THE CLASSIFICATION OF EXCEPTIONAL CDQL WEBS ON COMPACT COMPLEX SURFACES

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Abstract. Codimension one webs are configurations of finitely many codimension one foliations in general position. Much of the classical theory evolved around the concept of abelian relation: a functional relation among the first integrals of the foliations defining the web reminiscent of Abel's addition theorem in classical algebraic geometry. The abelian relations of a given web form a finite dimensional vector space with dimension (the rank of the web) bounded by Castelnuovo number $\pi(n,k)$ where n is the dimension of the ambient space and k is the number of foliations defining the web. A fundamental problem in web geometry is the classification of exceptional webs, that is, webs of maximal rank not equivalent to the dual of a projective curve. Recently, J.-M. Trépreau proved that there are no exceptional k-webs for $n \geq 3$ and $k \geq 2n$. In dimension two there are examples for arbitrary k and the classification problem is wide open.

In this paper, we classify the exceptional Completely Decomposable Quasi-Linear (CDQL) webs globally defined on compact complex surfaces. By definition, the CDQL $(k+1)$-webs are formed by the superposition of k linear foliations and one non-linear foliation. For instance, we show that up to projective transformations there are exactly four countable families and thirteen sporadic exceptional CDQL webs on $\mathbb{P}^2$.

Contents

1. Introduction and statement of the main results

1.1. **Codimension one webs of maximal rank.** A *germ of regular k-web* $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ of codimension one on $(\mathbb{C}^n, 0)$ is a collection of k germs of smooth holomorphic foliations $\mathcal{F}_i$ with tangent spaces in general position at the origin. By definition, the $\mathcal{F}_i$'s are the *defining foliations* of $\mathcal{W}$. If they are respectively induced by differentials 1-forms $\omega_1, \ldots, \omega_k$, then the *space of abelian relations* of $\mathcal{W}$ is the vector space

$$\mathcal{A}(\mathcal{W}) = \Big\{ \big(\eta_i\big)_{i=1}^k \in \Omega^1(\mathbb{C}^n,0)^k \;\Big|\; \sum_{i=1}^k \eta_i = 0 \;\text{ and }\; \forall i \; d\eta_i = 0\,,\; \eta_i \wedge \omega_i = 0 \Big\}.$$

If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$ then, after integration, the abelian relations can be read as functional equations of the form $\sum_{i=1}^k g_i(u_i) = 0$ for suitable germs of holomorphic functions $g_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

The dimension of $\mathcal{A}(\mathcal{W})$ is commonly called the *rank* of $\mathcal{W}$ and denoted by $rk(\mathcal{W})$. It is a theorem of Bol (for $n = 2$) and Chern (for $n \geq 3$) that

$$rk(\mathcal{W}) \leq \pi(n,k) = \sum_{j=1}^{\infty} \max\big(0, k - j(n-1) - 1\big). \tag{1}$$

A k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$ is of *maximal rank* if $rk(\mathcal{W}) = \pi(n, k)$. The integer $\pi(n, k)$ is the well-known Castelnuovo's bound for the arithmetic genus of irreducible and non-degenerated algebraic curves of degree k on $\mathbb{P}^n$.

One of the main topics of the theory of webs concerns the characterization of webs of maximal rank. It follows from Abel's Addition Theorem that all the webs $\mathcal{W}_C$ obtained from reduced Castelnuovo curves[1] C by projective duality are of maximal rank (see [36] for instance). The webs analytically equivalent to $\mathcal{W}_C$ for some non-degenerated projective curve C are the so called *algebraizable webs*.

It can be traced back to Lie the proof that all 4-webs on $(\mathbb{C}^2, 0)$ of maximal rank are algebraizable. In [7], Bol proved that a maximal rank k-web on $(\mathbb{C}^3, 0)$ is algebraizable when $k \geq 6$. Recently, building up on previous work by Chern and Griffiths [16], Trépreau extended Bol's result and established in [47] that k-webs of maximal rank on $(\mathbb{C}^n, 0)$ are algebraizable whenever $n \geq 3$ and $k \geq 2n$.

The non-algebraizable webs of maximal rank on $(\mathbb{C}^2, 0)$ are nowadays called *exceptional webs*. For almost 70 years there was just one example, due to Bol [8], of exceptional planar web in the literature. Recently a number of new examples have appeared, see [40, 44, 42, 32]. Despite these new examples, the classification problem for exceptional planar webs is wide open.

1.2. **Characterization of planar webs of maximal rank.** Although a classification seems out of reach, there are methods to decide if a given web has maximal rank. The first result in this direction is due to Pantazi [35]. It was published during the second world war and remained unknown to the practitioners of web theory until recently, see [40]. Unaware of this classical result, Hénaut [24] worked out an alternative approach to determine if a given web has maximal rank. Both approaches share in common the use of prolongations of differential systems to express the maximality of the rank by the vanishing of certain differential expressions determined by the defining equations of the web.

[1] That is, non-degenerate algebraic curve $C \subset \mathbb{P}^n$ such that $p_a(C) = \pi(n, k)$ where $k = deg(C)$.

It has to be noted that these results are wide generalizations of the classical criterion of Blaschke-Dubourdieu for the maximality of the rank of 3-webs. If $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ is a planar 3-web and the foliations $\mathcal{F}_i$ are defined by 1-forms ω_i satisfying $\omega_1 + \omega_2 + \omega_3 = 0$ then a simple computation ensures the existence of a unique 1-form γ such that $d\omega_i = \gamma \wedge \omega_i$ for $i = 1, 2, 3$. Although γ does depend on the choice of the 1-forms ω_i its differential $d\gamma$ is intrinsically attached to $\mathcal{W}$. It is the so called *curvature* $K(\mathcal{W})$ of $\mathcal{W}$. In [6] it is proved that a 3-web $\mathcal{W}$ has maximal rank if and only if $K(\mathcal{W}) = 0$.

Building on Pantazi's result, Mihăileanu gave in [33] a necessary condition for a planar k-web be of maximal rank : if $\mathcal{W}$ has maximal rank then $K(\mathcal{W}) = 0$. Now, the curvature $K(\mathcal{W})$ is the sum of the curvatures of all 3-subwebs of $\mathcal{W}$. Recently Hénaut, Ripoll and Robert (see [25, p.281],[43]) have rediscovered Mihăileanu necessary condition using Hénaut's approach.

As in the case of 3-webs the curvature is a holomorphic 2-form intrinsically attached to $\mathcal{W}$: it does not depend on the choice of the defining equations of $\mathcal{W}$. Another nice feature of the curvature is that it still makes sense, as a meromorphic 2-form, for global webs. More precisely if S is complex surface then a *global k-web* on S can be defined as an element $\mathcal{W} = [\omega]$ of $\mathbb{P}H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$ — where $\mathcal{N}$ denotes a line-bundle and $\mathrm{Sym}^k\Omega^1_S$ the sheaf of k-symmetric powers of differential 1-forms on S — subjected to the following two conditions: *(i)* the zero locus of ω has codimension at least two; *(ii)* $\omega(p)$ factors as the product of pairwise linearly independent 1-forms at some point $p \in S$. For $k = 1$ the condition *(ii)* is vacuous and we recover one of the usual definitions of foliations. When $k \geq 2$, the set where the condition *(ii)* does not hold is the *discriminant* of $\mathcal{W}$ and will be denoted by $\Delta(\mathcal{W})$. For $k \geq 3$, the curvature $K(\mathcal{W})$ is a global meromorphic 2-form on S with polar set contained in $\Delta(\mathcal{W})$.

Elementary arguments imply that the space of abelian relations of $\mathcal{W}$, in this global setup, is a local system over $S \setminus \Delta(\mathcal{W})$, see for instance [40, Théorème 1.2.2]. The rank of $\mathcal{W}$ appears now as the rank of the local system $\mathcal{A}(\mathcal{W})$.

One has to be careful when talking about defining foliations of a global web since these will make sense only in sufficiently small analytic open subsets of S. When it is possible to write globally $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ we will say that $\mathcal{W}$ is *completely decomposable*.

When S is a pseudo-parallelizable surface[2], a global k-web on S can be alternatively defined as an element $\mathcal{W} = [\omega]$ of the projective space $\mathbb{P}_{\mathbb{C}(S)}(\mathrm{Sym}^k\Omega^1_S)$ — where $\mathbb{C}(S)$ is the field of meromorphic functions on S and $\mathrm{Sym}^k\Omega^1_S$ denotes now the $\mathbb{C}(S)$-vector space of meromorphic k-symmetric powers of differential 1-forms on S— subjected to the condition that ω factors as the product of pairwise linearly independent 1-forms at some point of S.

1.3. **Mihăileanu necessary condition and $\mathcal{F}$-barycenters.** The present work stems from an attempt to understand geometrically Mihăileanu's necessary condition for the maximality of the rank. More precisely we try to understand the conditions imposed by the vanishing of the curvature on the behavior of $\mathcal{W}$ over its discriminant. It has to be mentioned that the idea of analyzing webs through

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[2] A complex manifold M of dimension n is called *pseudo-parallelizable* if it carries n global meromorphic 1-forms $\omega_1, \ldots, \omega_n$ with exterior product $\omega_1 \wedge \cdots \wedge \omega_n$ not identically zero.

theirs discriminants is not new, see [13] and [31]. More recently, [25] advocates the study of webs (decomposable or not) in neighborhoods of theirs discriminants.

Our result in this direction is stated in terms of $\beta_{\mathcal{F}}(\mathcal{W})$ — the $\mathcal{F}$-*barycenter* of a web $\mathcal{W}$. Suppose that S is a pseudo-parallelizable surface and $\mathcal{F} \in \mathbb{P}_{\mathbb{C}(S)}(\Omega^1_S)$ is a foliation on it. There is a naturally defined affine structure on $\mathbb{A}^1_{\mathcal{F}} = \mathbb{P}_{\mathbb{C}(S)}(\Omega^1_S) \setminus \mathcal{F}$. If $\mathcal{W} \in \mathbb{P}_{\mathbb{C}(S)}(\mathrm{Sym}^k \Omega^1_S)$ is a k-web not containing $\mathcal{F}$ as one of its defining foliations then it can be loosely interpreted as k points in $\mathbb{A}^1_{\mathcal{F}}$. The $\mathcal{F}$-barycenter of $\mathcal{W}$ is then the foliation $\beta_{\mathcal{F}}(\mathcal{W})$ defined by the barycenter of these k points in $\mathbb{A}^1_{\mathcal{F}}$. For a precise definition and some properties of $\beta_{\mathcal{F}}(\mathcal{W})$, see Sections 5 and 6.

