

Counterexamples to Calabi conjectures on minimal hypersurfaces cannot be proper

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Abstract

We give an estimate of the mean curvature of a complete submanifold lying inside a closed cylinder $B(r) \times \mathbb{R}^\ell$ in a product Riemannian manifold $N^{n-\ell} \times \mathbb{R}^\ell$. It follows that a complete hypersurface of given constant mean curvature lying inside a closed circular cylinder in Euclidean space cannot be proper if the circular base is of sufficiently small radius. In particular, any possible counterexample to Calabi's conjecture on complete minimal hypersurfaces cannot be proper. As another application of our method, we derive a result about the stochastic incompleteness of submanifolds with sufficiently small mean curvature.

1 Introduction

The Calabi problem in its original form, presented by Calabi [3] and promoted by Chern [4] about the same time, consisted on two conjectures about Euclidean minimal hypersurfaces. The first conjecture is that any complete minimal hypersurface of \mathbb{R}^n must be unbounded. The second and more ambitious conjecture asserted that any complete non-flat minimal hypersurface in \mathbb{R}^n has unbounded projections in every $(n - 2)$ -dimensional flat subspace.

Both conjectures turned out to be false for immersed surfaces in \mathbb{R}^3 . First Jorge and Xavier [9] exhibit a non-flat complete minimal surface lying between two parallel planes. Later on Nadirashvili [12] constructed a complete minimal surface inside a round ball in \mathbb{R}^3 .

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It was recently shown by Colding and Minicozzi [5] that both conjectures hold for embedded minimal surfaces. Their work involves the close relation between the Calabi conjectures and properness. Recall that an immersed submanifold in Euclidean space is *proper* if the pre-image of any compact subset of \mathbb{R}^n is compact. It is a consequence of their general result that a complete embedded minimal disk in \mathbb{R}^3 must be proper.

The immersed counterexamples to Calabi's conjectures discussed above are not proper. The example of Nadirashvili cannot be proper since from the definition a proper submanifold must be unbounded. The same conclusion hold for the other example but now the argument is not so easy, one has to use the strong half-space theorem due to Hoffman and Meeks [7].

The strong half-space theorem does not hold in \mathbb{R}^n for $n \geq 4$. In fact, the higher dimensional catenoids are between parallel hyperplanes. Hence, it is natural to ask if any possible higher dimensional counterexample to Calabi's second conjecture must be non-proper. In the special case of minimal immersion, it follows from the corollary of our main result that a complete hypersurface of \mathbb{R}^n , $n \geq 3$, with bounded projection in a two dimensional flat subspace cannot be proper (see Corollary 2.2 below).

As an application of our method, we generalize the results by Markvorsen [10] and Bessa and Montenegro [2] about stochastic incompleteness of minimal submanifolds to submanifolds of bounded mean curvature. In this respect, let us recall that a Riemannian manifold M is said to be *stochastically complete* if for some (and therefore, for any) $(x, t) \in M \times (0, +\infty)$ it holds that

$$\int_M p(x, y, t) dy = 1,$$

where $p(x, y, t)$ is the heat kernel of the Laplacian operator. Otherwise, the manifold M is said to be *stochastically incomplete* (for further details about this see, for instance, [6] or [14]).

An interesting problem in submanifold geometry is to understand stochastic completeness/incompleteness of submanifolds in terms of their extrinsic geometry. In [10] Markvorsen derived a mean time exit comparison theorem which implies that any bounded complete minimal submanifold of a Hadamard manifold N with sectional curvature $K_N \leq b \leq 0$ is stochastically incomplete. Recently, Bessa and Montenegro [2] considered minimal submanifolds of product spaces $N \times \mathbb{R}$, where N is a Hadamard manifold with $K_N \leq b \leq 0$, and proved a version of Markvorsen's result in this setting. In particular, they showed that complete cylindrically bounded minimal submanifolds of $N \times \mathbb{R}$ are stochastically incomplete. Here we extend these results to complete submanifolds with sufficiently small mean curvature lying inside a closed cylinder $B(r) \times \mathbb{R}^\ell$ in a product Riemannian manifold $N^{n-\ell} \times \mathbb{R}^\ell$.

2 The results

Part (a) of Theorem 2.1 below extends the main results given in [1] for compact hypersurfaces. Part (b) generalizes stochastic incompleteness results of [2] and [10] for minimal submanifolds.

