. By definition, there exist $z_1 \in \Psi^{-1}(U_1) \cap S^{su}$ and $z_2 \in L_1^u \cap S^{su}$ such that z_1 is in the local strong-unstable leaf of z_2, and z_2 in the local strong-stable leaf of z. Consequently, in view of our choices of U_1 and U_2,

$$\Psi(z_2) = h^u_{z_1,z_2}(\Psi(z_1)) \in U_2 \quad \text{and then} \quad \Psi(z) = h^s_{z_2,z}(\Psi(z_2)) \in U.$$

This proves claim (1) in the lemma. Claims (2) and (3) are clear from the construction: L is a local s-saturate of a subset of S^{su}. Notice also that

$$\lim_{n\to\infty} \lambda(S : G_n(x)) = 1,$$

because $L \supset L_2$. It follows, using Lemma 6.12, that x is a cu-julienne density point of the locally s-saturated set L. This gives claim (4) in the lemma. □

Now the proofs of Proposition 7.11 and Theorem 7.6 are complete. □

7.3. **Bi-essential continuity implies essential bi-continuity.** In this section we show how to adapt the previous arguments to prove the following proposition which, clearly, contains part (a) of Theorem E:

Proposition 7.13. *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle whose fiber is a polish metric space. For any bi-essentially continuous section $\Psi : M \to \mathcal{X}$ the set of points of measurable continuity is bi-saturated and $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-continuous.*

Remark 7.14. As introduced in Section 1, our definition of essential $*$-continuity, $* \in \{s,u\}$ is that the $*$-continuity property holds on some full measure subset S^*, uniformly on the neighborhood of every point. In formal terms: given x_0, $y_0 \in M$ and $\eta_0 \in P$ there exists $\rho > 0$ such that for any $\alpha > 0$ there exists $\beta > 0$ satisfying, for any x_1, $x_2 \in B(x_0,\rho) \cap S^*$ with $\Psi(x_1)$, $\Psi(x_2) \in B(\eta_0,\rho)$ and any y_1, $y_2 \in B(y_0,\rho) \cap S^*$,

$$\begin{aligned} &\operatorname{dist}(x_1,x_2) < \beta, \ \operatorname{dist}(y_1,y_2) < \beta, \ y_i \in \mathcal{W}^*_{\mathrm{loc}}(x_i) \text{ for } i = 1,2, \\ &\qquad \text{and } \operatorname{dist}(\Psi(x_1),\Psi(x_2)) < \beta \Longrightarrow \operatorname{dist}(\Psi(y_1),\Psi(y_2)) < \alpha \end{aligned} \tag{7.7}$$

(it is implicit the fiber bundle has been trivialized near x_0 and y_0). As a special case, corresponding to $x_0 = y_0$ and $x_1 = x_2 = y_2$, we get the following continuity property on strong leaves that will be used in the sequel:

$$\begin{aligned} &x_1, y_1 \in B(x_0,\beta/2) \cap S^*, \ \ y_1 \in \mathcal{W}^*_{\mathrm{loc}}(x_1), \ \ \Psi(x_1) \in B(\eta_0,\rho) \\ &\qquad \Longrightarrow \operatorname{dist}(\Psi(x_1),\Psi(y_1)) < \alpha \end{aligned} \tag{7.8}$$

(it is no restriction to suppose $\beta < \alpha < \rho$). Notice that if $\mathcal{X}$ is a locally compact fiber bundle with holonomies, then every essentially $*$-invariant section is essentially

$*$-continuous. That is because, in the locally compact case, Definition 7.1 implies the holonomies are locally uniformly continuous. Compare Remark 7.3.

Proof. Let $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$. We are going to show that $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}$ satisfies the s-continuity condition at $(x, y, \tilde{\Psi}(x))$. Dual arguments prove u-saturation and u-continuity. The combination of these two facts contains the conclusion of the proposition.

Denote $S^{su} = S^s \cap S^u$ be the intersection of the two full measures sets in Remark 7.14. Using (7.7) with $x_0 = x$, $y_0 = y$, and $\eta_0 = \tilde{\Psi}(x)$, we find that for any $\varepsilon \in (0, \rho)$ there exists $\delta \in (0, \varepsilon)$ such that

$$\begin{aligned} x_1, x_2 \in & B(x,\delta) \cap S^{su}, \quad y_1, y_2 \in B(y,\delta) \cap S^{su}, \quad y_i \in \mathcal{W}^s_{\mathrm{loc}}(x_i) \text{ for } i = 1, 2, \\ & \text{and } \Psi(x_1), \Psi(x_2) \in B(\tilde{\Psi}(x), 2\delta) \Longrightarrow \mathrm{dist}(\Psi(y_1), \Psi(y_2)) < \varepsilon. \end{aligned} \tag{7.9}$$

Using (7.8) with $x_0 = y_0 = x$ and $\eta_0 = \tilde{\Psi}(x)$, we find $\delta_1 \in (0, \delta)$ such that

$$\begin{aligned} x_1, z_1 \in B(x, \delta_1) \cap S^{su}, \quad z_1 \in \mathcal{W}^u_{\mathrm{loc}}(x_1), \quad \Psi(z_1) \in B(\tilde{\Psi}(x), \rho) \\ \Longrightarrow \mathrm{dist}(\Psi(x_1), \Psi(z_1)) < \delta. \end{aligned} \tag{7.10}$$

Define $A_\varepsilon = \Psi^{-1}\big(B(\tilde{\Psi}(x), \delta_1)\big) \cap B(x, \delta_1) \cap S^{su}$ and let A^u_ε be the intersection of S^{su} with the local u-saturate of A_ε inside $B(x, \delta_1)$. By definition, for any $x_1 \in A^u_\varepsilon$ there is $z_1 \in A_\varepsilon$ in the local strong-unstable leaf of x_1. This implies $\Psi(z_1) \in B(\tilde{\Psi}(x), \delta_1)$. Thus, we may use (7.10) to conclude that $\mathrm{dist}(\Psi(x_1), \Psi(z_1)) < \delta$ and, consequently,

$$\Psi(x_1) \in B(\tilde{\Psi}(x), 2\delta) \quad \text{for every } x_1 \in A^u_\varepsilon. \tag{7.11}$$

Let $\tilde{L}$ be the family of local strong-stable leaves through the points of A^u_ε, extended long enough along $W^s_{\mathrm{loc}}(x)$ so as to cross $B(y, \delta)$. Define $B_\varepsilon = \tilde{L} \cap B(y, \delta) \cap S^{su}$. In view of (7.11), we may use (7.9) to conclude that

$$\mathrm{dist}(\Psi(y_1), \Psi(y_2)) < \varepsilon \quad \text{for all } y_1,\, y_2 \in B_\varepsilon. \tag{7.12}$$

Of course, we may take the correspondences $\varepsilon \mapsto \delta \mapsto \delta_1$ to be monotone. Then the $\Psi(B_\varepsilon)$, $\varepsilon > 0$ are a monotone family of subsets of the fiber P, with diameter going to zero when ε goes to zero. Hence, since P is complete, there exists exactly one point η in the intersection of the closures of all these sets. By (7.12),

$$\Psi(B_\varepsilon) \subset B(\eta, \varepsilon) \quad \text{for every } \varepsilon > 0. \tag{7.13}$$

Using (7.8) with $x_0 = y_0 = y$ and $\eta_0 = \eta$, we find $\delta_2 \in (0, \delta)$ such that

$$\begin{aligned} y_1, w_1 \in B(y, \delta_2) \cap S^{su}, \quad w_1 \in \mathcal{W}^u_{\mathrm{loc}}(y_1), \quad \Psi(y_1) \in B(\eta, \rho) \\ \Longrightarrow \mathrm{dist}(\Psi(y_1), \Psi(w_1)) < \varepsilon. \end{aligned} \tag{7.14}$$

Let B^u_ε be the intersection of S^{su} with the local u-saturate of $B_\varepsilon \cap B(y, \delta_2)$ inside $B(y, \delta_2)$. By definition, for any $w_1 \in B^u_\varepsilon$ there exists $y_1 \in B_\varepsilon \cap B(y, \delta_2)$ in the same strong-unstable leaf. Property (7.13) ensures that $\Psi(y_1) \in B(\eta, \varepsilon)$. So, we may use (7.14) to conclude that $\mathrm{dist}(\Psi(y_1), \Psi(w_1)) < \varepsilon$, and so $\mathrm{dist}(\Psi(w_1), \eta) < 2\varepsilon$. This proves that

$$\Psi(B^u_\varepsilon) \subset B(\eta, 2\varepsilon). \tag{7.15}$$

According to Lemma 7.15 below, this implies that y is a Lebesgue density point of $\Psi^{-1}(B(\eta, 2\varepsilon))$. Since ε is arbitrary, it follows that η is a density value for Ψ at y, and so y is a point of measurable continuity. Therefore, $\mathrm{MC}(\Psi)$ is indeed s-saturated.

Lemma 7.15. *The point y is a Lebesgue density point of B^u_ε.*

Proof. Since x is a point of measurable continuity of Ψ, it is a Lebesgue density point of A_ε. Then x is also a density point of A^u_ε, because A_ε is contained in A^u_ε up to a zero measure subset. Since A^u_ε is essentially u-saturated, we may use Lemmas 6.8 through 6.11 to conclude that $\lim_{n\to\infty}\lambda(A^u_\varepsilon : G_n(x)) = 1$. Then x is a cu-julienne density point of $\tilde{L}$: this follows from the previous observation together with Lemma 6.12, because $\tilde{L}\cap B(x,\delta)$ is locally s-saturated and contains A^u_ε. Since cu-julienne density points of locally s-saturated sets are preserved by stable holonomy, as we have seen in Section 6.3, it follows that y is also a cu-julienne density point of $\tilde{L}$. Applying Lemmas 6.12 and 6.11, we deduce that $\lim_{n\to\infty}\lambda(\tilde{L} : D_n(y)) = 1$. This implies $\lim_{n\to\infty}\lambda(B^u_\varepsilon : D_n(y)) = 1$, because B^u_ε contains $\tilde{L}\cap B(y,\delta)$ up to a zero measure subset. Since B^u_ε is locally essentially u-saturated, we may use Lemmas 6.10 through 6.8 to conclude that y is a Lebesgue density point of B^u_ε. The proof of the lemma is complete. □

Now we only have to show that the Lebesgue density $\tilde{\Psi}$ is s-continuous on $\mathrm{MC}(\Psi)$. To this end, consider any $\bar{x}\in\mathrm{MC}(\Psi)$ satisfying $\mathrm{dist}(x,\bar{x})<\delta/2$ and $\mathrm{dist}(\tilde{\Psi}(x),\tilde{\Psi}(\bar{x}))<\delta/2$ and any $\bar{y}\in\mathcal{W}^s_{\mathrm{loc}}(\bar{x})$ with $\mathrm{dist}(y,\bar{y})<\delta/2$. Conducting the previous construction with $\bar{x}$, $\bar{y}$ in the place of x, y one finds sets $\bar{A}_\varepsilon$, $\bar{A}^u_\varepsilon\subset B(\bar{x},\delta)$ and $\bar{B}_\varepsilon$, $\bar{B}^u_\varepsilon\subset B(\bar{y},\delta)$. Define also

$$D_\varepsilon = \Psi^{-1}\big(B(\tilde{\Psi}(\bar{x}),\delta/2)\big)\cap B(\bar{x},\delta/2)\cap D$$