Theorem 1. *Let $\mathcal{F}$ be a foliation and $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_k$ be a k-web, $k \geq 2$, both defined on the same domain $U \subset \mathbb{C}^2$. Suppose that C is an irreducible component of* $\mathrm{tang}(\mathcal{F}, \mathcal{F}_1)$ *that is not contained in $\Delta(\mathcal{W})$. The curvature $K(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic over a generic point of C if and only if the curve C is $\mathcal{F}$-invariant or $\beta_{\mathcal{F}}(\mathcal{W}')$-invariant, where $\mathcal{W}' = \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_k$.*

Theorem 1 is the cornerstone of our approach to the classification of exceptional completely decomposable quasi-linear webs (CDQL webs for short) on compact complex surfaces.

1.4. **Linear webs and CDQL webs.** Linear webs are classically defined as the ones for which all the leaves are open subsets of lines. Here we will adopt the following global definition. A web $\mathcal{W}$ on compact complex surface S is *linear* if (a) the universal covering of S is an open subset $\widetilde{S}$ of $\mathbb{P}^2$; (b) the group of deck transformations acts on $\widetilde{S}$ by automorphisms of $\mathbb{P}^2$, and; (c) the pull-back of $\mathcal{W}$ to $\widetilde{S}$ is linear in the classical sense[3].

A *CDQL* $(k+1)$-*web* on a compact complex surface S is, by definition, the superposition of k linear foliations and one non-linear foliation.

It follows from [26, 29] that the only compact complex surfaces satisfying (a) and (b) are: the projective plane; surfaces covered by the unit ball; Kodaira primary surfaces; complex tori; Inoue surfaces; Hopf surfaces and principal elliptic bundles over hyperbolic curves with odd first Betti number.

If S is not $\mathbb{P}^2$ then the group of deck transformations is infinite. Because it acts on $\widetilde{S}$ without fixed points, every linear foliation on S is a smooth foliation. An inspection of Brunella's classification of smooth foliations [10] reveals that the only compact complex surfaces admitting at least two distinct linear foliations are the projective plane, the complex tori and the Hopf surfaces. Moreover the only Hopf surfaces admitting four distinct linear foliations are the primary Hopf surfaces H_α, $|\alpha| > 1$. Here H_α is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the map $(x, y) \mapsto (\alpha x, \alpha y)$.

The linear foliations on complex tori are pencils of parallel lines on theirs universal coverings. The ones on Hopf surfaces are either pencils of parallels lines or the pencil of lines through the origin of $\mathbb{C}^2$. In particular all completely decomposable linear webs on compact complex surfaces are algebraic[4] on theirs universal coverings.

[3]Alternatively one could assume that S admits a $(\mathbb{P}^2, \mathrm{PGL}(3, \mathbb{C}))$-structure and that $\mathcal{W}$ is linear in the local charts of this structure. Although more general, this definition does not seem to encompass more examples of linear webs. To avoid a lengthy case by case analysis of the classification of $(\mathbb{P}^2, \mathrm{PGL}(3, \mathbb{C}))$-structures [28] we opted for the more astringent definition above.

[4]Beware that algebraic here means that they are locally dual to plane curves. In the cases under scrutiny they are dual to certain products of lines.

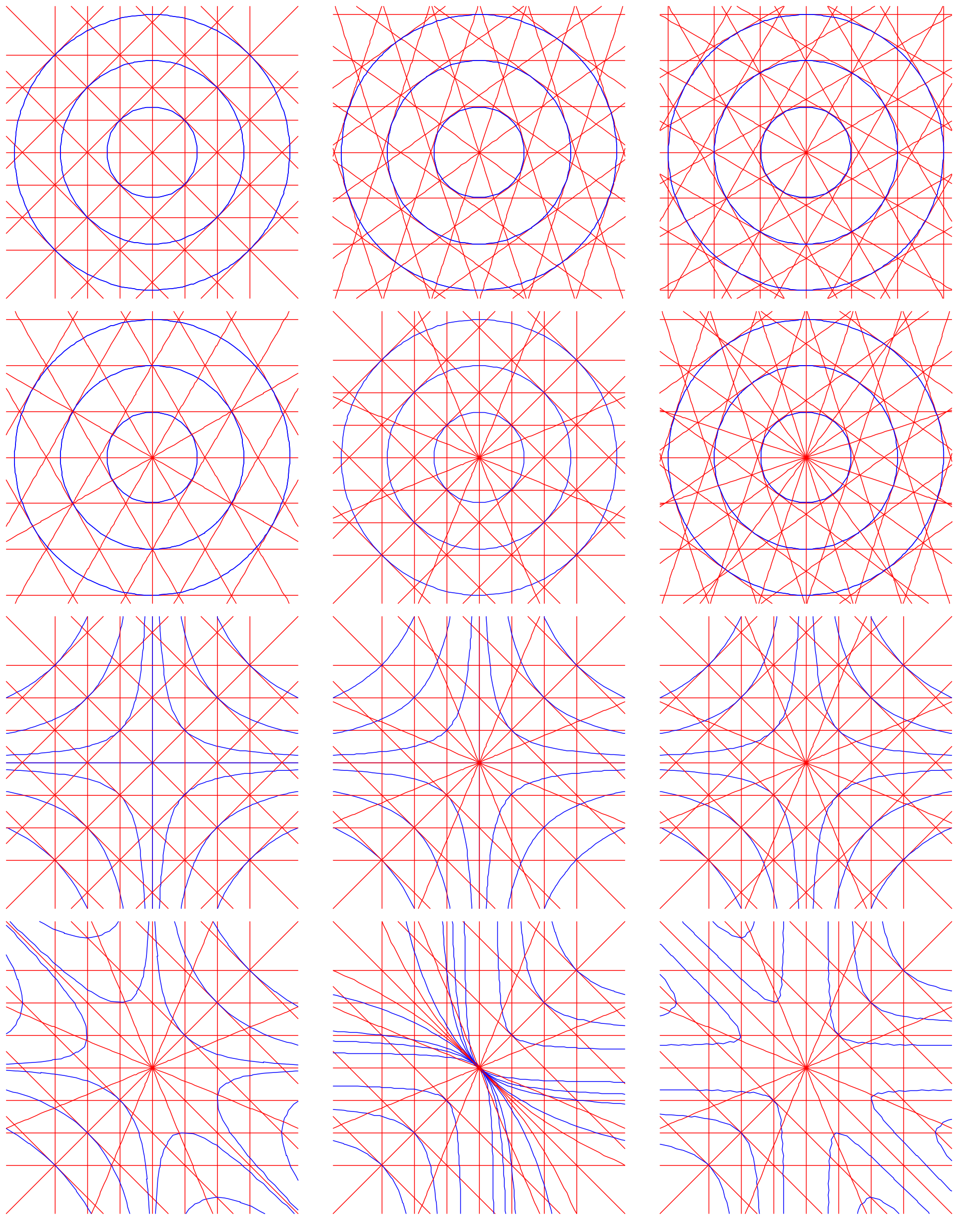

FIGURE 1. A sample of real models for exceptional CDQL webs on $\mathbb{P}^2$. In the first and second rows, the first three members of the infinite family $\mathcal{A}_I^k$ and $\mathcal{A}_{II}^k$ respectively. In the third row, from left to right, $\mathcal{A}_{III}^2$, $\mathcal{A}_{IV}^1$ and $\mathcal{A}_{IV}^2$. In the fourth row: $\mathcal{A}_5^a, \mathcal{A}_5^b$ and $\mathcal{A}_5^c$.

1.5. **Classification of exceptional CDQL webs on the projective plane.** On $\mathbb{P}^2$ the CDQL webs can be written as $\mathcal{W} \boxtimes \mathcal{F}$ where $\mathcal{W}$ is a product of pencil of lines and $\mathcal{F}$ is a non-linear foliation. These webs are determined by the pair $(\mathcal{P}, \mathcal{F})$ where $\mathcal{P} \subset \mathbb{P}^2$ is the set of singularities of the linear foliations defining $\mathcal{W}$. One key example is the already mentioned Bol's 5-web. It is the exceptional

CDQL 5-web on $\mathbb{P}^2$ with $\mathcal{F}$ equal to the pencil generated by two reduced conics intersecting transversely and $\mathcal{P}$ equal to the set of four base points of this pencil. Other examples of exceptional CDQL webs on the plane have appeared in [40, 44].

We will deduce from Theorem 1 a complete classification of exceptional CDQL webs on the projective plane. In succinct terms it can be stated as follows:

Theorem 2. *Up to projective automorphisms, there are exactly four infinite families and thirteen sporadic exceptional CDQL webs on $\mathbb{P}^2$.*

In suitable affine coordinates $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, the four infinite families are

$$\begin{aligned}
\mathcal{A}_I^k &= \big[(dx^k - dy^k)\big] \boxtimes \big[d(xy)\big] && \text{where } \ k \geq 4\,;\\
\mathcal{A}_{II}^k &= \big[(dx^k - dy^k)\,(xdy - ydx)\big] \boxtimes \big[d(xy)\big] && \text{where } \ k \geq 3\,;\\
\mathcal{A}_{III}^k &= \big[(dx^k - dy^k)\,dx\,dy\big] \boxtimes \big[d(xy)\big] && \text{where } \ k \geq 2\,;\\
\mathcal{A}_{IV}^k &= \big[(dx^k - dy^k)\,dx\,dy\,(xdy - ydx)\big] \boxtimes \big[d(xy)\big] && \text{where } \ k \geq 1.
\end{aligned}$$

The diagram below shows how these webs relate to each other in terms of inclusions for a fixed k. Moreover if k divides k' then $\mathcal{A}_I^k, \mathcal{A}_{II}^k, \mathcal{A}_{III}^k, \mathcal{A}_{IV}^k$ are subwebs of $\mathcal{A}_I^{k'}, \mathcal{A}_{II}^{k'}, \mathcal{A}_{III}^{k'}, \mathcal{A}_{IV}^{k'}$ respectively.

$$\begin{tikzcd}
 & \mathcal{A}_{II}^k \arrow[dr] & \\
\mathcal{A}_I^k \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{A}_{IV}^k\,. \\
 & \mathcal{A}_{III}^k \arrow[ur] &
\end{tikzcd}$$

All the webs above are invariant by the $\mathbb{C}^*$-action $t \cdot (x,y) = (tx, ty)$ on $\mathbb{P}^2$. Among the thirteen sporadic examples of exceptional CDQL webs on the projective plane, seven (four 5-webs, two 6-webs and one 7-web) are also invariant by the same $\mathbb{C}^*$-action. They are:

$$\begin{array}{lcllcl}
\mathcal{A}_5^a &=& \big[dx\,dy\,(dx+dy)\,(xdy - ydx)\big] &\boxtimes& \big[d\big(xy(x+y)\big)\big]\,;\\
\mathcal{A}_5^b &=& \big[dx\,dy\,(dx+dy)\,(xdy - ydx)\big] &\boxtimes& \Big[d\Big(\frac{xy}{x+y}\Big)\Big]\,;\\
\mathcal{A}_5^c &=& \big[dx\,dy\,(dx+dy)\,(xdy - ydx)\big] &\boxtimes& \Big[d\Big(\frac{x^2+xy+y^2}{xy\,(x+y)}\Big)\Big]\,;\\
\mathcal{A}_5^d &=& \big[dx\,(dx^3+dy^3)\big] &\boxtimes& \big[d\big(x(x^3+y^3)\big)\big]\,;\\
\mathcal{A}_6^a &=& \big[dx\,(dx^3+dy^3)\,(xdy - ydx)\big] &\boxtimes& \big[d\big(x(x^3+y^3)\big)\big]\,;\\
\mathcal{A}_6^b &=& \big[dx\,dy\,(dx^3+dy^3)\big] &\boxtimes& \big[d(x^3+y^3)\big]\,;\\
\mathcal{A}_7 &=& \big[dx\,dy\,(dx^3+dy^3)\,(xdy - ydx)\big] &\boxtimes& \big[d(x^3+y^3)\big]\,.
\end{array}$$