In the following we denote

$$C_b(t) = \begin{cases} \sqrt{b} \cot(\sqrt{b}t) & \text{if } b > 0, \ t < \pi/2\sqrt{b}, \\ 1/t & \text{if } b = 0, \\ \sqrt{-b} \coth(\sqrt{-b}t) & \text{if } b < 0. \end{cases}$$

Theorem 2.1 *Let $\varphi: M^m \rightarrow N^{n-\ell} \times \mathbb{R}^\ell$ be an isometric immersion of a complete Riemannian manifold M of dimension $m \geq \ell + 1$. Let $B_N(r)$ be the geodesic ball of $N^{n-\ell}$ centered at p with radius r . Given $q \in M$, assume that the radial sectional curvature K_N^{rad} along the radial geodesics issuing from $p = \pi_N(\varphi(q)) \in N^{n-\ell}$ is bounded as $K_N^{\text{rad}} \leq b$ in $B_N(r)$. Suppose that*

$$\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$$

for $r < \min\{\text{inj}_N(p), \pi/2\sqrt{b}\}$, where we replace $\pi/2\sqrt{b}$ by $+\infty$ if $b \leq 0$.

(a) *If $\varphi: M^m \rightarrow N^{n-\ell} \times \mathbb{R}^\ell$ is proper, then*

$$\sup_M |H| \geq \frac{(m-\ell)}{m} C_b(r). \quad (1)$$

(b) *If*

$$\sup_M |H| < \frac{(m-\ell)}{m} C_b(r), \quad (2)$$

then M is stochastically incomplete.

For Euclidean hypersurfaces of constant mean curvature we have the following consequence.

Corollary 2.2 *Let $\varphi: M^n \rightarrow \mathbb{R}^{n+1}$ be a complete hypersurface with constant mean curvature H . If $\varphi(M) \subset B_{\mathbb{R}^2}(r) \times \mathbb{R}^{n-1}$ and $|H| < 1/nr$, then φ cannot be proper.*

Observe that the assumption on the bound of the mean curvature cannot be weakened since $1/nr$ is the mean curvature of the cylinder $\mathbb{S}^1(r) \times \mathbb{R}^{n-1}$.

We point out that Martín and Morales [11] constructed examples of complete minimal surfaces properly immersed in the interior of a cylinder $B_{\mathbb{R}^2}(r) \times \mathbb{R}$. By the above result these surfaces cannot be proper in \mathbb{R}^3 .

3 The proofs

Let $\varphi: M^m \rightarrow N^n$ be an isometric immersion between Riemannian manifolds. Given a function $g \in C^\infty(N)$ we set $f = g \circ \varphi \in C^\infty(M)$. Since

$$\langle \text{grad}^M f, X \rangle = \langle \text{grad}^N g, X \rangle$$

for every vector field $X \in TM$, we obtain

$$\text{grad}^N g = \text{grad}^M f + (\text{grad}^N g)^\perp$$

according to the decomposition $TN = TM \oplus T^\perp M$. An easy computation using the Gauss formula gives the well-known relation (see e.g. [8])

$$\text{Hess}_M f(X, Y) = \text{Hess}_N g(X, Y) + \langle \text{grad}^N g, \alpha(X, Y) \rangle \quad (3)$$

for all vector fields $X, Y \in TM$, where α stands for the second fundamental form of φ . In particular, taking traces with respect to an orthonormal frame $\{e_1, \dots, e_m\}$ in TM yields

$$\Delta_M f = \sum_{i=1}^m \text{Hess}_N g(e_i, e_i) + \langle \text{grad}^N g, \vec{H} \rangle. \quad (4)$$

where $\vec{H} = \sum_{i=1}^m \alpha(e_i, e_i)$.

The first main ingredient of our proofs is the Hessian comparison theorem.

Theorem 3.1 *Let M^m be a Riemannian manifold and $x_0, x_1 \in M$ be such that there is a minimizing unit speed geodesic γ joining x_0 and x_1 and let $\rho(x) = \text{dist}(x_0, x)$ be the distance function to x_0 . Let $K_\gamma \leq b$ be the radial sectional curvatures of M along γ . If $b > 0$ assume $\rho(x_1) < \pi/2\sqrt{b}$. Then, we have $\text{Hess} \rho(x)(\gamma', \gamma') = 0$ and*

$$\text{Hess} \rho(x)(X, X) \geq C_b(\rho(x)) \|X\|^2 \quad (5)$$

where $X \in T_x M$ is perpendicular to $\gamma'(\rho(x))$.

The second main ingredient is the version proved by Pigola-Rigoli-Setti [14, Theorem 1.9] of the Omori-Yau maximum principle.

