THE RIEMANN PROBLEM FOR COMPOSITIONAL FLOWS IN POROUS MEDIA WITH MASS TRANSFER BETWEEN PHASES

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Abstract. We are interested in solving systems of conservation laws modeling multiphase fluid flows under the approximation of local thermodynamical equilibrium except at very localized places. This equilibrium occurs for states on sheets of a stratified variety called the “thermodynamical equilibrium variety”, obtained from thermodynamical laws. Strong deviation from equilibrium occurs in shocks connecting adjacent sheets of this variety.

We assume that fluids may expand and we model the physical problem by a system of equations where a velocity variable appears only in the flux terms, giving rise to a wave with “infinite” characteristic speed. We develop a general theory for fundamental solutions for this class of equations. We study all bifurcation loci, such as coincidence and inflection locus and develop a systematic approach to solve problems described by similar equations.

For concreteness, we exhibit the bifurcation theory for a representative system with three equations. We find the complete solution of the Riemann problem for two phase thermal flow in porous media with two chemical species; to simplify the physics, the liquid phase consists of a single chemical species. We give an example of steam and nitrogen injection into a porous medium, with applications to geothermal energy recovery.

1. Introduction

We are interested in systems of conservation laws modeling flows in porous media with mass conservation of each chemical species, under local thermodynamical equilibrium between phases. These models are called compositional models in petroleum science, see [10], [14], [24]. These systems of conservation laws can be used for either thermal or isothermal compositional models; in the former there is also an equation representing conservation of energy.

Such flows are modeled by systems of equations exhibiting a particular feature: a non-constant unknown u appears in the flux terms only:

$$\frac{\partial}{\partial t}G(V)+\frac{\partial}{\partial x}uF(V)=0. \tag{1.1}$$

The dependent variables in (1.1) are $V\in\Omega\subset\mathbb{R}^m$ and $u\in\mathbb{R}$. In the system of equations (1.1) the fluids can exist in several physical phases. The assumption of thermodynamical equilibrium restricts the variables to a stratified variety where the liquid and gaseous phases can coexist. In [19], we give a rigorous definition of this stratified variety. In that paper, we show how compositional models originate from systems of balance laws. We call each sheet (together with its particular evolution system) a “phase configuration”. The equilibrium variety is continuous and piecewise smooth; each sheet of the variety has its own variables

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V, its own system of conservation laws and its own interpretation. The unknowns V are called the "primary variables" of the sheet. The vector-valued functions G and F are smooth on each sheet, but only continuous at sheet junctions. Contrary to the classical Buckley-Leverett theory for two-phase flow in porous media, in these systems the total volume is not conserved, so the independent variable u representing volumetric flow rate is not constant in space. An important result is that we can obtain u in terms of the primary variables, hence the name "secondary variable" for u. We say that (V, u) lies in a phase configuration if V lies in this phase configuration.

In (1.1), generically, we have saturation variables, thermodynamical variables and the speed u. In our class of models, we assume that pressure variations are so small that they do not affect gas volume; the latter varies due to temperature changes or mass transfer between phases. These are realistic assumption for many flows in porous media, see [14]. Since the pressure is fixed, the main thermodynamical variables are the temperature T and the compositions of each phase.

We aim at a systematic theory to solve Riemann problems associated to (1.1). We will see that in such models rarefaction waves typically occur within each phase configuration. We assume that there is very fast mass transfer in the infinitesimally thin space between regions in adjacent phase configurations, so we propose shocks linking such configurations. In [11], [12], [13], a general bifurcation theory for the Riemann solutions of conservation laws was developed. Here, we generalize that theory for equations of type (1.1) possessing a variable u only in the flux term.

In Theorem 1 we summarize the theory on the solution structure and show that, up to a scaling, the Riemann solution can be obtained solely in terms of the primary variables; once the rarefaction and shock waves are found in the spaces of primary variables, they determine the secondary variable u, given either left or right Riemann data for u. In [23], the structure of a single combustion wave in which the speed u changes was examined; here u may vary in almost all waves.

In Section 2, we present the flow of nitrogen, steam and liquid water in a porous medium. This is a representative and interesting example for the theory because it exhibits a non-trivial stratified variety, as well as a new type of wave.

In Section 3, we generalize the Triple Shock Rule [11] and the Bethe-Wendroff theorem. These results isolate resonances and bifurcations. In Section 3.2 we define shocks connecting different configurations. In Section 4, we present the fundamental theorems for bifurcation theory. In Section 5, we obtain the elementary waves in each phase configuration: shocks, rarefactions or contact discontinuities. There exists rarefaction wave associated to evaporation in the two-phase configuration. We intend to explain this strange wave in a future paper.

In Section 6, we present the Riemann solution for the problem of geothermal energy recovery at moderate temperatures. In Section 7, we draw the conclusions. In A, we present the thermodynamical laws for the example.

2. Phase configurations in a specific model

Compositional models (1.1) in porous media are widely studied in Petroleum Engineering, see [14]. They describe flows in porous media where mass transfer of chemicals between phases, and possibly temperature changes need to be tracked. In [16], we have studied injection of steam and water in several proportions into a porous medium containing steam (gaseous H_2O), water (liquid H_2O) or a mixture. The thermodynamics in [16] is very simple. Here we consider the one-dimensional horizontal flow resulting the injection of steam and *nitrogen* in a porous rock cylinder, see also [18]; the thermodynamics for this system is non-trivial. We disregard gravity effects and heat conductivity; we assume that the pores in the rock are fully filled with fluids (one of the fluids is gaseous). Different fluid phases do not mix microscopically. Each saturation variable is the local fraction of the volume of a fluid phase relative to the total volume of the fluid phases. The rock has constant porosity φ and absolute permeability k (see A). We assume that the fluids are incompressible. This is a good approximation for liquid water; for the gaseous mixture of steam and nitrogen we assume that the gas density does not change due to pressure, but it is expansible and its density is a function of the temperature only; in other words, we assume that the pressure variations along the core are so small compared to the prevailing pressure that they do not affect the physical properties of the gas phase. The recovery of geothermal energy is used as an application of the model and theory developed in this paper.

2.1. **The concrete model.** The model used as example utilizes Darcy's law for multiphase flows, relating the pressure gradient in each fluid phase (water and gas) with its seepage speed:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p}{\partial x}. \tag{2.1}$$

The water and gas relative permeability functions $k_{rw}(s_w)$ and $k_{rg}(s_g)$ are considered to be functions of their respective saturations (see A); μ_w and μ_g are the viscosities of liquid and gaseous phases. Since we are interested in large scale problems, with flow rate far from zero we have disregarded capillarity effects (entailing equal pressures in all phases) as well as diffusive effects. The "fractional flows" for water and steam are saturation-dependent functions defined by:

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{rg}/\mu_g}, \qquad f_g = \frac{k_{rg}/\mu_g}{k_{rw}/\mu_w + k_{rg}/\mu_g}. \tag{2.2}$$

The saturations s_w and s_g add to 1. By (2.2) the same is true for f_w and f_g. Using Darcy's law (2.1), the definitions (2.2) yield:

$$u_w = uf_w, \qquad u_g = uf_g, \qquad \text{where} \quad u = u_w + u_g \quad \text{is the } \textit{total} \text{ or } \textit{Darcy velocity.} \tag{2.3}$$

We write the equations of conservation of total mass of water (liquid and gaseous H_2O) and nitrogen (gaseous N_2) as follows, [14]:

$$\frac{\partial}{\partial t}\varphi\left(\rho_W s_w + \rho_{gw} s_g\right) + \frac{\partial}{\partial x}u\left(f_w\rho_W + \rho_{gw}f_g\right) = 0. \tag{2.4}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn}s_g + \frac{\partial}{\partial x}uf_g\rho_{gn} = 0, \tag{2.5}$$

here ρ_W is the liquid water density, assumed to be constant, ρ_{gw} (ρ_{gn}) denote the concentration of vapor (nitrogen) in the gaseous phase (mass per unit gas volume).

To describe temperature variation, we formulate the energy conservation in terms of enthalpies, see [1], [2], as we ignore adiabatic compression and decompression effects. We neglect longitudinal heat conduction and heat losses to the surrounding rock. We assume also that the temperature T in water, solid and gas phases is the same. Thus the energy conservation is given by:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_W s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\Big(H_W f_w + H_g f_g\Big) = 0, \tag{2.6}$$

here $\hat{H}_r = H_r/\varphi$ and H_r, H_W and H_g are the rock, the liquid water and the gas enthalpies per unit volume; their expressions can be found in Eq. $(A.3)$.

The unknowns on the system are (subsets of) the variables T, s_g, ψ_{gw} and u; the phase configuration of the flow determines which unknowns are used, as explained in next sections. The quantity ψ_{gw} represents the composition (molar fraction) of the vapor in the gaseous phase; in Section 2.2.1, we derive the system of equations where ψ_{gw} is an unknown.

2.2. **Phase configurations in the example.** An innovative feature of our model deals with phase transitions. In [7], Colombo et. al. studied a problem with phase transitions in 2×2 systems of conservation laws. Their physical domain was formed by two disjoint sub-domains, which were called phases by Colombo: they are phase configurations in our nomenclature. The phase transition is the jump between states in different sub-domains. Our theory is physically more realistic because it includes also infinitesimally small phase transitions, as our sub-domains may be adjoining.

In this specific model, there are three main different phase configurations: a single-phase liquid configuration, *spl*, in which pores contain only liquid water; a single-phase gaseous configuration, *spg*, in which pores are filled with steam and nitrogen; and a two-phase configuration, *tp*, in which pores are filled with a mixture of liquid water, gaseous nitrogen and steam. In the latter case, the temperature is specified by the concentration of vapor in the gas through Clausius-Clapeyron law, as we will see. We assume that each configuration is in local thermodynamical equilibrium, so we can use Gibbs' phase rule, $f_G = c - p + 2$ represents Gibb's number of thermodynamical degrees of freedom, c and p are the number of chemical species and phases, respectively. As in our thermodynamical model the pressure is fixed, the remaining number of thermodynamical degrees of freedom is $f = f_G - 1$.

2.2.1. *Single-phase gaseous configuration - spg.* There are two chemical species (N_2 and H_2O) and one gaseous phase, i.e., $c = 2$ and $p = 1$, so the number of thermodynamical degrees of freedom is $f = 2$: temperature and gas composition. The only other unknown is u. We define the steam and nitrogen gas compositions ψ_{gw} and ψ_{gn} as follows, see [3], [18]:

$$\psi_{gw} = \rho_{gw}/\rho_{gW}(T), \quad \psi_{gn} = \rho_{gn}/\rho_{gN}(T), \quad \text{with} \quad \psi_{gw} + \psi_{gn} = 1, \tag{2.7}$$

where ρ_{gW} and ρ_{gN} are the densities of pure steam and pure nitrogen given by $(A.8)$. We assume that in the nitrogen and vapor there are no effects due to mixing so that the volumes of the components are additive, hence Eq. $(2.7.c)$.

Using Eqs. (2.7) we rewrite Eqs. (2.4)-(2.6) as follows:

$$\frac{\partial}{\partial t}\varphi\rho_{gW}\psi_{gw} + \frac{\partial}{\partial x}u\rho_{gW}\psi_{gw} = 0, \tag{2.8}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gN}\psi_{gn} + \frac{\partial}{\partial x}u\rho_{gN}\psi_{gn} = 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}\rho_{gW}h_{gW} + \psi_{gn}\rho_{gN}h_{gN}\right) + \frac{\partial}{\partial x}u\Big(\psi_{gw}\rho_{gW}h_{gW} + \psi_{gn}\rho_{gN}h_{gN}\Big) = 0; \tag{2.10}$$

where h_{gW} and h_{gN} are the enthalpies per mass unit of pure steam and nitrogen, which are functions of T given in $(A.1)$-$(A.2)$. Since the liquid water saturation vanishes, the water enthalpy H_W does not appear in the energy equation (2.10). For the gaseous phase enthalpy H_g, we use $(A.3.c)$ and ρ_{gw}, ρ_{gn} given by (2.7).

Remark 1. *The spg configuration consists of a 2-Dimensional (2-D) set of triplets* $(s_w = 0, T, \psi_{gw})$. *They form a sheet of the thermodynamical equilibrium stratified variety, where* $\psi_{gw} \leq \Gamma(T)$, *see* $(A.7.a)$, *Fig.* 2.1 *and* [3], [18], *for*

$$\Gamma(T) \equiv \rho_{gw}(T)/\rho_{gW}(T). \tag{2.11}$$

Here, $V = (T, \psi_{gw})$ *are the primary variables,* $s_w = 0$ *and* $\psi_{gn} = 1 - \psi_{gw}$.