Four of the remaining six sporadic exceptional CDQL webs (one k-web for each $k \in \{5,6,7,8\}$) share the same non-linear foliation $\mathcal{F}$: the pencil of conics through four points in general position. For them the set $\mathcal{P}$ is a subset of $\mathrm{sing}(\mathcal{F})$ containing the base points of the pencil. Up to automorphism of $\mathcal{F}$ there is just one choice for each possible cardinality. They all have been previously known (see [44]).

$$\begin{aligned}
\mathcal{B}_5 &= \Big[dx\,dy\,d\big(\tfrac{x}{1-y}\big)\,d\big(\tfrac{y}{1-x}\big)\Big] &&\boxtimes \Big[d\big(\tfrac{xy}{(1-x)(1-y)}\big)\Big]\,;\\
\mathcal{B}_6 &= \mathcal{B}_5 &&\boxtimes \big[d\,(x+y)\big]\,;\\
\mathcal{B}_7 &= \mathcal{B}_6 &&\boxtimes \Big[d\big(\tfrac{x}{y}\big)\Big]\,;\\
\mathcal{B}_8 &= \mathcal{B}_7 &&\boxtimes \Big[d\big(\tfrac{1-x}{1-y}\big)\Big]\,.
\end{aligned}$$

The last two sporadic CDQL exceptional webs (one 5-web and one 10-web) also share the same non-linear foliation: the Hesse pencil of cubics. Recall that this pencil is the one generated by a smooth cubic and its Hessian and that it is unique up to automorphisms of $\mathbb{P}^2$. These webs are (with $\xi_3 = \exp(2i\pi/3)$):

$$\begin{aligned}
\mathcal{H}_5 &= \Big[(dx^3+dy^3)\,d\big(\tfrac{x}{y}\big)\Big] \boxtimes \Big[d\big(\tfrac{x^3+y^3+1}{xy}\big)\Big]\,;\\
\mathcal{H}_{10} &= \Big[(dx^3+dy^3)\Big(\prod_{i=0}^{2} d\big(\tfrac{y-\xi_3^i}{x}\big)\Big)\Big(\prod_{i=0}^{2} d\big(\tfrac{x-\xi_3^i}{y}\big)\Big)\Big] \boxtimes \Big[d\big(\tfrac{x^3+y^3+1}{xy}\big)\Big]\,.
\end{aligned}$$

The web $\mathcal{H}_{10}$ shares a number of features with Bol's web $\mathcal{B}_5$. They both have a huge group of birational automorphisms (the symmetric group S_5 for $\mathcal{B}_5$ and Hesse's group G_{216} for $\mathcal{H}_{10}$), both are naturally associated to nets in the sense of Section 3.1 and their abelian relations can be expressed in terms of logarithms and dilogarithms.

Because they have parallel 4-subwebs whose slopes have non real cross-ratio the webs $\mathcal{A}^k_{III}$, $\mathcal{A}^k_{IV}$ for $k \geq 3$, $\mathcal{A}^d_5, \mathcal{A}^a_6, \mathcal{A}^b_6$ and $\mathcal{A}_7$ do not admit real models. The web $\mathcal{H}_{10}$ also does not admit a real model. To verify this fact, one possibility is to observe that the lines passing through two of the nine base points always contain a third and notice that this contradicts Sylvester-Gallai Theorem [17]: for every finite set of non collinear points in $\mathbb{P}^2_{\mathbb{R}}$ there exists a line containing exactly two points of the set. All the other exceptional CDQL webs admit real models. Some of them are pictured in Figure 1.

1.6. **Exceptional CDQL webs on Hopf surfaces.** The classification of CDQL webs on $\mathbb{P}^2$ admits as a corollary the classification of exceptional CDQL webs on Hopf surfaces.

Corollary 1. *Up to automorphism, the only exceptional CDQL webs on Hopf surfaces are quotients of the restrictions of the webs $\mathcal{A}^*_*$ to $\mathbb{C}^2 \setminus \{0\}$ by the group of deck transformations.*

The proof is automatic. One has just to remark that a foliation on a Hopf surface of type H_α when lifted to $\mathbb{C}^2 \setminus \{0\}$ gives rise to an algebraic foliation on $\mathbb{C}^2$ invariant by the $\mathbb{C}^*$-action $t \cdot (x, y) = (tx, ty)$.

1.7. **From global to local. . .** Although based on global methods, the classification of exceptional CDQL webs on $\mathbb{P}^2$ also yields information about the singularities of local exceptional webs.

Corollary 2. *Assume that $k \geq 4$. Let $\mathcal{W}$ be a smooth k-web and $\mathcal{F}$ be a foliation, both defined on $(\mathbb{C}^2, 0)$. If the $(k+1)$-web $\mathcal{W} \boxtimes \mathcal{F}$ has maximal rank then one of the following situations holds:*

(1) the foliation $\mathcal{F}$ is of the form $\big[H(x,y)(\alpha dx+\beta dy)+h.o.t.\big]$ where H is a non-zero homogeneous polynomial and $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$;

(2) the foliation $\mathcal{F}$ is of the form $\big[H(x,y)(ydx-xdy)+h.o.t.\big]$ where H is a non-zero homogeneous polynomial;

(3) $\mathcal{W}\boxtimes\mathcal{F}$ is exceptional and its first non-zero jet defines, up to linear automorphisms, one of the following webs

$$\mathcal{A}_I^k,\ \mathcal{A}_{III}^{k-2},\ \mathcal{A}_5^d\ (\textit{only when } k=4)\quad \textit{or}\quad \mathcal{A}_6^b\ (\textit{only when } k=5)\,.$$

In fact, as it will be clear from its proof, it is possible to state a slightly more general result in the same vein. Nevertheless the result above suffices for the classification of exceptional CDQL webs on complex tori.

1.8. **. . . and back: classification of exceptional CDQL webs on tori.** A CDQL web on a torus is the superposition of a non-linear foliation with a product of foliations induced by global holomorphic 1-forms. Since étale coverings between complex tori abound and because the pull-back of exceptional CDQL webs under these are still exceptional CDQL webs, we are naturally lead to extend the notion of isogenies between complex tori. Two webs $\mathcal{W}_1,\mathcal{W}_2$ on complex tori T_1,T_2 are *isogeneous* if there exist a complex torus T and étale morphisms $\pi_i:T\to T_i$ for $i=1,2$, such that $\pi_1^*(\mathcal{W}_1)=\pi_2^*(\mathcal{W}_2)$.

Theorem 3. *Up to isogenies, there are exactly three sporadic (one for each $k\in\{5,6,7\}$) and one continuous family (with $k=5$) of exceptional CDQL k-webs on complex tori.*

The elements of the continuous family are

$$\mathcal{E}_\tau=\big[\,dx\,dy\,(dx^2-dy^2)\,\big]\boxtimes\left[d\left(\frac{\vartheta_1(x,\tau)\vartheta_1(y,\tau)}{\vartheta_4(x,\tau)\vartheta_4(y,\tau)}\right)^2\right]$$

defined, respectively, on the torus E_τ^2 for arbitrary $\tau\in\mathbb{H}=\{z\in\mathbb{C}\,|\,\Im\mathrm{m}(z)>0\,\}$ where $E_\tau=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$. The functions ϑ_i involved in the definition are the classical Jacobi theta functions, see Example 4.1.

These webs first appeared in Buzano's work [11] but their rank was not determined at that time. They were later rediscovered in [42] where it is proved that they are all exceptional and that $\mathcal{E}_\tau$ is isogeneous to $\mathcal{E}_{\tau'}$ if and only if τ and τ' belong to the same orbit under the natural action on $\mathbb{H}$ of the $\mathbb{Z}/2\mathbb{Z}$ extension of $\Gamma_0(2)\subset\mathrm{PSL}(2,\mathbb{Z})$ generated by $\tau\mapsto-2\tau^{-1}$. Thus the continuous family of exceptional CDQL webs on tori is parameterized by a $\mathbb{Z}/2\mathbb{Z}$-quotient of the modular curve $X_0(2)$.

The sporadic CDQL 7-web $\mathcal{E}_7$ is strictly related to a particular element of the previous family. Indeed $\mathcal{E}_7$ is the 7-web on E_{1+i}^2

$$\mathcal{E}_7=\big[dx^2+dy^2\big]\boxtimes\mathcal{E}_{1+i}\,.$$

The sporadic CDQL 5-web $\mathcal{E}_5$ lives naturally in $E_{\xi_3}^2$ and can be described as

$$\big[dx\,dy\,(dx-dy)\,(dx+\xi_3^2\,dy)\big]\boxtimes\left[d\Big(\frac{\vartheta_1(x,\xi_3)\vartheta_1(y,\xi_3)\vartheta_1(x-y,\xi_3)\vartheta_1(x+\xi_3^2\,y,\xi_3)}{\vartheta_2(x,\xi_3)\vartheta_3(y,\xi_3)\vartheta_4(x-y,\xi_3)\vartheta_3(x+\xi_3^2\,y,\xi_3)}\Big)\right].$$

The sporadic CDQL 6-web $\mathcal{E}_6$ also lives in $E^2_{\xi_3}$ and is best described in terms of Weierstrass $\wp$-function.

$$\mathcal{E}_6 = \left[\, dx\, dy\, (dx^3 + dy^3)\right] \boxtimes \Big[\wp(x,\xi_3)^{-1}dx + \wp(y,\xi_3)^{-1}dy\Big]\,.$$

Although not completely evident from the above presentation, it turns out that the foliation $[\wp(x,\xi_3)^{-1}dx + \wp(y,\xi_3)^{-1}dy]$ admits a rational first integral, see Proposition 4.2.

A more geometric description of these exceptional *elliptic webs* will be given in Section 4 together with the proof that they are indeed exceptional.

The proof of Theorem 3 follows the same lines of the proof of Theorem 2 but with some twists. The key extra ingredients are Corollary 2 and the following (considerably easier) analogue for two dimensional complex tori of [39, Theorem 1].

Theorem 4. *If T is a two-dimension complex tori and $f : T \dashrightarrow \mathbb{P}^1$ a meromorphic map then the number of linear fibers of f, when finite, is at most six.*

For us the linear fibers of a rational map from a two-dimensional complex torus to a curve are the ones that are set-theoretically equal to a union of subtori.