Theorem 3.2 *Let M^m be a Riemannian manifold and assume that there exists a non-negative C^2 -function ψ satisfying the following requirements:*

$$\psi(x) \rightarrow +\infty \quad \text{as } x \rightarrow \infty$$

$$\exists A > 0 \quad \text{such that } |\text{grad} \psi| \leq A\sqrt{\psi} \quad \text{off a compact set}$$

$$\exists B > 0 \quad \text{such that } \Delta \psi \leq B\sqrt{\psi G(\sqrt{\psi})} \quad \text{off a compact set}$$

where G is a smooth function on $[0, +\infty)$ satisfying:

$$\begin{aligned} (i) \quad & G(0) > 0, & (ii) \quad & G'(t) \geq 0 \text{ on } [0, +\infty), \\ (iii) \quad & 1/\sqrt{G(t)} \notin L^1(0, +\infty), & (iv) \quad & \limsup_{t \rightarrow +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty. \end{aligned} \quad (6)$$

Then, given a function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M^m$ such that

$$u(x_k) > u^* - 1/k; \quad |\text{grad } u|(x_k) < 1/k; \quad \Delta u(x_k) < 1/k.$$

Observe that a function G satisfying the above conditions is

$$G(t) = (t + 2)^2(\log(t + 2))^2. \quad (7)$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: Define $\sigma : N^{n-\ell} \times \mathbb{R}^\ell \rightarrow [0, +\infty)$ by

$$\sigma(z, y) = \rho_{\mathbb{R}^\ell}(y),$$

where $\rho_{\mathbb{R}^\ell}(y) = \|y\|_{\mathbb{R}^\ell}$ is the distance function to the origin in \mathbb{R}^ℓ . Since φ is proper and $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, then the function $\psi(x) = \sigma \circ \varphi(x)$ satisfies $\psi(x) \rightarrow \infty$ as $\rho_M(x) = \text{dist}_M(q, x) \rightarrow +\infty$. Off a compact set, we now have

$$|\text{grad}^M \psi(x)| \leq |\text{grad}^{N \times \mathbb{R}^\ell} \sigma(\varphi(x))| = |\text{grad}^{\mathbb{R}^\ell} \rho_{\mathbb{R}^\ell}| = 1 \leq \sqrt{\psi(x)}.$$

To compute $\Delta_M \psi$ we start with bases $\{\partial/\partial\rho_N, \partial/\partial\theta_2, \dots, \partial/\partial\theta_{n-\ell}\}$ of TN and $\{\partial/\partial\rho_{\mathbb{R}^\ell}, \partial/\partial\gamma_2, \dots, \partial/\partial\gamma_\ell\}$ of $T\mathbb{R}^\ell$ (polar coordinates) orthonormal at $x \in M$. Then, we choose an orthonormal basis $\{e_1, \dots, e_m\}$ for $T_x M$ as follows

$$e_i = \alpha_i \frac{\partial}{\partial\rho_N} + \sum_{j=2}^{n-\ell} a_{ij} \frac{\partial}{\partial\theta_j} + \beta_i \frac{\partial}{\partial\rho_{\mathbb{R}^\ell}} + \sum_{t=2}^{\ell} b_{it} \frac{\partial}{\partial\gamma_t}.$$

Hence, we have

$$\text{Hess}_{N \times \mathbb{R}^\ell} \sigma(\varphi(x))(e_i, e_i) = \text{Hess}_{\mathbb{R}^\ell} \rho_{\mathbb{R}^\ell}(\pi_{\mathbb{R}^\ell} e_i, \pi_{\mathbb{R}^\ell} e_i) = \frac{1}{\sigma(\varphi(x))} \sum_{t=2}^{\ell} b_{it}^2 \leq \frac{1}{\psi(x)},$$

where $\pi_{\mathbb{R}^\ell}$ denotes the orthogonal projection onto $T\mathbb{R}^\ell$. Here, we are using

$$|e_i| = 1 = \alpha_i^2 + \sum_{j=2}^{n-\ell} a_{ij}^2 + \beta_i^2 + \sum_{t=2}^{\ell} b_{it}^2$$

that yields $\sum_{t=2}^{\ell} b_{it}^2 \leq 1$.

