TWO-DIMENSIONAL BLASCHKE PRODUCTS: DEGREE GROWTH AND ERGODIC CONSEQUENCES

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Abstract. We study the dynamics of Blaschke products in two dimensions, particularly the rates of growth of the degrees of iterates and the corresponding implications for the ergodic properties of the map.

1. Introduction

For dominant rational maps of compact, complex, Kahler manifolds there is a conjecture specifying the expected ergodic properties of the map depending on the relationship between the rates of growth of certain degrees under iteration of the map. (See Conjecture 1.1, as presented in [14], as well as the results towards this conjecture [15, 5, 6, 7].) We observe that the two-dimensional Blaschke products fit naturally within this conjecture, having examples falling into both of the major (dramatically distinct) cases that the conjecture gives for maps of a surface. We then analyze the dynamics of Blaschke products in each of these cases, relating it to the behavior predicted by this conjecture.

Furthermore, generic (in an appropriate sense) Blaschke products do not have the Julia set contained within $\mathbb{T}^2$. Rather, "the majority of it" is away from $\mathbb{T}^2$ within the support of the measure of maximal entropy. This is very different from the case of 1-dimensional Blaschke products for which the Julia set is the unit circle.

A (finite) Blaschke product is a map of the form

$$E(z) = \theta_0 \prod_{i=1}^{n} \frac{z - e_i}{1 - z\overline{e_i}},$$

where $n \geq 2$, $e_i \in \mathbb{C}$ with $|e_i| < 1$ for each $i = 1, \ldots, n$, and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$.

We consider Blaschke products in two variables that are of the form

$$f(z,w) = \left(\theta_1 \prod_{i=1}^{m} \frac{z - a_i}{1 - \bar{a}_i z} \prod_{i=1}^{n} \frac{w - b_i}{1 - \bar{b}_i z}, \theta_2 \prod_{i=1}^{m} \frac{z - c_i}{1 - \bar{c}_i z} \prod_{i=1}^{n} \frac{w - d_i}{1 - \bar{d}_i z} \right). \tag{1}$$

We will often denote the corresponding 1-variable Blaschke products by $A(z), B(w), C(z)$, and $D(w)$. (As in [19], we allow the case that some of the degrees m, n, p, and q can be 1).

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It is often helpful to describe the degrees of a given Blaschke product f by a matrix

$$N = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

Given any matrix N containing positive integers, for any choice of rotations θ_1, θ_2 and of zeros $a_1, \ldots, a_m$, $b_1, \ldots, b_n$, $c_1, \ldots, c_p$, and $d_1, \ldots, d_q$ (all of modulus less than 1) there is a Blaschke product. We denote the space of all such Blaschke products by $\mathcal{B}_N$ and we will call any $f \in \mathcal{B}_N$ a *Blaschke Product associated to* N. We typically will use the notation $\sigma \in \mathbb{D}^{m+n+p+q}$ to represent the collection of zeros $a_1, \ldots, d_q$. Notice that $\mathcal{B}_N$ can be identified with $\mathbb{D}^{m+n+p+q} \times \mathbb{T}^2$, an identification that we use when discussing sets of full measure on $\mathcal{B}_N$.

In the case that all of the zeros are equal to 0, a 2-dimensional Blaschke product becomes a *monomial map*

$$f(z, w) = (z^m w^n, z^p w^q), \tag{2}$$

whose dynamics was studied extensively in [11, 17]. For any N we will call this map the *monomial map associated to* N. (It is also interesting to note that monomial maps occur frequently "outside of dynamical systems", for example in the description of cusps for Inoue-Hirzebruch surfaces [10]).

One particularly nice reason to study Blaschke product is that they preserve the unit torus $\mathbb{T}^2 := \{(z, w) \; : \; |z| = |w| = 1\}$. In fact N describes the action of f_* on $\pi_1(\mathbb{T}^2)$ in terms of the obvious choice of generators for $\pi_1(\mathbb{T}^2)$. The topological entropy on $\mathbb{T}^2$ for any Blaschke product is the largest eigenvalue $c_1(N)$ of N.

As in [19], we will often consider the case in which f induces an orientation preserving diffeomorphism on $\mathbb{T}^2$. This holds if and only if $\det(N) = 1$. In this case, we refer to f as a *Blaschke product diffeomorphism.* (Generally such an f is only a diffeomorphism on $\mathbb{T}^2$, not globally on $\mathbb{P}^2$.) The corresponding monomial map induces a linear Anosov map on $\mathbb{T}^2$ and a Blaschke product whose zeros are sufficiently small will also induce an Anosov map on $\mathbb{T}^2$. Moreover, given a Blaschke product diffeomorphisms the restriction $f|_{\mathbb{T}^2}$ always has an invariant measure of maximal entropy $\mu_{\rm tor}$ of entropy $\log(c_1(N))$; see Lemma 3.3.

Given a rational map $g : \mathbb{P}^2 \to \mathbb{P}^2$, the *algebraic degree* $d_{\rm alg}(g)$ is the common degree of the homogeneous equations on $\mathbb{C}^3$ defining g. In some cases, the degree of iterates drops, $d_{\rm alg}(g^n) < (d_{\rm alg}(g))^n$, because a common factor appears in the homogeneous equations for g^n. (See [12]). However, a limiting degree called *the first dynamical degree*

$$\lambda_1(g) = \lim_{n\to\infty} (d_{\rm alg}(g^n))^{1/n} \tag{3}$$

always exists, describing the asymptotic rate of growth in the sequence $\{d_{\rm alg}(g^n)\}$, [20]. Note that $\lambda_1(g) \leq d_{\rm alg}(g)$.

There is another degree called the *topological degree* $d_{\rm top}(g)$ which is defined to be the number of preimages of a generic point under g.

The ergodic properties of g are believed to depend heavily on the relationship between $\lambda_1(g)$ and $d_{\text{top}}(g)$, as explained in [14]. We provide a brief summary in the case that the ambient space is a Kahler surface X.

Conjecture 1.1. *The ergodic properties of a rational map $g : X \to X$ are believed to fall into three cases:*

- *Case I:* **Large topological degree:** $d_{\text{top}}(g) > \lambda_1(g)$. *This case has been solved by* [15] *where it was shown that there is an ergodic invariant measure μ of maximal entropy* $\log(d_{\text{top}}(f))$. *The measure μ is not supported on hypersurfaces, it does not charge the points of indeterminacy, and the repelling points of f are equidistributed according to this measure.*
- *Case II:* **Small topological degree:** $d_{\text{top}}(g) < \lambda_1(g)$. *It is believed that there is an ergodic invariant measure μ of maximal entropy* $\log(\lambda_1(g))$ *that is not supported on hypersurfaces and does not charge the points of indeterminacy. Saddle-type points are believed to be equidistributed according to this measure. A recent series of preprints* [5, 6, 7] *has appeared where it is proven that these expected properties hold, provided that certain technical hypotheses are met.*
- *Case III:* **Equal degrees:** $d_{\text{top}}(g) = \lambda_1(g)$. *Little is known or conjectured in this case.*

The following concept is essential to computing dynamical degrees:

Definition 1.2. *A rational mapping $f : X \to X$ of a Kahler surface X is called* algebraically stable *if there is no integer n and no hypersurface V so that each component of $f^n(V)$ is contained within the indeterminacy set $I(f)$.*

For the case $X = \mathbb{P}^2$, see [21, p. 109] and more generally, see [4].