1.9. **Plan of the Paper.** The remaining of the paper can be roughly divided in four parts. The first goes from Section 2 to Section 4 and is devoted to prove that all the webs presented in the Introduction are exceptional. The highlights are Theorems 3.1 and Theorem 4.1 that show that the webs $\mathcal{B}_5, \mathcal{H}_{10}, \mathcal{E}_\tau, \mathcal{E}_5, \mathcal{E}_6$ and $\mathcal{E}_7$ are exceptional thanks to essentially the same reason. Their abelian relations are expressed in terms of logarithms, dilogarithms and their elliptic counterparts. Sections 5, 6 and 7 form the second part of the paper which is devoted to the study of the $\mathcal{F}$-barycenter of a web. Besides the proof of Theorem 1 of the Introduction it also contains a very precise description of the barycenters of decomposable linear webs centered at linear foliations on $\mathbb{P}^2$. This description lies at the heart of our approach to the classification of exceptional CDQL webs on $\mathbb{P}^2$. The third part of the paper goes from Section 8 to Section 10 and contains the classification of exceptional CDQL webs on the projective plane. Finally the fourth and last part is contained in the last two sections and deals with the classification of exceptional CDQL webs on complex tori. Beside this classification it also contains the proofs of Corollary 2 and Theorem 4.

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2. Abelian relations for CDQL webs invariant by $\mathbb{C}^*$-actions

We start things off with the following well-known proposition.

Proposition 2.1. *Let $\mathcal{W}$ be a linear k-web of maximal rank and $\mathcal{F}$ be a non-linear foliation on $(\mathbb{C}^2, 0)$. The $(k+1)$-web $\mathcal{W} \boxtimes \mathcal{F}$ is exceptional if and only if it has maximal rank and $k \geq 4$.*

Proof. For $k \leq 3$, all $(k+1)$-webs of maximal rank are algebraizable thanks to Lie's Theorem. Suppose that $k \geq 4$ and let $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a biholomorphism algebraizing $\mathcal{W}\boxtimes\mathcal{F}$. Since $\mathcal{W}$ has maximal rank, $\varphi^*\mathcal{W}$ must be algebraic. According to [23] (see also [5, p. 247]) the biholomorphism φ must be the restriction of a projective transformation. It follows that $\varphi^*\mathcal{F}$ is non-linear and consequently it cannot exist an algebraization of $\mathcal{W} \boxtimes \mathcal{F}$. □

As a corollary one sees that in order to prove that a CDQL k-web $\mathcal{W}$ is exceptional, when $k \geq 5$, it suffices to verify that it has maximal rank. The most obvious way to accomplish this task is to exhibit a basis of the space of its abelian relations. In general, the explicit determination of $\mathcal{A}(\mathcal{W})$ is a fairly difficult problem. To our knowledge, the only general method available is Abel's method for solving functional equations (see [1] and [40, Chapitre 2]). It assumes the knowledge of first integrals for the defining foliations of $\mathcal{W}$ and it tends to involve rather lengthy computations.

In particular cases there are more efficient ways to determine the space of abelian relations. For instance, if the web admits an infinitesimal automorphism then the results of [32] reduce the problem to plain linear algebra. In Section 2.1 we recall the results of [32] and use them in Sections 2.2 and 2.3 to deal with the CDQL webs invariant by $\mathbb{C}^*$-actions described in the Introduction. We point out that the content of Section 2.1 plays a decisive role in the classification of exceptional CDQL webs of degree one carried out in Section 9.

2.1. Webs with infinitesimal automorphisms.

Let $\mathcal{F}$ be a regular foliation on $(\mathbb{C}^2, 0)$ induced by a 1-form ω. We say that a vector field X is an *infinitesimal automorphism* of $\mathcal{F}$ if $L_X\omega \wedge \omega = 0$. When such infinitesimal automorphism X is transverse to $\mathcal{F}$, that is when $\omega(X) \neq 0$, then the 1-form

$$\eta = \frac{\omega}{i_X\omega}$$

is closed and satisfies $L_X\eta = 0$. By definition, the integral

$$u(z) = \int_0^z \eta$$

is the *canonical first integral* of $\mathcal{F}$ (with respect to X).

Assume now that $\mathcal{W}$ is a regular k-web on $(\mathbb{C}^2, 0)$ induced by 1-forms $\omega_1, \ldots, \omega_k$ and let X be an infinitesimal automorphism of all the defining foliations of $\mathcal{W}$.

The Lie derivative L_X induces a linear map

$$\begin{aligned} L_X : \mathcal{A}(\mathcal{W}) \quad &\to \quad \mathcal{A}(\mathcal{W}) \\ (\eta_1, \ldots, \eta_k) \quad &\mapsto \quad (L_X\eta_1, \ldots, L_X\eta_k)\,. \end{aligned} \tag{2}$$

The study of this linear map leads to the following proposition.

Proposition 2.2. *Let $\lambda_1, \ldots, \lambda_\tau \in \mathbb{C}$ be the eigenvalues of the map L_X acting on $\mathcal{A}(\mathcal{W})$ corresponding to minimal eigenspaces with dimensions $\sigma_1, \ldots, \sigma_\tau$. The abelian relations of $\mathcal{W}$ are of the form*

$$P_1(u_1)\, e^{\lambda_i\, u_1}\, du_1 + \cdots + P_k(u_k)\, e^{\lambda_i\, u_k}\, du_k = 0$$

where $P_1, \ldots, P_k$ are polynomials of degree less or equal to σ_i. Moreover the abelian relations corresponding to eigenvectors are precisely the ones for which the polynomials P_i are constant.

Proposition 2.2 suggests an effective method to determine $\mathcal{A}(\mathcal{W})$ from the study of the linear map (2). For details see [32]. It also follows from the study of (2) the main result of [32].

Theorem 2.1. *Let $\mathcal{W}$ be a k-web which admits a transverse infinitesimal automorphism X. Then*

$$rk(\mathcal{W} \boxtimes \mathcal{F}_X) = rk(\mathcal{W}) + (k-1)\,.$$

In particular, $\mathcal{W}$ is of maximal rank if and only if $\mathcal{W} \boxtimes \mathcal{F}_X$ is also of maximal rank.

Below we will make use of Theorem 2.1 to prove that certain webs have maximal rank without giving a complete list of their abelian relations. Nevertheless, the proof of Theorem 2.1 (see [32]) is constructive and the interested reader can easily determine a complete list of the abelian relations.

2.2. **Four infinite families.** Recall the definition of the webs $\mathcal{A}_I^k, \mathcal{A}_{II}^k, \mathcal{A}_{III}^k, \mathcal{A}_{IV}^k$:

$$\begin{aligned}
\mathcal{A}_I^k &= \big[(dx^k - dy^k)\big] \boxtimes \big[d(xy)\big] && \text{where } k \geq 4\,;\\
\mathcal{A}_{II}^k &= \big[(dx^k - dy^k)\,(xdy - ydx)\big] \boxtimes \big[d(xy)\big] && \text{where } k \geq 3\,;\\
\mathcal{A}_{III}^k &= \big[(dx^k - dy^k)\,dx\,dy\big] \boxtimes \big[d(xy)\big] && \text{where } k \geq 2\,;\\
\mathcal{A}_{IV}^k &= \big[(dx^k - dy^k)\,dx\,dy\,(xdy - ydx)\big] \boxtimes \big[d(xy)\big] && \text{where } k \geq 1.
\end{aligned}$$

The exceptionality of these webs follows from the next proposition.

Proposition 2.3. *For all $k \geq 1$ the webs $\mathcal{A}_I^k, \mathcal{A}_{II}^k, \mathcal{A}_{III}^k$ and $\mathcal{A}_{IV}^k$ have maximal rank.*

Proof. Let $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be the radial vector field. Note that it is an infinitesimal automorphism of all the webs above. Moreover

$$\mathcal{A}_{II}^k = \mathcal{A}_I^k \boxtimes \mathcal{F}_R \quad \text{and} \quad \mathcal{A}_{IV}^k = \mathcal{A}_{III}^k \boxtimes \mathcal{F}_R\,.$$

It follows from Theorem 2.1 that $\mathcal{A}_{II}^k$ (resp. $\mathcal{A}_{IV}^k$) has maximal rank if and only if $\mathcal{A}_I^k$ (resp. $\mathcal{A}_{III}^k$) also does.

To prove that $\mathcal{A}_I^k$ has maximal rank consider the linear automorphism of $\mathbb{C}^2$, $\varphi(x,y) = (x, \xi_k y)$. Consider also the induced automorphism of the vector space $\mathbb{C}_{2k-2}[x,y]$ of homogeneous polynomials of degree $2k-2$:

$$\begin{aligned}
\varphi^* : \mathbb{C}_{2k-2}[x,y] &\longrightarrow \mathbb{C}_{2k-2}[x,y]\\
p &\longmapsto p \circ \varphi\,.
\end{aligned}$$

For $k = 1$ there is nothing to prove: every 2-web has maximal rank. Assume that $k \geq 2$. If $\xi_k = \exp(2\pi i/k)$ then the (ξ_k^{k-1})-eigenspace of φ^* has dimension one and is generated by $(xy)^{k-1}$.

If $V \subset \mathbb{C}_{2k-2}[x,y]$ denotes the vector subspace generated by the homogeneous polynomials $(x - \xi_k^i y)^{2k-2}$ with i ranging from 0 to $k-1$, then φ^* preserves V and the characteristic polynomial of ${\varphi^*}_{|V}$ is equal to $t^k - 1$. It follows that there exists $p \in V \setminus \{0\}$ such that $\varphi^* p = (\xi_k^{k-1})\, p$. Since the eigenspace of φ^* associated to the eigenvalue ξ_k^{k-1} has dimension one, p must be a complex multiple of $(xy)^{k-1}$. Therefore, there exist complex constants $\mu_1, \ldots, \mu_k$ such that

$$(xy)^{k-1} = \sum_{i=1}^{k} \mu_i \, (x - \xi_k^i y)^{2k-2} \, .$$

This identity can interpreted as an abelian relation of $\mathcal{A}_I^k$. If we apply the second-order differential operator $\frac{\partial^2}{\partial x \partial y}$ to it we obtain another abelian relation

$$(k-1)^2 (xy)^{k-2} = \sum_{i=1}^{k} \mu_i (2k-2)(2k-1) \xi_k^i (x - \xi_k^i y)^{2(k-1)-2} \, .$$

When $k \geq 3$, this abelian relation is clearly linearly independent from the previous one. Iteration of this procedure shows that

$$\dim \frac{\mathcal{A}(\mathcal{A}_I^k)}{\mathcal{A}\big([dx^k - dy^k]\big)} \geq k - 1 \, .$$

Since $[dx^k - dy^k]$ is an algebraic k-web its rank is $(k-1)(k-2)/2$. Thus $\dim \mathcal{A}(\mathcal{A}_I^k) = k\,(k-1)/2$ and $\mathcal{A}_I^k$ is indeed of maximal rank. Theorem 2.1 implies that the $(k+2)$-web $\mathcal{A}_{II}^k$ is also of maximal rank.