Then D_ε is non-empty, since $\bar{x}\in\mathrm{DP}(D_\varepsilon)$, and it is contained in $A_\varepsilon\cap\bar{A}_\varepsilon$. It follows that $A^u_\varepsilon\cap\bar{A}^u_\varepsilon$, $B^u\cap\bar{B}^u$, and $B^u_\varepsilon\cap\bar{B}^u_\varepsilon$ are all non-empty. Then, in view of (7.15) and the corresponding fact for $\bar{x}$, the diameter of $\Psi(B^u_\varepsilon)\cup\Psi(\bar{B}^u_\varepsilon)$ is bounded by 4ε. It follows that $\mathrm{dist}(\Psi(y),\tilde{\Psi}(\bar{y}))\le 4\varepsilon$, because the closure $\Psi(B^u_\varepsilon)\cup\Psi(\bar{B}^u_\varepsilon)$ contains both $\tilde{\Psi}(y)$ and $\tilde{\Psi}(\bar{y})$. This proves that $\tilde{\Psi}$ is s-continuous on $\mathrm{MC}(\Psi)$. The proof of Proposition 7.13 is complete. □

8. Accessibility and continuity

Now we suppose that f is accessible, in addition to being center bunched. Then $\mathrm{MC}(\Psi)=M$ for every bi-essentially invariant function, because the set of measurable continuity points is bi-saturated, and so the Lebesgue density $\tilde{\Psi}$ is defined on the whole M. Thus, part (b) of Theorem D is now a consequence of the following result, that we are going to prove next:

Theorem 8.1. *Let $f: M\to M$ be a partially hyperbolic accessible diffeomorphism and $\mathcal{X}$ be a continuous fiber bundle with stable and unstable holonomies. Then any bi-invariant section $\tilde{\Psi}: M\to\mathcal{X}$ is continuous.*

Moreover, Theorem 8.1 is a consequence of part (b) of Theorem E since, by Remark 7.3, every bi-invariant section is bi-continuous. In the sequel we prove Theorem E(b).

8.1. **Access sequences.** The main ingredient in the proof of Theorem E(b) is to show that small open sets can be reached by "nearby" su-paths starting from a fixed point in M. For the precise statement, to be given in Proposition 8.3, we need the following

Definition 8.2. Let z, $w \in M$. An *access sequence connecting z to w* is a finite sequence of points $[y_0, y_1, \dots, y_n]$ such that $y_0 = z$, $y_j \in \mathcal{W}^*(y_{j-1})$ for $1 \le j \le n$ where each $* \in \{s, u\}$, and $y_n = w$.

Proposition 8.3. *Given $x_0 \in M$, there is $w \in M$ and there is an access sequence $[y_0(w), \dots, y_N(w)]$ connecting x_0 to w and satisfying the following property: for any $\varepsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that for every $z \in B(w, \delta)$ there exists an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z and such that*

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \text{and} \quad \operatorname{dist}_{\mathcal{W}^*}(y_{j-1}(z), y_j(z)) < L \quad \text{for } j = 1, \dots, N$$

where $\operatorname{dist}_{\mathcal{W}^*}$ denotes the distance along the strong (either stable or unstable) leaf common to the two points.

8.2. **Proof of continuity.** Here we deduce Theorem E(b) from Proposition 8.3. Since the section $\tilde{\Psi}$ is bi-continuous, it suffices to prove it is continuous at *some* point in order to conclude that it is continuous everywhere.

Fix $x_0 \in M$ and then let $w \in M$ and $[y_0(w), y_1(w), \dots, y_N(w)]$ be an access sequence connecting x_0 to w such as in Proposition 8.3. We are going to prove that $\tilde{\Psi}$ is continuous at w. Take the fiber bundle $\pi : \mathcal{X} \to M$ to be trivialized on the neighborhood of every node $y_j(w)$, via local coordinates. Let $V \subset P$ be any neighborhood of $\tilde{\Psi}(w) = \tilde{\Psi}(y_N(w))$. Since $\tilde{\Psi}$ is bi-continuous, we may find numbers $\varepsilon_j > 0$ and neighborhoods V_j of $\tilde{\Psi}(y_j(w))$ such that $V_N = V$ and

$$\begin{aligned} x \in B(y_{j-1}(w), \varepsilon_j), \quad y \in B(y_j(w), \varepsilon_j), \quad y \in \mathcal{W}^{*_j}(x), \quad \tilde{\Psi}(x) \in V_{j-1} \\ \Longrightarrow \tilde{\Psi}(y) \in V_j \end{aligned} \tag{8.1}$$

for every $j = 1, \dots, N$. Let $\varepsilon = \min\{\varepsilon_j : 1 \le j \le N\}$. Using Proposition 8.3 we find $\delta > 0$ and, for each $z \in B(w, \delta)$, an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z, with

$$y_j(z) \in B(y_j(w), \varepsilon) \subset B(y_j(w), \varepsilon_j) \quad \text{for } j = 1, \dots, N. \tag{8.2}$$

We may suppose $\delta < \varepsilon$. Consider any $z \in B(w, \delta)$. Clearly, $\tilde{\Psi}(x) = \tilde{\Psi}(y_0(z)) \in V_0$. Then, we may use (8.1)-(8.2) inductively to conclude that $\tilde{\Psi}(y_j(z)) \in V_j$ for every $j = 1, \dots, N$. The last case, $j = N$, gives $\tilde{\Psi}(z) \in V$. We have shown that $\tilde{\Psi}(B(w, \delta)) \subset V$. This proves that $\tilde{\Psi}$ is continuous at w, as claimed.

In this way, we reduced the proof of Theorem E(b) to proving Proposition 8.3.

8.3. **Non-injective parametrizations.** In this section we prepare the proof of Proposition 8.3, that will be given in the next section.

8.3.1. *Exhaustion of accessibility classes.* Fix any point $x_0 \in M$. For each $r \in \mathbb{N}$, we consider the following sequence of sets $K_{r,n}$, $n \in \mathbb{N}$:

$$K_{r,1} = \{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0, y) \le r\} \quad \text{and}$$
$$K_{r,n} = \bigcup_{x \in K_{r,n-1}} \{y \in \mathcal{W}^*(x) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\}, \quad \text{for } n \ge 2,$$

where $* = s$ when n is odd, and $* = u$ when n is even. That is, $K_{r,n}$ is the set of points that can be reached from x_0 using an access sequence with n legs whose lengths do not exceed r.

Lemma 8.4. *Every $K_{r,n}$ is closed in M and, hence, compact.*

Proof. It is clear from the definition that $K_{r,1}$ is closed. The general case follows by induction. Suppose $K_{r,n-1}$ is closed, and let z belong to the complement of $K_{r,n}$. Then, by definition,

$$Z = \{y \in \mathcal{W}^*(z) : \mathrm{dist}_{\mathcal{W}^*}(x, y) \leq r\}$$

does not intersect the closed set $K_{r,n-1}$. It follows that $U \cap K_{r,n} = \emptyset$ for some neighborhood U of the set Z. By continuity of strong (stable or unstable) foliations, and their induced Riemannian metrics, for every point w in a neighborhood of z,

$$\{y \in \mathcal{W}^*(z) : \mathrm{dist}_{\mathcal{W}^*}(x, y) \leq r\} \subset U$$

and hence, the set on the left hand side is disjoint from $K_{r,n-1}$. This proves that points w in that neighborhood of z do not belong to $K_{r,n}$ either. Thus, $K_{r,n}$ is indeed closed. □

By definition, the union of $K_{r,n}$ over all (r, n) is the accessibility class of x_0. Since we are assuming that f is accessible, this union is the whole manifold:

$$M = \bigcup_{r,n\in\mathbb{N}} K_{r,n}.$$

Since M is a Baire space, it follows that $K_{r,n}$ has non-empty interior for some r and n, that we consider fixed from now on.

Our immediate goal is to define a continuous "parametrization" (non-injective)

$$\Psi_n : \mathfrak{K}_{r,n} \to K_{r,n} \tag{8.3}$$

of the set $K_{r,n}$ by a convenient compact subspace $\mathfrak{K}_{r,n}$ of a Euclidean space, that we are going to introduce in the sequel. Let d_s and d_u denote the dimensions of the strong-stable leaves and the strong-unstable leaves, respectively. This Euclidean space will be the alternating product of $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_u}$, with n factors, each of which parametrizing one leg of the access sequence. The case $n = 2$ is described in Figure 3.

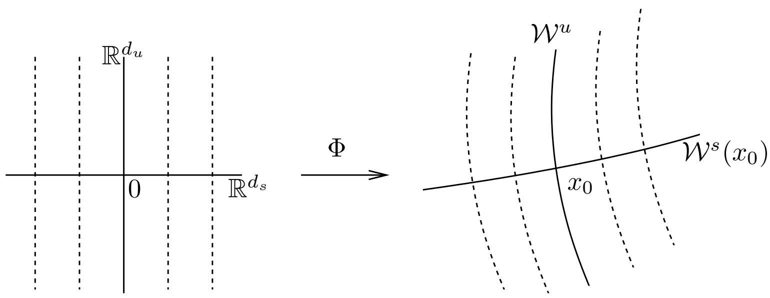

FIGURE 3.

8.3.2. *Fiber bundles induced by local strong leaves.* The following lemma will be useful in the construction of (8.3). The whole point with the statement is that U does not need to be small. The diffeomorphisms in the statement are as regular as the partially hyperbolic diffeomorphism f itself.

Lemma 8.5. *For any contractible space A, any continuous function $\Psi : A \to M$, and any symbol $* \in \{s, u\}$, there exists a homeomorphism*

$$\Theta : A \times \mathbb{R}^{d_*} \to \{(a, y) : a \in A \text{ and } y \in \mathcal{W}^*_{\text{loc}}(\Psi(a))\}$$

that maps every $\{a\} \times \mathbb{R}^{d_*}$ diffeomorphically to $\{a\} \times \mathcal{W}^*_{\text{loc}}(\Psi(a))$ and satisfies $\Theta(a, 0) = (a, \Psi(a))$ for all $a \in A$.

Proof. We consider the case $* = s$. Since $\mathcal{W}^s$ is a continuous lamination with smooth leaves (see [24]), for each $p \in M$ we may find a neighborhood U_p and a continuous map

$$\Phi_p : U_p \times \mathbb{R}^{d_s} \to M$$

such that $\Phi_p(x, 0) = x$ and $\Phi_p(x, \cdot)$ maps $\mathbb{R}^{d_s}$ diffeomorphically to $\mathcal{W}^s_{\text{loc}}(x)$, for every $x \in U_p$. Using these maps we may endow the set

$$F_s = \{(x, y) : x \in M \text{ and } y \in \mathcal{W}^s_{\text{loc}}(x)\}$$

with the structure of a continuous fiber bundle over M, with local charts

$$U_p \times \mathbb{R}^{ds} \to \{(x, y) : x \in U_p \text{ and } y \in \mathcal{W}^s_{\text{loc}}(x)\} \quad (x, v) \mapsto (x, \Phi_p(x, v)).$$

Then $F^s_\Psi = \{(a, y) : a \in A \text{ and } y \in \mathcal{W}^s_{\text{loc}}(\Psi(a))\}$ also has a fiber bundle structure, with local coordinates

$$\Theta_p : \Psi^{-1}(U_p) \times \mathbb{R}^{d_s} \to \{(a, y) : \Psi(a) \in U_p \text{ and } y \in \mathcal{W}^s_{\text{loc}}(\Psi(a))\}$$

given by $\Theta_p(a, v) = (a, \Phi_p(\Psi(a), v))$. This fiber bundle admits the space of diffeomorphisms of $\mathbb{R}^{d_s}$ that fix the origin as a structural group: all coordinate changes along the fibers belong to this group. The core of the proof is the general fact (see [15, Chapter 4,Theorem 9.9]) that, for any topological group G, any fiber bundle over a contractible paracompact space that has G as a structural group is G-trivial. When applied to F^s_Ψ this result means that there exists a global chart

$$\Theta : A \times \mathbb{R}^{d_s} \to \{(a, y) : a \in A \text{ and } y \in \mathcal{W}^s_{\text{loc}}(\Psi(a))\}, \quad \Theta(a, v) = (a, \Phi(a, v))$$

such that every $\Phi(a, \cdot)$ maps $\mathbb{R}^{d_s}$ to the strong-stable leaf through $\Psi(a)$, and every $\Phi(a, \cdot)^{-1} \circ \Phi_p(\Psi(a), \cdot)$ is a diffeomorphism that fixes the origin of $\mathbb{R}^{d_s}$. The latter gives that $\Phi(a, 0) = \Phi_p(\Psi(a), 0) = \Psi(a)$ for all $a \in A$. □

8.3.3. *Construction of non-injective parametrizations.* Now we construct $\mathfrak{K}_{r,n}$ and Ψ as in (8.3). Let $l \geq 1$ be fixed such that, for any $x \in M$,

$$\begin{aligned} &\{y \in \mathcal{W}^s(x) : \operatorname{dist}_{\mathcal{W}^s}(x, y) \leq 2r\} \subset f^{-l}\big(\mathcal{W}^s_{\text{loc}}(f^l(x))\big) \\ &\{y \in \mathcal{W}^u(x) : \operatorname{dist}_{\mathcal{W}^u}(x, y) \leq 2r\} \subset f^l\big(\mathcal{W}^s_{\text{loc}}(f^{-l}(x))\big). \end{aligned} \tag{8.4}$$

Our argument is somewhat more transparent when $l = 0$, and so the reader should find it convenient to keep that case in mind throughout the construction.