We now summarize the main results of this paper:

Let $\widetilde{\mathbb{P}}^2$ be $\mathbb{P}^2$ blown-up at the two points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ on the line at infinity. (See [13] for details on the blow-up procedure). In Section 2 we prove:

Theorem 1.3. *For any N, there exists a generic family $\mathcal{B}'_N \subset \mathcal{B}_N$ of total measure from which f extents to an algebraically stable map of $\widetilde{\mathbb{P}}^2$.*

(The explicit characterization of $\mathcal{B}'_N$ is given in Section 2).

From this we can conclude

Theorem 1.4. *For any $f \in \mathcal{B}'_N$ we have that $\lambda_1(f) = c_1(N)$, where $c_1(N)$ is the leading eigenvalue of N, i.e.*

$$\lambda_1(f) = c_1(N) = \frac{m + q + \sqrt{(m-q)^2 + 4nq}}{2}.$$

In fact a lower bound $\lambda_1(f) \geq c_1(f)$ exists for all $f \in \mathcal{B}_N$, leading us to the central question of the paper:

Question 1. *Does every Blaschke product $f \in \mathcal{B}_N$ have dynamical degree $\lambda_1(f) = c_1(N)$?*

This question is discussed in greater detail in Remark 3.

We conclude Section 2 by showing that for any matrix of degrees N there is an open dense set of full measure $\mathcal{B}''_N \subset \mathcal{B}_N$ so that for all $f \in \mathcal{B}''_N$ we have $d_{\rm top}(f) = mq + np$. Therefore,

Theorem 1.5. *For any matrix of degrees N there is a full measure set $\hat{\mathcal{B}}_N = \mathcal{B}'_N \cap \mathcal{B}''_N$ in $\mathcal{B}_N$ so that for $f \in \hat{\mathcal{B}}_N$ we have $d_{\rm top}(f) > \lambda_1(f)$.*

That is, generic Blaschke products are in Case I of Conjecture 1.1. For these maps the results from [15] apply providing for the existence of a measure of maximal entropy μ having entropy $\log(d_{\rm top}(f))$. This measure is supported away from the invariant torus $\mathbb{T}^2$ and the repelling points of f are are equidistributed according to μ. More details are discussed in Section 3.

In Section 4 we show that for some N there are also $f \in \mathcal{B}_N$ for which the opposite inequality holds: $d_{\rm top}(f) < \lambda_1(f)$. Hence some Blaschke products also fall into Case II of Conjecture 1.1. In Section 4 we provide examples and study the relationship between the dynamics of these maps and the expected behavior.

None of these examples (that we have found) satisfy the hypothesis of Theorem 1.3, hence we do not know precisely their dynamical degree, rather we utilize the lower bound $\lambda_1(f) \geq c_1(N)$ that is computed in Proposition 2.4. Many of the examples in Section 4 induce a diffeomorphism of $\mathbb{T}^2$ in which case there is an invariant measure $\mu_{\rm tor}$ supported on $\mathbb{T}^2$ of entropy $\log(c_1(f))$. It would be interesting to check whether $\mu_{\rm tor}$ is of maximal entropy $\log(\lambda_1(f))$ (see [9]). It would also be interesting to see how these maps fit within the framework presented in [5, 6, 7], in particular whether the theorems within apply.

We conclude in Section 5 with a discussion of perturbations of monomial maps which are holomorphic on $\mathbb{C}^2$ and $\mathbb{P}^2$, respectively. The latter form of perturbation provide a bifurcation similar to the one found in the present paper for Blaschke products.

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2. Computation of degrees

2.1. **Dynamical degree.**

The following five paragraphs are a summary of the work of many other authors (see the references within) and none of it is original to this paper. We provide it as a brief summary of the technique that we will use for computing the dynamical degree of Blaschke products.

One can recasts the dynamical degree (3) as:

$$\lambda_1(f) = \limsup (r_1((f^n)^*))^{1/n}, \tag{4}$$

where $r_1((f^n)^*)$ is the spectral radius of the linear action of $(f^n)^*$ on $H_a^{1,1}(X,\mathbb{R})$. Here, $H_a^{1,1}(X,\mathbb{R})$ is the part of the $(1,1)$ cohomology that is generated by algebraic curves in X, see [15, Prop 1.2(iii)]. (We consider the cohomology class $[D]$ of an algebraic curve D in the sense of closed-positive $(1,1)$ currents.) When $X = \mathbb{P}^2$ this definition agrees with (3) and this new definition is invariant under birational conjugacy (see [15, Prop 1.5]).

This clarifies one motivation for the definition of algebraic stability: if $f : X \to X$ is algebraically stable then, according to [4, Thm 1.14], one has that the action of $f^* : H_a^{1,1}(X,\mathbb{R}) \to H_a^{1,1}(X,\mathbb{R})$ is well-behaved: $(f^n)^* = (f^*)^n$. In this case, (4) simplifies to

$$\lambda_1(f) = r_1(f). \tag{5}$$

Given a rational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ that is not algebraically stable, a typical way to compute $\lambda_1(f)$ is as follows. One tries to find an appropriate finite sequence of blow-ups at certain points in $\mathbb{P}^2$ in an attempt to obtain a new surface X on which the extension $\tilde{f}$ of f is algebraically stable. Note that in this approach $\tilde{f}$ and f are birationally conjugate using the canonical projection $\pi : X \to \mathbb{P}^2$, and hence $\lambda_1(f) = \lambda_1(\tilde{f})$.

A surface X that is birationally equivalent to $\mathbb{P}^2$ is called a *rational surface*. In this paper we will always construct X using the strategy described in the previous paragraph, so it will be an ongoing assumption that any surface X is rational (unless otherwise explicitly stated). Given a rational surface X, $H_a^{1,1}(X)$ coincides with the full cohomology $H^{1,1}(X)$, allowing us a further simplification.