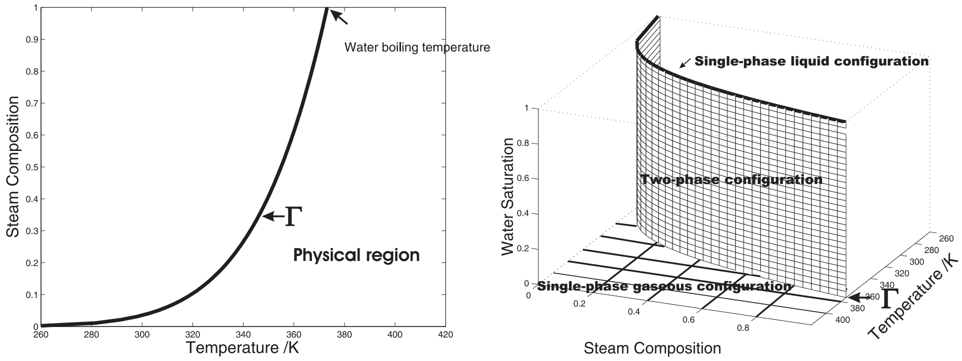

FIGURE 2.1. *a) left:* The physical region for the *spg* configuration of steam and nitrogen is formed by the pairs (T, ψ_{gw}) satisfying $\psi_{gw} \leq \Gamma(T)$. The solid graph $\Gamma(T)$ represents the composition of a mixture of nitrogen and steam in equilibrium with liquid water, see Remark 1. *b) right:* Phase space for (s_W, ψ_{gw}, T) and phase configurations sheets: single phase liquid (*spl*), single phase gaseous (*spg*), and the two-phase (*tp*) configurations. The union of the sheets for *spl*, *spg* and *tp* sheets is the stratified variety.

2.2.2. *Single-phase liquid configuration - spl.* There is one chemical species (H_2O) and one phase, so $f = 1$ and the single thermodynamical degree of freedom is the temperature. The liquid water does not expand and composition changes have no volumetric effects, so that the total Darcy velocity u is independent of position. Eqs. (2.4), (2.5) are satisfied trivially. As rock and liquid water have constant heat capacity, see $(A.3.a)$ and $(A.3.c)$, (2.6) reduces to:

$$\frac{\partial}{\partial t}T + \lambda_T^W \frac{\partial T}{\partial x} = 0, \quad \text{where} \quad \lambda_T^W = \frac{u_W}{\varphi} \frac{C_W}{C_W + \hat{C}_r}, \tag{2.12}$$

where we use u_W to indicate that the velocity u is spatially constant in the *spl* region; here C_W is the water heat capacity and $\hat{C}_r$ is the rock heat capacity C_r divided by φ. All heat capacities are in energy per unit mass.

The *spl* configuration consists of a 1-D set of triplets $(s_w = 1, T, \psi_{gw}(T))$, see Fig. 2.1. Here, $V = T$ is the primary variable, $s_w = 0$ and $\psi_{gw} = \Gamma(T)$. Actually ψ_{gw} is unnecessary, however we specify it as in the *tp* configuration to ensure continuity of the equilibrium variety, see Fig. 2.1.*b*.

2.2.3. *Two-phase configuration - tp.* There are two chemical species (N_2 and H_2O), $c = 2$, and two phases (liquid water and gas), $p = 2$; so $f = 1$ and the free thermodynamical variable is temperature. The steam composition ψ_{gw} is prescribed as function of temperature, from $\psi_{gw}(T) = \Gamma(T)$, for Γ defined by (2.11). One obtains ψ_{gn} from Eq. (2.7.*c*).

The *tp* configuration consists of a 2-D set of triplets $(s_w, T, \psi_{gw}(T))$, see Fig. 2.1. Here, $V = (s_w, T)$ are the primary variables and $\psi_{gw} = \Gamma(T)$.

Fig. 2.1.*b* shows the three phase configurations in the variables (T, ψ_{gw}, s_w), i.e., the whole equilibrium variety. As basic variable, we arbitrarily choose s_w, not s_g.

3. THEORY OF RIEMANN SOLUTIONS FOR COMPOSITIONAL FLOWS

We are interested in the Riemann-Goursat problem associated to (1.1) with initial data with initial and boundary data of the form:

$$\begin{cases} (V,u)_L, & \text{if } x = 0,\ t \geq 0, \\ (V,\cdot)_R, & \text{if } x > 0,\ t = 0. \end{cases} \tag{3.1}$$

The injection point is $x = 0$. If the characteristic speeds are positive, we may expect that this problem can be regarded as a Riemann problem with initial data $(V, u)_L$ for $x < 0$ defined by the injection boundary data at $x = 0$. We prove that in (3.1) one Darcy speed should be provided as boundary data. The speed $u_L > 0$ is specified at the injection point. In the next sections we show that u_R can be obtained in terms of u_L and the primary variables.

Conservation laws with Riemann data exhibit self-similar solutions. In the plane (x, t), the rarefactions are continuous solutions, while the shocks are the discontinuous ones. Despite the absence of the unknown u in the accumulation terms, we will prove that the Riemann solution associated to Eq. (1.1) still consists of a sequence of elementary waves, rarefactions and shocks; for the classical definitions of these waves, see [8], [26]. In our problems the rarefaction waves reside in individual sheets of the equilibrium variety, while shocks have left and right states in single sheet or in contiguous sheets of the equilibrium variety.

3.1. **Characteristic speeds.**

In each phase configuration, systems of conservation laws in appropriate form must be used to find the characteristic speeds. If we assume that the solution is sufficiently smooth, we differentiate all equations in (1.1) with respect to their variables, obtaining systems of the form:

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} V \\ u \end{pmatrix} = 0, \tag{3.2}$$

where the matrices $B(V)$ and $A(V)$ are the derivatives of $G(V)$ and $uF(V)$ with respect to the primary variables V and to u. Since $G(V)$ does not depend on u, the last column in the matrix B is zero. For the pair of primary and secondary variables $W = (V, u)$, the characteristic values $\lambda^i := \lambda^i(W)$ and vectors $\boldsymbol{r}^i := \boldsymbol{r}^i(W)$, (where i is the family label; characteristic values increase with i):

$$A\boldsymbol{r}^i = \lambda^i B\boldsymbol{r}^i, \quad \text{where } \lambda^i \text{ is obtained by solving} \quad det(A - \lambda^i B) = 0. \tag{3.3}$$

Similarly the left eigenvectors $\boldsymbol{\ell}^i = (\ell_1^i, \ell_2^i, \ell_3^i)$ satisfy:

$$\boldsymbol{\ell}^i A = \lambda^i \boldsymbol{\ell}^i B, \quad \text{for the same } \lambda^i. \tag{3.4}$$

Remark. Here we derive the formulae for a general 3×3 system because it is the smallest non-trivial system of type (1.1); it will be used in the *tp* configuration. For such a system, generically the right and left eigenvectors have three components, corresponding to the two primary variables and to the secondary variable.

Hereafter, the word "eigenvector" means "right eigenvector". The following lemma characterizes the eigenpairs, see [17]; for another derivation, see [6].

Lemma 1. *Assume that* $u \neq 0$ *and that the flux vector* $F(V)$ *does not vanish for any* V*. For each eigenvalue* λ *the corresponding right and left eigenvectors in the generalized eigenvalue problem* (3.3) *have the form:*

$$\lambda = u\vartheta(V), \;\; \boldsymbol{r} = (g_1(V), g_2(V), ug_3(V))^T \;\; \text{and} \;\; \boldsymbol{\ell} = (\ell_1(V), \ell_2(V), \ell_3(V)), \tag{3.5}$$

where ϑ*,* g_l *and* ℓ_l *(for* $l = 1, 2, 3$*) are functions of* V *only. Moreover, there are at most* 2 *eigenvalues and eigenvectors associated to this problem with* 3 *equations.*

In [17], we generalize Lemma 1 for n equations.

For any $\vartheta \in \mathbb{R}$, let us define the following 3×2 matrix:

$$C(V;\vartheta) = \frac{\partial F}{\partial V}\,(V) - \vartheta \frac{\partial G}{\partial V}(V); \tag{3.6}$$

we also define the following 3×3 matrix, where $F(V)$ is the flux column vector:

$$A - \lambda B = \Big(uC(V;\lambda/u) \;\Big|\; F(V)\Big). \tag{3.7}$$

Proof of Lemma 1**:** The eigenvalues λ of (3.7) are the roots of $det(A - \lambda B) = 0$. Since $u \neq 0$, we divide the first 2 columns of (3.7) by u and set $\vartheta = \lambda/u$, obtaining the characteristic equation:

$$det\Big(C(V;\vartheta) \;\Big|\; F(V)\Big) = 0. \tag{3.8}$$

Since Eq. (3.8) depends on V only, the rescaled eigenvalues ϑ are functions of V, i.e., $\vartheta = \vartheta(V)$. Clearly the eigenvalues λ have the form (3.5.a).

The eigenvectors are solutions of $(A - \lambda B)\boldsymbol{r} = 0$, or $\Big(uC(V;\vartheta) \;\Big|\; F(V)\Big)\boldsymbol{r} = 0$. For each V there is a neighborhood $\mathcal{B}$ in which $F_k \neq 0$ for some $k \in \{1, 2, 3\}$, so:

$$r_3 = u\frac{1}{F_k(V)}\sum_{l=1}^{2} C_{kl} r_l, \qquad \text{with } \; C_{kl} = C_{kl}(V;\vartheta(V)), \tag{3.9}$$

where C_{kl} is the (k,l) element of $C(V;\vartheta)$ and $\boldsymbol{r} = (r_1, r_2, r_3)^T$. By suitable compactness arguments we can extend the result (3.9) for all $V \in \Omega$.

Substituting r_3 given by Eq. (3.9) into $(A - \lambda B)r = 0$, we obtain a linear system in the unknowns r_l for $l = 1, 2$. We can cancel u in this system, showing that the r_l for $l = 1, 2$ depend on V only. So (3.5.b) follows from (3.9).

In order to determine $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$, we impose $\boldsymbol{\ell} \cdot \Big(uC(V;\vartheta) \;\Big|\; F(V)\Big) = 0$, so that $\boldsymbol{\ell}$ solves the following system of three equations:

$$uC_{1l}\ell_1 + uC_{2l}\ell_2 + uC_{3l}\ell_3 = 0, \quad \text{for } \; l = 1, 2, \tag{3.10}$$

$$F_1\ell_1 + F_2\ell_2 + F_3\ell_3 = 0. \tag{3.11}$$

We divide Eqs. (3.10) by u and we obtain a system with coefficients that depend only on the variables V, leading to (3.5.c). □

Following the classical construction for fixed (i) and using $(3.5.b)$, we define locally the **integral curves** in $W=(V,u)$ space as solutions of:

$$\left(\frac{dV}{d\xi},\frac{du}{d\xi}\right)=\boldsymbol{r},\ \text{ i.e, }\ \frac{dV_1}{d\xi}=g_1(V),\ \ \frac{dV_2}{d\xi}=g_2(V)\ \text{ and }\ \frac{du}{d\xi}=ug_3(V); \tag{3.12}$$

if the parametrization by ξ is chosen to satisfy:

$$\xi=\lambda(W(\xi)),\qquad \text{so that }\ \nabla\lambda(W(\xi))\cdot\boldsymbol{r}(W(\xi))=1, \tag{3.13}$$

and ξ, λ increase together, the integral curve defines a **rarefaction curve**. A rarefaction curve $W(\xi)$ parametrizes a rarefaction wave in the (x,t) plane, provided:

$$\xi=x/t=\lambda(W(x/t)). \tag{3.14}$$

Lemma 2. *Assume that* $\nabla\lambda\cdot r(V)\neq 0$ *and* $r_3(V)\neq 0$. *Generically, there exists a normalization for the eigenvector* $\boldsymbol{r}$ *in terms of* V *and* u^-, *so that* $(3.13.b)$ *is true.*

Proof: Assume that λ is a eigenvalue with geometric multiplicity 1. Since $r_3\neq 0$, then $g_3\neq 0$, we solve $(3.3.a)$ to obtain an eigenvector $\boldsymbol{r}$ of form $(3.5.b)$, with:

$$g_1(V)=\alpha_1(V)g_3(V),\quad g_2(V)=\alpha_2(V)g_3(V),\quad \text{where} \tag{3.15}$$

$$\alpha_1=\frac{C_{21}F_1-C_{11}F_2}{C_{22}C_{11}-C_{12}C_{21}}\quad\text{and}\quad \alpha_2=\frac{C_{12}F_1-C_{22}F_2}{C_{22}C_{11}-C_{12}C_{21}}. \tag{3.16}$$

See $(3.9.b)$ for the definition of C_{kl}. If λ has geometric multiplicity two, we can obtain the same result with minor modifications.

Using the form of λ given in $(3.5.a)$ and (3.15), we see that imposing $\nabla\lambda\cdot\boldsymbol{r}=1$ is equivalent to:

$$\left(u\frac{\partial\vartheta}{\partial V_1},u\frac{\partial\vartheta}{\partial V_2},\vartheta\right)\cdot(\alpha_1(V)g_3(V),\alpha_2(V)g_3(V),ug_3(V))=1,\ \text{ or}$$

$$u\beta(V)g_3(V)=1,\quad\text{where}\quad \beta(V)=\frac{\partial\vartheta}{\partial V_1}\alpha_1+\frac{\partial\vartheta}{\partial V_2}\alpha_2+\vartheta. \tag{3.17}$$

Differentiating $(3.17.a)$ with respect to ξ along an integral curve, we obtain:

$$\frac{du}{d\xi}\beta(V)g_3(V)+u\left(\nabla_V\beta(V)\cdot\frac{dV}{d\xi}\right)g_3(V)+u\beta(V)\left(\nabla_V g_3(V)\cdot\frac{dV}{d\xi}\right)=0, \tag{3.18}$$

where ∇_V is the gradient with respect to the variables V.