Define $E_1 = \{y \in M : f^l(y) \in \mathcal{W}^s_{\text{loc}}(f^l(x_0))\}$ and $\Phi_1 : E_1 \to M$ to be the inclusion. Notice that E_1 is contractible and $\Phi_1(E_1)$ contains $K_{r,1}$. Since E_1 is a smooth disc, there exists an diffeomorphism $\Theta_1 : \mathbb{R}^{d_s} \to E_1$ with $\Theta_1(0) = x_0$. Then

$$\Psi_1 = \Phi_1 \circ \Theta_1 : \mathbb{R}^{d_s} \to M$$

is a continuous function whose image contains $K_{r,1}$. Notice that the pre-image $\mathfrak{K}_{r,1} = \Psi_1^{-1}(K_{r,1})$ is compact: $K_{r,1} = \{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0, y) \leq r\}$ and we have a factor 2 in (8.4). Next, define

$$E_2 = \big\{(a, y) : a \in \mathbb{R}^{d_s} \text{ and } f^{-l}(y) \in \mathcal{W}^u_{\text{loc}}(f^{-l}(\Psi_1(a)))\big\}$$

and $\Phi_2 : E_2 \to M$, $\Phi_2(a,y) = y$. Notice that $\Phi_2(E_2)$ contains $K_{r,2}$. Using Lemma 8.5 with $A = \mathbb{R}^{d_s}$, $\Psi = f^{-l} \circ \Psi_1$, and $* = u$, we find a homeomorphism

$$\Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\}$$

that maps each $\{a\} \times \mathbb{R}^{d_u}$ diffeomorphically to $\{a\} \times \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))$ and satisfies $\Theta_2(a,0) = (a, f^{-l}(\Psi_1(a)))$. Clearly, the map

$$\Gamma_2 : \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\} \to E_2, \quad \Gamma_2(a,y) = (a, f^l(y))$$

is a homeomorphism, and $\Gamma_2(\Theta_2(a,0)) = (a, \Psi_1(a))$. Then

$$\Psi_2 = \Phi_2 \circ \Gamma_2 \circ \Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to M$$

is a continuous map whose image contains $K_{r,2}$. Moreover, Ψ_2 may be viewed as a continuous extension of Ψ_1, because

$$\Psi_2(a,0) = \Phi_2(\Gamma_2(\Theta_2(a,0))) = \Phi_2(a, \Psi_1(a)) = \Psi_1(a)$$

for all $a \in \mathbb{R}^{d_s}$. In general, $\Psi_2^{-1}(K_{r,2})$ needs not be compact. However,

$$\mathfrak{K}_{r,2} = \left\{(a,b) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} : a \in \mathfrak{K}_{r,1} \text{ and } \mathrm{dist}_{\mathcal{W}^u}(\Psi_2(a,0), \Psi_2(a,b)) \le r\right\}$$

is compact and satisfies $\Psi_2(\mathfrak{K}_{r,2}) = K_{r,2}$. Repeating this procedure, we construct continuous maps

$$\Psi_j : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times \cdots \times \mathbb{R}^{d_*} \to M$$

(there are j factors, and so $* = u$ if j is even and $* = s$ if j is odd), contractible sets E_j, and compact sets $\mathfrak{K}_{r,j}$ such that each Ψ_j is a continuous extension of Ψ_{j-1}, in the previous sense, and $\Psi_j(\mathfrak{K}_{r,j}) = K_{r,j}$. We stop this procedure for $j = n$. The corresponding map Ψ_n is the non-injective parametrization announced in (8.3).

8.4. **Selection of nearby access sequences.** Now we prove Proposition 8.3. We need the following general fact about regular values of continuous functions.

Definition 8.6. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map between topological spaces $\mathcal{A}$ and $\mathcal{B}$. A point $x \in \mathcal{A}$ is *regular* for Φ, if for every neighborhood $\mathcal{V}$ of x we have $\Phi(x) \in \Phi(\mathcal{V})^\circ$. A point $y \in B$ is a *regular value* of Φ if every point of $\Phi^{-1}(y)$ is regular.

Proposition 8.7. *Let $\mathcal{A}$ be a compact metrizable space and $\mathcal{B}$ a locally compact Hausdorff space. If $\Phi : \mathcal{A} \to \mathcal{B}$ is continuous then the set of regular values of Φ is residual.*

Proof. We are going to prove that the image of the set of non-regular points is meager. The assumptions imply that $\mathcal{A}$ admits a countable base $\mathcal{T}$ of open sets, and the map Φ is closed. If x is a non-regular point of Φ, then there exists $\mathcal{V} \in \mathcal{T}$ such that $\Phi(x)$ does not belong to the interior of $\Phi(\overline{\mathcal{V}})$. Therefore, $\Phi(x)$ belongs to the closed set $\partial\Phi(\overline{\mathcal{V}})$, which has empty interior because $\Phi(\overline{\mathcal{V}})$ is closed. Then, the image of non-regular points is a subset of the meager set $\bigcup\left\{\partial\Phi(\overline{\mathcal{V}}) : \mathcal{V} \in \mathcal{T}\right\}$. □

We apply this proposition to the continuous map $\Psi_n : \mathfrak{K}_{r,n} \to K_{r,n}$. Recall that, by construction, the image $K_{r,n}$ has non empty interior. Then, in particular, Ψ_n has some regular value $w \in K_{r,n}$. Let $(a_1, \ldots, a_n) \in \mathfrak{K}_{r,n}$ be any point in $\mathfrak{K}_{r,n}$ such that $\Psi_n(a_1, \ldots, a_n) = w$. Let $\varepsilon > 0$ be as in the statement of the proposition. Since the functions Ψ_1, Ψ_2, ..., Ψ_n are continuous, there exists $\rho > 0$ such that if $|a_j - b_j| < \rho$, for $j = 1, \ldots, n$, then

$$\mathrm{dist}(\Psi_j(a_1, \ldots, a_j), \Psi_j(b_1, \ldots, b_j)) < \varepsilon \tag{8.5}$$

for all $j = 1, \dots, n$. Using that the point $(a_1, \dots, a_n)$ is regular (Definition 8.6), we get that the image $\Psi_n(V)$ of the neighborhood

$$V = \mathfrak{K}_{r,n} \cap \{(b_1, \dots, b_n) : |a_j - b_j| < \rho, \text{ for } j = 1, \dots, n\}$$

has w in its interior. In other words, there exists $\delta > 0$ such that $B(w, \delta) \subset \Psi_n(V)$. Consider any point $z \in B(w, \delta)$. Then there exists $(b_1(z), \dots, b_n(z)) \in V$ such that $z = \Psi_n(b_1(z), \dots, b_n(z))$. Define

$$y_j(z) = \Psi_j(b_1(z)), \dots, y_j(z))$$

for $j = 1, \dots, n$, and $y_0(z) = w$. Then $[y_1(z), \dots, y_n(z)]$ is an access sequence connecting x_0 to z. The inequalities (8.5) mean that

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \text{for } j = 1, \dots, n.$$

Moreover, since $\Psi_n(b_1(z), \dots, b_n(z)) \in K_{r,n}$, the distance between every $y_{j-1}(z)$ and $y_j(z)$ along their common strong (stable or unstable) leaf does not exceed r. Proposition 8.3 follows taking $L = r$ and $N = n$.

9. Generic linear cocycles over partially hyperbolic maps

In this section we prove Theorem A. Let us begin by giving an outline of the proof. We take the vector bundle to be trivial $\mathcal{V} = M \times \mathbb{K}^d$. This greatly simplifies the presentation, but is not really necessary for our arguments, which are mostly local: for obtaining the conclusion we consider modifications of the cocycle supported in a neighborhood of certain special points (the pivots, see Proposition 9.8), where triviality holds anyway, by definition. Let $\mathbb{K}_x = \{x\} \times \mathbb{K}^d$ be the fiber of $\mathcal{V}$ and $\mathbb{P}(\mathbb{K}_x) = \{x\} \times \mathbb{P}(\mathbb{K})$ be the fiber of the projective bundle $\mathbb{P}(\mathcal{V})$ over the point x. We call *loop* of $f : M \to M$ at $x \in M$ any access sequence $\gamma = [y_0, \dots, y_n]$ connecting a point $x \in M$ to itself, that is, such that $y_0 = y_n = x$. Then we denote

$$H_\gamma = H^{*_n}_{y_{n-1},y_n} \circ \cdots \circ H^{*_j}_{y_{j-1},y_j} \circ H^{*_1}_{y_0,y_1} : \mathbb{P}(\mathbb{K}_x) \to \mathbb{P}(\mathbb{K}_x)$$

where $*_j \in \{s, u\}$ is the symbol of the strong leaf common to the nodes y_{j-1} and y_j. Theorem B(b) implies that if $\lambda_+(F) = \lambda_-(F)$ then any F-invariant probability measure m that projects down to μ admits a disintegration $\{m_z : z \in M\}$ such that

$$(H_\gamma)_* m_x = m_x \quad \text{for any loop } \gamma. \tag{9.1}$$

We consider *loops with slow recurrence*, for which some node y_r, that we call *pivot*, is slowly accumulated by the orbits of all the nodes including its own. Using perturbations of the cocycle supported on a small neighborhood of the pivot, we prove that the map $F \mapsto H_\gamma$ assigning to each cocycle the corresponding holonomy over the loop is a submersion. In fact, we are able to consider several independent loops with slow recurrence, $\gamma_1, \dots, \gamma_m$, and prove that the map

$$F \mapsto (H_{\gamma_1}, \dots, H_{\gamma_m})$$

is a submersion. Consequently, for typical cocycles, the matrices H_{γ_i} are in general position, and so they have no common invariant probability in the projective space. This shows that for typical cocycles the condition (9.1) fails and, hence, the extremal Lyapunov exponents are distinct.