Suppose that one has created such a new surface X so that $\tilde{f} : X \to X$ is algebraically stable. Then, $\lambda_1(f) = \lambda_1(\tilde{f}) = r_1(\tilde{f}^*)$, where $r_1(\tilde{f}^*)$ is the spectral radius of the action $\tilde{f}^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$. This latter number can be computed by considering the pull-backs f^* of appropriate finite set of curves that form a basis for $H^{1,1}(X,\mathbb{R})$. Nice descriptions of this procedure, and explicit examples, are demonstrated in [2, 1, 3] and the references therein. (In fact the latter two of these references work in terms of $Pic(X)$, rather than $H^{1,1}(X)$, but the technique is essentially the same.)

In fact, such a modification does not exist for all rational maps. In [11] it was shown that for certain monomial maps (with some negative powers) there is no finite sequence of blow-ups that one can do, starting with $\mathbb{P}^2$, in order to obtain a surface X on which the map is algebraically stable. However, in the case that f is a monomial map with all positive powers (as assumed in this paper) it was shown in [11] that one can always find a toric surface $\check{X}$ on which f becomes algebraically stable. However, in this case, $\check{X}$ is obtained first by blowing-up $\mathbb{P}^2$ *and then extending to a ramified cover* (so that it is typically no longer a rational surface). Luckily, our generic Blaschke products will be much simpler. See Question 3 at the end of this section.

We begin by writing a Blaschke product $f \in \mathcal{B}_N$ in homogeneous coordinates $[Z : W : T]$, with $T = 0$ corresponding to the line at infinity with respect to the usual affine coordinates (z, w). We write

$$f([Z : W : T]) = [f_1(Z, W, T) : f_2(Z, W, T) : f_3(Z, W, T)]$$

with

$$\begin{aligned} f_1(Z,W,T) &= \theta_1 \prod_{i=1}^{m}(Z - a_iT)\prod_{i=1}^{n}(W - b_iT)\prod_{i=1}^{p}(T - Z\overline{c_i})\prod_{i=1}^{q}(T - W\overline{d_i}) \\ f_2(Z,W,T) &= \theta_2 \prod_{i=1}^{p}(Z - c_iT)\prod_{i=1}^{q}(W - d_iT)\prod_{i=1}^{m}(T - Z\overline{a_i})\prod_{i=1}^{n}(T - W\overline{b_i}) \\ f_3(Z,W,T) &= \prod_{i=1}^{m}(T - Z\overline{a_i})\prod_{i=1}^{n}(T - W\overline{b_i})\prod_{i=1}^{p}(T - Z\overline{c_i})\prod_{i=1}^{q}(T - W\overline{d_i}) \end{aligned} \tag{6}$$

so that $d_{\mathrm{alg}}(f) = m + n + p + q$.

Given $f \in \mathcal{B}_N$, we will call the lines $Z - a_iT = 0$ the *zeros* of A and denote the union of such lines by $Z(A)$. Similarly, we will call the lines $T - Z\overline{a_i} = 0$ the *poles* of A, denoting the union of such lines by $P(A)$. The collections of lines $Z(B), P(B), Z(C), P(C), Z(D)$, and $P(D)$ are all defined similarly.

From (6) we see that the "vertical" lines from $P(A)$ and the "horizontal" lines from $P(B)$ are collapsed to the point at infinity $[1 : 0 : 0]$. In a similar way $P(C)$ and $P(D)$ are collapsed to $[0 : 1 : 0]$.

For the monomial map associated to N, the lines of zeros are also collapsing lines, each collapsing to $[0 : 0 : 1]$. This is not the case for typical Blaschke products—for example when all of the zeros $a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p, d_1, \ldots, d_q$ are distinct—these lines no longer collapse. In addition, for monomial maps, the line at infinity $T = 0$ is typically collapsed to either $[1 : 0 : 0]$ or $[0 : 1 : 0]$, but when all of the zeros for a Blaschke product are different from zero this no longer happens.

The points of indeterminacy for f are precisely the points for which all three coordinates of (6) vanish:

Lemma 2.1. *Any $f \in \mathcal{B}_N$ has 2 points of indeterminacy on the line at infinity: $[1 : 0 : 0]$ and $[0 : 1 : 0]$.*

Remark 1. *Blaschke products $f \in \mathcal{B}_N$ are never algebraically stable on $\mathbb{P}^2$: As mentioned previously the lines in $P(A) \cup P(B)$ collapse under f to $[1 : 0 : 0]$ and the lines in $P(C) \cup P(D)$ collapse to $[0 : 1 : 0]$.*

We note that there are typically quite a large number of additional indeterminant points in $\mathbb{C}^2$. In fact each of the intersection points from

$$\begin{gathered} Z(A) \cap P(B),\ P(A) \cap Z(B), \\ Z(C) \cap P(D),\ P(C) \cap Z(D),\ \text{and} \\ (P(A) \cup P(B)) \cap (P(C) \cup P(D)) \end{gathered}$$

is a point of indeterminacy. In particular, assuming that all of the zeros are distinct, there are $2(mn + pq) + (mq + np)$ such points of indeterminacy in $\mathbb{C}^2$.

Given $f \in \mathcal{B}_N$ we write I_f to denote the indeterminacy points of f and C_f to denote the critical set of f (within which are all of the collapsing curves of f). Let $P_f = P(A) \cup P(B) \cup P(C) \cup P(D)$ be the union of all lines of poles for f.

Lemma 2.2. *Given any f_1 and $f_2 \in \mathcal{B}_N$ differing by rotations (but with the same zeros: $\sigma_1 = \sigma_2$), we have the following:*

- $I_{f_1} = I_{f_2}$,
- $C_{f_1} = C_{f_2}$, *and*
- $P_{f_1} = P_{f_2}$.

Proof. For each f_1 and f_2, the indeterminacy points are given by the points where the corresponding lift F_1 or F_2, respectively, to $\mathbb{C}^3$ has all three coordinates vanishing. The rotation multiplies the first two coordinates of each map by non-zero constants θ_1 and θ_2, hence has no affect on the indeterminacy points.

Similarly, any rotation of f by factors θ_1, θ_2 (non-zero) changes $\det(Df)$ by the non-zero factor $\theta_1\theta_2$, so the critical curves are unaffected.

The third item follows similarly. □

We now prove Theorem 1.3. The appropriate set of generic maps $\mathcal{B}'_N$ will be defined within the proof.

Proof of Theorem 1.3: The strategy of proof is as follows. For any $f \in \mathcal{B}_N$ the two indeterminant points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are unavoidable since all of the lines of poles collapse to these two points. Let I_f^0 be the set of remaining indeterminant points, all of which are in $\mathbb{C}^2$. We first select an appropriate full-measure subset $\mathcal{B}'_N \subset \mathcal{B}_N$ from we can ensure that any collapsing curve (other than the lines of poles) does not have orbit landing in I_f^0 or on one of the lines of poles. (Landing on the lines of poles is dangerous since they are mapped to the indeterminant points $[1 : 0 : 0]$ and $[0 : 1 : 0]$). We then blow up $\mathbb{P}^2$ at $[1 : 0 : 0]$ and $[0 : 1 : 0]$ obtaining $\widetilde{\mathbb{P}^2}$ and show that every $f \in \mathcal{B}'_N$ extends to algebraically stable map on $\widetilde{\mathbb{P}^2}$.