Since $\psi(x) \rightarrow \infty$ as $\rho_M(x) = \text{dist}_M(q, x) \rightarrow +\infty$, off a compact set we may assume that

$$|\vec{H}|(x) = m|H|(x) \leq \sqrt{\psi(x)G(\sqrt{\psi(x)})}$$

where $G(t)$ is given by (7). Otherwise, $\sup_M |H| = +\infty$ and there is nothing to prove. Besides, off a compact set we also have that

$$\frac{1}{\psi(x)} \leq \sqrt{\psi(x)G(\sqrt{\psi(x)})}.$$

Hence, from (4) we have off a compact set that

$$\begin{aligned} \Delta_M \psi(x) &= \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}^\ell} \sigma(\varphi(x))(e_i, e_i) + \langle \text{grad}^{N \times \mathbb{R}^\ell} \sigma(\varphi(x)), \vec{H}(x) \rangle \\ &\leq \frac{m}{\psi(x)} + m|H|(x) \\ &\leq (m+1)\sqrt{\psi(x)G(\sqrt{\psi(x)})}. \end{aligned}$$

Therefore, by Theorem 3.2 the Omori-Yau maximum principle holds on M .

Define $\rho: N^{n-\ell} \times \mathbb{R}^\ell \rightarrow \mathbb{R}$ by

$$\rho(z, y) = \rho_N(z) = \text{dist}_N(p, z)$$

and $u: M^m \rightarrow \mathbb{R}$ by

$$u(x) = \rho \circ \varphi(x).$$

Since $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, we have that $u^* = \sup_M u \leq r < \infty$. Therefore, by the maximum principle there is a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M^m$ such that

$$u(x_k) > u^* - 1/k; \quad |\text{grad } u|(x_k) < 1/k; \quad \Delta u(x_k) < 1/k.$$

Hence, we have

$$\frac{1}{k} > \Delta u(x_k) = \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}^\ell} \rho(\varphi(x_k))(e_i, e_i) + \langle \text{grad}^{N \times \mathbb{R}^\ell} \rho(\varphi(x_k)), \vec{H}(x_k) \rangle \quad (8)$$

where $\{e_1, \dots, e_m\}$ is an orthonormal basis for $T_{x_k} M$. Start with an orthonormal basis $\{\partial/\partial\rho_N, \partial/\partial\theta_2, \dots, \partial/\partial\theta_{n-\ell}\}$ for TN and standard coordinates $\{y_1, \dots, y_\ell\}$ for \mathbb{R}^ℓ . Then, choose an orthonormal basis for $T_{x_k} M$ as follows

$$e_i = \alpha_i \frac{\partial}{\partial \rho_N} + \sum_{j=2}^{n-\ell} a_{ij} \frac{\partial}{\partial \theta_j} + \sum_{t=1}^{\ell} c_{it} \frac{\partial}{\partial y_t}.$$

Using Theorem 3.1, a straightforward computation yields

$$\begin{aligned}
\text{Hess}_{N \times \mathbb{R}^\ell} \rho(\varphi(x_k))(e_i, e_i) &= \text{Hess}_{N \rho_N}(z(x_k))(\pi_{TN} e_i, \pi_{TN} e_i) \\
&= \sum_{j=2}^{n-\ell} a_{ij}^2 \text{Hess}_{N \rho_N}(z(x_k))(\partial/\partial\theta_j, \partial/\partial\theta_j) \\
&\geq \sum_{j=2}^{n-\ell} a_{ij}^2 C_b(r) \\
&= \left(1 - \alpha_i^2 - \sum_{t=1}^{\ell} c_{it}^2\right) C_b(r)
\end{aligned} \tag{9}$$

since

$$|e_i| = 1 = \alpha_i^2 + \sum_{j=2}^{n-\ell} a_{ij}^2 + \sum_{t=1}^{\ell} c_{it}^2,$$

where π_{TN} denotes the orthogonal projection onto TN . Therefore,

$$\sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}^\ell} \rho(\varphi(x_k))(e_i, e_i) \geq \left(m - \sum_i \alpha_i^2 - \sum_{i,t} c_{it}^2\right) C_b(r). \tag{10}$$

At x_k , we have

$$\text{grad}^{N \times \mathbb{R}^\ell} \rho(\varphi(x_k)) = \text{grad } u(x_k) + (\text{grad}^{N \times \mathbb{R}^\ell} \rho(\varphi(x_k)))^\perp$$

and hence

$$|\text{grad } u|^2(x_k) = \sum_{i=1}^m \left\langle \frac{\partial}{\partial \rho_N}, e_i \right\rangle = \sum_i \alpha_i^2 < 1/k^2. \tag{11}$$

Taking into account $|\text{grad}^{N \times \mathbb{R}^\ell} \rho| = |\text{grad}^N \rho_N| = 1$, from (8) and (10) we obtain

$$\frac{1}{k} > \left(m - \sum_i \alpha_i^2 - \sum_{i,t} c_{it}^2\right) C_b(r) - m \sup_M |H|.$$