Let us also point out that these arguments extend, more or less directly, to $\mathrm{SL}(d, \mathbb{K})$-valued cocycles (see Remarks 9.9 and 9.15), so that the statement of the theorem remains valid restricted to the subspace $\mathcal{S}^{r,\alpha}(M, d, \mathbb{K})$ of cocycles with

$\det F_x = 1$ at every point. It would be interesting to investigate the case of G-valued cocycles for more general subgroups of $\mathrm{GL}(d,\mathbb{K})$, for instance the symplectic group.

9.1. **Accessibility with slow recurrence.** An important step is to prove that loops with slow recurrence do exist. Beforehand, let us give the precise definition.

Definition 9.1. A family $\{\gamma_1,\dots,\gamma_m\}$ of loops $\gamma_i = [y_0^i,\dots,y_{n(i)}^i]$ has *slow recurrence* if there exists $c>0$ and for each $1\le i\le m$ there exists $0<r(i)<n(i)$ such that, for all $i,l=1,\dots,m$, all $0\le j\le n(i)$, and all $k\in\mathbb{Z}$,

$$\operatorname{dist}\big(f^k(y_j^i), y_{r(l)}^l\big) \ge c/(1+k^2)$$

with the exception of $k=0$ when $(i,j)=(l,r(l))$.

It is convenient to distinguish access sequences $[y_0,y_1,\dots,y_n]$ according to the nature of the last leg: we speak of accessibility *s-sequence* if y_{n-1} and y_n belong to the same strong-stable leaf, and we speak of accessibility *u-sequence* if y_{n-1} and y_n belong to the same strong-unstable leaf. Let d_s and d_u be the dimensions of the strong-stable leaves and strong-unstable leaves, respectively.

Proposition 9.2. *For any $m\ge 1$ and any $(x_1,\dots,x_m)\in M^m$, there exists a family γ_i of loops with slow recurrence, where each γ_i is a loop at x_i.*

The proof of this proposition requires a number of preparatory results.

Lemma 9.3. *Given any finite set $\{w_1,\dots,w_n\}\subset M$, any $y\in M$, and any symbol $*\in\{s,u\}$, there exists a full Lebesgue measure subset of points $w\in\mathcal{W}^*_{\mathrm{loc}}(y)$ such that*

$$\operatorname{dist}(f^k(w_j),w) \ge c/(1+k^2) \tag{9.2}$$

for some $c>0$ and for all $1\le j\le n$ and all $k\in\mathbb{Z}$.

Proof. Consider $*=s$: the case $*=u$ is analogous. Since local strong-stable leaves are a continuous family of C^2 embedded disks, there exists a constant $D_1>0$ such that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y)}\big(\mathcal{W}^s_{\mathrm{loc}}(y)\cap B(z,c/(1+k^2))\big) \le D_1(c/(1+k^2))^{d_s}$$

for any $z\in M$. Thus, the Lebesgue measure of the subset of points $w\in\mathcal{W}^s_{\mathrm{loc}}(y)$ not satisfying inequality (9.2) for some fixed $c>0$ is bounded by

$$\sum_{j=1}^n\sum_{k\in\mathbb{Z}} D_1 c^{d_s}(1+k^2)^{-d_s} \le D_2\, c^{d_s} \quad \text{with } D_2 = nD_1\sum_{k\in\mathbb{Z}}(1+k^2)^{-d_s} < \infty.$$

Making $c\to 0$, we conclude that the inequality (9.2) is indeed satisfied by Lebesgue almost every point in $\mathcal{W}^s_{\mathrm{loc}}(y)$. $\square$

Corollary 9.4. *Given any $m\ge 1$, any $(x_1,\dots,x_m)\in M^m$, and any $*\in\{s,u\}$, then for every $(z_1,\dots,z_m)$ in a full Lebesgue measure subset of M^m there exist $c>0$ and accessibility $*$-sequences $[y_0^i,\dots,y_{n(i)}^i]$ connecting x_i to z_i such that*

$$\operatorname{dist}(f^k(y_j^i),z_l) \ge c/(1+k^2)$$

for all $i,l=1,\dots,m$, all $0\le j<n(i)$, and all $k\in\mathbb{Z}$.

Proof. Consider $* = s$: the case $* = u$ is analogous. Since the strong-stable foliation is absolutely continuous, it suffices to prove that, given any points $y_i \in M$, $1 \le i \le m$, the conclusion holds on a full Lebesgue measure subset of points $z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)$, $1 \le i \le m$. Now, by the accessibility assumption, there exist accessibility sequences $[y^i_0, \dots, y^i_{r(i)}]$ connecting x_i to y_i. Consider each z_i in the full Lebesgue measure subset of $\mathcal{W}^s(y_i)$ given by Lemma 9.3, applied to the finite set

$$\left\{y^i_j : 1 \le i \le m \text{ and } 0 \le j \le r(i)\right\}.$$

and the point $y = y_i$. Then the accessibility s-sequences $[y^i_0, \dots, y^i_{k(i)}, z_i]$ satisfy the conditions in the conclusion. In view of the observation at the beginning, this proves the corollary. □

Lemma 9.5. *For any $m \ge 1$ and any $(y_1, \dots, y_m) \in M^m$, there exists a full Lebesgue measure subset of $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \ge c/(1 + k^2)$$

for some $c > 0$ and for all $i, l = 1, \dots, m$ and all $k \ge 0$, except $k = 0$ when $i = l$. The statement remains true if one replaces $\mathcal{W}^s_{\mathrm{loc}}$ by $\mathcal{W}^u_{\mathrm{loc}}$ and $k \ge 0$ by $k \le 0$.

Proof. It is clear that each strong-stable leaf contains at most one periodic point. As an easy consequence we get that, that given any $\kappa \ge 1$, there exists a full Lebesgue measure subset of $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that $f^k(z_i) \ne z_l$ for all $i, l = 1, \dots, m$ and all $0 \le k < \kappa$, except $k = 0$ when $i = l$. Then the condition in the statement holds, for some $c > 0$, restricted to iterates $0 \le k < \kappa$. Let us focus on $k \ge \kappa$. For each $i, l = 1, \dots, m$, define

$$E^k_{i,l} = \left\{z \in \mathcal{W}^s_{\mathrm{loc}}(y_l) : \operatorname{dist}(f^k(z_i), z_l) < 1/(1 + k^2) \text{ for some } z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)\right\}.$$

The diameter of $f^k(\mathcal{W}^s_{\mathrm{loc}}(y_i))$ is bounded by $C_1\theta^k$, where $C_1 > 0$ is some uniform constant and $\theta < 1$ is an upper bound for the contraction function $\nu(x)$ in (2.1). Consequently,

$$\operatorname{diam}(E^k_{i,l}) \le C_1\theta^k + 2/(1 + k^2) \le C_2/(1 + k^2)$$

for another uniform constant $C_2 > 0$. It follows that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y_l)}\Big(\bigcup_{i=1}^{m} \bigcup_{k=\kappa}^{\infty} E^k_{i,l}\Big) \le m \sum_{k=\kappa}^{\infty} C_2(1 + k^2)^{-d_s}.$$

On the one hand, the right hand side of this expression goes to 0 when κ goes to infinity. On the other hand, in view of our previous observations, for any $\kappa \ge 1$, Lebesgue almost every $(z_1, \dots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ with

$$z_l \notin \bigcup_{i=1}^{m} \bigcup_{k=\kappa}^{\infty} E^k_{i,l}$$

satisfies the conclusion of the lemma for some $c \in (0, 1)$. This proves that the subset of $(z_1, \dots, z_m)$ for which the conclusion of the lemma does not hold has zero Lebesgue measure, as claimed. □

Corollary 9.6. *For any $m \ge 1$, and every $(z_1, \dots, z_m)$ in a full Lebesgue measure subset of M^m, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \ge c/(1 + k^2)$$

for all $i, l = 1, \ldots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$.

Proof. It suffices to prove that the conditions obtained replacing $k \in \mathbb{Z}$ by either $k \geq 0$ or $k \leq 0$ are satisfied on full Lebesgue measure subsets of M^m, and then take the intersection of these two subsets. We consider the case $k \geq 0$, as the other one is analogous. Suppose there is a positive Lebesgue measure subset of $(z_1, \ldots, z_m) \in M^m$ for which the condition is *not* satisfied: the forward orbit of some z_i accumulates some z_l faster than $c/(1+k^2)$ for any $c > 0$. Then, since M is covered by the foliation boxes of the strong-stable foliation, there exist foliation boxes U_i, $1 \leq i \leq m$ such that this exceptional subset intersects $U = U_1 \times \cdots \times U_m$ on a positive Lebesgue measure subset. The domain U is foliated by the products $\mathcal{W}^s_{\text{loc}}(y_1) \times \cdots \times \mathcal{W}^s(y_m)$ of local strong-stable leaves. We denote this foliation as $\mathcal{W}^{s,m}$. Given any holonomy maps $h_i : \Sigma_i^1 \to \Sigma_i^2$ between cross-sections to the strong-stable foliation $\mathcal{W}^s$ inside U_i, the products $\Sigma^j = \Sigma_1^j \times \cdots \times \Sigma_m^j$ are cross-sections to $\mathcal{W}^{s,m}$, and the holonomy map of $\mathcal{W}^{s,m}$ is

$$h : \Sigma^1 \to \Sigma^2, \quad h(z_1, \ldots, z_m) = (h_1(z_1), \ldots, h_m(z_m)).$$

Since all the h_i are absolutely continuous, so is h: the Jacobians are related by $Jh(z_1, \ldots, z_m) = Jh_1(z_1) \cdots Jh_m(z_m)$. This absolute continuity property implies that every positive Lebesgue measure subset of U intersects $\mathcal{W}^s_{\text{loc}}(y_1) \times \cdots \mathcal{W}^s_{\text{loc}}(y_m)$ on a positive Lebesgue measure subset, for a subset of $(y_1, \ldots, y_m)$ with positive Lebesgue measure. In particular, the exceptional set intersects some leaf of $\mathcal{W}^{s,m}$ on a positive Lebesgue measure subset. This contradicts Lemma 9.5, and this contradiction proves the corollary. □

Corollary 9.7. *For any $m \geq 1$, any $(x_1, \ldots, x_m) \in M^m$, and any $* \in \{s, u\}$, and a full Lebesgue measure set D_* of $(z_1, \ldots, z_m) \in M^m$, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1+k^2) \tag{9.3}$$

for all $i, l = 1, \ldots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$, and there exist accessibility $*$-sequences $[y_0^i, \ldots, y_{n(i)}^i]$ connecting x_i to z_i, for $1 \leq i \leq m$ such that

$$\operatorname{dist}(f^k(y_j^i), z_l) \geq c/(1+k^2) \tag{9.4}$$

for all $i, l = 1, \ldots, m$, all $0 \leq j < n(i)$, and all $k \in \mathbb{Z}$.

Proof. Just take the intersections of the full Lebesgue measure subsets given in Corollary 9.4, for $* \in \{s, u\}$, and in Corollary 9.6. □

Proof of Proposition 9.2. Given $m \geq 1$ and $(x_1, \ldots, x_m) \in M^m$, let D_s and D_u be the full Lebesgue measure sets given by Corollary 9.7, and then consider

$$(z_1, \ldots, z_m) \in D_s \cap D_u \, .$$

The corollary yields, for each $1 \leq i \leq m$, an accessibility s-sequence $[y_0^i, \ldots, y_{r(i)}^i]$ and an accessibility u-sequence $[w_0^i, \ldots, w_{t(i)}^i]$ connecting x_i to z_i. Then

$$\gamma_i = [y_0^i, \ldots, y_{r(i)}^i = w_{t(i)}^i, \ldots, w_0^i]$$

is a loop at x_i, and properties (9.3)-(9.4) mean that the family $\{\gamma_1, \ldots, \gamma_m\}$ of loops has slow recurrence. □

9.2. **Holonomies on loops with slow recurrence.** As we pointed out before, the tangent space at each point $B \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is naturally identified with the Banach space of $C^{r,\alpha}$ maps from M to the space of linear maps in $\mathbb{K}^d$. This means that we may view the tangent vectors $\dot{B}$ as $C^{r,\alpha}$ functions assigning to each $z \in M$ a linear map $\dot{B}(z) : \mathbb{K}_z \to \mathbb{K}_{f(z)}$.