To define $\mathcal{B}'_N$, we first restrict to those $f \in \mathcal{B}_N$ for which all of the zeros from σ are non-zero and distinct. A consequence of this definition is that I_f^0 is disjoint from the z and w-axes. Similarly, within $\mathbb{C}^2$, the lines of poles P_f are also disjoint from the z and w-axes.

Fix such an f. We will use the dynamics of f to define the appropriate rotation values θ_1 and θ_2.

Let I_f^0 denote the indeterminacy points of f that are in $\mathbb{C}^2$ and let $C_1, \ldots, C_k$ denote the curves of f that collapse to any points in $\mathbb{C}^2$, if any such curves exist. (Recall that there are always curves collapsing to $[1 : 0 : 0]$ and $[0 : 1 : 0]$, which we do not include here.) By Lemma 2.2, both of these sets are independent of any rotation of f by θ_1, θ_2.

We let $\xi_1, \ldots, \xi_k$ be the images of these collapsing curves and Σ be their union. These points are typically changed by a non-zero rotation.

We now construct the set of admissible rotations θ_1, θ_2:

Given a finite set $S \subset \mathbb{C}^2$ we denote by $(\theta_1, \theta_2) \cdot S$ the rotation of S by θ_1 and θ_2 in the z and w coordinates, respectively. Similarly we write $f_{(\theta_1,\theta_2)}$ for the composition of f followed by the rotation by (θ_1, θ_2) and by $f^n_{(\theta_1,\theta_2)}$ as the n-th iterate of $f_{(\theta_1,\theta_2)}$.

Let

$$\Omega_0 = \{(\theta_1, \theta_2) \in \mathbb{T}^2 \,:\, (\theta_1, \theta_2) \cdot \Sigma \cap (I^0_f \cup P_f) = \emptyset\}$$

and inductively we let

$$\Omega_n = \{(\theta_1, \theta_2) \in \Omega_n \,:\, f^n_{(\theta_1,\theta_2)}(\Sigma) \cap (I^0_f \cup P_f) = \emptyset\}.$$

For each n, Ω_n is the complement of a set of measure zero within $\mathbb{T}^2$. This follows because for each $f(f^{n-1}_{(\theta_1,\theta_2)}(\xi_j))$ the set of all rotated points is either a torus, a circle, or the origin, and hence can intersect I^0_f in at most a single choice of (θ_1, θ_2). (We are using I^0_f does not intersect the axes $z = 0$ or $w = 0$.) Similarly, since each of the lines of poles is not on either of the axes, so the torus (or circle respectively) can at most intersect this line in a circle (or point), which is of measure zero on the torus (or circle).

Thus, the set $\Omega = \cap \Omega_n$ is the complement of a countable union of sets of measure zero, and hence a set of total measure. We will denote by $\mathcal{B}'_N$ the set of all such maps.

We now blow up $[1:0:0]$ and $[0:1:0]$ obtaining $\widetilde{\mathbb{P}^2}$. Recall that the blow up of $\mathbb{C}^2$ at $(0,0)$ is

$$\widetilde{\mathbb{C}}^2_{(0,0)} = \big\{((w,t), l) \in \mathbb{C}^2 \times \mathbb{P}^1 \,:\, (z,t) \in l\big\}$$

There is a canonical projection $\pi : \widetilde{\mathbb{C}}^2_{(0,0)} \to \mathbb{C}^2$ and the fiber $E_{(0,0)} = \pi^{-1}((0,0))$ is referred to as the *exceptional divisor*. See [13]. In fact, this definition is coordinate independent so that the notion of blowing up a complex surface X at a point $p \in X$ is well-defined. The exceptional divisor above p will be denoted by E_p.

For $f \in \mathcal{B}'_N$ we check that f extends continuously (and hence holomorphically) to the blow-up at $[1:0:0]$. The calculation at $[0:1:0]$ is identical, and we omit it. We write f in the affine coordinates $w = W/Z$ and $t = T/Z$ so that the point of indeterminacy $[1:0:0]$ is at the origin with respect to these coordinates.

We work in the chart $(t, \lambda) \mapsto (\lambda t, t, \lambda) \in \widetilde{\mathbb{C}}^2_{(0,0)}$. With domain in this chart and codomain in the typical chart $(z,w) = (Z/T, W/T)$ we find that f induces:

$$(t,\lambda) \mapsto \left(\prod_{i=1}^{m} \frac{1 - a_i t}{t - \overline{a_i}} \prod_{i=1}^{n} \frac{\lambda t - b_i t}{t - \overline{b_i}\lambda t} \;,\; \prod_{i=1}^{m} \frac{1 - c_i t}{t - \overline{c_i}} \prod_{i=1}^{n} \frac{\lambda t - d_i t}{t - \overline{d_i}\lambda t} \right)$$

so that the extension to $E_{[1:0:0]}$ is given by taking the limit $t \to 0$:

$$\lambda \mapsto \left(\prod_{i=1}^{m} \frac{-1}{\overline{a_i}} \prod_{i=1}^{n} \frac{\lambda - b_i}{1 - \overline{b_i}\lambda} \;,\; \prod_{i=1}^{p} \frac{-1}{\overline{c_i}} \prod_{i=1}^{q} \frac{\lambda - d_i}{1 - \overline{d_i}\lambda} \right).$$

The calculation can also be done in the coordinates $\lambda' = \frac{1}{\lambda}$, where one sees that the extension is continuous to all of $E_{[1:0:0]}$, hence holomorphic. (We are essentially using that none of the zeros a_i or c_i are equal to 0.) Since the extension is non-constant with respect to λ, the extension of f sends $E_{[1:0:0]}$ to a non-trivial rational curve.

The blow-up at $[0:1:0]$ follows similarly and the extension of f also sends $E_{[0:1:0]}$ to a non-trivial rational curve. We denote by $\widetilde{f}:\widetilde{\mathbb{P}^2}\to\widetilde{\mathbb{P}^2}$ this extension of f to the space $\widetilde{\mathbb{P}^2}$ that is obtained by doing both blow-ups.