It follows using (11) that

$$\frac{1}{k} + \frac{C_b(r)}{k^2} + m \sup_M |H| \geq \left(m - \sum_{i,t} c_{it}^2\right) C_b(r). \tag{12}$$

Observe now that

$$\sum_{i,t} c_{it}^2 = \sum_{t=1}^{\ell} \sum_{i=1}^m c_{it}^2 = \sum_{t=1}^{\ell} |\text{grad}(y_t \circ \varphi)|^2 \leq \ell,$$

since $|\text{grad}(y_t \circ \varphi)|^2 \leq |\text{grad}^{\mathbb{R}^\ell} y_t|^2 = 1$. Thus,

$$\left(m - \sum_{i,t} c_{it}^2\right) \geq (m - \ell)$$

and we have letting $k \rightarrow +\infty$ in (12) that

$$m \sup_M |H| \geq (m - \ell) C_b(r).$$

This concludes the proof of the first part of Theorem 2.1.

For the proof of the second part, we make use of the following characterization of stochastic completeness given in [13] (see [14, Theorem 3.1]): A Riemannian manifold M is stochastically complete if and only if for every $u \in C^2(M)$ with $u^* = \sup u < \infty$ there exists a sequence $\{x_k\}$ such that $u(x_k) > u^* - 1/k$ and $\Delta u(x_k) < 1/k$ for every $k \geq 1$.

Suppose that M is stochastically complete. Define $g: N^{n-\ell} \times \mathbb{R}^\ell \rightarrow \mathbb{R}$ by

$$g(z, y) = \hat{g}(z) = \phi_b(\rho_N(z))$$

where

$$\phi_b(t) = \begin{cases} 1 - \cos(\sqrt{b}t) & \text{if } b > 0, \quad t < \pi/2\sqrt{b}, \\ t^2 & \text{if } b = 0, \\ \cosh(\sqrt{-b}t) & \text{if } b < 0. \end{cases}$$

Then $f = g \circ \varphi$ is a smooth bounded function on M . Thus there exists a sequence of points $\{x_k\}$ in M such that

$$f(x_k) > f^* - 1/k \quad \text{and} \quad \Delta f(x_k) < 1/k$$

for $k \geq 1$, where $f^* = \sup_M f \leq \phi_b(r) < \infty$. Similar as before, we have

$$\begin{aligned} \text{Hess}_{N \times \mathbb{R}^\ell} g(\varphi(x_k))(e_i, e_i) &= \text{Hess}_N \hat{g}(z(x_k))(\pi_{TN} e_i, \pi_{TN} e_i) \\ &= \phi_b''(r_k) \alpha_i^2 + \phi_b'(r_k) \sum_{j=2}^{n-\ell} a_{ij}^2 \text{Hess}_N \rho_N(z(x_k))(\partial/\partial\theta_j, \partial/\partial\theta_j) \\ &\geq \phi_b''(r_k) \alpha_i^2 + \phi_b'(r_k) C_b(r_k) \sum_{j=2}^{n-\ell} a_{ij}^2 \\ &= \phi_b''(r_k) \alpha_i^2 + \phi_b'(r_k) C_b(r_k) \left(1 - \alpha_i^2 - \sum_{t=1}^{\ell} c_{it}^2\right) \\ &= \phi_b'(r_k) C_b(r_k) \left(1 - \sum_{t=1}^{\ell} c_{it}^2\right) \end{aligned}$$

since $\phi_b''(t) - \phi_b'(t)C_b(t) = 0$. Here, we are writing $r_k = \rho_N(z(x_k))$. Therefore,

$$\begin{aligned} \frac{1}{k} > \Delta f(x_k) &= \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}^\ell} g(e_i, e_i) + \langle \text{grad}^{N \times \mathbb{R}^\ell} g, \vec{H} \rangle \\ &\geq \phi_b'(r_k) C_b(r_k) \left(m - \sum_{i,t} c_{it}^2 \right) + \phi_b'(r_k) \langle \text{grad}^{N \times \mathbb{R}^\ell} \rho_N, \vec{H} \rangle \\ &\geq \phi_b'(r_k) ((m - \ell) C_b(r_k) - m \sup |H|). \end{aligned}$$

Finally, since $\lim_{k \rightarrow \infty} \phi_b'(r_k) > 0$, letting $k \rightarrow \infty$ we have

$$\sup |H| \geq \frac{(m - \ell)}{m} C_b(r).$$

Proof of Corollary 2.2: If φ is proper in \mathbb{R}^{n+1} , from part (a) of Theorem 2.1 we would have $|H| \geq 1/nr$, and that is a contradiction. ■

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