Let $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ fiber bunched. As we gave seen in Section 3.2, there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that every $B \in \mathcal{U}$ is fiber bunched. Then, for any loop $\gamma = [y_0, \dots, y_n]$ at a point $x \in M$, and any $0 \le k < l \le n$, we have linear holonomy maps

$$H_{B,\gamma,k,l} = H^{*_l}_{B,y_{l-1},y_l} \circ \dots \circ H^{*_{k+1}}_{B,y_k,y_{k+1}} : \mathbb{K}_{y_k} \to \mathbb{K}_{y_l}.$$

Furthermore, all the maps $B \mapsto H_{B,\gamma,k,l}$ are C^1 on $\mathcal{U}$. In particular, the derivative of $B \mapsto H_{B,\gamma} = H_{B,\gamma,0,n}$ is given by

$$\partial_B H_{B,\gamma} : \dot{B} \mapsto \sum_{l=1}^{n} H_{B,\gamma,l,n} \big[\partial_B H_{B,\gamma,l-1,l}(\dot{B})\big] H_{B,\gamma,0,l-1}. \tag{9.5}$$

The main result in this section is

Proposition 9.8. *Let $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ be fiber bunched and $\mathcal{U}$ be a neighborhood as above. For each $x \in M$ and $m \ge 1$, let $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence. Then*

$$\mathcal{U} \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_m}) \in \mathrm{GL}(d,\mathbb{K}_x)^m$$

is a submersion: the derivative is surjective at every point, even restricted to the subspace of tangent vectors $\dot{B}$ supported on a small neighborhood of the pivots.

In the proof we use (9.5) together with the expressions for the $\partial_B H_{B,\gamma,l-1,l}(\dot{B})$ given in Propositions 3.7 and 3.9. The idea is quite simple. Perturbations in the neighborhood of the pivots affect the holonomies over all the loop legs, of course. However, Corollaries 3.8 and 3.10 show that the effect decreases exponentially fast with time, and slow recurrence means that the first iterates need not be considered. Combining these two ideas one shows (Corollary 9.12) that the derivative is a small perturbation of its term of order zero. The latter is easily seen to be surjective (Lemma 9.13), and then the same is true for any small perturbation.

Remark 9.9. Essentially the same arguments yield an $\mathrm{SL}(d,\mathbb{K})$-version of this proposition: the map $\mathcal{U} \cap \mathcal{S}^{r,\alpha}(M,d,\mathbb{K}) \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_m}) \in \mathrm{SL}(d,\mathbb{K}_x)^m$ is a submersion. Clearly, it remains true that the derivative is a small perturbation of its term of order zero. Then the main point is to observe that the restriction of the operator S in Lemma 9.13 maps $T_B\mathcal{S}^{r,\alpha}(M,d,\mathbb{K})$ surjectively to $T_{H_{B,\gamma}}\,\mathrm{SL}(d,\mathbb{K}_x)$.

Before getting into the details, let us make an easy observation that allows for some simplification of our notations. If $\gamma = [y_0, \dots, y_n]$ is a loop with slow recurrence then so is $\bar{\gamma} = [y_n, \dots, y_0]$, and $H_{B,\bar{\gamma}}$ is the inverse of $H_{B,\gamma}$. Hence, the statement of the proposition is not affected if one reverses the orientation of any γ_i as described. So, it is no restriction to suppose that every loop γ has the orientation for which the pivot y_r satisfies

$$y_r \in \mathcal{W}^s(y_{r-1}) \cap \mathcal{W}^u(y_{r+1}), \tag{9.6}$$

and we do so in all that follows.

Lemma 9.10. *Let $\gamma = [y_0, \dots, y_n]$ be a loop with slow recurrence and y_r be the corresponding pivot. Then, there is $\tau > 0$ such that for any small $\varepsilon > 0$ and any tangent vector $\dot{B}$ supported on $B(y_r, \varepsilon)$,*

$$\|\partial_B H_{B,\gamma,l-1,l}(\dot{B})\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \textit{for any } l \neq r, \textit{ and}$$
$$\|\partial_B H_{B,\gamma,r-1,r}(\dot{B}) + B(y_r)^{-1}\dot{B}(y_r)H^s_{B,y_{r-1},y_r}\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}.$$

Proof. By Definition 9.1, there exists $c > 0$ such that

$$\operatorname{dist}(f^k(y_l), y_r) \ge c/(1+k^2) \quad \text{for all } (l,k) \in \{0,\dots,n\} \times \mathbb{Z},\ (l,k) \neq (r,0).$$

Consider $\varepsilon < c$. Then $B(y_r, \varepsilon)$ contains no other node of the loop. Moreover, for any $0 \le l \le n$ and any $k \ge 1$,

$$f^k(y_l) \in B(y_r, \varepsilon) \Longrightarrow |k| \ge t(\varepsilon), \quad \text{where } t(\varepsilon) = \sqrt{c/\varepsilon - 1}.$$

Let us denote by $\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot{B})$ the t-tail of the derivative, that is, the sum over $i \ge t$ in Proposition 3.7 (case $*_l = s$) or Proposition 3.9 (case $*_l = u$). Then, for any $\dot{B} \in T_B\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ supported in $B(y_r, \varepsilon)$, the expression in Proposition 3.7 becomes

$$\partial_B H_{B,\gamma,l-1,l}(\dot{B}) = \partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot{B}) \tag{9.7}$$

for all $l \neq r$, and

$$\partial_B H_{B,\gamma,r-1,r}(\dot{B}) = -B(y_r)^{-1}\dot{B}(y_r)H^s_{B,y_{r-1},y_r} + \partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot{B}) \tag{9.8}$$

for $l = r$. This applies to the loop legs with symbol $*_l = s$. Observing that the sum in Proposition 3.9 does not include the term $i = 0$, we conclude that (9.7) extends to all loop legs with symbol $*_l = u$. Next, by Corollaries 3.8 and 3.10,

$$\|\partial_B H_{B,\gamma,l-1,l,t}(\dot{B})\| \le C_5(a)\,\theta^t\,\|\dot{B}\|_{0,\beta}, \tag{9.9}$$

for every $1 \le l \le n$ and any $t \ge 0$, where a is an upper bound for the distances between consecutive loop nodes. Choose any $\tau < c$. The lemma follows directly from (9.7), (9.8), (9.9) with $t = t(\varepsilon)$, because $t(\varepsilon) < \sqrt{\tau/\varepsilon}$ for all small $\varepsilon > 0$. $\square$

Corollary 9.11. *Let $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence and $y_{r(i)}$, $1 \le i \le m$ be the corresponding pivots. Then there exists $\tau > 0$ such that, for any small $\varepsilon > 0$, any $1 \le j \le m$, and any tangent vector $\dot{B}$ supported on $B(y_r^j, \varepsilon)$, $r = r(j)$*

$$\|\partial_B H_{B,\gamma_i,l-1,l}(\dot{B})\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \textit{for all } (i,l) \neq (j,r), \textit{ and}$$
$$\|\partial_B H_{B,\gamma_j,r-1,r}(\dot{B}) + B(y_r^j)^{-1}\dot{B}(y_r^j)H^s_{B,y_{r-1}^j,y_r^j}\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}.$$

Proof. The case $i = j$ is contained in Lemma 9.10. The cases $i \neq j$ follow from the same arguments, observing that

$$\operatorname{dist}(f^k(y_l^i), y_r^j) \ge c/(1+k^2) \quad \text{for every } k \in \mathbb{Z}$$

and so $f^k(y_l^i) \in B(y_r^j, \varepsilon)$ implies $|k| \ge t(\varepsilon)$, for every $0 \le l \le n(i)$. $\square$

Corollary 9.12. *Let $\gamma_i = [y_0^i, y_1^i, \dots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence, and $y_{r(i)}$, $1 \le i \le m$ be the corresponding pivots. Then, there* *exists* $K_1 > 0$ *such that, for any small* $\varepsilon > 0$*, any* $1 \le j \le m$*, and any tangent vector* $\dot B$ *supported on* $B(y_r^j, \varepsilon)$*,* $r = r(j)$

$$\|\partial_B H_{B,\gamma_i}(\dot B)\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot B\|_{0,\beta} \quad \textit{for all } i \ne j, \textit{ and}$$

$$\|\partial_B H_{B,\gamma_j}(\dot B) + H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1} \dot B(y_r^j) H_{B,\gamma_j,0,r}\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot B\|_{0,\beta}$$

Proof. This follows from replacing in (9.5) the estimates in Corollary 9.11. By Proposition 3.4(5), the factors $H_{B,\gamma_i,0,l-1}$ and $H_{B,\gamma_i,l,n(i)}$ are bounded by some uniform constant K_2 that depends only on the loops. Then, for every $i \ne j$, Corollary 9.11 and the relation (9.5) gives

$$\|\partial_B H_{B,\gamma_i}(\dot B)\| \le \sum_{l=1}^{n(i)} K_2^2 \|\partial_B H_{B,\gamma,l-1,l}(\dot B)\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot B\|_{0,\beta},$$

as long as we choose $K_1 \ge K_2^2 \max_i n(i)$. This gives the first part of the corollary. Now we consider $i = j$. For the same reasons as before, all but one term in the expression (9.5) are bounded by $K_2^2 \theta^{\sqrt{\tau/\varepsilon}} \|\dot B\|_{0,\beta}$. The possible exception is

$$H_{B,\gamma_j,r,n(j)} \left[\partial_B H_{B,\gamma_j,r-1,r}(\dot B)\right] H_{B,\gamma_j,0,r-1},$$

corresponding to $l = r$. By Corollary 9.11, this last expression differs from

$$\begin{aligned} - H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1} \dot B(y_r^j) H^s_{B,y_{r-1}^j,y_r^j} H_{B,\gamma_j,0,r-1} = \\ - H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1} \dot B(y_r^j) H_{B,\gamma_j,0,r} \end{aligned}$$

by a term bounded by $K_2^2 \theta^{\sqrt{\tau/\varepsilon}} \|\dot B\|_{0,\beta}$. This completes the proof. □

Lemma 9.13. *Let* $\gamma = [y_0, \dots, y_n]$ *be a loop at* $x \in M$ *and* $0 < r < n$ *be fixed. Then the linear map*

$$\begin{array}{rccc} S: & T_B \mathcal{G}^{r,\alpha}(M,d,\mathbb{K}) & \to & T_{H_{B,\gamma}} \mathrm{GL}(d,\mathbb{K}_x) \simeq \mathcal{L}(\mathbb{K}_x^d, \mathbb{K}_x^d) \\ & \dot B & \mapsto & -H_{B,\gamma,r,n} B(y_r)^{-1} \dot B(y_r) H_{B,\gamma,0,r} \end{array}$$

is surjective, even restricted to the subspace of tangent vectors $\dot B$ *vanishing outside some neighborhood of* y_r*. More precisely, there exists* $K_3 > 0$ *such that for* $0 < \varepsilon < 1$ *and* $\Theta \in \mathcal{L}(\mathbb{K}^d, \mathbb{K}^d)$ *there exists* $\dot B_\Theta \in T_B \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ *vanishing outside* $B(y_r, \varepsilon)$ *and such that* $S(\dot B_\Theta) = \Theta$ *and* $\|\dot B_\Theta\|_{0,\beta} \le K_3 \varepsilon^{-\beta} \|\Theta\|$*.*