Having blown up $[1:0:0]$ and $[0:1:0]$ we will now observe that each of the lines of poles from $P(A)\cup P(B)$ covers $E_{[1:0:0]}$ with non-zero degree and each of the lines of poles from $P(C)\cup P(D)$ covers $E_{[0:1:0]}$ with non-zero degree. *In particular $\widetilde{f}$ does not collapse any of these lines to points.*

The calculation is the same for each line, so we show it for $z=\frac{1}{\overline{a_1}}$. If we parameterize this line by $w=W/T$, then the image in coordinate $\rho=\frac{w'}{t}$ (here $w'=W/Z$) can be found by substituting $W=w$, $Z=\frac{1}{\overline{a_1}}$, and $T=1$ into the quotient $f_2(Z,W,T)/f_3(Z,W,T)$. We obtain

$$\rho(w)=\frac{\prod_{i=1}^{p}(1/\overline{a_1}-c_i)\prod_{i=1}^{q}(w-d_i)}{\prod_{i=1}^{p}(1-\overline{c_i}/\overline{a_1})\prod_{i=1}^{q}(1-w\overline{d_i})}.$$

This is a rational map of degree q if $a_1\neq c_i$ for any $i=1,\dots,p$, which holds by hypothesis that $f\in\mathcal{B}_N'$. (In fact one can do the same calculation in the other coordinate charts on the line $z=\frac{1}{\overline{a_1}}$ and on $E_{[1:0:0]}$, but the result will also be a rational map of degree q in those coordinates, as well.)

Similar calculations show that under $\widetilde{f}$, each of the lines of poles from $P(A)$ covers $E_{[1:0:0]}$ with degree q and each of the lines the poles from $P(B)$ cover $E_{[1:0:0]}$ with degree p. The poles from $P(C)$ cover $E_{[0:1:0]}$ with degree n and the poles from $P(D)$ with degree m.

Let $\widetilde{X}$ by the blow-up of complex surface X at point p and $\pi:\widetilde{X}\to X$ be the corresponding projection. Given an algebraic curve $D\subset X$ there are two natural ways to "lift" D to $\widetilde{X}$: the *total transform* and the *proper transform*. The total transform is just $\pi^{-1}(D)$ while the proper transform is obtained by the closure $\overline{\pi^{-1}(D\setminus\{p\})}$. Clearly when $p\notin D$ there is no difference, however when $p\in D$ they differ by the exceptional divisor $E_p\subset\widetilde{X}$. In the case the many points have been blown-up the analogous definitions hold, see [13].

We now check that the only collapsing curves for $\widetilde{f}:\widetilde{\mathbb{P}^2}\to\widetilde{\mathbb{P}^2}$ are the proper transforms of the curves $C_1,\dots,C_k$ that are collapsed under f to points in $\mathbb{C}^2$. In fact any collapsing curve must be either the proper transform of a collapsing curve for $f:\mathbb{P}^2\to\mathbb{P}^2$ or be one of the exceptional divisors $E_{[1:0:0]}$ or $E_{[0:1:0]}$. Since $\widetilde{f}$ maps each of $E_{[1:0:0]}$ or $E_{[0:1:0]}$ to a non-trivial rational curve, neither is a collapsing curve. Furthermore, we have just checked that the lines from $P(A)\cup P(B)\cup P(C)\cup P(D)$ are no longer collapsed by $\widetilde{f}$. All that remains are the proper transforms of $C_1,\dots,C_k$.

By the choice of $f\in\mathcal{B}_N'$ we have that the orbits of these collapsing curves avoid the indeterminant points as well as all of the lines of poles. Therefore, under the extension $\widetilde{f}$, their orbits cannot land on $E_{[1:0:0]}$, $E_{[0:1:0]}$, or the line at infinity. Thus, the orbits under $\widetilde{f}$ coincide with those under f, and they do not hit points in I_f^0, which are the only indeterminant points for $\tilde{f}$. $\square$

Question 2. *Is any rotation of the maps in $\mathcal{B}'_N$ also algebraically stable on $\widetilde{\mathbb{P}^2}$?*

Proof of Theorem 1.4. We can now compute $\lambda_1(f) = \lambda_1(\widetilde{f})$ as the spectral radius of the action of $\widetilde{f}^*$ on $H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right)$.

Let $\widetilde{L}_v \subset \widetilde{\mathbb{P}^2}$ be the proper transform of the vertical line $L_v := \{Z = 0\}$ and let $\widetilde{L}_h$ be the proper transform of the horizontal line $L_h := \{W = 0\}$. We choose the fundamental classes $[\widetilde{L}_v]$, $[E_{[0:1:0]}]$ and $[E_{[1:0:0]}]$ as our basis of $H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right)$. It will be useful in our calculation to express $[\widetilde{L}_h]$ in terms of this basis.

Lemma 2.3. *We have that:*

$$[\widetilde{L}_h] \sim [\widetilde{L}_v] + [E_{[0:1:0]}] - [E_{[1:0:0]}].$$

Proof. Both $[L_v]$ and $[L_h]$ are cohomologous in $\mathbb{P}^2$ so that their total transforms $\pi^*([L_v]) = [\widetilde{L}_v] + [E_{[0:1:0]}]$ and $\pi^*([L_h]) = [\widetilde{L}_h] + [E_{[1:0:0]}]$ are cohomologous, as well. $\square$

We have that $f : \mathbb{P}^2 \to \mathbb{P}^2$ maps the lines of zeros $Z(A), Z(B)$ and the lines of poles $P(C), P(D)$ to L_v. However, after blowing up $[0 : 1 : 0]$ the lines of poles cover $E_{[0:1:0]}$ so that they should be considered as part of $\widetilde{f}^*[E_{[0:1:0]}]$ and not part of $\widetilde{f}^*[\widetilde{L}_v]$.

To see this more formally we write $\widetilde{f}$ with domain in the affine coordinates $z = Z/T, w = W/T$ and image in the coordinates $t = T/W$, $\lambda = Z/T$. These image coordinates are chosen so that when $t = 0$, λ parameterizes $E_{[0:1:0]}$ (except for one point). The second coordinate of the the image $(t, \lambda) = \widetilde{f}(z, w)$ is given by:

$$\lambda \;=\; \frac{\Pi_{i=1}^m (z - a_i) \Pi_{i=1}^n (w - b_i)}{\Pi_{i=1}^m (1 - z\bar{a}_i) \Pi_{i=1}^n (1 - w\bar{b}_i)} \tag{7}$$

In the (t, λ) coordinates $\widetilde{L_v}$ is given by $\lambda = 0$. Therefore,

$$\widetilde{f}^*(\widetilde{L_v}) \sim m[\widetilde{L_v}] + n[\widetilde{L_h}] \tag{8}$$

because of the m factors of $z - a_i$ and n factors of $w - b_i$ in the numerator of Equation (7). Using Lemma 2.3 we find:

$$\widetilde{f}^*[\widetilde{L}_v] \sim (m + n)[\widetilde{L}_v] + n[E_{[0:1:0]}] - n[E_{[1:0:0]}].$$