Using (3.12) for the derivatives and then (3.15), we can rewrite (3.18) as:

$$u\big(g_3\beta g_3+(\nabla_V\beta\cdot(\alpha_1g_3,\alpha_2g_3))\,g_3+\beta\,(\nabla_V g_3\cdot(\alpha_1g_3,\alpha_2g_3))\,\big)=0. \tag{3.19}$$

From (3.17), we have that $u\beta(V)g_3(V)\neq 0$, so we divide (3.19) by $u\beta g_3^2$ obtaining:

$$\alpha_1\frac{\partial\ln(\beta g_3)}{\partial V_1}+\alpha_2\frac{\partial\ln(\beta g_3)}{\partial V_2}=-1, \tag{3.20}$$

which is a linear PDE for βg_3, with initial condition given by:

$$(\beta g_3)\,(V^-)=1/u^-. \tag{3.21}$$

We can solve the linear PDE (3.20) by using the method of characteristics. Since $\nabla\lambda\cdot\boldsymbol{r}(V(^-))\neq 0$, the initial data are prescribed along a non-characteristic curve.

We obtain the characteristic curve in (V_1, V_2) space parametrized by ς:

$$\frac{dV_1}{d\varsigma} = \alpha_1(V),\ \frac{dV_2}{d\varsigma} = \alpha_2(V),\ \frac{d\ln(\beta g_3)}{d\varsigma} = -1,\ \text{where } V(\varsigma = 0) = V^-, \tag{3.22}$$

and satisfying (3.21). Notice from (3.12) and (3.15) that this characteristic curve is the projection of the integral curve in the space (V_1, V_2).

Because α_1 and α_2 do not vanish simultaneously, so we can solve (3.22.a, b) in the space (V_1, V_2), obtaining $V(\varsigma, V^-) = (V_1(\varsigma, V^-), V_2(\varsigma, V^-))$. Using the initial condition (3.21), from Eq. (3.22.c), $g_3(V)$ is written as:

$$g_3(V) = \frac{1}{u^-\beta}\exp\left(-(\varsigma - \varsigma^-)\right). \tag{3.23}$$

Then g_1 and g_2 are obtained from (3.15). To complete the proof of Lemma, we need to prove that g_1, g_2 and g_3 defined by Eqs. (3.15) and (3.23) yield $\nabla\lambda \cdot r = 1$.

Since $r_3(V) \neq 0$, we can obtain a eigenvector $\boldsymbol{r}$, such that (3.12) becomes (3.22.a), (3.22.b) and $du/d\xi = u$.(Notice that here ξ becomes ς.) From the latter equations, u can be written as:

$$u = u^-\exp\left(\varsigma - \varsigma^-\right). \tag{3.24}$$

From Eq. (3.17) notice that $\nabla\lambda \cdot r = u\beta(V)g_3(V)$. Using g_3 from (3.23) and u from Eq. (3.24), we obtain $\nabla\lambda \cdot r = 1$, so the Lemma is proved. □

Lemma 2 is stated similarly for dimension $n > 3$. Using Lemma 2, we have:

Proposition 1. *Assume that near $W^- = (V^-, u^-)$ the eigenvector $\boldsymbol{r}$ associated to a certain family forms a vector field. We calculate the primary variables V on the rarefaction curve from V^-, as functions of V and W^- only, i.e., first we calculate the integral curve in the primary variables by solving* (3.12.b)-(3.12.c) *in the classical way to obtain $V(\xi)$ satisfying $V(\xi^-) = V^-$ with $\xi^- = \lambda(W^-) = x^-/t^-$, and then we complete the calculation for the secondary variable u in terms of $V(\xi)$ from:*

$$u(\xi) = u^- exp(\gamma(\xi)), \quad with \quad \gamma(\xi) = \int_{\xi^-}^{\xi} g_3(V(\eta))d\eta, \tag{3.25}$$

where g_3 is obtained as r_3/u, see (3.5.b)*; $u = u^-$ for $\xi = \xi^-$.*

Proof: Using (3.12) and (3.13) with $g_l(V) = r_l$ for $l = 1, 2$ we obtain V independently of u by solving the system of ordinary differential equations (3.12.b),(3.12.c), for $\xi^- = \lambda(W^-)$ and $V(\xi^-) = V^-$.

After obtaining $V(\xi)$, we use the expression for the last component of $\boldsymbol{r}$ in (3.5.b) to solve $du/d\xi = ug_3$, yielding (3.25). Lemma 1 asserts that λ has the form $u\vartheta(V)$, so using Eqs. (3.13) and (3.25), we obtain ξ implicitly as:

$$\xi = u^-\vartheta(V(\xi))exp(\gamma(\xi)). \tag{3.26}$$

Since ξ depends only on u^- and V on integral curves, the proof is complete. □

Definition 1. *Only the first* 2 *coordinates of the eigenvectors are pertinent to define the integral curves in the space of primary variables V. Therefore, for stating and proving the* *generalized Bethe-Wendroff Theorem in Section* 4*, it is useful to define for any i-family the following quantities that do not depend on u:*

$$\tilde{\boldsymbol{r}}^i = (r_1^i, r_2^i), \quad \tilde{\boldsymbol{\ell}}^i = (\ell_1^i, \ell_2^i) \quad \textit{and} \quad \tilde{\lambda}^{i,+}(V^+) := \lambda_i(W^+)/u^+. \tag{3.27}$$

Proposition 2. *Assume that* $u^- \neq 0$*, so we can perform the change of variables:*

$$\xi = x/t \longrightarrow \tilde{\xi} = x/\big(u^- t\big). \tag{3.28}$$

Any rarefaction wave in (1.1) *can be written in the space of variables* (χ, t)*, where* $\chi = x/u^-$*, i.e., in these space-time coordinates the rarefaction wave projected on the space of primary variables is independent of* u*.*

Proof: Performing the change of variable $(x,t) \longrightarrow (\chi, t)$, we rewrite (3.2) as:

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + \frac{1}{u^-}A\frac{\partial}{\partial \chi}\begin{pmatrix} V \\ u \end{pmatrix} = 0. \tag{3.29}$$

To obtain the characteristic speeds and vector, we need to solve:

$$A\hat{\boldsymbol{r}}^i = u^-\hat{\lambda}^i B\hat{\boldsymbol{r}}^i \quad \text{where } \hat{\lambda}^i \text{ is obtained by solving} \quad det(A - u^-\hat{\lambda}^i B) = 0. \tag{3.30}$$

From (3.3) and (3.30), we know that $\hat{\lambda}^i$ and $\hat{\boldsymbol{r}}^i$ satisfy:

$$\hat{\lambda}^i = \lambda^i/u^- \quad \text{and} \quad \hat{\boldsymbol{r}}^i = \boldsymbol{r}^i. \tag{3.31}$$

The rarefaction wave is obtained by solving:

$$\left(\frac{dV_1}{d\tilde{\xi}}, \frac{dV_2}{d\tilde{\xi}}, \frac{du}{d\tilde{\xi}}\right) = (g_1, g_2, ug_3), \ \text{ for } \tilde{\xi} = \frac{\chi}{t} = \hat{\lambda}(W(\tilde{\xi})), \ V(u^-\tilde{\xi}^-) = V^-, \tag{3.32}$$

From Eqs. (3.5.a), (3.31.b) and (3.32.b), it follows that:

$$\tilde{\xi}^- = \vartheta(V(\tilde{\xi}^-)). \tag{3.33}$$

Solving the ODE for $du/d\tilde{\xi}$ in (3.32.a) and using (3.3.a) and (3.31.a), we obtain:

$$\tilde{\xi} = \chi/t = \hat{\lambda}(W(\tilde{\xi})) = \vartheta(\tilde{V}(\tilde{\xi}))exp(\gamma(\tilde{\xi})), \ \text{ for } \gamma(\tilde{\xi}) = \int_{u^-\tilde{\xi}^-}^{u^-\tilde{\xi}} g_3(V(\eta))d\eta. \tag{3.34}$$

Thus in the (χ, t) space, the ODE's for V_1 and V_2 in (3.32) do not depend on u, so the rarefaction curve does not depend on u. The speed keeps the form (3.25). □

Remark. A rarefaction wave connecting adjoining sheets of the stratified variety occurs occasionally; such a connection can happen only when there is equality between characteristic speeds of the rarefaction curve branches at the left and right sheets. Such a rarefaction curve should be continuous; however generically its derivative is discontinuous at the boundary between the sheets.

3.2. **Shock waves.** Shocks are certain discontinuities in the solution of the PDE's. The discontinuities are the pairs $(W^-; W^+)$ such that the function $\mathcal{H} = \mathcal{H}(W^-; W^+)$ defined as:

$$\mathcal{H} := v\left(G^+ - G^-\right) - u^+F^+ + u^-F^-, \tag{3.35}$$

vanishes. Here $W^- = (V^-, u^-)$ and $W^+ = (V^+, u^+)$ are the states on the left and right sides of the discontinuity; $v = v(W^-, W^+)$ is the discontinuity propagation speed; $G^- = G^-(V^-)$, $(G^+ = G^+(V^+))$ and $F^- = F^-(V^-)$, $(F^+ = F^+(V^+))$ are the accumulation and fluxes at the left (right) of the discontinuity. When the states $(-)$ and $(+)$ lie in the same phase configuration, the conserved quantities, accumulations and fluxes arise from a system of conservation laws in a single sheet; while if these states $(-)$ and $(+)$ lie in different configurations, the conserved quantities, accumulations and fluxes arise from different systems of conservation laws, defined in two sheets, so they have different expressions. For shocks contained in a single sheet, usually the velocity u varies and the formulae have the same form as the formulae describing shocks between sheets.

For a fixed $W^- = (V^-, u^-)$ the **Rankine-Hugoniot** curve (RH curve) parametrizes the discontinuous solutions of Eqs. (2.4)-(2.6); it consists of the $W^+ = (V^+, u^+)$ that satisfy $\mathcal{H}(W^-; W^+) := 0$. We specify the state (V^-, u^-) on the left hand side, but at the right u^+ cannot be specified, and we will see that it is obtained from the condition $\mathcal{H}(W^-; W^+) = 0$. We denote the RH curve starting at the state W^- by $\mathcal{RH}(W^-)$.

In this work we assume an extension of hyperbolicity, namely, except on the certain curves in state space where eigenvalues coincide, the system (1.1) is hyperbolic in the primary variables, i.e., there exists a basis of characteristic vectors for each state V, see [17]. Also, only connected branches of the RH curve are considered (i.e., branches that contain the $(-)$ state), see [17]. Thus we use the following criterion due to Liu, [21], [22], to define admissible discontinuities, or shocks:

Definition 2. *For a fixed W^-, we call a* ***shock curve*** *each connected part $\{W\}$ formed by the W^+ in the $\mathcal{RH}(W^-)$, such that $v(W^-, W^+) < v(W^-, W)$, where $W \in \mathcal{RH}(W^-)$ between W^- and W^+. In (x,t) space, each point of the shock curve represents a* ***shock wave****. The shock curve parametrizes the $(+)$ states of admissible shocks between the fixed $(-)$ and $(+)$ states.*

3.2.1. *Properties of the Rankine-Hugoniot curve.* For a fixed $W^- = (V^-, u^-)$ in a configuration, the RH curve, or $\mathcal{RH}(W^-)$, is obtained setting (3.35) to zero, i.e., for $k = 1, 2, 3$:

$$v[G_k] = u^+ F_k^+ - u^- F_k^-, \tag{3.36}$$

where $[G_k] = G_k^+ - G_k^-$, $G_k^\pm = G_k^\pm(V^\pm)$ and $F_k^\pm = F_k^\pm(V^\pm)$. We rewrite Eq. (3.36) as a linear system:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} \begin{pmatrix} v \\ u^+ \\ u^- \end{pmatrix} = 0. \tag{3.37}$$

We define the unordered pairs $\mathbb{K} = \{\{2,1\}, \{1,3\}, \{3,2\}\}$. We utilize the notation:

$$\mathcal{Y}_{kj} = F_k^+ F_j^- - F_j^+ F_k^-, \;\; \mathcal{X}_{kj}^+ = F_k^+[G_j] - F_j^+[G_k] \;\text{ and }\; \mathcal{X}_{kj}^- = F_k^-[G_j] - F_j^-[G_k]. \tag{3.38}$$

For a non-trivial solution of the system (3.37), the determinant of the matrix in Eq. (3.37) must vanish; this yields another form of the Rankine-Hugoniot curve denoted by $\mathcal{RH}(V^-)$, namely for each V^- it the set of V^+ satisfying $\mathcal{H}_V(V^-, V^+) = 0$ with:

$$\mathcal{H}_V := [G_1]\mathcal{Y}_{32} + [G_2]\mathcal{Y}_{13} + [G_3]\mathcal{Y}_{21}. \tag{3.39}$$

Generically, $\mathcal{RH}(V^-)$ is a 1-D structure, see B, independently of the number of equations, see [17].

There are two primary variables in V^+ for the *spg* and the *tp*, so in both cases $\mathcal{RH}(V^-)$ consists of the union of two curves through V^-, see [17]. In the *spl* there is only one scalar equation with the temperature as the primary variable, so $\mathcal{RH}(T^-)$ is the whole physical range of the temperature axis.