Proof. Let $\tau : M \to [0,1]$ be a $C^{r,\alpha}$ function vanishing outside $B(y_r, \varepsilon)$ and such that $\tau(y_r) = 1$ and the Hölder constant $H_\beta(\tau) \le 2\varepsilon^{-\beta}$. For $\Theta \in \mathcal{L}(\mathbb{K}^d, \mathbb{K}^d)$, define $\dot B_\Theta \in T_B \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ by

$$\dot B_\Theta(w) = B(y_r)\, H_{B,\gamma,r,n}^{-1}\, \Theta\, B(y_r)^{-1}\, \tau(w)\, B(w)\, H_{B,\gamma,0,r}^{-1}.$$

Notice that $\dot B_\Theta(y_r) = B(y_r)\, H_{B,\gamma,r,n}^{-1}\, \Theta\, H_{B,\gamma,0,r}^{-1}$ and so $S(\dot B_\Theta) = \Theta$. Moreover,

$$\|\dot B_\Theta\|_{0,0} \le \|H_{B,\gamma,r,n}^{-1}\|\, \|H_{B,\gamma,0,r}^{-1}\|\, \|B(y_r)\|\, \|B(y_r)^{-1}\|\, \|B\|_{0,0}\, \|\Theta\|. \tag{9.10}$$

For any $w_1, w_2 \in M$ the norm of $\dot B_\Theta(w_1) - \dot B_\Theta(w_2)$ is bounded by

$$\begin{aligned} &\|H_{B,\gamma,r,n}^{-1}\|\, \|H_{B,\gamma,0,r}^{-1}\|\, \|B(y_r)\|\, \|B(y_r)^{-1}\| \\ &\qquad \Big(\|\tau(w_1) - \tau(w_2)\| \|B(w_1)\| + |\tau(w_2)| \|B(w_1) - B(w_2)\| \Big)\, \|\Theta\|. \end{aligned}$$

Consequently, the Hölder constant $H_\beta(\dot B_\Theta)$ of $\dot B_\Theta$ is bounded above by

$$\|H_{B,\gamma,r,n}^{-1}\| \, \|H_{B,\gamma,0,r}^{-1}\| \, \|B(y_r)\| \, \|B(y_r)^{-1}\| \, \big(2\varepsilon^{-\beta}\|B\|_{0,0} + H_\beta(B)\big) \, \|\Theta\|. \tag{9.11}$$

Adding the inequalities (9.10) and (9.11), and taking

$$K_3 = \|H_{B,\gamma,r,n}^{-1}\| \, \|H_{B,\gamma,0,r}^{-1}\| \, \|B(y_r)\| \, \|B(y_r)^{-1}\| \, \|B\|_{0,\beta},$$

one obtains $\|\dot B_\Theta\|_{0,\beta} \le K_3 e^{-\beta}\|\Theta\|$. □

Proof of Proposition 9.8. For each $1 \le j \le m$, let S_j be the operator associated to $\gamma = \gamma_j$ as in Lemma 9.13. Let Θ_j be any element of the unit sphere in $\mathcal{L}(\mathbb{K}_x, \mathbb{K}_x)$. By Lemma 9.13, for any small $\varepsilon > 0$ there exists a tangent vector $\dot B(j, \Theta_j)$ supported in $B(y^j_{r(j)}, \varepsilon)$ such that

$$S_j\big(\dot B(j,\Theta_j)\big) = \Theta_j \quad \text{and} \quad \|\dot B(j,\Theta_j)\| \le K_3 e^{-\beta}.$$

By Corollary 9.12, the norm of

$$(\partial_B H_{B,\gamma_1}, \dots, \partial_B H_{B,\gamma_j}, \dots, \partial_B H_{B,\gamma_m})(\dot B) - (0, \dots, 0, S_j(\dot B), 0, \dots, 0)$$

is bounded above by $K_3 \theta^{\sqrt{\tau/\varepsilon}}\|\dot B\|$, for any tangent vector supported in $B(y^j_{r(j)}, \varepsilon)$. For $\dot B = \dot B(j, \Theta_j)$ this gives that

$$\|(\partial_B H_{B,\gamma_1}, \dots, \partial_B H_{B,\gamma_j}, \dots, \partial_B H_{B,\gamma_m})(\dot B(j,\Theta_j)) - (0, \dots, 0, \Theta_j, 0, \dots, 0)\|$$

is bounded by $K_1 K_3 \theta^{\sqrt{\tau/\varepsilon}} e^{-\beta}$. Assume $\varepsilon > 0$ is small enough so that

$$K_1 K_3 \theta^{\sqrt{\tau/\varepsilon}} e^{-\beta} < 1/(2m).$$

Then for any $\Theta = (\Theta_1, \dots, \Theta_m)$ with Θ_j in the unit sphere of $\mathcal{L}(\mathbb{K}_x, \mathbb{K}_x)$ we find a tangent vector $\dot B(\Theta) = \sum_{j=1}^m \dot B(j, \Theta_j)$ supported on the ε-neighborhood of the pivots and such that

$$\|\big(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m}\big)\big(\dot B(\Theta)\big) - \Theta\| < 1/2.$$

This implies that the image of the derivative $(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m})$ is the whole target space $\mathcal{L}(\mathbb{K}^d_x, \mathbb{K}^d_x)^m$, as claimed. □

9.3. **Invariant measures of generic matrices.** Finally, we prove Theorem A. The only missing ingredient is

Proposition 9.14. *Given $\ell \ge 1$, let $G_{2\ell}$ be the set of $(\mathtt{A}_1, \dots, \mathtt{A}_{2\ell}) \in \mathrm{GL}(d, \mathbb{K})^{2\ell}$ such that there exists some probability η in $\mathbb{P}(\mathbb{C})$ invariant under the action of $\mathtt{A}_i$ for every $1 \le i \le 2\ell$. Then $G_{2\ell}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\ge \ell$.*

Remark 9.15. The arguments that we are going to present remain valid if one replaces $\mathrm{GL}(d, \mathbb{K})$ by the subgroup $\mathrm{SL}(d, \mathbb{K})$ of matrices with determinant 1: just note that the curves $\mathtt{B}(t)$ defined in (9.13) and (9.17) lie in $\mathrm{SL}(d, \mathbb{K})$ if the initial matrix $\mathtt{A}$ does. Thus, the proposition holds for $\mathrm{SL}(d, \mathbb{K})$ as well.

Let us assume this proposition for a while, and use it to conclude the proof of the theorem in the complex case. Let $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ be fiber bunched. Fix any $\ell \ge 1$ and $x \in M$. By Proposition 9.2 there is a family γ_i, $1 \le i \le 2\ell$, of loops at x with slow recurrence. By Proposition 9.8, the map

$$\mathcal{U} \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_{2\ell}}) \in \mathrm{GL}(d, \mathbb{K}_x)^{2\ell}$$

is a submersion, where $\mathcal{U}$ is a neighborhood of A independent of ℓ. Let $\mathcal{Z}$ be the pre-image of $G_{2\ell}$ under this map. Then $\mathcal{Z}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$.

We claim that $\lambda_-(B,\mu) < \lambda_+(B,\mu)$ for all $B \in \mathcal{U} \setminus \mathcal{Z}$. Indeed, suppose the equality holds, and let m be any $\mathbb{P}(F_B)$-invariant probability that projects down to μ. By Theorem B, the measure m admits a disintegration $\{m_z : z \in M\}$ which is invariant under strong-stable holonomies $h^s = \mathbb{P}(H^s)$ and strong-unstable holonomies $h^u = \mathbb{P}(H^u)$, on the whole manifold M. In particular,

$$\mathbb{P}(H_{B,\gamma_i})_* m_x = m_x \quad \text{for every } 1 \leq i \leq 2\ell. \tag{9.12}$$

This contradicts the definition of $G_{2\ell}$, and this contradiction proves our claim. Let $\mathcal{Z}_0$ be the set of fiber bunched $B \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ for which $\lambda_-(B,\mu) = \lambda_+(B,\mu)$. We have shown that any fiber bunched $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ admits a neighborhood $\mathcal{U}$ such that, for any $\ell \geq 1$, there exists a nowhere dense subset $\mathcal{Z}$ of $\mathcal{U}$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ and such that $\mathcal{Z}_0 \cap \mathcal{U} \subset \mathcal{Z}$. Thus, the closure of $\mathcal{Z}_0$ has infinite codimension and, in particular, is nowhere dense.

The proof of Theorem A has been reduced to proving Proposition 9.14. The proof of the proposition is presented in the next two sections.

9.3.1. *Complex case.* Let S be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d,\mathbb{C})$ whose eigenvalues are all distinct in norm. Then, S is an open and dense subset of $\mathrm{GL}(d,\mathbb{C})$ whose complement is contained in a finite union of closed manifolds of positive codimension. We use the following fact about variation of eigenvectors inside S:

Lemma 9.16. *Let $\mathtt{A} \in S$. Then there exist C^∞ functions $\lambda_i : S_{\mathtt{A}} \to \mathbb{C}$ and $v_i : S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)$ defined on an open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$, for each $1 \leq i \leq d$, such that $v_i(\mathtt{B})$ is the direction of an eigenvector of $\mathtt{B}$ associated to the eigenvalue $\lambda_i(\mathtt{B})$, for any $B \in S_{\mathtt{A}}$. Furthermore, the map $S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)^d$, $\mathtt{B} \mapsto (v_1(\mathtt{B}),\dots,v_d(\mathtt{B}))$ is a submersion.*

Proof. Since each eigenvalue $\lambda_i(\mathtt{A})$ is a simple root of the polynomial $\det(\mathtt{A}-\lambda\,\mathrm{id})$, it has a C^∞ continuation $\lambda_i(\mathtt{B})$ for all nearby matrices, given by the implicit function theorem. Denote $L_i(B) = \mathtt{B} - \lambda_i(\mathtt{B})\,\mathrm{id}$. It depends smoothly on $\mathtt{B} \in S_{\mathtt{A}}$ and, since $\lambda_i(B)$ remains a simple eigenvalue of $\mathtt{B}$, it has rank $d-1$. Since the entries of $\mathrm{adj}(L_i(\mathtt{B}))$ are cofactors of $L_i(\mathtt{B})$, the adjoint is a non-zero matrix that also varies in a C^∞ fashion with $\mathtt{B}$. Moreover,

$$L_i(\mathtt{B}) \cdot \mathrm{adj}(L_i(\mathtt{B})) = \det(L_i(\mathtt{B}))\,\mathrm{id} = 0.$$

This means that any nonzero column of $\mathrm{adj}(L_i(B))$ is an eigenvector for $L_i(\mathtt{B})$, depending in a C^∞ fashion on the matrix, and so we may use it to define a function $v_i(\mathtt{B})$ as in the statement. To check that the derivative of v at $\mathtt{A}$ is onto just consider any differentiable curve $(-\varepsilon,\varepsilon) \ni t \mapsto (\beta_1(t),\dots,\beta_d(t))$ such that $\beta_i(0) = v_i(\mathtt{A})$ for all $i = 1,\dots,d$. Define $P(t) = [\beta_1(t),\dots,\beta_d(t)]$, that is, $P(t)$ is the matrix whose column vectors are the $\beta_i(t)$. Then define

$$\mathtt{B}(t) = P(t)\,\mathrm{diag}[\lambda_1(\mathtt{A}),\dots,\lambda_d(\mathtt{A})]P(t)^{-1}. \tag{9.13}$$

Then, $\mathtt{B}(0) = \mathtt{A}$ and $v(\mathtt{B}(t)) = (\beta_1(t),\dots,\beta_d(t))$ for all t. In particular, the derivative $Dv(A)$ maps $\mathtt{B}'(0)$ to $(\beta_1'(0),\dots,\beta_d'(0))$. So, the derivative is indeed surjective. $\square$

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1, \dots, \mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ matrices $\mathtt{A}_i$ whose eigenvalues all have distinct norms. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. By Lemma 9.16, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(v_j(\mathtt{A}_i)\big)_{1 \leq j \leq d,\ 1 \leq i \leq \ell+1} \in \mathbb{P}(\mathbb{C}^d)^{d(\ell+1)}$$

is a submersion. Consequently, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that

$$v_a(\mathtt{A}_i) \neq v_b(\mathtt{A}_{\ell+1}) \quad \text{for every } a,\, b \in \{1, \dots, d\}. \tag{9.14}$$

Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_* \eta = \eta \quad \text{for every } 1 \leq l \leq 2\ell. \tag{9.15}$$

Consider $l = i$, as in (9.14), and also $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, η must be a convex combination of Dirac masses supported on the eigenspaces of $\mathtt{A}_i$. For the same reason, η must be supported on the set of eigenspaces of $\mathtt{A}_{\ell+1}$. However, (9.14) means that these two sets are disjoint, and so we reached a contradiction. This contradiction proves Proposition 9.14 in the complex case.