The proof that $\mathcal{A}_{III}^k$ and $\mathcal{A}_{IV}^k$ are of maximal rank is analogous. As before, it suffices to show that the $(k+3)$-web $\mathcal{A}_{III}^k$ has maximal rank.

Consider now the induced automorphism φ^* on the space $\mathbb{C}_{2k}[x, y]$ of homogeneous polynomials of degree $2k$. The 1-eigenspace of φ^* has dimension three and is generated by x^{2k}, y^{2k} and $(xy)^k$. If $V \subset \mathbb{C}_{2k}[x, y]$ denotes now the vector subspace generated by the polynomials $(x - \xi_k^i y)^{2k}$ with $i = 0, \ldots, k-1$, then the characteristic polynomial of $\varphi^*_{|V}$ is also equal to $t^k - 1$. Thus there exists an abelian relation of $\mathcal{A}_{III}^k$ of the form

$$(xy)^k = \sum_{i=1}^{k} \mu_i (x - \xi_k^i y)^{2k} + \mu_{k+1} x^{2k} + \mu_{k+2} y^{2k} \, .$$

Applying the operator $\frac{\partial^2}{\partial x \partial y}$ and iterating as above one deduces that

$$\dim \frac{\mathcal{A}(\mathcal{A}_{III}^k)}{\mathcal{A}\big([dx\, dy\, (dx^k - dy^k)]\big)} \geq k \, .$$

Taking into account the logarithmic abelian relation

$$\log(xy) = \log x + \log y$$

we conclude that $\dim \frac{\mathcal{A}(\mathcal{A}_{III}^k)}{\mathcal{A}([dx\, dy\, (dx^k - dy^k)])} \geq k+1$. Since $[dxdy(dx^k - dy^k)]$ has rank $k\,(k+1)/2$, it follows that the $(k+3)$-web $\mathcal{A}_{III}^k$ also has maximal rank. $\square$

2.3. **The seven sporadic exceptional CDQL webs invariant by $\mathbb{C}^*$-actions.** Recall from the Introduction the other seven webs invariant by the $\mathbb{C}^*$-action $t \cdot (x, y) \mapsto (tx, ty)$:

$$
\begin{array}{lcll}
\mathcal{A}_5^a & = & \big[dx\,dy\,(dx+dy)\,(xdy-ydx)\big] & \boxtimes \; \big[d\big(xy(x+y)\big)\big]\,; \\
\mathcal{A}_5^b & = & \big[dxdy(dx+dy)(xdy-ydx)\big] & \boxtimes \; \Big[d\big(\tfrac{xy}{x+y}\big)\Big]\,; \\
\mathcal{A}_5^c & = & \big[dxdy(dx+dy)(xdy-ydx)\big] & \boxtimes \; \Big[d\big(\tfrac{x^2+xy+y^2}{xy(x+y)}\big)\Big]\,; \\
\mathcal{A}_5^d & = & \big[dxdy(dx+dy)(dx-\xi_3 dy)\big] & \boxtimes \; \big[d\big(xy(x+y)(x-\xi_3 y)\big)\big]\,; \\
\mathcal{A}_6^a & = & \big[dxdy(dx+dy)(dx-\xi_3 dy)(xdy-ydx)\big] & \boxtimes \; \big[d\big(xy(x+y)(x-\xi_3 y)\big)\big]\,; \\
\mathcal{A}_6^b & = & \big[dxdy(dx^3+dy^3)\big] & \boxtimes \; \big[d(x^3+y^3)\big]\,; \\
\mathcal{A}_7 & = & \big[dxdy(dx^3+dy^3)(xdy-ydx)\big] & \boxtimes \; \big[d(x^3+y^3)\big]\,.
\end{array}
$$

Of course, they all share the same infinitesimal automorphism: the radial vector field R. Because

$$
\mathcal{A}_6^a = \mathcal{A}_5^d \boxtimes \mathcal{F}_R \qquad \text{and} \qquad \mathcal{A}_7 = \mathcal{A}_6^b \boxtimes \mathcal{F}_R\,,
$$

Theorem 2.1 implies that the maximality of the rank of $\mathcal{A}_6^a$ (resp. $\mathcal{A}_7$) is equivalent to the maximality of the rank of $\mathcal{A}_5^d$ (resp. $\mathcal{A}_6^b$). Thus, to prove that all the seven webs above are exceptional, it suffices to prove that $\mathcal{A}_5^a, \mathcal{A}_5^b, \mathcal{A}_5^c, \mathcal{A}_5^d, \mathcal{A}_6^b$ have maximal rank. For this sake we list below a basis for a subspace of the space of abelian relations of these webs that is transversal to the space of abelian relations of the maximal linear subweb contained in each of them.

2.3.1. *Abelian Relations for $\mathcal{A}_5^a$.* If $g_0 = xy(x+y)$, $g_1 = x$, $g_2 = y$, $g_3 = x+y$ and $g_4 = \frac{x}{y}$ then the sought abelian relations for $\mathcal{A}_5^a$ are

$$
\begin{array}{lcllllllll}
\ln g_0 & = & \ln g_1 & + & \ln g_2 & + & \ln g_3 & & \\
\ln^2 g_0 & = & 3\ln^2 g_1 & + & 3\ln^2 g_2 & + & 3\ln^2 g_3 & - & \varphi(g_4) \\
3\,g_0 & = & -g_1^3 & - & g_2^3 & + & g_3^3 & &
\end{array}
$$

where $\varphi(t) = \ln^2 t + \ln^2(t+1) + \ln^2(t^{-1}+1)$.

2.3.2. *Abelian Relations for $\mathcal{A}_5^b$.* If $g_0 = xy/(x+y)$ and g_1, g_2, g_3, g_4 are as above then

$$
\begin{array}{lcllllllll}
\ln g_0 & = & \ln g_1 & + & \ln g_2 & - & \ln g_3 & & \\
\ln^2 g_0 & = & \ln^2 g_1 & + & \ln^2 g_2 & - & 3\ln^2 g_3 & - & \varphi(g_4) \\
g_0^{-1} & = & g_1^{-1} & + & g_2^{-1} & & & &
\end{array}
$$

where $\varphi(t) = \ln^2 t - \ln^2(t+1) - \ln^2(t^{-1}+1)$.

2.3.3. *Abelian Relations for $\mathcal{A}_5^c$.* If $g_0 = (x^2+xy+y^2)/\big(xy(x+y)\big)$ and g_1, g_2, g_3, g_4 are as above then

$$
\begin{array}{lcllllllll}
\ln g_0 & = & & & & + & \ln g_3 & + & \ln(g_4 + g_4^{-1} + 1) \\
g_0 & = & g_1^{-1} & + & g_2^{-1} & - & g_3^{-1} & & \\
g_0^2 & = & g_1^{-2} & + & g_2^{-2} & - & g_3^{-2}. & &
\end{array}
$$

2.3.4. *Abelian Relations for $\mathcal{A}_5^d$.* Notice that $\mathcal{A}_5^d$ is equivalent to

$$\big[dx\,dy\,(dx+dy)\,(dx-\xi_3 dy)\big]\boxtimes\big[d\big(xy(x+y)(x-\xi_3 y)\big)\big]$$

under a linear change of coordinates. If $g_0=xy(x+y)(x-\xi_3 y)$, $g_4=x-\xi_3\,y$ and g_1,g_2,g_3 are as above then

$$\begin{array}{rcccccccc}
\ln g_0 & = & \ln g_1 & + & \ln g_2 & + & \ln g_3 & + & \ln g_4\\
12\,g_0 & = & (-2-\xi_3)\ g_1^4 & + & (1+2\,\xi_3)\,g_2^4 & + & (1-\xi_3)\ g_3^4 & + & (1+2\,\xi_3)\,g_4^4\\
28\,g_0^2 & = & (1+\xi_3)\ {g_1}^8 & - & {g_2}^8 & - & \xi_3\,{g_3}^8 & - & {g_4}^8\,.
\end{array}$$

2.3.5. *Abelian Relations for $\mathcal{A}_6^b$.* If $g_0=x^3+y^3$, $g_4=x+\xi_3\,y$, $g_5=x+\xi_3^2\,y$ and g_1,g_2,g_3 are as above then

$$\begin{array}{rcccccccccc}
g_0 & = & {g_1}^3 & + & {g_2}^3 & & & & & & \\
\ln g_0 & = & & & & & \ln g_3 & + & \ln g_4 & + & \ln g_5\\
30\,g_0^2 & = & 27\,{g_1}^6 & + & 27\,{g_2}^6 & + & {g_3}^6 & + & {g_4}^6 & + & {g_5}^6\\
84\,{g_0}^3 & = & 81\,{g_1}^9 & + & 81\,{g_2}^9 & + & {g_3}^9 & + & {g_4}^9 & + & {g_5}^9\,.
\end{array}$$

3. Abelian relations for planar webs associated to nets

The determination of $\mathcal{A}(\mathcal{B}_5)$ is due to Bol, see [7]. The determination of $\mathcal{A}(\mathcal{B}_6),\mathcal{A}(\mathcal{B}_7)$ and $\mathcal{A}(\mathcal{B}_8)$ is treated in [44] (see also [40, 41] for the determination of $\mathcal{A}(\mathcal{B}_6)$ and $\mathcal{A}(\mathcal{B}_7)$ through Abel's method). In this section we will prove that $\mathcal{H}_5$ and $\mathcal{H}_{10}$ — the two remaining exceptional CDQL webs on $\mathbb{P}^2$ presented in the Introduction — have maximal rank. We adopt here an approach similar to the one used by Robert in [44] and that can be traced back to [22]. We look for the abelian relations among k-uples of Chen's iterated integrals of logarithmic 1-forms with poles on certain hyperplane arrangements. It turns out that this particular class of webs carry logarithmic and dilogarithmic abelian relations thanks to purely combinatorial reasons.

3.1. **Webs associated to nets.** Let $r\geq 3$ be an integer. Recall from [48] that a *r-net* in $\mathbb{P}^2$ is a pair $(\mathcal{L},\mathcal{P})$ where $\mathcal{L}$ is a finite set of lines partitioned into r disjoint subsets $\mathcal{L}=\sqcup_{i=1}^r\mathcal{L}_i$ and $\mathcal{P}$ is a finite set of points subjected to the two conditions:

(1) for every $i\neq j$ and every $\ell\in\mathcal{L}_i$, $\ell'\in\mathcal{L}_j$, we have that $\ell\cap\ell'\in\mathcal{P}$;
(2) for every $p\in\mathcal{P}$ and every $i=1,2,\ldots,r$, there exists a unique $\ell\in\mathcal{L}_i$ passing through p.