Solving the system (3.37), we obtain u^+ and v as functions of V^-, V^+ and u^-:

$$u^+ = u^- \frac{\mathcal{X}^-_{kj}}{\mathcal{X}^+_{kj}}, \quad u^+ = u^- \frac{\sum_{\{p,q\}\in\mathbb{K}} \mathcal{X}^-_{pq}\mathcal{X}^+_{pq}}{\sum_{\{p,q\}\in\mathbb{K}} \left(\mathcal{X}^+_{pq}\right)^2}, \quad v = u^- \frac{\mathcal{Y}_{jk}}{\mathcal{X}^+_{kj}}, \tag{3.40}$$

for any $\{k,j\} \in \mathbb{K}$. Eq. (3.40.b) is obtained from (3.40.a). It is useful for numerical calculations; there is a similar expression for v. Of course, Eqs. (3.40.a) and (3.40.c) are valid if the corresponding denominators are non-zero, while (3.40.b) requires that just one term in the denominator is non-zero. In [17] we give the definition of *regular RH curve*: for such a locus (3.40.b) is well defined if $V^+ \neq V^-$, because we assume that for each V in the primary variable space there is a $\{k,j\} \in \mathbb{K}$ such that the inequality $\mathcal{X}^-_{kj} \neq 0$ is satisfied.

Remark. In the definitions that follow, all wave structures can be obtained in the space of primary variables V. Using $u^+ = u^+(V^-, u^-; V^+)$ and $v := v(V^-, u^-; V^+)$, we define:

$$\mathcal{Z}(V^-;V^+) = \frac{u^+}{u^-}, \quad \tilde{v}^-(V^-;V^+) := \frac{v}{u^-} \quad \text{and} \quad \tilde{v}^+(V^-;V^+) := \frac{v}{u^-\mathcal{Z}(V^-;V^+)}. \tag{3.41}$$

Lemma 3. *Assume that u^- is positive and that all RH curves are regular. If the Darcy speed u^- is modified while V^- and V^+ are kept fixed, the Darcy speed u^+ as well as the shock waves are rescaled in the (x,t) plane, while the values of V are preserved in the rescaled shocks.*

Proof: Performing the change of variable $(x,t) \longrightarrow (\chi,t)$ in $\mathcal{H} = 0$, for $\chi = x/u^-$ and $\mathcal{H}$ given by (3.35), we obtain:

$$\frac{v}{u^-}(G^+ - G^-) = \frac{u^+}{u^-}F^+ - F^-. \tag{3.42}$$

The result follows from the relationships in (3.41), because $\mathcal{Z}$ and $\tilde{v}^-$ depend only on the primary variables V. □

3.3. **Wave Sequences and Riemann Solutions.** In the Riemann solution, we distinguish different states W and V by a subindex between parenthesis to avoid confusion with vector components. The left and right states are indicated only by the subindex L and R and the $(-)$ and $(+)$ states by the superscript $-$ and $+$.

A Riemann solution is a sequence of elementary waves w_k (shocks and rarefactions) and states $W_{(k)} = (V,u)_{(k)}$ for $k = 1, \cdots, m$, with increasing wave speeds. We will see that we can determine the Riemann solution in the space of primary variables V, without taking into account the secondary variable u. The values of the latter variable along the Riemann solutions are fully determined by the values of the primary variables supplemented by a boundary condition on u. From Prop. 2 and Lemma 3, a minor modification of the proof of Lemma 3 yields, see [17]:

Theorem 1. *Assume that u_L is positive and that the hypotheses of Lemma 3 hold. If the Darcy speed u_L in the initial data (3.1) is modified while V_L and V_R are kept fixed in the Riemann problem for (1.1), the Darcy speed u_R as well as the Riemann solutions are rescaled in the sense that the solution in the $(x/u_L,t)$ plane does not change with u_L, i.e., the values of V in the wave sequence are preserved. Then:*

$$u_R = u_L \prod_{l=1}^{\varrho_1} exp(\gamma_l) \prod_{m=1}^{\varrho_2} \mathcal{Z}_m, \tag{3.43}$$

We could specify u_R instead of u_L; a formula similar to (3.43) is valid. In the Proposition above, $\mathcal{Z}_m$ is given by (3.41.a) for the m-th shock wave. Similarly γ_l is given by (3.25.b) for $\xi = \xi^+$ computed along the l-th rarefaction curve. The integers $\varrho 1$ and ϱ_2 are the number of shock and rarefaction waves, respectively.

Remark. Theorem 1 says that the Riemann solution can be obtained in each phase configuration first in the primary variables V. Then the Darcy speed can be obtained at any point of the space (V, u) in terms of V and u_L by an equation analogous to Eq. (3.43). Because of this fact, we omit the speed u in the figures.

4. Bifurcation theory for Riemann solutions

Riemann solutions bifurcate non-trivially for general systems of conservation laws that violate the hypotheses of Lax Theorem, see [11], [12], [13].

4.1. **Fundamental theorems for bifurcation theory.**

We recall an important theorem for bifurcation of Riemann problems for systems of conservation laws of standard form, the *Triple Shock Rule* [11]:

Proposition 3. *For systems of conservation laws* $\partial G(V)/\partial t + \partial F(V)/\partial x = 0$*, consider three states* V^M*,* V^+ *and* V^-*. Assume that* $V^- \in \mathcal{RH}(V^+)$*,* $V^M \in \mathcal{RH}(V^-)$ *and* $V^+ \in \mathcal{RH}(V^M)$*, with speeds* $v^{+,-}$*,* $v^{-,M}$ *and* $v^{M,+}$*. Then, either:*

(1) $v^{+,-} = v^{M,-} = v^{+,M}$*; or*
(2) $G(V^+) - G(V^-)$ *and* $G(V^+) - G(V^M)$ *are linearly dependent.*

Instead of the Triple Shock Rule, for (1.1) the *Quadruple Shock Rule* holds:

Proposition 4. *Assume that there are two phase configurations labelled by* I *and* II*, with a common boundary. Consider four states:* (V^-, u^-) *in* I*,* (V^+, u^+) *in* II*;* (V^M, u^M)*,* (V^*, u^*) *free to be in* I *and* II*. Assume that the RH condition is satisfied by the following pairs of states:*
(i) (V^-, u^-) *and* (V^+, u^+) *with speed* $v^{-,+}$*,*
(ii) (V^-, u^-) *and* (V^M, u^M) *with speed* $v^{-,M}$*,*
(iii) (V^M, u^M) *and* (V^*, u^*) *with speed* $v^{M,*}$*,*
such that two speeds coincide, i.e., at least one of the following equalities is satisfied:

$$\text{either} \quad v^{-,+} = v^{-,M} \quad \text{or} \quad v^{-,+} = v^{M,*} \quad \text{or} \quad v^{-,M} = v^{-,+}. \tag{4.1}$$

If the following conditions (a) *to* (c) *are satisfied:*
(a) $G(V^+) - G(V^-)$ *and* $G(V^*) - G(V^M)$ *are linearly independent (LI);*
(b) V^+ *and* V^* *have one component* V_k *with coinciding values;*
(c) $\partial \mathcal{H}_V / \partial V_j \neq 0$ *for all* $j \neq k$ *for all* $V \in \mathcal{RH}(V^M)$*, see Eq.* (3.39)*;*
then:

(1) $V^* = V^+$;
(2) $u^* = u^+$;
(3) *all three speeds are equal:* $\quad v^{-,+} = v^{-,M} = v^{M,*}. \quad (4.2)$

Proof: The RH conditions (3.35) for (V^-, u^-)-(V^+, u^+), (V^-, u^-)-(V^M, u^M) and (V^M, u^M)-(V^*, u^*) are respectively:

$$v^{-,+}(G^+ - G^-) = u^+F^+ - u^-F^-, \tag{4.3}$$

$$v^{-,M}(G^M - G^-) = u^M F^M - u^-F^-, \tag{4.4}$$

$$v^{M,*}(G^* - G^M) = u^*F^* - u^M F^M. \tag{4.5}$$

Assume now that Eq. (4.1.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (4.3) and (4.4) and subtracting the resulting equations, we obtain:

$$v(G^+ - G^M) = u^+F^+ - u^M F^M. \tag{4.6}$$

Notice that Eqs. (4.6) and (4.5) define implicitly the RH curve by $\mathcal{H}_V(V^M; V^+) = 0$ in the variables V^M and V^+. Since the RH curve depends solely on V^M and the accumulation and flux functions, we obtain that both RH curves defined by Eqs. (4.6) and (4.5) coincide. From the condition (b) the states V^+ and V^* have a coinciding coordinate V_k. From the conditions (a) and (c), the implicit function theorem ensures that we can write the components V_j in terms of V_k. Thus there exists a single V with component V_k satisfying (4.5) and (4.6), so V^* and V^+ are equal.

Now from Eq. (3.40.b), we notice for a fixed u^- that the Darcy and shock speeds depend solely on V^- and V^+. From Eqs. (4.5) and (4.6), we can see that the $(-)$ and $(+)$ states are the same for each expression and that they define the same RH curve, so $u^* = u^+$ and Eq. (4.2) is satisfied.

The other cases are proved similarly. □

Another important bifurcation theorem for Riemann solutions is the *Bethe-Wendroff Theorem*, see [27]. We extend this result for the velocity-dependent system (1.1), including shocks connecting different sheets of the stratified variety, obtaining the *generalized Bethe-Wendroff Theorem*; its proof is given in Appendix C.1.

Proposition 5. *Assume that F and G are $\mathcal{C}^2$. Let $(W^+; W^-; v)$ be a shock between different phase configurations. Assume that $\boldsymbol{\ell}^i(V^+) \cdot [G] \neq 0$ and that for all $W \in \mathcal{RH}(W)$ with W between W^- and W^+ the inequality $\nabla\lambda(W) \cdot r(W) \neq 0$ is satisfied. Then v has a critical point at W^+ (and $\tilde{v}^+(V^-; V^+)$ has a critical point at V^+), if and only if:*

$$\tilde{v}^+(V^-; V^+) = \tilde{\lambda}^{i,+}(V^+) \qquad \textit{for} \quad i = 1 \textit{ or } 2, \tag{4.7}$$

where $\tilde{\lambda}^{i,+}(V)$ is given by Eq. (3.27) and $\tilde{v}^+(V^-; V^+)$ by (3.41). In this case the tangent to the RH curve in the space $\{W\}$ is the characteristic vector $\boldsymbol{r}^i$ at W^+; similarly in the space $\{V\}$ the tangent to the RH curve is $\tilde{\boldsymbol{r}}^i$ at V^+.

4.2. **Bifurcation loci.**

Assume that in no RH curve there are no higher-order degeneracies (described in B). For conservation laws standard form, there are loci which induce topological change in the Riemann solution, such as: secondary bifurcation, coincidence, double contact, inflection, hysteresis and interior boundary contact, see [12], [13]. For our class of problems, such loci also exist and are equally important.

4.2.1. *Secondary bifurcation locus.* This locus is defined in the space $W = (V, u)$ for conservation laws (1.1), however we will see that it suffices to study it in the space of primary variables V. The RH curve for a fixed V^- is obtained implicitly by $\mathcal{H}_V(V^-; V^+) = 0$, where $\mathcal{H}_V : \mathbb{R}^4 \longrightarrow \mathbb{R}$, is given in Eq. (3.39) and V^+ are primary variables. At some pairs (V^-, V^+) for fixed V^-, this implicit expression fails to define a curve for V^+. Following [12], we call the set of points V^+ where there is potential for failure the **secondary bifurcation locus** in V; (we recall that $V^+ = V^-$ is the primary bifurcation). From the implicit function theorem, for each $(-)$ state, it consists of the $(+)$ states such that:

$$\mathcal{H}_V(V^-;V^+) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{H}_V}{\partial V_j^+} = 0, \quad \text{for} \quad j = 1 \text{ and } 2. \tag{4.8}$$

The following theorem yields an equivalent expression for the secondary bifurcation, which is remarkably similar to the expression for standard conservation laws. This Proposition is proved for $n = 3$ in *Appendix C*.2.

Proposition 6. *Let the dimension of V be 3, and let the system* (1.1) *be hyperbolic in a neighborhood of V^+. A state $V^+ \in \mathcal{RH}(V^-)$ belongs to the secondary bifurcation locus for the family i if:*

$$\tilde{v}^+(V^-;V^+) = \tilde{\lambda}^{i,+}(V^+) \quad \textit{and} \quad \boldsymbol{\ell}^i(V^+) \cdot [G] = 0. \tag{4.9}$$

Remark. The secondary bifurcation locus in the space W is a ruled surface in u. In other words, the RH curve for a fixed W^- is obtained implicitly by $\mathcal{H}_V(V^-;V^+) = 0$ and in addition to conditions (4.8) we have $\partial \mathcal{H}_V / \partial u^+ = 0$, which is trivially satisfied because the RH curve, Eq. (3.39), depends only on V.

4.2.2. *Inflection locus and coincidence locus.* The rarefaction curves are useful to construct rarefaction waves where the characteristic speed varies monotonically, see [8], [12], [21]; the inflection locus is the curve where the monotonicity fails, thus rarefaction curves stop at this curve. Any state $W = (V, u)$ on the inflection locus satisfies $\nabla \lambda^i(W) \cdot \boldsymbol{r}^i(W) = 0$. However, the Darcy speed can be isolated in this equality; indeed, in the space of primary variables, the inflection locus of family i, for $i = 1, 2$, consists of the states V satisfying the equation:

$$\nabla_V \tilde{\lambda}^i(V) \cdot \tilde{\boldsymbol{r}}^i(V) = -\tilde{\lambda}^i(V) g_3(V); \tag{4.10}$$

this equality is equivalent to $\beta(V) = 0$ in Eq. (3.17).