Suppose that we had parameterized the image of $\widetilde{f}$ in the other natural set of coordinates in a neighborhood of $E_{[0:1:0]}$, given by $\hat{z} = Z/W$ and $\eta = T/Z$. Then, the total transform $\pi^{-1}(L_v) = \widetilde{L_v} \cup E_{[0:1:0]}$ is given by $\hat{z} = 0$. The first coordinate of the image $(\hat{z}, \eta) = \widetilde{f}(z, t)$ is given by

$$\hat{z} = \frac{\theta_1 \prod_{i=1}^m (z - a_i) \prod_{i=1}^n (w - b_i) \prod_{i=1}^p (1 - z\overline{c_i}) \prod_{i=1}^q (1 - w\overline{d_i})}{\Pi_{i=1}^p (z - c_i) \Pi_{i=1}^q (w - d_i) \Pi_{i=1}^m (1 - z\bar{a}_i) \Pi_{i=1}^n (1 - w\bar{b}_i)}$$

so that

$$\widetilde{f}^* \left([L_v] + [E_{[0:1:0]}]\right) \sim m[L_v] + n[L_h] + p[L_v] + q[L_h].$$

By subtracting (8) and using Lemma 2.3 we find:

$$\widetilde{f}^*([E_{[0:1:0]}]) \sim (p+q)[\widetilde{L}_v] + q[E_{[0:1:0]}] - q[E_{[1:0:0]}].$$

A similar calculation gives that

$$\widetilde{f}^*([E_{[1:0:0]}]) \sim (m+n)[\widetilde{L}_v] + n[E_{[0:1:0]}] - n[E_{[1:0:0]}].$$

Therefore, in terms of the basis $\left\{[\widetilde{L}_v], [E_{[0:1:0]}], [E_{[1:0:0]}]\right\}$ we have $\widetilde{f}^*$ given by:

$$\begin{bmatrix} (m+n) & (p+q) & (m+n) \\ n & q & n \\ -n & -q & -n \end{bmatrix}.$$

Therefore, $\lambda_1\left(\widetilde{f}\right) = r_1\left(\widetilde{f}^*\right)$ is the largest eigenvalue of this matrix, which happens to coincide with $c_1(N)$. Since dynamical degrees are invariant under birational conjugacy with $\widetilde{f}$ and f conjugate under the projection π we find $\lambda_1(f) = \lambda_1\left(\widetilde{f}\right) = c_1(N)$, as well. $\square$

Remark 2. *Observe that for $f \in \mathcal{B}'_N$ we have $\lambda_1(f)$ coincides with $c_1(N)$ which is the asymptotic rate of maximal growth of f_*^n on $H_1(\mathbb{T}^2)$.*

Question 3. *As mentioned earlier, in* [11] *it is shown that for any matrix of degrees N with positive coefficients there is some toric surface $\check{X}_N$ on which monomial map corresponding to N becomes algebraically stable. Do all $f \in \mathcal{B}_N$ extend to algebraically stable maps on $\check{X}_N$, as well? This would be particularly helpful for studying bifurcations within the family.*

While Theorem 1.4 may only hold for $f \in \mathcal{B}'_N$, the following lower bound for $\lambda_1(f)$ holds on all of $\mathcal{B}_N$.

Proposition 2.4. *For any Blaschke product $f \in \mathcal{B}_N$ we have that $\lambda_1(f) \geq c_1(N)$.*

Proof. We use the definition given in Equation (3) for $\lambda_1(f)$.

Consider the basis $\{[\gamma_1], [\gamma_2]\}$ for $H_1(\mathbb{T}^2)$ generated by the unit circle γ_1 in the plane $w = 0$ and the unit circle γ_2 in the plane $z = 0$. As noted earlier, the action $f_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ with respect to this basis is given by multiplication by the matrix N.

We will show that $\deg(f^n) \geq ||N^n||_\infty$, i.e. that $\deg(f^n)$ grows at least as fast as the largest element of N^n. This suffices to prove the assertion since $||N^n||_\infty \geq a \cdot c_1(N)^n$ for some positive constant a.

Notice that f_* acts "stably" on $H_1(\mathbb{T}^2)$ in the sense that the action of f_*^n is given by N^n with respect to the previously mentioned basis. Consider now the largest element of N^n, which we suppose (for the moment) is the $(1,1)$ element. Then $f_*^n([\gamma_1]) = k[\gamma_1]$ where $k \geq a \cdot c_1(N)^n$.

Write f in affine coordinates (z,w). We will show that the first coordinate of f^n is a rational function of degree at least k in z. This is sufficient to give that any homogeneous expression for f^n has degree at least k, as well.

Let π be the projection $\pi(z,w)=z$ so that the first coordinate of f^n is given by $\pi \circ f^n$. Also let $\iota(z)=(z,1)$. The iterate f^n is holomorphic on the open bidisc $\mathbb{D}\times\mathbb{D}$ because f^n forms a normal family there. Then $\pi \circ f^n \circ \iota : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a holomorphic function preserving the unit circle. By the previous homological considerations this map has degree k on the circle, so by a standard theorem it must be a (one variable) Blaschke product of degree k with no poles inside of $\mathbb{D}$. This gives a lower bound for the degree in z of the first coordinate of f by $k \geq a \cdot c_1(N)^n$.

In the case that some other element than the $(1,1)$ element of N^n were largest, an identical proof works by choosing ι to be the appropriate inclusion and π to be the appropriate projection.

For each n the same argument can be applied to show that one of the affine coordinates of f^n is a rational function of degree at least $a \cdot c_1(N)^n$. The same holds for the homogeneous expression for f^n, giving $d_{\text{alg}}(f^n) \geq a \cdot c_1(N)^n$, which is sufficient for the desired bound on $\lambda_1(f)$. □

Remark 3. *In Question 1 from the introduction we asked: Does every Blaschke product $f \in \mathcal{B}_N$ have dynamical degree $\lambda_1(f)=c_1(N)$? This could likely follow from either:*

(i) *Lower-semicontinuity of $\lambda_1(f)$ over all $f \in \mathcal{B}_N$ (since the equality holds on a set of full measure and the lower bound holds everywhere), or*
(ii) *If the cohomological calculation in the proof of Theorem 1.4 gives an upper bound for $\lambda_1(f)$ for those maps f that do not extend to an algebraically stable map on $\widetilde{\mathbb{P}^2}$. (Such a bound would be similar to the fact that for a map f on $\mathbb{P}^2$ one has $\lambda_1(f) \leq d_{\text{alg}}(f)$.)*

2.2. **Topological degree.** We let $\mathcal{B}''_N$ be the set of Blaschke products for which all of the zeros from σ are distinct and none of the zeros are critical for their corresponding one-variable Blaschke factor. I.e. $A'(a_i) \neq 0$ for all i, and similarly for B, C, and D.