The definition implies that $\mathcal{P}$ has cardinality m^2 and that the cardinalities of the sets $\mathcal{L}_i$ do not depend on i and are all equal to $m=\mathrm{C}ard(\ell\cap\mathcal{P})$ for any $\ell\in\mathcal{L}$. We say that $\mathcal{L}$ is a *(r,m)-net.*

For every pair $(\alpha,\beta)\in\{1,\ldots,r-1\}^2$, we have a function

$$n_\alpha^\beta:\mathcal{L}_\alpha\times\mathcal{L}_r\to\mathcal{L}_\beta$$

that assigns to $(\ell,\ell')\in\mathcal{L}_\alpha\times\mathcal{L}_r$ the line in $\mathcal{L}_\beta$ passing through $\ell\cap\ell'$. Notice that for a fixed $\ell\in\mathcal{L}_r$ the functions $n_\alpha^\beta(\cdot,\ell):\mathcal{L}_\alpha\to\mathcal{L}_\beta$ are bijective.

It follows from the definition of a r-net (*cf.* [48]) that there exists a rational function $F:\mathbb{P}^2\dashrightarrow\mathbb{P}^1$ of degree m with r values $c_1,\ldots,c_r\in\mathbb{P}^1$ for which $F^{-1}(c_i)$ can be identified with $\mathcal{L}_i$. Although there is some ambiguity in the definition of F (we can compose it with an automorphism of $\mathbb{P}^1$) the induced foliation is uniquely determined and will be denoted by $\mathcal{F}(\mathcal{L})$. Similarly, if $(\mathcal{L},\mathcal{P})$ is a (r,m)-net then we will denote by $\mathcal{W}(\mathcal{L})$ the CDQL (m^2+1)-web $\mathcal{W}(\mathcal{P})\boxtimes\mathcal{F}(\mathcal{L})$, where $\mathcal{W}(\mathcal{P})$ is the completely decomposable linear m^2-web formed by the superposition of the pencils of lines through the points of $\mathcal{P}$.

Among the thirteen sporadic examples of exceptional CDQL webs presented in the Introduction, two are webs associated to nets. The first one is Bol's web $\mathcal{B}_5$ which is associated to a $(3,2)$-net with $\mathcal{P}$ equal to four points in general position and $\mathcal{L}$ equal to the set of lines joining any two of them. In this case $\mathcal{F}(\mathcal{L})$ is the pencil of conics through the four points. The other example is the CDQL 10-web $\mathcal{H}_{10}$. It is associated to a $(4,3)$-net with $\mathcal{P}$ equal to the set of base points of the Hesse pencil, $\mathcal{L}$ equal to the set of lines through any two of them and $\mathcal{F}(\mathcal{L})$ equal to the Hesse pencil.

The result below implies that both $\mathcal{B}_5$ and $\mathcal{H}_{10}$ are exceptional.

Theorem 3.1. *If $\mathcal{L}$ is a (r,m)-net then*

$$rk\big(\mathcal{W}(\mathcal{L})\big) \geq \frac{(m^2-1)(m^2-2)}{2} + (r-1)^2 - 1\,.$$

In particular if $\mathcal{L}$ is a $(3,2)$-net or a $(4,3)$-net then $\mathcal{W}(\mathcal{L})$ has maximal rank.

Proof. Since the m^2-subweb $\mathcal{W}(\mathcal{P})$ is linear, it has maximal rank. To prove the theorem it suffices to show that

$$\dim \frac{\mathcal{W}(\mathcal{L})}{\mathcal{W}(\mathcal{P})} \geq (r-1)^2 - 1\,.$$

Set $\mathcal{L}_i = \{\ell_1^{(i)},\ldots,\ell_m^{(i)}\}$ and let $L_j^{(i)}$ be a linear homogenous polynomial in $\mathbb{C}[x,y,z]$ defining $\ell_j^{(i)}$. Let $p_{ij} = \ell_i^{(1)} \cap \ell_j^{(r)}$ and $\mathcal{L}_{ij}$ be the subset of $\mathcal{L}$ formed by the lines through p_{ij}. Notice that $\mathcal{P} = \cup_{i,j}\{p_{ij}\}$.

For a suitable choice of the linear forms $L_j^{(i)}$ the rational function $F:\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ associated to the arrangement satisfies

$$F - c_\alpha = \frac{\prod_{i=1}^m L_i^{(\alpha)}}{\prod_{i=1}^m L_i^{(r)}} \tag{3}$$

for every $\alpha < r$.

Let $V = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log \mathcal{L}))$ (resp. $V_{ij} = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log \mathcal{L}_{ij}))$) be the vector space of logarithmic 1-forms with poles in $\mathcal{L}$ (resp. $\mathcal{L}_{ij}$). Every element in V_{ij} vanishes when restricted to the leaves of the foliation $\mathcal{L}_{p_{ij}}$ induced by the pencil of lines through p_{ij}. If $\ell_{n^\beta_\alpha(i,j)}$ denotes the line $n^\beta_\alpha(\ell_i^{(\alpha)},\ell_j^{(\beta)})$ then the logarithmic 1-forms

$$\frac{dL^{(\alpha)}_{n_1^\alpha(i,j)}}{L^{(\alpha)}_{n_1^\alpha(i,j)}} - \frac{dL_j^{(r)}}{L_j^{(r)}}$$

with α ranging from 1 to $r-1$, can be taken as a basis of V_{ij}.

It can be promptly verified that the union of the subspaces $V_{ij} \subset V$ spans V. Indeed V is generated by elements of the form $\omega = \frac{dL}{L} - \frac{dL'}{L'}$ where L, L' are linear forms cutting out $\ell,\ell' \in \mathcal{L}$. If $\ell\cap\ell' = p_{ij} \in \mathcal{P}$ then $\omega \in V_{ij}$. Otherwise ℓ and ℓ' belongs to the same set $\mathcal{L}_i$. Then we choose $\ell'' \in \mathcal{L}_j$, $j \neq i$, and write

$$\omega = \left(\frac{dL}{L} - \frac{dL''}{L''}\right) - \left(\frac{dL'}{L'} - \frac{dL''}{L''}\right)\,.$$

It follows from (3) that the 1-forms $\frac{dF}{F-c_\alpha}$, $\alpha = 1, \ldots, r-1$, belong to V. Therefore there exists $\omega_{ij}^{(\alpha)} \in V_{ij}$ such that

$$\frac{dF}{F-c_\alpha} + \sum_{i,j} \omega_{ij}^{(\alpha)} = 0 \, .$$

These equations can be interpreted as elements of $\mathcal{A}(\mathcal{W}(\mathcal{L}))$. Since the 1-forms $\frac{dF}{F-c_\alpha}$ are linearly independent, the classes of these equations span a $(r-1)$-dimensional subspace $A_0 \subset \frac{\mathcal{A}(\mathcal{W}(\mathcal{L}))}{\mathcal{A}(\mathcal{W}(\mathcal{P}))}$.

If $\alpha \neq \beta$ then $\bigcup_{j=1}^m \ell_{n_\alpha^\beta(i,j)} = \mathcal{L}_\beta$ for every fixed $i \in \{1, \ldots, m\}$. Using this fact to work out the expansion of $\dfrac{dF}{F-c_\alpha} \otimes \dfrac{dF}{F-c_\beta}$ yields

$$\begin{aligned} \frac{dF}{F-c_\alpha} \otimes \frac{dF}{F-c_\beta} \quad &= \underbrace{\sum_{i,j} \left(\frac{dL_i^{(\alpha)}}{L_i^{(\alpha)}} - \frac{dL_j^{(r)}}{L_j^{(r)}} \right) \otimes \left(\frac{dL^{(\beta)}_{n_\alpha^\beta(i,j)}}{L^{(\beta)}_{n_\alpha^\beta(i,j)}} - \frac{dL_j^{(r)}}{L_j^{(r)}} \right)}_{\in \bigoplus V_{ij} \otimes V_{ij}} \\ &+ \quad \sum_{i \neq j} \frac{dL_i^{(r)}}{L_i^{(r)}} \otimes \frac{dL_j^{(r)}}{L_j^{(r)}} - (r-2) \sum_i \left(\frac{dL_i^{(r)}}{L_i^{(r)}} \right)^{\otimes 2} \, . \end{aligned}$$

Thus for ordered pairs (α, β) and (γ, δ) with distinct entries in $\{1, \ldots, r-1\}$, one have an identity

$$\left(\frac{dF}{F-c_\alpha} \otimes \frac{dF}{F-c_\beta} - \frac{dF}{F-c_\gamma} \otimes \frac{dF}{F-c_\delta} \right) + \sum_{i,j} \omega_{ij}^{(\alpha\beta\gamma\delta)} = 0$$

for suitable $\omega_{ij}^{(\alpha\beta\gamma\delta)} \in V_{ij} \otimes V_{ij}$. It follows from Chen's theory of iterated integrals (see [15, Theorem 4.1.1] and [44, Théorème 2.1]) that after integration these identities can be interpreted as elements of $\mathcal{A}(\mathcal{L})$. [5]

Moreover their classes span a subspace $A_1 \subset \frac{\mathcal{A}(\mathcal{W}(\mathcal{L}))}{\mathcal{A}(\mathcal{W}(\mathcal{P}))}$ of dimension $(r-1)(r-2)-1$. Since $A_0 \cap A_1 = 0$, the theorem follows. $\square$

It has to be noted that Theorem 3.1 has a rather limited scope. Indeed, the Hesse net is the only r-net in $\mathbb{P}^2$ known with $r \geq 4$ and recently J. Stipins has proved that there is no r-net in $\mathbb{P}^2$ if $r \geq 5$ (for a proof that there is no r-net in $\mathbb{P}^2$ when $r \geq 6$, see [39]). Nevertheless Theorem 3.1 might give some clues on how to approach the problem about the abelian relations of webs associated to hyperplane arrangements proposed in [39]. We refer to this paper and the references therein for further examples of nets.

The maximality of the rank of $\mathcal{H}_5$ follows from similar reasons. If $\mathcal{L}$ is the Hesse arrangement of lines then an argument similar to the one used in the proof of Theorem 3.1 shows that $V = H^0(\mathbb{P}^2, \Omega^1(\log \mathcal{L}))$ can be generated by logarithmic 1-forms inducing the defining foliations of the maximal linear subweb of $\mathcal{H}_5$. Since

[5] Notice that on the universal covering of $\mathbb{P}^2 \setminus |\mathcal{L}|$, one have
$\int \left(\frac{dF}{F-c_\alpha} \otimes \frac{dF}{F-c_\beta} - \dfrac{dF}{F-c_\gamma} \otimes \dfrac{dF}{F-c_\delta} \right) = \log(F-c_\alpha) \frac{dF}{F-c_\beta} - \log(F-c_\gamma) \frac{dF}{F-c_\delta} \, .$

the Hesse pencil has four linear fibers it follows that

$$\dim \frac{\mathcal{A}(\mathcal{H}_5)}{\mathcal{A}\big([(xdy-ydx)\,(dx^3+dy^3)]\big)} \geq 3\,.$$

Consequently $\mathcal{H}_5$ has maximal rank.