There are two important types of speed coincidence: coincidence between eigenvalues and coincidence between eigenvalues and shock speeds. For example, the coincidence between eigenvalues for a system of form (1.1) with two eigenvalues is $\lambda^1(W) = \lambda^2(W)$, or $\vartheta^1(V) = \vartheta^2(V)$ in terms of primary variables.

Similarly we can define other important bifurcation loci in the space of primary variables V: double contact locus, interior boundary contact (extension of the boundary), left or right characteristic shocks and hysteresis, see [17]. See [11], [13] for the definition of these loci for conservation laws in the classical form.

5. ELEMENTARY WAVES FOR THE NITROGEN-STEAM MODEL

The elementary waves are the basic ingredients in the Riemann solution. In the previous section we obtained some important results to construct the solution and proved that we can obtain it in the space of primary variables. Thus in the sections that follow, we summarize the structures for the Riemann solution in the space of primary variables only. We utilize the software MATLAB to draw all curves and solutions.

The theory developed in this paper has applications in a class of problems with two phases and two chemical species. We study properties associated to bifurcation of rarefaction waves for the class of equations of type (1.1) modeling two phases and two chemical species with mass transfer between phases. The matrix $A-\lambda B$ defined in (3.7) has the general form:

$$\left(\begin{array}{ccc} a_{11}\theta_s & a_{12}\theta + ua_{11}\frac{\partial f}{\partial T} & a_{13}+fa_{11} \\ a_{21}\theta_s & a_{22}\theta + ua_{21}\frac{\partial f}{\partial T} & a_{23}+fa_{21} \\ a_{31}\theta_s & a_{32}\theta + ua_{31}\frac{\partial f}{\partial T} + ub_{32} - \lambda c_{32} & a_{33}+fa_{31} \end{array}\right). \tag{5.1}$$

In the most important applications the variables are temperature and saturation or gas composition. So our unknowns in the system are denoted by T and s. Then in (5.1), a_{ij} are functions of T for all i,j; b_{32} and c_{32} are constants and θ_s and θ are:

$$\theta_s = u\frac{\partial f}{\partial s} - \lambda\varphi, \qquad \theta = uf - \lambda\varphi s. \tag{5.2}$$

Remark. In the *spg* we substitute s by ψ_{gw} and f by ψ_{gw}.

After some algebra, we obtain the following:

Lemma 4. *The matrix $A-\lambda B$ of form (5.1) has two eigenvalues of the form:*

$$\lambda_s = \frac{u}{\varphi}\frac{\partial f}{\partial s}, \quad \textit{and} \quad \lambda_e = \frac{u}{\varphi}\frac{f\Phi(T)+\xi_1}{s\Phi(T)+\xi_2}, \quad \textit{with eigenvectors } \boldsymbol{r}_s \textit{ and } \boldsymbol{r}_e. \tag{5.3}$$

Here $\xi_1(T) = -b_{32}a_{11}D$ and $\xi_2(T) = -c_{32}a_{11}D$, with $D = D(T) = a_{13}a_{21} - a_{23}a_{11}$; $\Phi(T) = (a_{22}a_{11} - a_{12}a_{21})(a_{23}a_{31} - a_{33}a_{21}) - D(a_{22}a_{31} - a_{32}a_{22})$. Notice that the eigenvector associated to λ_s is $\boldsymbol{r}_s = (1,0,0)$. This line field has Buckley-Leverett type, as it is associated to changes in saturation with constant temperature.

Remark The temperature T does not change along Buckley-Leverett waves. Under our assumption all entries a_{ij} of (5.1) depend only on T. Thus the waves associated to the eingenpair $(\lambda_e, \boldsymbol{r}_e)$ are called tie lines, see [9], [10], as on these waves the phase compositions are constant.

Remark 2. *Substituting the eigenvalue λ_e in Eq. (3.3.a), we obtain that when θ defined in (5.2.b) vanishes $\boldsymbol{r}_e$ satisfies:*

$$\boldsymbol{r}_e = (r_e^1, r_e^2, r_e^3) = (\partial f/\partial T, -\theta_s/u, 0) \tag{5.4}$$

The set of points $\mathbb{C}_{s,e}$ where $\lambda_s = \lambda_e$ is called the coincidence locus. Another structure is the set of states $\mathbb{I}_{s,e}$ satisfying the equality:

$$\frac{f(s,T)}{s} = \frac{\xi_1(T)}{\xi_2(T)}. \tag{5.5}$$

A similar structured appeared in [4], where it was called HISW. The difference between the HISW and $\mathbb{I}_{s,e}$ is the presence of two chemical species in the left state of the latter, while there was only one chemical species in the former.

Remark Substituting λ_e given by (5.3.b) in θ given by Eq. (5.2.b) and using (5.5), we obtain that $\theta = 0$ on $\mathbb{I}_{s,e}$.

We have the following results:

Lemma 5. *On the coincidence locus $\mathbb{C}_{s,e}$, the derivative $\partial\lambda_e/\partial s$ vanishes. Moreover the eigenvector $\boldsymbol{r}_e$ coincides with $\boldsymbol{r}_s$.*

Proof: Differentiating λ_e from (5.3) with respect to s we have $\partial\lambda_e/\partial s =$

$$\frac{u}{\varphi}\frac{(\partial f/\partial s)\,\Phi\,(s\Phi+\xi_2) - (f\Phi+\xi_1)\,\Phi}{(s\Phi+\xi_2)^2} = \frac{\Phi}{s\Phi+\xi_2}\left(\frac{u}{\varphi}\frac{\partial f}{\partial s} - \lambda_e\right) = \frac{\Phi}{s\Phi+\xi_2}\left(\lambda_e - \lambda_s\right) \qquad (5.6)$$

On $\mathbb{C}_{s,e}$, we have $\lambda_s = \lambda_e$, so $\partial\lambda_e/\partial\lambda_s = 0$. Since the eigenvalues coincide, $\boldsymbol{r}_e = \boldsymbol{r}_s = (1,0,0)$ and the result follows. $\square$

Remark. For $L \in \mathbb{C}_{s,e}$, the $RH(L)$ curve has two branches associated to λ_s and λ_e, which have the same tangent $\boldsymbol{r}_e$ at the initial state L.

To state the next result, we define $\mathcal{I}_e$, the inflection locus associated to $(\lambda_e, \boldsymbol{r}_e)$:

$$\mathcal{I}_e = \{(s,T) \text{ such that } \nabla\lambda_e\cdot\boldsymbol{r}_e = 0\}. \qquad (5.7)$$

Lemma 6. *The loci* $\mathbb{C}_{s,e}$ *and* $\mathbb{I}_{s,e}$ *are contained in* $\mathcal{I}_e$*.*

Proof: The inflection $\mathcal{I}_e$ is given by equality $\nabla\lambda_e\cdot\boldsymbol{r}_e = 0$. Now:

$$\nabla\lambda_e\cdot\boldsymbol{r}_e = \frac{\partial\lambda_e}{\partial s}r_e^1 + \frac{\partial\lambda_e}{\partial T}r_e^2 + \frac{\partial\lambda_e}{\partial u}r_e^3. \qquad (5.8)$$

From Remark 2 and Lemma 5, we see that $r_e^3 = 0$ on $\mathbb{C}_{s,e}$ and $\mathbb{I}_{s,e}$, then Eq. (5.8) reduces to:

$$\nabla\lambda_e\cdot\boldsymbol{r}_e = \frac{\partial\lambda_e}{\partial s}r_e^1 + \frac{\partial\lambda_e}{\partial T}r_e^2. \qquad (5.9)$$

From Lemma 5 we have $\nabla\lambda_e\cdot\boldsymbol{r}_e = 0$ on $\mathbb{C}_{s,e}$, so that $\mathbb{C}_{s,e}$ is contained in $\mathcal{I}_e$.

To prove that $\mathbb{I}_{s,e}$ is contained in $\mathcal{I}_e$, we calculate $\partial\lambda_e/\partial T$ for (s,T) on $\mathbb{I}_{s,e}$:

$$\frac{\partial\lambda_e}{\partial T} = \frac{\Phi}{s\Phi+\xi_2}\frac{\partial f}{\partial T} + \frac{1}{(s\Phi+\xi_2)^2}\left(\Phi'\left(\xi_2 f - s\xi_1\right) + \Phi(s\xi_1'\Phi - \xi_2' f) + \xi_1'\xi_2 - \xi_1\xi_2'\right), \qquad (5.10)$$

where $'$ indicates derivatives with respect to T. Using (5.5) and $\xi_i' = \xi_i\,(a_{11}'/a_{11}) - \xi_i\,(D'/D)$, Eq. (5.10) reduces to:

$$\frac{\partial\lambda_e}{\partial T} = \frac{\Phi}{s\Phi+\xi_2}\frac{\partial f}{\partial T}, \qquad (5.11)$$

Substituting (5.6), (5.11) in Eq. (5.9) and using (5.4) we have:

$$\nabla\lambda_e\cdot\boldsymbol{r}_e = \frac{\Phi}{s\Phi+\xi_2}\left(\lambda_s - \lambda_e\right)\frac{\partial f}{\partial T} - \frac{\Phi}{s\Phi+\xi_2}\frac{\partial f}{\partial T}\theta_s = 0. \qquad (5.12)$$

$\square$

5.1. **Elementary waves in the single-phase gaseous configuration.**

The *spg* configuration in Section 2.2.1 is described by (2.8)-(2.10).

1. Rarefaction waves. There are two eigenvalues and eigenvectors. The first eigenvalue is labeled as λ_c, because the composition ψ_{gw} changes but the speed u and the temperature are constant along the corresponding wave; the eigenpair is given by:

$$\lambda_c = u_c/\phi, \qquad \boldsymbol{r}_c = (1,0,0)^T, \qquad (5.13)$$

which corresponds to fluid transport; this wave is actually a contact discontinuity with constant u_c.

The other eigenpair is labelled as $(\lambda_T, \boldsymbol{r}_T)$, because the temperature changes on the corresponding rarefaction waves; on this wave u changes and the composition ψ_{gw} is constant; it is given by:

$$\lambda_T = \left(1 - \hat{C}_r T/\mathbb{F}\right) u/\varphi \quad \text{and} \quad \boldsymbol{r}_T = (0, \mathbb{F}, u\hat{C}_r). \tag{5.14}$$

Here $\mathbb{F} := \mathbb{F}(T) = T\psi_{gw}\rho_{gW}h'_{gW} + T\psi_{gn}\rho_{gN}h'_{gN} + T\hat{C}_r$. The temperature-dependent functions ρ_{gW}, ρ_{gN}, h'_{gW}, h'_{gN} and the constant $\hat{C}_r$ are positive; ψ_{gw} and ψ_{gn} are non-negative, so $\mathbb{F}(T)$ is positive and $\lambda_T < \lambda_c$ in the physical range.

2. Inflection locus for thermal waves. The gaseous thermal inflection locus, $\mathcal{I}_T$ consist of (ψ_{gw}, T) in *spg* satisfying $\nabla\lambda_T \cdot \boldsymbol{r}_T = 0$ (or equivalently, Eq. (4.10)).

We plot the physical region and $\mathcal{I}_T$ in Fig. 5.1.*a*, showing the signs of $\nabla\lambda_T \cdot r_T$. We plot the horizontal rarefaction lines associated to λ_c, (5.13), and the vertical rarefaction lines associated to λ_T, (5.14), in Fig. 5.1.*b*.

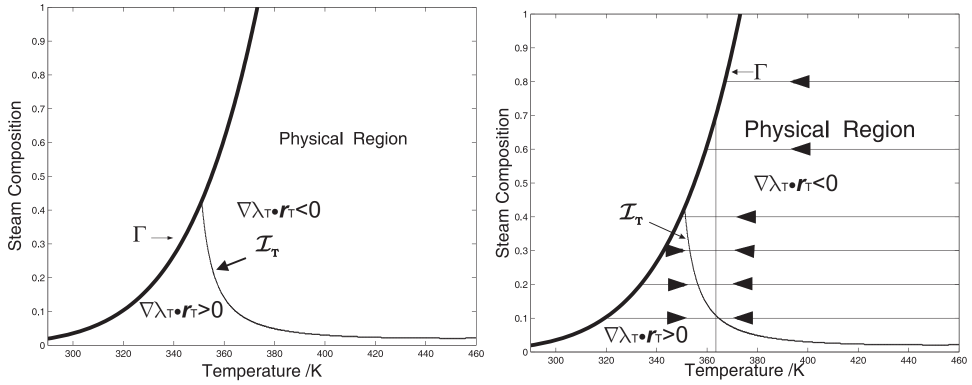

FIGURE 5.1. *a)-left:* The single-phase gaseous configuration, the curve Γ and the inflection locus $\mathcal{I}_T$. *b)-right:* Rarefaction curves. The horizontal rarefaction curves are associated to λ_T; we indicate with an arrow the direction of increasing speed. The vertical lines are contact discontinuity lines associated to λ_c, in which ψ_{gw} changes, T and u are constant; λ_c is constant in each curve.

3. Shock waves. The *RH* curve is obtained from Eq. $\mathcal{H}_V(V^-, V^+) = 0$ given by (3.39). The wave associated to λ_c is a contact discontinuity.