It is straightforward that $\mathcal{B}''_N$ is an open dense subset of $\mathcal{B}_N$ having total measure. Furthermore it is invariant under rotations by θ_1, θ_2.

Proposition 2.5. *For $f \in \mathcal{B}''_N$, the topological degree $d_{top}(f) = mq + np$.*

Proof. It suffices to count the preimages of any point that is not a critical value of f. For $f \in \mathcal{B}''_N$, the origin $(0,0)$ is not a critical value. This follows because the preimages of $(0,0)$ in $\mathbb{C}^2$ are precisely the collection of points from $Z(A) \cap Z(D)$ and $Z(B) \cap Z(C)$. Substituting into $Jf = A'(z)B(w)C(z)D'(w) - A(z)B'(w)C'(z)D(w)$ we see that the definition of $\mathcal{B}''_N$ prevents the Jf from vanishing on these points.

Since the zeros are distinct for $f \in \mathcal{B}''_N$ we have $mq+np$ such points. In $\mathbb{P}^2$ the line at infinity $T=0$ is forward invariant, so that there are no additional preimages of $(0,0)$ that are not in $\mathbb{C}^2$. Then, this total number of preimages of the non-critical value $(0,0)$ is $d_{\text{top}}(f)$. □

Remark 4. *The monomial map associated to N has topological degree* $\det(N)$*, which is always lower than the result of Proposition 2.5.*

2.3. **Example.** When

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

for $f \in \mathcal{B}'_N \cap \mathcal{B}''_N$ we have $\lambda_1(f) = \frac{3+\sqrt{5}}{2} \approx 2.62$ and $d_{\text{top}}(f) = 3$. Note that $d_{\text{top}}(f) > \lambda_1(f)$.

3. Case I: $d_{\text{top}}(f) > \lambda_1(f)$

Consider $\hat{\mathcal{B}}_N := \mathcal{B}'_N \cap \mathcal{B}''_N$, which has total measure in $\mathcal{B}_N$ (but it may not be open). Throughout this section it will be a standing hypothesis that every $f \in \hat{\mathcal{B}}_N$.

Proof of Theorem 1.5. As observed in Theorem 1.4 any $f \in \mathcal{B}'_N$ we have $\lambda_1(f) = \frac{m+q+\sqrt{(m-q)^2+4nq}}{2}$. For any $f \in \mathcal{B}''_N$ we have $d_{\text{top}}(f) = mq + np$ so that

$$\begin{aligned} \lambda_1(f) &= \frac{m+q+\sqrt{(m-q)^2+2np}}{2} < \frac{m+q+\sqrt{(m-q)^2}+2\sqrt{np}}{2} \\ &= \frac{m+q+|m-q|}{2} + \sqrt{np} < mq + np = d_{\text{top}}(f). \end{aligned}$$

∎

Therefore on the generic set $\hat{\mathcal{B}}_N \subset \mathcal{B}_N$ the Blaschke products fall into Case I of the conjecture and [15, Thm 2.1] gives an ergodic invariant measure μ of maximal entropy $\log(d_{\text{top}}(f))$.

Proposition 3.1. $\mathbb{T}^2$ *is isolated in the recurrent set of* f*.*

Since $\text{supp}(\mu)$ is also within the recurrent set and not contained within $\mathbb{T}^2$ this gives that $\mathbb{T}^2$ is isolated from $\text{supp}(\mu)$, as well.

Proof. In Proposition 3.8 from [19] it was shown that that any point in the unit bidisk $\mathbb{D}^2$ is in the basin of an (semi) attracting point e and any point in the "symmetric bidisk" $(\mathbb{C} \setminus \bar{\mathbb{D}})^2$ is in the basin of attraction for an (semi) attracting point e'. The orbit of any point in a neighborhood of $\mathbb{T}^2$ is within $W^s(\mathbb{T}^2) \cup W^s(e) \cup W^s(e')$, so that $\mathbb{T}^2$ is isolated in the recurrent set. ∎

Proposition 3.2. *There are points in* $W^s(\mathbb{T}^2)$ *accumulating to* $\text{Supp}(\mu)$*.*

Proof. As a consequence of [15, Thm 3.1] for any x not in the postcritical set $PC(f)$ if we start with the Dirac mass δ_x, then the sequence of measures

$$\frac{1}{(d_{\text{top}}(f))^n} f^{n*} \delta_x \tag{9}$$

converges weakly to μ. Because $PC(f)$ is pluripolar [15] it follows that the set of x for which this convergence does not hold cannot locally separate points. (See [16].)

Since $W^s(\mathbb{T}^2)$ is a 3-dimensional real manifold, thus separating points, there must be points $x \in W^s(\mathbb{T}^2) \setminus PC(f)$. Some preimages of these points converge to $\mathrm{Supp}(\mu)$. $\square$

Question 4. *Are there points $x \in \mathbb{T}^2$ for which the weighted sequence of preimages (9) converges weakly to μ, as above? Furthermore, is $\mathbb{T}^2 \subset \mathrm{PC}(f)$?*

Recall from the introduction that when $\det(N) = 1$, f induces a diffeomorphism on $\mathbb{T}^2$ and that we call such maps Blaschke product diffeomorphisms.

Lemma 3.3. *Any Blaschke product diffeomorphism f has unique measure of maximal entropy μ_{tor} for $f|_{\mathbb{T}^2}$. The entropy of this measure is $\log(c_1(N))$, where $c_1(N)$ is the leading eigenvalue of N.*

Note that here the meaning of "maximal" is with respect to invariant measures supported on $\mathbb{T}^2$. We have already observed that generic Blaschke products typically have an invariant measure μ of higher entropy $\log(d_{\mathrm{top}}(f)) > \log(c_1(f))$ supported away from $\mathbb{T}^2$.

Proof. Given N, let us consider the set $\mathcal{B}_N$. From Corollary 3.3 in [19] follows that any Blaschke product diffeomorphism has a dominated splitting on $\mathbb{T}^2$. Therefore, from [18, Thm F] it follows that the topological entropy is constant in $\mathcal{B}_N$. Since the monomial associated to N has topological entropy $\log(c_1(N))$ on $\mathbb{T}^2$ (using that it induces a linear Anosov map on $\mathbb{T}^2$), the same holds for any Blaschke product diffeomorphism in $\mathcal{B}_N$.