3.2. **Explicit abelian relations for $\mathcal{H}_5$.** Alternatively, one can also establish directly that the rank of $\mathcal{H}_5$ is maximal. Indeed the functions $g_0=(x^3+y^3+1)/(xy)$, $g_1=\xi_3\,x+y$, $g_2=x+y$, $g_3=x+\xi_3\,y$ and $g_4=x/y+y/x$ are first integrals of $\mathcal{H}_5$ and they verify the abelian relations:

$$\begin{aligned}
\ln\left(\tfrac{g_0-3}{g_0-3\,\xi_3}\right) &= \ln\left(\tfrac{g_1+(\xi_3)2}{g_1+1}\right) + \ln\left(\tfrac{g_2+1}{g_2+\xi_3}\right) + \ln\left(\tfrac{g_3+(\xi_3)2}{g_3+1}\right)\\
\ln\left(\tfrac{g_0-3\,\xi_3}{g_0-3\,(\xi_3)2}\right) &= \ln\left(\tfrac{g_1+1}{g_1+\xi_3}\right) + \ln\left(\tfrac{g_2+\xi_3}{g_2+(\xi_3)2}\right) + \ln\left(\tfrac{g_3+1}{g_3+\xi_3}\right)\\
\ln\big(\xi_3\,(g_0-3)\big) &= -\ln\big(g_1+(\xi_3)2\big) + \ln\left(\tfrac{g_2 2}{1+g_2}\right) - \ln(g_3+(\xi_3)2) - \ln\left(g_4+\tfrac{1}{2}\right).
\end{aligned}$$

These abelian relations span a 3-dimensional vector space $\mathcal{A}_1$ such that

$$\mathcal{A}(\mathcal{H}_5)=\mathcal{A}\big([(xdy-ydx)\,(dx^3+dy^3)]\big)\oplus\mathcal{A}_1.$$

4. Abelian relations for the elliptic CDQL webs

In this section we will prove that the elliptic CDQL webs presented in the introduction are exceptional. The abelian relations of them that do not come from the maximal linear subweb are all captured by Theorem 4.1 below.

The analogy with Theorem 3.1 is evident and probably not very surprising for the specialists in polylogarithms since, according to the terminology of Beilinson and Levine [4], the integrals $\int dz$ and $\int d\log\vartheta$ (ϑ being a theta function) can be considered as elliptic analogs of the classical logarithm and dilogarithm.

4.1. **Rational maps on complex tori with many linear fibers.** Let T be a two-dimensional complex torus and $F: T \dashrightarrow \mathbb{P}^1$ be a meromorphic map. We will say that a fiber $F^{-1}(\lambda)$ is *linear* if it is supported on a union of subtori.

Notice that each subtorus E of T determines a unique linear foliation with E and its translates being the leaves. We will say that a linear web $\mathcal{W}$ on T supports a fiber $F^{-1}(\lambda)$ if it contains all the linear foliations determined by the irreducible components of $F^{-1}(\lambda)$.

Theorem 4.1. *Let $\mathcal{F}$ be the foliation induced by a meromorphic map $F: T \dashrightarrow \mathbb{P}^1$. If $\mathcal{W}$ is a linear k-web with $k\geq 3$ that supports m distinct linear fibers of F, then*

$$\dim\frac{\mathcal{A}(\mathcal{W}\boxtimes\mathcal{F})}{\mathcal{A}(\mathcal{W})}\geq m-1\,.$$

Before proving Theorem 4.1 let us briefly review some basic facts about theta functions. For details see for instance [19, Chapitre IV]. If V is a complex vector space and $\Gamma\subset V$ is a lattice then a theta function associated to Γ is any entire function ϑ on V such that for each $\gamma\in\Gamma$ there exists a linear form a_γ and a constant b_γ such that

$$\vartheta(z+\gamma)=\exp\big(2\,i\,\pi\,(a_\gamma(z)+b_\gamma)\big)\vartheta(z)\quad\text{for every } z\in V.$$

Any effective divisor on the complex torus $T = V/\Gamma$ is the zero divisor of some theta function. Moreover if the divisors of two theta functions, say ϑ and $\tilde{\vartheta}$, coincide then their quotient is a trivial theta function, that is

$$\frac{\vartheta}{\tilde{\vartheta}}(z) = \exp\big(P(z)\big)$$

where $P : V \to \mathbb{C}$ is a polynomial of degree at most two.

Example 4.1. If $(\mu,\nu) \in \{0,1\}^2$ and $\tau \in \mathbb{H}$ then the entire functions on $\mathbb{C}$

$$\vartheta_{\mu,\nu}(x,\tau) = \sum_{n=-\infty}^{+\infty} (-1)^{n\nu} \exp\Big(i\,\pi\big(n+\frac{\mu}{2}\big)^2\tau + 2\,i\,\pi\big(n+\frac{\mu}{2}\big)x\Big)\,.$$

satisfy the following relations

$$(4) \qquad \begin{array}{rcl} \vartheta_{\mu,\nu}(x+1,\tau) & = & (-1)^{\mu}\vartheta_{\mu,\nu}(x,\tau) \\ \vartheta_{\mu,\nu}(x+\tau,\tau) & = & (-1)^{\nu}\exp\big(-i\pi(2z+\tau)\big)\,\vartheta_{\mu,\nu}(x,\tau)\,. \end{array}$$

It is then clear that they are examples of theta functions with respect to the lattice $\mathbb{Z}\oplus\mathbb{Z}\tau \subset \mathbb{C}$. The theta functions ϑ_i that appeared in the Introduction are nothing more than

$$\vartheta_1 = -i\,\vartheta_{1,1}, \quad \vartheta_2 = \vartheta_{1,0}, \quad \vartheta_3 = \vartheta_{0,0} \quad \text{and} \quad \vartheta_4 = \vartheta_{0,1}.$$

If E_τ denotes the elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ then the zero divisors of the functions $\vartheta_i = \vartheta_i(\cdot,\tau)$ are

$$\big(\vartheta_1\big)_0 = 0, \quad \big(\vartheta_2\big)_0 = \frac{1}{2}, \quad \big(\vartheta_3\big)_0 = \frac{1+\tau}{2} \quad \text{and} \quad \big(\vartheta_4\big)_0 = \frac{\tau}{2}\,.$$

Proof of Theorem 4.1. With notation as above, suppose that $T = V/\Gamma$. If $F^{-1}(\lambda)$ is a linear fiber then one can write

$$F^{-1}(\lambda) = D_1^\lambda + \cdots + D_{r(\lambda)}^\lambda$$

where each divisor D_i^λ (for $i = 1,\ldots,r(\lambda)$) is supported on a union of translates of a subtori E_i^λ. Therefore there exist complex vector spaces V_i^λ of dimension one, linear maps $p_i^\lambda : V \to V_i^\lambda$ and lattices $\Gamma_i^\lambda \subset V_i^\lambda$ such that

(1) $p_i^\lambda(\Gamma) \subset \Gamma_i^\lambda$;
(2) D_i^λ is the pull-back by the map $[p_i] : T \to V_i^\lambda/\Gamma_i^\lambda$ of a divisor on $V_i^\lambda/\Gamma_i^\lambda$.

Notice that p_i^λ can be interpreted as a linear form on V and its differential dp_i^λ as a 1-form defining the linear foliation determined by E_i^λ.

Composing F with an automorphism of $\mathbb{P}^1$ we can assume that the linear fibers are $F^{-1}(\lambda_1), F^{-1}(\lambda_2), \ldots, F^{-1}(\lambda_{m-1})$ and $F^{-1}(\infty)$. Thus, for j ranging from 1 to $m-1$, we can write

$$(5) \qquad F - \lambda_j = \exp\big(P_j(z)\big)\,\frac{\prod_i [p_i^{\lambda_j}]^*\vartheta_i^{\lambda_j}}{\prod_i [p_i^{\infty}]^*\vartheta_i^{\infty}}$$

where the P_j's are polynomials of degree at most two and $\vartheta_i^{\lambda_j}$ are theta functions on $V_i^{\lambda_j}$ associated to the lattices $\Gamma_i^{\lambda_j}$. Taking the logarithmic derivative of (5), we obtain

$$(6) \qquad \frac{dF}{F-\lambda_j} = dP_j(z) + \sum_i [p_i^{\lambda_j}]^* d\log\vartheta_i^{\lambda_j} - \sum_i [p_i^{\infty}]^* d\log\vartheta_i^{\infty}\,.$$

Since $\mathcal{W}$ is a k-web with $k \geq 3$ there exist three pairwise linearly independent linear forms p_1, p_2, p_3 among the $p_i^{\lambda_j}$ such that dP_j can be written as a linear combination of $dp_1, dp_2, p_1dp_1, p_2dp_2, p_3dp_3$. It follows that (6) is an abelian relation for $\mathcal{W} \boxtimes \mathcal{F}$. Since the logarithmic 1-forms $\frac{dF}{F-\lambda_1}, \ldots, \frac{dF}{F-\lambda_{m-1}}$ are linearly independent over $\mathbb{C}$, the abelian relations described in (6) are also linearly independent and generate a subspace of $\mathcal{A}(\mathcal{W} \boxtimes \mathcal{F})$ of dimension $m-1$ intersecting $\mathcal{A}(\mathcal{W})$ trivially. The theorem follows. $\square$

In the next three subsections we will derive the exceptionality of the CDQL webs $\mathcal{E}_5^\tau, \mathcal{E}_5, \mathcal{E}_6$ and $\mathcal{E}_7$ from Theorem 4.1. Along the way a clear geometric picture of these webs will emerge.

4.2. **The harmonic 5-webs $\mathcal{E}_5^\tau$ and the superharmonic 7-web $\mathcal{E}_7$.** For $\tau \in \mathbb{H}$, let E_τ be the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and T_τ be the complex torus E_τ^2. For every $\tau \in \mathbb{H}$, the 5-web $\mathcal{E}_5^\tau = [dx\, dy\, (dx^2 - dy^2)\, dF_\tau]$ is naturally defined on T_τ where

$$F_\tau(x,y) = \left(\frac{\vartheta_1(x,\tau)\vartheta_1(y,\tau)}{\vartheta_4(x,\tau)\vartheta_4(y,\tau)} \right)^2 . \tag{7}$$

The 7-web $\mathcal{E}_7 = [dx\, dy\, (dx^2 - dy^2)\, (dx^2 + dy^2)\, dF_{1+i}]$ in its turn is naturally defined on T_{1+i}.