The thermal shock occurs with ψ_{gw} constant; the shock and Darcy speeds are obtained by using Eqs. (3.40), see [17] for the actual expression. The thermal shocks and rarefaction curves are contained in the horizontal lines in Fig. 5.1.*b*.

5.2. **Elementary waves in the single-phase liquid configuration.** For the *spl* given in Section 2.2.2, Eq. (2.12) is linear, so a single wave is associated to λ_T^W, given by (2.12.*b*). This wave is a contact discontinuity and there are no genuine rarefaction or shock waves.

5.3. **Elementary waves in the two-phase configuration.** The *tp* configuration is described in Section 2.2.3.

1. Rarefaction waves. We have two waves. The first one is an isothermal wave, defined by the Buckley-Leverett (*BL*) characteristic speed and characteristic vector:

$$\lambda_s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}, \qquad \boldsymbol{r}_s = (1, 0, 0)^T, \tag{5.15}$$

On the corresponding rarefaction curve T and u are constant, only the saturation s_g changes, hence the subscript s.

The other eigenpair corresponds to an evaporation wave, where the temperature, saturation and speed change; it is denoted by $(\lambda_e, \boldsymbol{r}_e)$, see [17].

In Fig. 5.3 we see that in the region where $\lambda_s > \lambda_e$, temperature, gas saturation and u increase along the rarefaction wave, while in the region where $\lambda_s < \lambda_e$ temperature and u increase and the gas saturation decreases. This is an evaporation wave, hence the subscript e in λ_e, $\boldsymbol{r}_e$.

2. Shock waves. For V^- fixed, the RH curve defined by $\mathcal{H}_V(V^-, V^+) = 0$, with $\mathcal{H}_V$ given by (3.39). For the isothermal branch of the RH curve the shock speed is

$$v = \frac{u^-}{\varphi} \frac{f_w(s_w^+, T) - f_w(s_w^-, T)}{s_w^+ - s_w^-}.$$

On this wave, the Darcy speed is constant. This is the BL shock; for a fixed V^-, the RH and rarefaction curves of saturation waves lie on the same vertical lines in Fig. 5.3. The other branch of the RH is a condensation shock, drawn in Fig. 5.4.

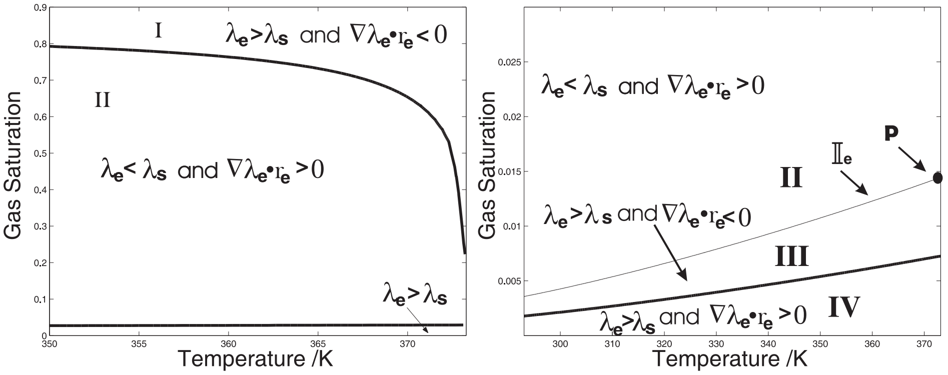

FIGURE 5.2. *a)-left:* Coincidence locus $\mathbb{C}_{s,e}$. Relative sizes of λ_s and λ_e in a sheet of the tp state space. The almost horizontal coincidence locus $\lambda_e = \lambda_s$ is not drawn in scale, because it is very close to the axis $S_g = 0$. *b)-right:* A zoom of the region below the lower coincidence locus. In both figures, all curves form $\mathcal{I}_e$, subdividing the tp configuration in four parts; the locus $\mathbb{I}_e$ is the defined satisfying Eq. (5.16.b), the point P is the intersection of $\mathbb{I}_e$ with the boundary of the TS plane at the water boiling temperature.

From Lemma 6 and Eq. (5.5) we know that the states satisfying

$$\lambda_e = \lambda_s \qquad \text{or} \qquad \frac{f_g}{s_g} = \frac{C_W}{C_W + \hat{C}_r} \tag{5.16}$$

lie on the inflection locus $\mathcal{I}_e$. Moreover, one can prove that the states in $\mathcal{I}_e$ satisfy one or the other equation in (5.16). Thus a stronger version of Lemma 6 holds:

Lemma 7. *In the tp configuration, Lemma* 6 *is stronger:* $\mathcal{I}_e = \mathbb{C}_{s,e} \bigcup \mathbb{I}_e$.

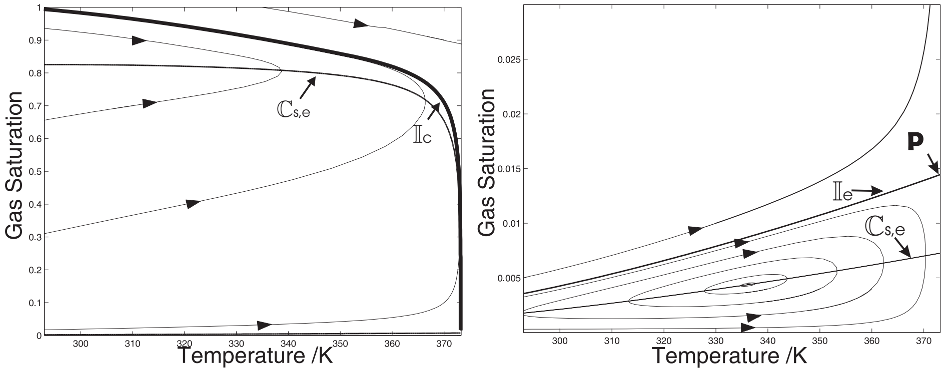

Figure 5.3. *a)-left:* The rarefaction curves projected in the (T, s_g) plane. The thin curve without arrows is $\mathbb{C}_{s,e}$. The bold curve $\mathbb{I}_c$ is an invariant curve for the rarefaction field, namely the integral curve through P of Fig. 5.2.*b*. *b)-right:* The rarefaction curves in the regions III and IV shown in the Fig. 5.2.*b*. The arrows indicate the direction of increasing speed.

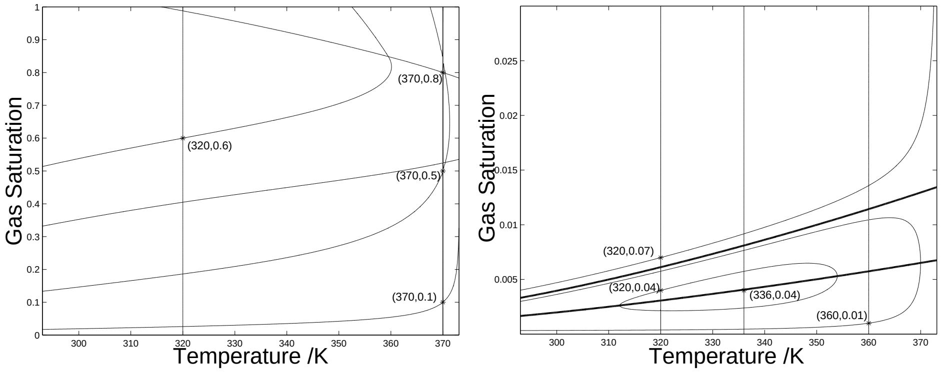

Figure 5.4. *a)-left:* The RH curves projected in the (T, s_g) plane for the marked $(-)$ states: $(320, 0.6)$, $(370, 0.1)$, $(370, 0.5)$ and $(370, 0.8)$. The vertical line is the isothermal wave: BL shock and rarefaction. The other branch is the condensation wave. *b)- right:* The region shown in Fig. 5.2.*b* is a zoom of the bottom of 5.2.*a*. Each RH curve is formed by a non-isothermal curve and a vertical line which is the isothermal BL of the RH curve. For the state $(336, 0.04)$ the Rankine-Hugoniot curve reduces to the isothermal line only.

6. The Riemann solution for geothermal energy recovery

We present an example of Riemann solution for the specific model of Section 2. Other examples are found in [17]. We consider the injection of a two-phase mixture of water, steam and nitrogen into a rock containing superheated steam ($\psi_R = 1$) at a temperature $T_R > T^b$. The injection boundary condition and initial data are:

$$\begin{cases} (s_g, \psi(T), T, u)_L & \text{if } x = 0 \text{ (the injection point)}, \qquad \text{with} \qquad u_L > 0, \\ (s_g = 1, 1, T, \cdot)_R & \text{if } x > 0 \end{cases} \tag{6.1}$$

In the tp configuration the inflection locus $\mathcal{I}_e$ consist of two curves, which divide the tp configuration space into 4 regions, labeled as I, II, III and IV, see Fig. 5.2.

In the *spg* configuration, we fix the right state R. We can subdivide each of the regions $I-IV$ in subregions $\mathcal{L}_1$-$\mathcal{L}_6$ with the property that follows: for any L in a given subregion the Riemann problems with data L and R of form (6.1) have the same sequence of waves, see Figs. 6.1 and 6.2.*a*. The corresponding Riemann solutions are described in Section 6.2, where we give the solution only for $V_L \in \mathcal{L}_3$ and $\mathcal{L}_4$. In the section that follows we indicate the Riemann solution in the space of variables V. In that section, we use some results of [16] to obtain the Riemann solution.

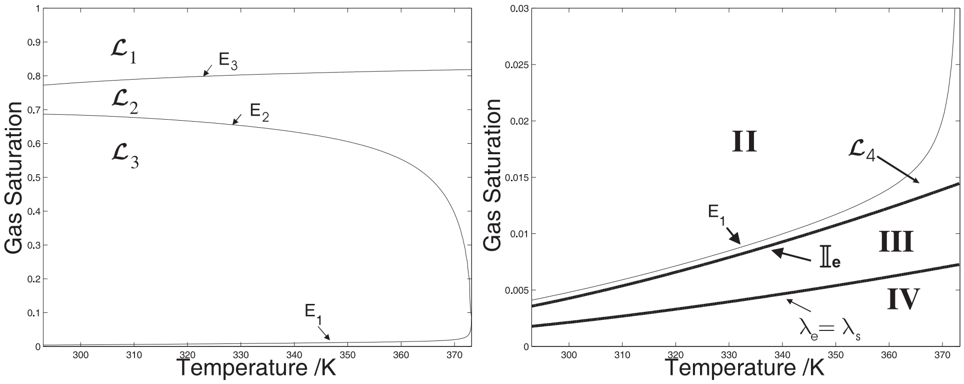

FIGURE 6.1. *a)-left:* The 3 subregions in I and II for a Riemann data of form (6.1). The region $\mathcal{L}_4$ lies below E_1. *b)-right:* Zoom of regions II to IV. Each curve is explained in Section 6.1. The locus $\mathbb{I}_e$ satisfies (5.16.*b*); it is an invariant curve for shock and rarefaction curves.

6.1. **Subdivision of** ***tp*****.** We obtain the curves that bound the subregions $\mathcal{L}_k$: for E_1, see Fig. 6.1; for E_2, see Fig. 6.1.*a*; for E_3, see Fig. 6.1.*a*.

The evaporation wave speed λ_e for states in $\mathcal{L}_2$ and $\mathcal{L}_3$ in the *tp* is larger than the thermal wave speed λ_T in the *spg*. Requiring geometrical compatibility in the wave sequence, there is no thermal wave (rarefaction or shock) after the evaporation shock in the *spg*, i.e., all shocks from $\mathcal{L}_2$, $\mathcal{L}_3$ to *spg* reach states (ψ_{gw}, T_R).

1. Curve E_2*.* This curve consists of the states V^- in the *tp* where the characteristic speed $\lambda_e(V^-)$ equals the shock speed $v^{BG}(V^-;V^+)$ for $V^+ = (T_R, \psi_{gw})$ in the *spg*. E_2 is an extension of the physical boundary. Here ψ_{gw} is obtained numerically.

2. Curve E_1*.* We find the Riemann solution for left states in I or II. To satisfy the geometrical compatibility, we choose first the slowest possible wave. In II, this is the evaporation rarefaction wave, which is used to connect states from lower to higher temperatures, see Sec. 5.3. This can be done until the characteristic speed at its rightmost state E_2 equals the speed of the shock between configurations.

We define E_1 as the evaporation rarefaction curve that crosses E_2 at the boiling temperature. Notice that for left states in II above E_1, the evaporation curve always crosses E_2, while for states below E_1 the evaporation curve never crosses E_2.

For states below E_1 the evaporation curve reaches the boundary of *tp*, which represents pure steam at boiling temperature.

3. Curve E_3. For states V^- above E_2, the evaporation speed is larger than $v^{BG}(V^-;V^+)$, for $V^+ = (T_R, \psi_{gw})$; in Figs. 5.2.*a* and 6.2.*a* we show the relative sizes of v^{BG}, λ_e and λ_s. When s_g tends to 1 the BL shock speed is zero, so there are curves $v^{BL} = v^{BG}$, $v^{BL} = v^T$

and $v^{BG} = v^T$. From Prop. 4, we obtain a bifurcation curve E_3 where $v^T = v^{BL} = v^{BG}$. For states above this curve, the BL shock is slower than v^{BG} and v^T, so there is no direct shock between the *tp* and the *spg* configurations.