Moreover, from [19, Thm 3.10] it follows that generic Blaschke product diffeomorphisms are Axiom-A on $\mathbb{T}^2$ with a unique non-trivial homoclinic class. From [18, Thm E], it follows that any Blaschke product diffeomorphism on $\mathbb{T}^2$ is conjugate to one of these Axiom-A maps of $\mathbb{T}^2$ and therefore it has a unique non-trivial homoclinic class. Therefore, on one hand, we conclude that any Blaschke product diffeomorphisms has a unique non-trivial homoclinic class in the torus so that the topological entropy in $\mathbb{T}^2$ (which by previous paragraph is $\log(c_1(N))$) is equal to the topological entropy of the diffeomorphisms restricted to this homoclinic class. On the other hand, since the homoclinic class is conjugate to a hyperbolic one, it follows that it has a unique measure of maximal entropy with support in the homoclinic class. $\square$

From the characterization of the dynamics of $f|_{\mathbb{T}^2}$ given by [19, Thm 3.10], in combination with the classical theory of measures of maximal entropy for hyperbolic sets, it follows that either $W^s(\mu_{\mathrm{tor}})$ is dense in $\mathbb{T}^2$ or there is a fixed point that is attracting within $\mathbb{T}^2$. In the former case, the same proof as above shows that there are points in $W^s(\mu_{\mathrm{tor}})$ whose preimages converge to $\mathrm{Supp}(\mu)$. In the latter case, it is unclear if this result holds.

Remark 5. *Saddle sets for globally holomorphic maps were studied by* [8]*, where it was shown showed that the topological entropy of the saddle set is bounded above by $\log(d_{\mathrm{alg}}(f))$, with equality holding if and only if the saddle set is terminal. The saddle set given by $\mathbb{T}^2$ that is discussed above conforms with a possible generalization to meromorphic maps of the result of* [8]*, since $\mathbb{T}^2$ is terminal and has topological entropy $\log(\lambda_1(f))$.*

Question 5. *For some large set $U \subset \hat{\mathcal{B}}_N$ is it true that* $\mathrm{Supp}(\mu)$ *is uniform hyperbolic? Similarly, is $f \in U$ Axiom-A?*

4. Case II: $\lambda_1(f) > d_{\mathrm{top}}(f)$

Within $\mathcal{B}_N$, the monomial map f (associated to N) provides an example with $\lambda_1(f) > d_{\mathrm{top}}(f)$. This follows because for any monomial map $d_{\mathrm{top}}(f) = \det(N) < c_1(N) \leq \lambda_1(f)$, where we are using Proposition 2.4 in the last inequality.

However there are many examples of of Blaschke products f that are not monomial and have $\lambda_1(f) > d_{\mathrm{top}}(f)$. We present one specific family, for concreteness:

Example 1. *For $a \neq b \neq c$, consider the family*

$$f_{a,b,c}(z,w) = \left(\theta_1 \left(\frac{z-a}{1-\bar{a}z}\right)^5 \left(\frac{w-b}{1-\bar{b}w}\right)^2,\ \theta_2 \frac{z-a}{1-\bar{a}z} \cdot \frac{z-c}{1-\bar{c}z} \cdot \frac{w-b}{1-\bar{b}z}\right).$$

Members of this family are not in $\hat{\mathcal{B}}_N$ because of the repeated zeros, however one can directly check that $d_{\mathrm{top}}(f_{a,b,c}) = 5$. Then, the lower bound from Proposition 2.4 gives $\lambda_1(f) \geq c_1(N) = \frac{6+\sqrt{32}}{2} > d_{\mathrm{top}}(f_{a,b,c})$.

It is interesting to note that for this example f is a diffeomorphism on $\mathbb{T}^2$ so that because $d_{\mathrm{top}}(f) = 5$ there are many preimages of $\mathbb{T}^2$ in $\mathbb{P}^2$ that are away from $\mathbb{T}^2$. This raises the question of whether there exists a dynamically non-trivial invariant set outside of $\mathbb{T}^2$. Furthermore, since we only know the lower bound $\lambda_1(f_{a,b,c}) \geq c_1(f)$, it might be possible that the topological entropy of $f_{a,b,c}$ is greater than $\log(c_1(N))$ *and even that there is a measure of maximal entropy (of entropy greater than* $\log(c_1(N))$*) outside of $\mathbb{T}^2$.*

Remark 6. *It would be interesting to determine if the Blaschke products for which $\lambda_1(f) > d_{\mathrm{top}}(f)$ satisfy the hypothesis of* [5, 6, 7].

Question 6. *Are there examples of Blaschke products with $\lambda_1(f) > d_{\mathrm{top}}(f)$ for which we have equality $\lambda_1(f) = c_1(N)$? This would be interesting because in this case (using the bound on entropy from* [9]*) the measure μ_{tor} is an invariant measure of maximal entropy for f contained within $\mathbb{T}^2$.*

Remark 7. *Within $\mathcal{B}_N$ moving the zeros can result in the change from Case II to Case I, resulting in a big change in the dynamics. Aside from degree considerations, it would be interesting to know the mechanism(s) for this bifurcation.*

5. Holomorphic perturbations of monomial maps

The Blaschke products in $\mathcal{B}_N$ can be considered as a certain type of perturbation (depending real analytically on the zeros σ) of the monomial map associated to N that results in rational maps that are not globally holomorphic on $\mathbb{C}^2$ (and hence not on $\mathbb{P}^2$, either.)

Alternative perturbations can be considered that are holomorphic on $\mathbb{C}^2$ and on $\mathbb{P}^2$. For example, one can always perturb each component of a monomial map by adding small constants $\epsilon = (\epsilon_1, \epsilon_2)$:

$$f_\epsilon(z, w) = (z^2 w + \epsilon_1, zw + \epsilon_2)$$

The resulting map is still holomorphic on $\mathbb{C}^2$, but does not typically extend holomorphically to $\mathbb{P}^2$ because of a miss-match in top degrees of the two terms.

If, instead one prefers to do a perturbation of a monomial map resulting in a map that is globally holomorphic on $\mathbb{P}^2$, one typically must add a small perturbation to the high degree terms in one of the factors. For example

$$g_\epsilon(z, w) = (z^2 w, zw + \epsilon(z^3 + w^3)),$$

which, for $\epsilon \neq 0$ are globally holomorphic on $\mathbb{P}^2$. (In this sense the perturbation g_ϵ is more natural than f_ϵ.)

For the remainder of the discussion we consider only the case of maps that induce diffeomorphism on $\mathbb{T}^2$ (i.e. $\det(N) = 1$) because the makes the hyperbolic theory on $\mathbb{T}^2$ easier. (Much of the following holds in general.)

Because the monomial map on $\mathbb{T}^2$ is hyperbolic, for small enough ϵ either form of perturbation produces a continuation of $\mathbb{T}^2$ to an invariant hyperbolic set for f_ϵ or g_ϵ. However the second form of perturbation creates another invariant set of larger entropy $\log(d^2)$ (using either [15], or or earlier work as referenced in [21, Section 3]). Therefore for the perturbations g_ϵ, the type of analysis performed in Sections 2 and 3 for the Blaschke products becomes nearly trivial.

In fact the bifurcation that occurs for g_ϵ is somewhat similar to the one discussed in Remark 7.

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