9.3.2. *Real case.* The proof for real matrices is a bit more complicated due to the possibility of complex conjugate eigenvalues. In particular, the set of matrices whose eigenvalues are all distinct in norm is not dense. This difficulty has been met before by Bonatti, Gomez-Mont, Viana [5], and we use a similar approach in dimensions $d \geq 3$. For $d = 2$ we use a different argument, based on the conformal barycenter construction of Douady, Earle [10].

For each $r, s \geq 0$ with $r + 2s = d$, let $S(r, s)$ be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d, \mathbb{R})$ having r real eigenvalues, and s pairs of (strictly) complex conjugate eigenvalues, such that all the eigenvalues that do not belong to the same complex conjugate pair have distinct norms. Every $S(r, s)$ is open and their union $S = \cup_{r,s} S(r, s)$ is an open and dense subset of $\mathrm{GL}(d, \mathbb{R})$ whose complement is contained in a finite union of closed submanifolds with positive codimension. Let $\mathrm{Grass}(k, d)$ denote the k-dimensional Grassmannian of $\mathbb{R}^d$, for $1 \leq k \leq d$. In what follows we often think of elements of $\mathrm{Grass}(2, d)$ as subsets of $\mathrm{Grass}(1, d) = \mathbb{P}(\mathbb{R}^d)$.

Lemma 9.17. *Let $\mathcal{F} = \big\{[(r_1, \dots, r_d)e^{i\theta}] \in \mathbb{P}(\mathbb{C}^d) : \theta \in [0, 2\pi], (r_1, \dots, r_d) \in \mathbb{R}^d\big\}$. Then $\mathcal{F}$ is closed in $\mathbb{P}(\mathbb{C}^d)$ and the map $\Psi : \mathbb{P}(\mathbb{C}^d) \setminus \mathcal{F} \to \mathrm{Grass}(2, d)$ defined by $\Psi(v) = \mathrm{Span}\,\{\mathrm{Re}(v), \mathrm{Im}(v)\}$ is a submersion.*

Proof. First, we recall the usual local charts in $\mathrm{Grass}(2, d)$. Let $e_1, \dots, e_d$ the canonical base of $\mathbb{R}^d$ and $1 \leq i < j \leq d$ be fixed. For any $d \times 2$ matrix $\mathtt{A}$ we denote by $\varphi(\mathtt{A})$ the 2×2 matrix formed by the ith and jth rows of $\mathtt{A}$ and by $\varphi^*(\mathtt{A})$ the $(d-2) \times 2$ matrix formed by the other rows of $\mathtt{A}$. Let $U_{i,j}$ be the open set of planes $L \in \mathrm{Grass}(2, d)$ such that the orthogonal projection of L to $\mathrm{Span}\,\{e_i, e_j\}$ is an isomorphism. This means that if $L \in U_{i,j}$ with $L = \mathrm{Span}\,\{v_1, v_2\}$ then $\varphi(\mathtt{A}_L)$ is invertible, where $\mathtt{A}_L = [v_1, v_2]$ is the matrix whose columns are the vectors v_1,

v_2. Then the map $\phi : U_{i,j} \to \mathbb{R}^{2(d-2)}$ defined by $\phi(L) = \varphi^*(\mathtt{A}_L)\varphi(\mathtt{A}_L)^{-1}$, where we identify $(d-2)\times 2$ matrices with points in $\mathbb{R}^{2(d-2)}$, is a local chart in the Grassmannian.

Now, note that v, $\overline{v} \in \mathbb{C}^d$ are linearly independent if and only if $v \in \mathbb{P}(\mathbb{C}^d)\backslash\mathcal{F}$. Moreover, in that case $\mathrm{Re}(v), \mathrm{Im}(v)$ are $\mathbb{C}$-linearly independent and, in particular, $\Psi(v)$ is well defined. It is clear from its expression in local charts that Ψ is differentiable. Moreover, still in local charts, its derivative is given by

$$D\Psi(v)\dot{v} = \varphi^*(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1} - \varphi^*(\mathtt{A})\varphi(\mathtt{A})^{-1}\varphi(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1},$$

where $\dot{v} \in T_v\mathbb{P}(\mathbb{C}^d)$, $\mathtt{A} = [\mathrm{Re}(v), \mathrm{Im}(v)]$ and $\dot{\mathtt{A}} = [\mathrm{Re}(\dot{v}), \mathrm{Im}(\dot{v})]$. Let $\dot{\mathtt{B}}$ be in the tangent space $T_{\Psi(v)}\,\mathrm{Grass}(2,d)$. Then $\dot{\mathtt{B}}$ is a $(d-2)\times 2$ matrix with real entries. Let $\dot{\mathtt{A}}_{\dot{\mathtt{B}}}$ be the $d \times 2$ matrix defined by $\varphi^*(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = \dot{\mathtt{B}}\varphi(\mathtt{A})$ and $\varphi(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = 0$. Since, $\dot{\mathtt{A}}_{\dot{\mathtt{B}}} = [\dot{v}_1, \dot{v}_2]$, we have that $D\Psi(v)(\dot{v}_1 + i\dot{v}_2) = \dot{\mathtt{B}}$. This finishes the proof of the lemma. □

Lemma 9.18. *Let $\mathtt{A} \in S(r,s)$. Then there exists an open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$ and there exist C^∞ functions*

$$\lambda_j : S_{\mathtt{A}} \to \mathbb{R}, \quad \xi_j : S_{\mathtt{A}} \to \mathrm{Grass}(1,d), \quad \textit{for } 1 \le j \le r, \textit{ and}$$
$$\mu_k : S_{\mathtt{A}} \to \mathbb{C}\setminus\mathbb{R}, \quad \eta_k : S_{\mathtt{A}} \to \mathrm{Grass}(2,d), \quad \textit{for } 1 \le k \le s,$$

such that $\xi_j(\mathtt{B})$ is the eigenspace of $\mathtt{B}$ associated to the eigenvalue $\lambda_j(\mathtt{B})$, and $\eta_k(\mathtt{B})$ is the characteristic space associated to the conjugate pair of eigenvalues $\mu_k(\mathtt{B})$ and $\bar{\mu}_k(\mathtt{B})$. Furthermore, the map

$$S_{\mathtt{A}} \to \mathrm{Grass}(1,d)^r \times \mathrm{Grass}(2,d)^s, \quad \mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$$

is a submersion.

Proof. Existence and regularity of the eigenvalues λ_j and μ_k follow from the implicit function theorem. Moreover, the arguments in Lemma 9.16 imply that if $v_j(\mathtt{B})$ is an eigenvector associated to the eigenvalue $\lambda_j(\mathtt{B})$, for $j = 1,\dots,r$, and $v_{r+2k-1}(\mathtt{B}), v_{r+2k}(\mathtt{B})$ are eigenvectors associated to $\mu_k(\mathtt{B})$, $\bar{\mu}_k(\mathtt{B})$, respectively, for $k = 1,\dots,s$, then the map Φ defined by

$$\Phi(\mathtt{B}) = (v_1(\mathtt{B}),\dots,v_r(\mathtt{B}), v_{r+1}(\mathtt{B}),\dots,v_{r+2s}(\mathtt{B})) \in \mathbb{P}(\mathbb{R}^d)^r \times \mathbb{P}(\mathbb{C}^d)^s \tag{9.16}$$

is C^∞. We are going to show that this map is a submersion on some open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$. For this, it is sufficient to show that the derivative $D\Phi(A)$ is onto. Consider any differentiable curve $(-\varepsilon,\varepsilon) \ni t \mapsto (\beta_1(t),\dots,\beta_{r+s}(t))$ such that $\beta_j(0) = v_j(\mathtt{A})$ for $j = 1,\dots,r$ and $\beta_{r+k}(0) = v_{r+2k-1}(\mathtt{A})$ for $k = 1,\dots,s$. Define

$$\begin{aligned} &P(t) = [\beta_1(t),\dots,\beta_r(t),\beta_{r+1},\bar{\beta}_{r+1},\dots,\beta_{r+s},\bar{\beta}_{r+s}], \textit{and} \\ &\mathtt{B}(t) = P(t)\ \mathrm{diag}[\lambda_1(\mathtt{A}),\dots,\lambda_r(\mathtt{A}),\mu_1(\mathtt{A}),\bar{\mu}_1(\mathtt{A}),\dots,\mu_s(\mathtt{A}),\bar{\mu}_s(\mathtt{A})]\,P(t)^{-1}. \end{aligned} \tag{9.17}$$

Observe that $t \mapsto \mathtt{B}(t)$ is a curve in $\mathrm{GL}(d,\mathbb{R})$, with $\mathtt{B}(0) = \mathtt{A}$. Observe also that $\Phi(\mathtt{B}(t)) = (\beta_1(t),\dots,\beta_{r+s}(t)$ for all $t \in (-\varepsilon,\varepsilon)$, and so $D\Phi(A)$ maps $\mathtt{B}'(0)$ to the vector $(\beta_1'(0),\dots,\beta_{r+s}'(0))$. So, the derivative is indeed surjective. Finally, define

$$\xi_j(\mathtt{B}) = v_j(\mathtt{B}) \quad \text{for } j = 1,\dots,r \text{ and}$$
$$\eta_k(\mathtt{B}) = \mathrm{Span}\,\{\mathrm{Re}(v_{r+2k-1}), \mathrm{Im}(v_{r+2k-1})\} \ \text{ for } k = 1,\dots,s.$$

Clearly these maps are C^∞. Moreover, since (9.16) is a submersion, Lemma 9.17 implies that $\mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$ is a submersion. □

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1, \dots, \mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ values of i such that $\mathtt{A}_i \in S$, that is, $\mathtt{A}_i \in S(r_i, s_i)$ for r_i and s_i. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. By Lemma 9.18, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \Big(\xi_j(\mathtt{A}_i)_{1 \leq j \leq r_i},\ \eta_k(\mathtt{A}_i)_{1 \leq k \leq s_i}\Big)_{1 \leq i \leq \ell+1} \tag{9.18}$$

is a submersion.