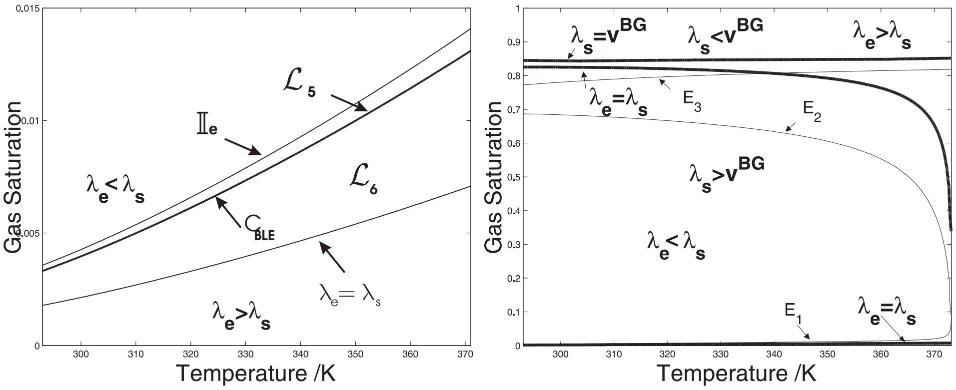

FIGURE 6.2. *a)-left:* Subregions $\mathcal{L}_5$ and $\mathcal{L}_6$. $\mathcal{C}_{BLE}$ is the coincidence curve $v^b_{g,w} = \lambda_e$. For states in $\mathcal{L}_5$, we have that $v^b_{g,w} < \lambda_e$; for states in $\mathcal{L}_6$, we have that $v^b_{g,w} > \lambda_e$. *b)-right:* The relevant curves for the Riemann solution for the left states in the *tp* configuration.

6.2. **The Riemann solution.** The notation $V_L = (s_g, T)_L$ in the *tp*, $V_R = (1, T)_R$ in the *spg* will be used. All the variables for the intermediate states are written out. The states are labelled by 1, 2, and so on. We use the following nomenclature: evaporation rarefaction R_E; BL shock S_{BL} and rarefaction R_{BL}; shock S_{BG} between the *tp* and the *spg* regions; compositional contact discontinuity S_C; the vaporization shock S_{VS} from the boiling region (in the *tp* configuration, for $T = T^b$) to the steam region (in the *spg* configuration, for $\psi_{gw} = 1$), both defined in [16].

1. V_L in $\mathcal{L}_4$. There is a R_E from V_L up to $V_{(1)} = (s_L, T^b)$ in the *spg*, where T^b is the boiling temperature. For this state there is no nitrogen so $\psi_{gw} = 1$. The solution after the intermediate state $V_{(1)}$ was found in [16]: there is a R_{BL} up to $V_{(2)} = (s^\dagger, T^b)$ in the *spg*; $s^\dagger$ is obtained implicitly from $\lambda(s^\dagger) = v^{VS}$, where v^{VS} is the shock speed of S_{VS}. We use the notation for wave sequences established in Section 3.3. The solution consists of the waves R_e, R_{BL} and S_{VS} with sequence:

$$V_L \xrightarrow{R_e} V_{(1)} \xrightarrow{R_{BL}} V_{(2)} \xrightarrow{S_{VS}} V_R. \tag{6.2}$$

V_L in $\mathcal{L}_3$. There is a R_E from V_L up to $V_{(1)} = (s^*, T^*)$ in the *tp*, satisfying $\lambda_e(V_{(1)}) = v^{BL}(V_{(1)}; V_{(2)})$, with $V_{(2)} = (\psi_{gw}, T_R)$ in the *spg*. We have determined ψ_{gw} in Section 6.1(1). Finally, there is a compositional contact discontinuity at temperature T_R with speed v^C from $V_{(2)}$ to V_R. The solution consists of the waves R_E, S_{BG} and S_C with sequence:

$$V_L \xrightarrow{R_e} V_{(1)} \xrightarrow{S_{BG}} V_{(2)} \xrightarrow{S_C} V_R. \tag{6.3}$$

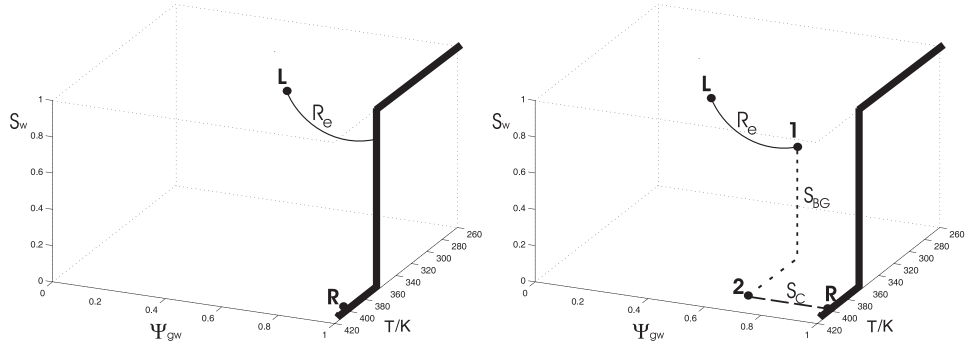

Figure 6.3. Riemann solutions in phase space, omitting the surface shown in Fig. 2.1. *a) left:* Solution (6.2) for $V_L \in \mathcal{L}_4$. *b) right:* Solution (6.3) for $V_L \in \mathcal{L}_3$. The numbers 1 and 2 indicate the intermediate states V in the wave sequence.

7. Summary

We have described the Riemann solutions for the injection of a nitrogen/steam/water mixture into a porous rock filled with steam above the boiling temperature. The set of solutions depends L^1-continuously on the Riemann data.

We have extended the known bifurcation theory to 3×3 systems of conservation laws with an unknown velocity u appearing only in the flux term; these systems represent multiphase multicomponent flow under thermodynamical equilibrium. This equilibrium occurs on a stratified variety in state space.

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Appendix A. Physical quantities; symbols and values

The steam enthalpy $h_{gW}[J/kg]$ as a function of temperature is approximated by

$$\begin{aligned} h_{gW}(T) = &-2.20269\times 10^{7} + 3.65317\times 10^{5}T - 2.25837\times 10^{3}T^{2} + 7.3742T^{3} \\ &- 1.33437\times 10^{-2}T^{4} + 1.26913\times 10^{-5}T^{5} - 4.9688\times 10^{-9}T^{6} - \overline{h}_{w}. \end{aligned} \tag{A.1}$$

The nitrogen enthalpy $h_{gN}[J/kg]$ as a function of temperature is approximated by

$$h_{gN}(T) = 975.0T + 0.0935T^{2} - 0.476\times 10^{-7}T^{3} - \overline{h}_{gN}. \tag{A.2}$$

The constants $\overline{h}_w$ and $\overline{h}_{gN}$ are chosen so that $h_w(T)$, $h_{gN}(T)$ vanish at a reference temperature $\overline{T} = 293K$. In the range $[290K, 500K]$, h_{gW} and h_{gN} are almost linear.

The rock enthalpy H_r, $\hat{H}_r$, water and gaseous enthalpies per mass unit H_W and H_g are are given by:

$$H_r = C_r(T-\bar{T}),\ \ \hat{H}_r = H_r/\varphi,\ \ H_W = \rho_W h_w \ \text{ and } \ H_g = \rho_{gw}h_{gW} + \rho_{gn}h_{gN}. \tag{A.3}$$

The temperature dependent liquid water viscosity μ_w $[Pas]$ is approximated by

$$\mu_w = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425 \times 10^7}{T^3} - \frac{2.17342 \times 10^9}{T^4} + \frac{1.86935 \times 10^{11}}{T^5}. \tag{A.4}$$

We assume that that the viscosity of the gas is independent of the composition.

$$\mu_g = 1.826\,4 \times 10^{-5} \left(T/T^b\right)^{0.6}. \tag{A.5}$$

The water saturation pressure as a function of temperature is given as

$$\begin{aligned} p^{sat} = 10^3(&-175.776 + 2.29272T - 0.0113953T^2 + 0.000026278T^3 \\ &- 0.0000000273726T^4 + 1.13816 \times 10^{-11}T^5)^2 \end{aligned} \tag{A.6}$$

The graph of this function looks like a growing parabola.

From the ideal gas law, the corresponding concentrations $\rho_{gw}(T)$, $\rho_{gn}(T)$ are:

$$\rho_{gw}(T) = M_W\ p^{sat}/\left(RT\right),\ \rho_{gn}(T) = M_N\ \left(p_{at} - p^{sat}\right)/\left(RT\right), \tag{A.7}$$

where the gas constant $R = 8.31[J/mol/K]$. The pure phase densities are:

$$\rho_{gW}(T) = M_W p_{at}/\left(RT\right), \qquad \rho_{gN}(T) = M_N p_{at}/\left(RT\right). \tag{A.8}$$

Here M_W and M_N are the nitrogen and water molar masses.

The relative permeability functions k_{rw} and k_{rg} are considered to be quadratic functions of their respective reduced saturations, i.e.

$$k_{rw} = \left\{ \begin{array}{c} 0.5\left(\frac{s_w - s_{wc}}{1 - s_{wc} - s_{gr}}\right)^2 \\ 0 \end{array} \right., \ k_{rg} = \left\{ \begin{array}{cl} 0.95\left(\frac{s_g - s_{gr}}{1 - s_{wc} - s_{gr}}\right)^2 & \text{for} \quad s_{wc} \le s_w \le 1, \\ 1 & \text{for} \qquad s_w < s_{wc}. \end{array} \right. \tag{A.9}$$

Table 2, Summary of physical parameters and variables

Physical quantity	*Symbol*	*Value*	*Unit*
Water, steam relative permeabilities	k_{rw}, k_{rg}	Eq. (A.9) .	[m^3/m^3]
Pressure	p_{at}	1.0135×10^5.	[Pa]
Water saturation pressure	p^{sat}	Eq. $(A.6)$.	[Pa]
Water, gaseous phase velocity	u_w, u_g	Eq. (2.1) .	[m^3/(m^{2}s)]
Total Darcy velocity	u	$u_w + u_g$, Eq. (2.3) .	[m^3/(m^{2}s)]
Rock and water heat capacities	C_r and C_W	2.029×10^6 and 4.018×10^6.	[J/(m^3 K)]
Connate water saturation	s_{wc}	0.15.	[m^3/m^3]
Boiling and reference temperatures	T^b T_{ref}	$373.15K$, $293K$.	[K]
Water, gaseous phase viscosity	μ_w, μ_g	Eq. (A.4) , Eq. (A.5) .	[Pa s]
Steam and nitrogen densities	ρ_{gw}, ρ_{gn}	Eq. $(A.7.a)$, $(A.7.b)$.	[kg/m^3]
Constant water density	ρ_W	998.2.	[kg/m^3]
Nitrogen and water molar masses	M_N, M_W	0.28, 0.18	$[kg/mol]$
Rock porosity	φ	0.38.	[m^3/m^3]

Appendix B. Higher order degeneracies of the RH Curve

Theorem 1 is valid if the denominator of Eqs. (3.43) is non-zero for some $\{k,j\} \in \mathbb{K}$ (more generically it is necessary that $(3.40.a)$, $(3.40.c)$ is non-zero for some $\{k,j\} \in \mathbb{K}$). So it is necessary to study the behavior of the solution when $\mathcal{X}^+_{kj} = 0$ for all $\{k,j\} \in \mathbb{K}$. For a fixed pair $\{k',j'\} \in \mathbb{K}$.

It is easy to prove that:

Lemma 8. *Let* $\{V^-, V^+\}$ *satisfy* $\mathcal{H}_V(V^-; V^+) = 0$*, where* $\mathcal{H}_V$ *is given by Eq.* (3.39)*. If* $\mathcal{X}^+_{k'j'} = 0$*, then one of following conditions is satisfied:*

$$(i) \quad \mathcal{Y}_{k'j'} = 0 \qquad or \qquad (ii) \quad \mathcal{X}^+_{12} = \mathcal{X}^+_{31} = \mathcal{X}^+_{23} = 0. \tag{B.1}$$

From this Lemma it follows immediately that:

Corollary 1. *Let* $\{V^-, V^+\}$ *satisfy* $\mathcal{H}_V(V^-; V^+) = 0$*. If* $\mathcal{X}^+_{kj} = \mathcal{X}^+_{k'j'} = 0$ *for two index pairs* $\{k, j\}$ *and* $\{k', j'\}$ *in* $\mathbb{K}$*, then it vanishes for all pairs.*

Proposition 7. *If* $\mathcal{X}^+_{kj} = 0$ *for all* $\{k, j\} \in \mathbb{K}$ *and* $(F^+_1, F^+_2, F^+_3) \neq 0$*, we obtain:*

$$[G_l] = \varrho_1 F^+_l \qquad and \qquad F^-_l = \varrho_2 F^+_l, \quad for\ l = 1, 2, 3. \tag{B.2}$$

where ϱ_1 *and* ϱ_2 *are constants depending on* $[G]$*,* F^- *and* F^+*, which are given in the proof of this Proposition. Moreover, for* $\mathcal{Z}$ *defined in* (3.41.*a*)*, the shock speed* v *satisfies:*

$$v = u^- \frac{\mathcal{Z}(V^-; V^+) - \varrho_2}{\varrho_1}. \tag{B.3}$$

Proof: Let $u^- > 0$. Since $\mathcal{X}^+_{kj} = 0$ for all $\{k, j\} \in \mathbb{K}$, it follows that:

$$\mathcal{X}^+_{23} e_1 + \mathcal{X}^+_{31} e_2 + \mathcal{X}^+_{12} e_3 = 0, \tag{B.4}$$

where e_l for $l = 1, 2, 3$ is the canonical basis for $\mathbb{R}^3$. Eq. $(B.4)$ can be written as:

$$(F^+_1, F^+_2, F^+_3) \times ([G_1], [G_2], [G_3]) = 0, \tag{B.5}$$

where × represents the outer product. Since Eq. $(B.5)$ is satisfied, it follows that $[G]$ is parallel to F^+, so there is a constant ϱ_1 so Eq. $(B.2.a)$ is satisfied. Substituting $[G] = \varrho_1 F^+$ into the RH condition (3.35), we obtain, for l = 1, 2, 3:

$$v \varrho_1 F^+_l = u^+ F^+_l - u^- F^-_l, \tag{B.6}$$

If $F^-_l = 0$ for some $l = 1, 2, 3$, so $v = u^+/\varrho_1$. If $F^-_l \neq 0$ for all $l = 1, 2, 3$, multiplying Eq. $(B.6)$ for $l = 1$ by F^-_2 and $(B.6)$ for $l = 2$ by $-F^-_1$ and adding, it follows that $v \varphi \varrho_1 \mathcal{Y}_{12} = u^+ \mathcal{Y}_{12}$. Let us assume temporarily that $\mathcal{Y}_{12} \neq 0$, so Eq. $(B.6)$ yields $v = u^+/\varrho_1$. Substituting $v = u^+/\varrho_1$ into Eq. $(B.6)$ we obtain $u^- F^-_1 = 0$. Since $F^-_1 \neq 0$ generically, it follows that $u^- = 0$, which is false. So $\mathcal{Y}_{12} = 0$.

Similar calculations show that $\mathcal{Y}_{12} = \mathcal{Y}_{23} = \mathcal{Y}_{31} = 0$, so there exists a constant ϱ_2 such that $F^- = \varrho_2 F^+$. Eq. $(B.3)$ can be obtained by substituting $[G] = \varrho_1 F^+$ and $F^- = \varrho_2 F^+$ in the RH condition (3.35). □

With some modifications, we can prove that Proposition 7 is valid for all $n > 3$.

Remark If $(F^+_1, F^+_2, F^+_3) = 0$ and $([G_1], [G_2], [G_3]) \neq 0$, it is easy to prove for $l = 1, 2, 3$ that:

$$F^-_l = \rho_3 [G_l] \quad \text{and} \quad v = u^- \rho_3, \tag{B.7}$$

where ρ_3 is a constant that depends on $[G]$ and F^-.

Corollary 2. *The states* $\{V^-, V^+\}$ *satisfying the RH condition* (3.35) *for which* $\mathcal{X}^+_{kj} = 0$ *for all* $\{k, j\} \in \mathbb{K}$*, satisfy also:*

$$\mathcal{Y}_{kj} = 0, \quad \forall\ \{k, j\} \in \mathbb{K}. \tag{B.8}$$

We notice that the system $(B.8)$ has always the trivial solution $V^+ = V^-$.

APPENDIX C. PROOF OF THE BETHE-WENDROFF EXTENSION AND CHARACTERIZATION OF THE SECONDARY BIFURCATION.

C.1. **Proof of Prop. 5.** Assuming that the RH curve can be parametrized by ζ in a neighborhood of (V^+, u^+), so that $(V(\zeta^+), u(\zeta^+)) = (V^+, u^+)$; we can write the RH condition as:

$$v(G(V(\zeta)) - G^-) = u(\zeta)F(V(\zeta)) - u^- F^-, \tag{C.1}$$

where $v := v(\zeta)$. Differentiating $(C.1)$ with respect to ζ we obtain:

$$\frac{dv}{d\zeta}(G(V(\zeta)) - G^-) + v\frac{\partial G(V(\zeta))}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(u(\zeta)F(V(\zeta)))}{\partial W}\frac{dW}{d\zeta}, \tag{C.2}$$

Setting $\zeta = \zeta^+$, such that $(V(\zeta^+), u(\zeta^+)) = (V^+, u^+) = W(\zeta^+) = W^+$, Eq. $(C.2)$ yields:

$$[G]\frac{dv}{d\zeta} + v\frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(uF)}{\partial W}\frac{dW}{d\zeta}, \tag{C.3}$$

where $W = (V, u)$ and $[G] = G^+ - G^-$. Assume first that (4.7) is satisfied. Notice that if $\tilde{v}^+(V^-, V^+) = \tilde{\lambda}^+$, then for $(V^+, u = u^+)$, we have $\lambda = u^+\tilde{\lambda}^+$ and $v(V^-, u^-; V^+) = \lambda(V^+, u^+)$ (we dropped the family index i). Substituting $\lambda = u^+\tilde{\lambda}^+$ and $v(V^-, u^-; V^+) = \lambda(V^+, u^+)$ in $(C.3)$ we obtain at $(+) = (V^+, u^+)$ the expression $(C.3)$ with $v := u\lambda/u$. Let ℓ the left eigenvector associated to $\tilde{\lambda}^+$ at (V^+, u^+) ; taking the inner product of $(C.3)$ with $\boldsymbol{\ell}$, we obtain:

$$\boldsymbol{\ell}\cdot[G]\frac{dv}{d\zeta} + \boldsymbol{\ell}\cdot\left(u\frac{\lambda}{u}\frac{\partial G}{\partial W} - \frac{\partial\,(uF)}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{C.4}$$

Since $\boldsymbol{\ell}$ is an eigenvector associated to λ, the second term of $(C.4)$ is zero and:

$$\boldsymbol{\ell}\cdot[G]\frac{dv}{d\zeta} = 0.$$

Since by hypothesis $\boldsymbol{\ell}\cdot[G] \neq 0$, we obtain that $dv/d\zeta = 0$ and the shock speed is critical. On other hand, assume that v has an extremum, so $dv/d\zeta = 0$ and Eq. $(C.3)$ reduces to:

$$v\frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(uF)}{\partial W}\frac{dW}{d\zeta}, \quad \text{or} \quad \left(\frac{\partial\,(uF)}{\partial W} - v\frac{\partial G}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{C.5}$$

Notice that $dW/d\zeta$ is parallel to the eigenvector $\boldsymbol{r}$ at W^+. If W has $n+1$ components, the first n components of the vector $dW/d\zeta$ and of the eigenvector $\tilde{\boldsymbol{r}}^i$ at V^+ are proportional.

The converse it is true, because Eq. $(C.5)$ has a solution if, only if,

$$v(V^-, u^-; V^+) = \lambda(V^+, u^+) \qquad \text{so} \qquad \tilde{v}^+(V^-; V^+) = \tilde{\lambda}^+(V^+).$$

□

C.2. **Proof of Prop. 6.** The proof given below is valid for the case $n = 3$.

We drop the family index i. Assume that (4.8) is satisfied, where the $\partial\mathcal{H}_V/\partial V_j^+$ for $j = 1, 2$ are:

$$\left(F^- \times \frac{\partial F^+}{\partial V_j^+}\right)[G]^T + (\mathcal{Y}_{32}, \mathcal{Y}_{13}, \mathcal{Y}_{21})\left(\frac{\partial G^+}{\partial V_j^+}\right)^T,$$

where × is the outer product. Rearranging the terms, we can rewrite $\partial\mathcal{H}_V/\partial V_j^+$ as:

$$\frac{\partial\mathcal{H}_V}{\partial V_j^+} = \left(\mathcal{X}_{32}^-, \mathcal{X}_{13}^-, \mathcal{X}_{21}^-\right)\left(\frac{\partial F^+}{\partial V_j^+}\right)^T + \left(\mathcal{Y}_{32}, \mathcal{Y}_{13}, \mathcal{Y}_{21}\right)\left(\frac{\partial G^+}{\partial V_j^+}\right)^T. \tag{C.6}$$

Recall that $\lambda^+ = u^+\tilde{\lambda}^+$, so $\partial_W(uF) - \lambda\partial_W G$ at (V^+, u^+) is:

$$\begin{pmatrix} u^+C_{1,1}(V^+;\tilde{\lambda}^+) & u^+C_{1,2}(V^+;\tilde{\lambda}^+) & F_1^+ \\ u^+C_{2,1}(V^+;\tilde{\lambda}^+) & u^+C_{2,2}(V^+;\tilde{\lambda}^+) & F_2^+ \\ u^+C_{3,1}(V^+;\tilde{\lambda}^+) & u^+C_{3,2}(V^+;\tilde{\lambda}^+) & F_3^+ \end{pmatrix}. \tag{C.7}$$

Setting $\tilde{v}^+(V^-;V^+) = \tilde{\lambda}^+(V^+)$ in $(C.7)$ and assuming that $\mathcal{X}_{kj}^- \neq 0$ for all $\{k,j\} \in \mathbb{K}$, we use Eqs. $(3.40.a)$ and (3.41) to write an equivalent but convenient expression for $\tilde{v}^+$ in each matrix element, we can rewrite matrix $(C.7)$ as:

$$\begin{pmatrix} u^+C_{1,1}(V^+;\mathbb{Y}_{32}) & u^+C_{1,2}(V^+;\mathbb{Y}_{32}) & F_1^+ \\ u^+C_{2,1}(V^+;\mathbb{Y}_{13}) & u^+C_{2,2}(V^+;\mathbb{Y}_{13}) & F_2^+ \\ u^+C_{3,1}(V^+;\mathbb{Y}_{21}) & u^+C_{3,2}(V^+;\mathbb{Y}_{21}) & F_3^+ \end{pmatrix}, \quad \text{where} \quad \mathbb{Y}_{kj} = \frac{\mathcal{Y}_{kj}}{\mathcal{X}_{kj}}. \tag{C.8}$$

Since $\partial\mathcal{H}_V/\partial V_j^+ = 0$ for $j = 1, 2$, from Eq. $(C.6)$, it follows that for $\boldsymbol{\ell}$ given by

$$\boldsymbol{\ell} = \left(\mathcal{X}_{32}^-,\ \ \mathcal{X}_{13}^-,\ \ \mathcal{X}_{21}^-\right), \tag{C.9}$$

the inner products of columns 1 and 2 of $(C.8)$ by $\boldsymbol{\ell}$ are zero. Since at (V^-, V^+) the expression $\mathcal{H}_V(V^-)$ vanishes, after some calculations we obtain that $\boldsymbol{\ell}\cdot(F_1^+, F_2^+, F_3^+) = 0$, so that $\boldsymbol{\ell}$ is a left eigenvector of the system. Since the system (1.1) is hyperbolic in the region around W^+, there is only one left eigenvector associated to this eigenvalue, thus all left eigenvectors of $(C.8)$ with $\lambda(V^+) = v(V^-, V^+)$ are parallel to $\boldsymbol{\ell}$. Notice that $\boldsymbol{\ell}\cdot([G_1],[G_2],[G_3]) = 0$, because this equality satisfies the RH condition $\mathcal{H}_V = 0$, with $\mathcal{H}_V$ given by (3.39).

If $\mathcal{X}_{k'j'}^- = 0$ for a pair $\{k', j'\} \in \mathbb{K}$, using Lemma 8, the relationships $\mathcal{Y}_{k'j'}^- = 0$ or $\left(\mathcal{X}_{32}^-, \mathcal{X}_{13}^-, \mathcal{X}_{21}^-\right) = (0,0,0)$ follow; the latter implies that $\boldsymbol{\ell}^i(V^+)\cdot[G] = 0$.

In other hand, if there exists a $\mathcal{X}_{kj}^- \neq 0$ for some $\{k,j\} \in \mathbb{K}$ (for concreteness we set $k = 1$ and $j = 3$; the other cases can be proved similarly), we have $\mathcal{X}_{13}^- \neq 0$ and the matrix $(C.8)$ has the form (a_{kj}) for $k, j = 1, 2, 3$, where:

$$(a_{11}, a_{12}, a_{13}) = \left(u^+C_{1,1}(V^+;\mathbb{Y}_{13}), u^+C_{1,2}(V^+;\mathbb{Y}_{13}), F_1^+\right);$$

Substituting $\boldsymbol{\ell}$ in $(C.9)$ by the following vector:

$$\boldsymbol{\ell} = \left(0,\ \ F_1^-[G_3] - F_3^-[G_1],\ \ F_2^-[G_1] - F_1^-[G_2]\right), \tag{C.10}$$

it is easy to prove that $\boldsymbol{\ell}$ lies in the kernel of the transpose of the matrix $(C.8)$.

Assume that $\mathcal{X}_{kj}^- = 0$ for all $\{k,j\} \in \mathbb{K}$. Since $\boldsymbol{\ell}$ is a left eigenvector of the matrix $(C.8)$ it follows that $\boldsymbol{\ell}\cdot(F_1^+, F_2^+, F_3^+) = 0$. From Eq. $(B.2.a)$ in Proposition 7, we see that $([G_1],[G_2],[G_3]) = \rho_1(F_1^+, F_2^+, F_3^+)$ for any constant $\rho_1 \in \mathbb{R}$, so:

$$\boldsymbol{\ell}\cdot[G] = \boldsymbol{\ell}\cdot(\rho_1 F^+) = \rho_1\boldsymbol{\ell}\cdot F^+ = 0.$$

The converse can be proved similarly by reversing the order of the calculations. ☐

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