Assume first that $d \geq 4$, and so $\dim \mathbb{P}(\mathbb{R}^d) \geq 3$. Since the $\xi_j(\mathtt{A})$ are points and the $\eta_k(\mathtt{A})$ are lines in the projective space, it follows that there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\mathtt{A} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that

$$\xi_a(\mathtt{A}_i) \neq \xi_b(\mathtt{A}_{\ell+1}) \tag{9.19}$$

$$\xi_a(\mathtt{A}_i) \notin \eta_c(\mathtt{A}_{\ell+1}) \quad \text{and} \quad \xi_b(\mathtt{A}_i) \notin \eta_d(\mathtt{A}_{\ell+1}) \tag{9.20}$$

$$\eta_c(\mathtt{A}_i) \cap \eta_d(\mathtt{A}_{\ell+1}) = \emptyset \tag{9.21}$$

for every $1 \leq a \leq r(\mathtt{A}_i)$, $1 \leq b \leq r(\mathtt{A}_{\ell+1})$, $1 \leq c \leq s(\mathtt{A}_i)$, and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_* \eta = \eta \quad \text{for every } 1 \leq l \leq 2\ell. \tag{9.22}$$

Consider both $l = i$, as in (9.19)–(9.21), and $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, apart from the complex conjugate pairs, the measure η must be supported on

$$\Sigma(\mathtt{A}_i) = \bigcup_{j=1}^{r} \{\xi_j(\mathtt{A}_i)\} \cup \bigcup_{k=1}^{s} \eta_k(\mathtt{A}_i).$$

Analogously, η must be supported on $\Sigma(\mathtt{A}_{\ell+1})$. However, conditions (9.19)–(9.21) mean that the two sets $\Sigma(\mathtt{A}_i)$ and $\Sigma(\mathtt{A}_{\ell+1})$ are disjoint. This contradiction proves the proposition in any dimension $d \geq 4$.

For $d = 3$ the projective space $\mathbb{P}(\mathbb{R}^3)$ is only 2-dimensional, and so one can not force a pair of 1-dimensional submanifolds $\eta_k(\mathtt{A})$ to be disjoint, as required in (9.21). However, the argument can easily be adapted to cover the 3-dimensional case as well. Firstly, one replaces (9.21) by

$$\eta_c(\mathtt{A}_i) \neq \eta_d(\mathtt{A}_{\ell+1}) \tag{9.23}$$

for every $1 \leq c \leq s(\mathtt{A}_i)$ and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. (Both (9.21) and (9.23) are void if either $s(\mathtt{A}_i) = 0$ or $s(\mathtt{A}_{\ell+1}) = 0$; the only other possibility is $s(\mathtt{A}_i) = s(\mathtt{A}_{\ell+1}) = 1$, with $c = d = 1$.) Then the argument proceeds as before, except that we may no longer have disjointness: when $s = 1$,

$$\Sigma(\mathtt{A}_i) \cap \Sigma(\mathtt{A}_{\ell+1}) = \eta_1(\mathtt{A}_i) \cap \eta_1(\mathtt{A}_{\ell+1})$$

consists of exactly one point in projective space. Then η must be a Dirac measure supported on this point. However, in view of (9.22), this would have to be a fixed point of $\mathtt{A}_i$ contained in $\eta_1(\mathtt{A}_i)$, which is impossible because the eigenspace $\eta_i(\mathtt{A}_i)$ contains no invariant line. Thus, we reach a contradiction also in this case.

Now we deal with the case $d = 2$. Let $\mathcal{Z}_1$ be as in the previous cases: for every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ values of i such that $\mathtt{A}_i \in S = S(2,0) \cup S(0,1)$. As before, it is no restriction to assume that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. There are three cases to consider:

First, suppose there exist $1 \le i, j \le \ell + 1$ such that $\mathtt{A}_i \in S(2,0)$, that is, it has two real (distinct) eigenvalues, and $\mathtt{A}_j \in S(0,1)$, that is, it has a pair of complex eigenvalues. We claim that in this case $\underline{\mathtt{A}}$ can not belong to $G_{2\ell}$. Indeed, on the one hand, any probability measure η on $\mathbb{P}(\mathbb{R}^2)$ which is invariant under $\mathtt{A}_i \in S(2,0)$ must be a convex combination of Dirac masses at the two eigenspaces. On the other hand, the action of $\mathtt{A}_j \in S(0,1)$ on the projective space is a rotation whose angle is *not* a multiple of π, and so it admits no such invariant measure.

Next, suppose all the matrices are hyperbolic: $\mathtt{A}_i \in S(2,0)$ for all $1 \le i \le \ell$. In this case one can use precisely the same argument as we did before in higher dimensions (conditions (9.20) and (9.21)-(9.23) become void). One finds a closed subset $\mathcal{Z}_2$ contained in a finite union of submanifolds with codimension $\ge \ell$ such that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

Finally, suppose all the matrices are elliptic: $\mathtt{A}_i \in S(0,1)$ for all $1 \le i \le \ell$. Recall that every matrix $\mathtt{A} \in \mathrm{GL}(2,\mathbb{R})$ with positive determinant induces an automorphism $h_{\mathtt{A}}$ of the Poincaré half plane $\mathbb{H}$:

$$\mathtt{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longrightarrow \quad h_{\mathtt{A}}(z) = \frac{az+b}{cz+d}. \tag{9.24}$$

The action of $\mathtt{A}$ on the projective plane may be identified with the action of $h_{\mathtt{A}}$ on the boundary of $\mathbb{H}$, via

$$\partial\mathbb{H} \to \mathbb{P}(\mathbb{R}^2), \qquad x \mapsto [(x,1)]$$

(including $x = \infty$) so that $\mathbb{P}(\mathtt{A})$-invariant measures on the projective plane may be seen as $h_{\mathtt{A}}$-invariant measures sitting on the real axis. It is also easy to check that $h_{\mathtt{A}}$ has a fixed point in the open disc $\mathbb{H}$ if and only if $\mathtt{A} \in S(0,1)$. Define $\phi(\mathtt{A})$ to be this (unique) fixed point. It is easy to see that the $\mathtt{A} \mapsto \phi(\mathtt{A})$ is a C^∞ submersion: just use the explicit expression for the fixed point extracted from (9.24). The key feature is the following consequence of a classical construction of Douady, Earle [10]:

Lemma 9.19. *If $\mathtt{A}$, $\mathtt{B} \in S(0,1)$ have some common invariant probability measure μ on $\partial\mathbb{H}$ then $\phi(\mathtt{A}) = \phi(\mathtt{B})$.*

Proof. It is clear that elliptic matrices have no invariant measures with atoms of mass larger than 1/3: such atoms would correspond to periodic points of $\mathtt{A}$ in the projective plane with period 1 or 2, which would contradict the definition of $S(0,1)$. In Proposition 1 of [10] a map $\mu \mapsto B(\mu)$ is constructed that assigns to each probability measure μ with no atoms of mass $\ge 1/2$ (see Remark 2 in [10, page26]) a point $B(\mu)$ in the half plane $\mathbb{H}$, in such a way that

$$B(h_*\mu) = h(B(\mu)) \quad \text{for every automorphism } h : \mathbb{H} \to \mathbb{H}.$$

When μ is $\mathtt{A}$-invariant this implies $h_{\mathtt{A}}(B(\mu)) = B((h_{\mathtt{A}})_*\mu) = B(\mu)$, and so the *conformal barycenter* $B(\mu)$ must coincide with the fixed point $\phi(\mathtt{A})$ of the automorphism $h_{\mathtt{A}}$. Thus, if μ is a common invariant measure then $\phi(\mathtt{A}) = B(\mu) = \phi(\mathtt{B})$. $\square$

It follows from the previous observations that the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(\phi(\mathtt{A}_i)\big)_{1 \le i \le \ell+1} \in \mathbb{H}^{\ell+1}.$$

is a submersion. Hence, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V}\setminus\mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V}\setminus(\mathcal{Z}_1\cup\mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that $\phi(\mathtt{A}_i) \neq \phi(\mathtt{A}_{\ell+1})$. Thus, we may apply Lemma 9.19 to conclude that if $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. In other words, $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

The proofs of Proposition 9.14 and Theorem A are now complete.

References

[1] F. Abdenur and M. Viana, *Flavors of partial hyperbolicity.* Preprint www.preprint.impa.br 2008.

[2] J. F. Alves, C. Bonatti, and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math., 140 (2000), pp. 351–398.

[3] A. Avila and M. Viana, *Extremal Lyapunov exponents of smooth cocycles.* Preprint www.preprint.impa.br 2008.

[4] C. Bonatti, L. J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, vol. 102 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, 2005.

[5] C. Bonatti, X. Gómez-Mont, and M. Viana, *Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20 (2003), pp. 579–624.

[6] C. Bonatti and M. Viana, *Lyapunov exponents with multiplicity 1 for deterministic products of matrices*, Ergod. Th. & Dynam. Sys, 24 (2004), pp. 1295–1330.

[7] M. Brin and Y. Pesin, *Partially hyperbolic dynamical systems*, Izv. Acad. Nauk. SSSR, 1 (1974), pp. 177–212.

[8] K. Burns, C. Pugh, M. Shub, and A. Wilkinson, *Recent results about stable ergodicity*, in Smooth ergodic theory and its applications (Seattle WA, 1999), vol. 69 of Procs. Symp. Pure Math., Amer. Math. Soc., 2001, pp. 327–366.

[9] K. Burns and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, Annals of Math.

[10] A. Douady and J. C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math., 157 (1986), pp. 23–48.

[11] H. Furstenberg, *Non-commuting random products*, Trans. Amer. Math. Soc., 108 (1963), pp. 377–428.

[12] B. Hasselblatt and Y. Pesin, *Partially hyperbolic dynamical systems*, in Handbook of dynamical systems. Vol. 1B, Elsevier, 2006, pp. 1–55.

[13] F. R. Hertz, J. R. Hertz, and R. Ures, *A survey on partially hyperbolic dynamics.* Preprint www.arxiv.org 2006.

[14] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds*, vol. 583 of Lect. Notes in Math., Springer Verlag, 1977.

[15] D. Husemoller, *Fibre bundles*, Graduate texts in Mathematics, Springe-Verlag, 1994.

[16] J. Kingman, *The ergodic theorem of subadditive stochastic processes*, J. Royal Statist. Soc., 30 (1968), pp. 499–510.

[17] F. Ledrappier, *Positivity of the exponent for stationary sequences of matrices*, in Lyapunov exponents (Bremen, 1984), vol. 1186 of Lect. Notes Math., Springer, 1986, pp. 56–73.

[18] V. I. Oseledets, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), pp. 197–231.

[19] C. Pugh and M. Shub, *Ergodicity of Anosov actions*, Invent. Math., 15 (1972), pp. 1–23.

[20] ———, *Stably ergodic dynamical systems and partial hyperbolicity*, J. Complexity, 13 (1997), pp. 125–179.

[21] ———, *Stable ergodicity and julienne quasi-conformality*, J. Europ. Math. Soc., 2 (2000), pp. 1–52.

[22] V. A. Rokhlin, *On the fundamental ideas of measure theory*, A. M. S. Transl., 10 (1952), pp. 1–52. Transl. from Mat. Sbornik 25 (1949), 107–150.

[23] W. Rudin, *Real and complex analysis*, McGraw-Hill, 3 ed., 1987.

[24] M. Shub, *Global stability of dynamical systems*, Springer Verlag, 1987.

[25] M. Viana, *Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents*, Annals of Math., 167 (2008), pp. 643–680.

[26] A. Wilkinson, *Livsič theory for partially hyperbolic maps*. Preprint 2008.

E-mail address: artur@math.sunysb.edu
E-mail address: jimmath@impa.br
E-mail address: viana@impa.br