For every $\alpha, \beta \in \mathrm{End}(E_\tau)$, denote by $E_{\alpha,\beta}$ the elliptic curve described by the image of the morphism

$$\begin{array}{rccl} \varphi_{\alpha,\beta} : & E_\tau & \longrightarrow & T_\tau \\ & x & \longmapsto & (\alpha \cdot x, \beta \cdot x). \end{array} \tag{8}$$

For example $E_{1,0}$ is the *horizontal* elliptic curve through $0 \in T_\tau$, $E_{0,1}$ is the *vertical* one, $E_{1,1}$ is the *diagonal* and $E_{1,-1}$ is the *anti-diagonal*. The translation of $E_{\alpha,\beta}$ by an element $(a,b) \in T_\tau$ will be denoted by $L_{(a,b)}E_{\alpha,\beta}$.

Let $D_1 = E_{1,0} + E_{0,1}$ and $D_2 = L_{(0,\tau/2)}E_{1,0} + L_{(\tau/2,0)}E_{0,1}$ be divisors in T_τ. Notice that the rational function F_τ is such that $\mathrm{div}(F_\tau) = 2D_1 - 2D_2$. The indeterminacy set of F_τ is

$$\mathrm{Indet}(F_\tau) = \big\{ (\tau/2, 0), (0, \tau/2) \big\}.$$

Blowing-up the two indeterminacy points of F_τ we obtain a surface $\widetilde{T_\tau}$ containing two pairwise disjoint divisors $\widetilde{D_1}$, $\widetilde{D_2}$: the strict transforms of D_1 and D_2 respectively. Let $D_3 = L_{(\tau/2,0)}E_{1,1} + L_{(0,\tau/2)}E_{1,-1}$. The pairwise intersection of the supports of the divisors D_1, D_2 and D_3 are all equal, that is

$$|D_1| \cap |D_3| = |D_2| \cap |D_3| = |D_1| \cap |D_2| = \mathrm{Indet}(F_\tau)\,.$$

Therefore $\widetilde{D_3}$, the strict transform of D_3, is a divisor in $\widetilde{T_\tau}$ with support disjoint from the supports of $\widetilde{D_1}$ and $\widetilde{D_2}$. The lifting of F_τ to $\widetilde{T_\tau}$ must map the support of $\widetilde{D_3}$ to $\widetilde{F_\tau}\big(\widetilde{T_\tau} - (|\widetilde{D_1}| \cup |\widetilde{D_2}|)\big) = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. The maximal principle implies that the image must be a point. Since $\widetilde{D_3} \cdot \widetilde{D_3} = 0$, D_3 must be a connected component of a fiber of F_τ. Since D_3 is numerically equivalent to $2D_1$ and $2D_2$ it turns out that $\widetilde{D_3}$ is indeed a fiber of F_τ. Moreover, because $\widetilde{D_3}$ is connected and reduced, the generic fiber of F_τ is irreducible. In particular, the linear equivalence class of the divisor $\frac{1}{2}\mathrm{div}(F_\tau) = D_1 - D_2$ is a non-trivial 2-torsion point in $\mathrm{Pic}_0(T_\tau)$.

So far we have proved that F_τ has at least three linear fibers. For generic τ it can be verified that 3 is the exact number of linear fibers of F_τ. But if $\tau = 1+i$ then

$$D_4 = L_{((1+i)/2,0)}E_{1,i} + L_{(0,(1+i)/2)}E_{1,-i}$$

is such that $|D_1| \cap |D_4| = |D_2| \cap |D_4| = |D_3| \cap |D_4| = |D_1| \cap |D_2| = \mathrm{Indet}(F_{1+i})$. The arguments above imply that F_{1+i} has at least 4 linear fibers.

Theorem 4.1 can be applied to the 5-webs $\mathcal{E}_5^\tau = [dx\, dy\, (dx^2 - dy^2)\, dF_\tau]$ (resp. to the 7-web $\mathcal{E}_7 = [dx\, dy\, (dx^2 - dy^2)\, (dx^2 + dy^2)\, dF_{1+i}]$) to ensure that $rk(\mathcal{E}_5^\tau) \geq 3 + 2 = 5$ (resp. $rk(\mathcal{E}_7) \geq 10 + 3 = 13$). To prove that $\mathcal{E}_5^\tau$ and $\mathcal{E}_7$ are exceptional it remains to find two extra abelian relations for the latter web and one for the former.

The *missing* abelian relations are also captured by Theorem 4.1. The point is that the torsion of $D_1 - D_2$ is *hiding* two extra linear fibers of F_{1+i} and one extra linear fiber of F_τ. More precisely, since $D_1 - D_2$ is a non-trivial 2-torsion element of $\mathrm{Pic}_0(T_\tau)$, there exists a complex torus X_τ, an étale covering $\rho : X_\tau \to T_\tau$ and a rational function $G_\tau : X_\tau \dashrightarrow \mathbb{P}^1$, with irreducible generic fiber, fitting in the commutative diagram:

$$\begin{tikzcd} X_\tau \arrow[r, "\rho"] \arrow[d, dashed, "G_\tau"'] & T_\tau \arrow[d, dashed, "F_\tau"] \\ \mathbb{P}^1 \arrow[r, "z \mapsto z^2"] & \mathbb{P}^1 \end{tikzcd} .$$

The étale covering above can be assumed to lift to the identity over the universal covering of X_τ and T_τ. In other words, if $T_\tau = \mathbb{C}^2/\Gamma_\tau$ for some lattice $\Gamma_\tau \subset \mathbb{C}^2$ then X_τ is induced by a sublattice of Γ_τ. In particular

$$\rho^*\big[dxdy(dx^k - dy^k)\big] = \big[dxdy(dx^k - dy^k)\big] \quad \text{for every } k \geq 1.$$

Clearly G_τ has at least four linear fibers supported by the linear web $[dxdy(dx^2 - dy^2)]$ and G_{1+i} has at least six linear fibers supported by the linear web $[dxdy(dx^4 - dy^4)]$. Theorem 4.1 implies that $\rho^*\mathcal{E}_5^\tau$ and $\rho^*\mathcal{E}_7$ are webs of maximal rank. Since the rank is locally determined the same holds for $\mathcal{E}_5^\tau$ and $\mathcal{E}_7$.

4.3. **The equianharmonic 5-web $\mathcal{E}_5$.** Let $F : T_{\xi_3} \dashrightarrow \mathbb{P}^1$ be the rational function

$$F = \frac{\vartheta_1(x,\xi_3)\, \vartheta_1(y,\xi_3)\, \vartheta_1(x-y,\xi_3)\, \vartheta_1(x+\xi_3^2\, y,\xi_3)}{\vartheta_2(x,\xi_3)\, \vartheta_3(y,\xi_3)\, \vartheta_4(x-y,\xi_3)\, \vartheta_3(x+\xi_3^2\, y,\xi_3)} \,. \tag{9}$$

Proposition 4.1. *The function F has four linear fibers on T_{ξ_3}. Moreover each of these fibers is supported on the linear 4-web $\mathcal{W} = [dx\, dy\, (dx - dy)\, (dx + \xi_3^2\, dy)]$.*

Proof. Consider the divisor $D_1 = E_{1,0} + E_{0,1} + E_{1,1} + E_{1,-\xi_3}$. Notice that D_1 can be given by the vanishing of

$$f_1(x,y) = \vartheta_1(x,\xi_3)\, \vartheta_1(y,\xi_3)\, \vartheta_1(x-y,\xi_3)\, \vartheta_1(x+\xi_3^2 y,\xi_3).$$

The complex torus T_{ξ_3} has sixteen 2-torsion points and the support of D_1 contains thirteen of them. The 2-torsion points that are not contained in $|D_1|$ are

$$p_2 = \left(\frac{1}{2}, \frac{1+\xi_3}{2}\right), \quad p_3 = \left(\frac{\xi_3}{2}, \frac{1}{2}\right) \quad \text{and} \quad p_4 = \left(\frac{1+\xi_3}{2}, \frac{\xi_3}{2}\right).$$

If we set $D_i = L_{p_i} D_1$ (the translation of D_1 by p_i) for $i = 2,3,4$, then the support of $D_i \cap D_j$ (with $j \neq i$) does not depend on (i,j) and is the set of 12 non-trivial 2-torsion points of T_{ξ_3} contained in D_1. Notice that D_2 can be given by the vanishing of

$$f_2(x,y) = \vartheta_2(x,\xi_3)\,\vartheta_3(y,\xi_3)\,\vartheta_4(x-y,\xi_3)\,\vartheta_3(x+\xi_3^2 y,\xi_3).$$

The quotient $F = f_1(x,y)/f_2(x,y)$ is the rational function we are interested in.

Blowing up the 12 indeterminacy points of F one sees that the strict transforms of the divisors D_i are connected and pairwise disjoint divisors of self-intersection zero. This is sufficient to prove that each of the divisors D_i is a linear fiber of F and that F has generic fiber irreducible as in the analysis of the webs $\mathcal{E}_5^\tau$ and $\mathcal{E}_7$. Clearly each one of these fibers is supported on the linear web $\mathcal{W}$. $\square$

The proposition above combined with Theorem 4.1 implies at once that the web

$$\mathcal{E}_5 = [dx\,dy\,(dx-dy)\,(dx+\xi_3^2\,dy)] \boxtimes [dF]$$

is exceptional.

4.4. **The equianharmonic 6-web $\mathcal{E}_6$.** It remains to analyze the 6-web

$$\mathcal{E}_6 = \big[dx\,dy\,(dx+dy)\,(dx+\xi_3\,dy)\,(dx+\xi_3^2\,dy)\big] \boxtimes \big[dx/\wp(x)+dy/\wp(y)\big]$$

on $T_{\xi_3} = E^2_{\xi_3}$. We will proceed exactly as in the previous cases.

Proposition 4.2. *The foliation $\mathcal{F} = [dx/\wp(x)+dy/\wp(y)]$ on T_{ξ_3} admits a rational first integral $F : T_{\xi_3} \dashrightarrow \mathbb{P}^1$ with generic fiber irreducible and with three linear fibers, one reduced and two of multiplicity three. Moreover these three linear fibers are supported on the linear web $\mathcal{W} = [dx\,dy\,(dx+dy)\,(dx+\xi_3\,dy)\,(dx+\xi_3^2\,dy)]$.*

Proof. Recall that if $\Gamma \subset \mathbb{C}$ is a lattice then the Weierstrass $\wp$-function associated to Γ is defined as

$$\wp(z,\Gamma) = \frac{1}{z^2} + \sum_{\gamma\in\Gamma\setminus\{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right). \tag{10}$$

It is an entire meromorphic function with poles of order two on Γ and for a fixed Γ, the function $\wp(\cdot,\Gamma)$ is Γ-periodic, that is $\wp(\cdot,\Gamma)$ descends to a meromorphic function on the elliptic curve $E(\Gamma) = \mathbb{C}/\Gamma$ with a unique pole of order two at zero.

Recall also that $\wp$ is homogeneous of degree -2, that is, for any $\mu \in \mathbb{C}^*$

$$\wp(\mu z,\mu\Gamma) = \mu^{-2}\wp(z,\Gamma)\,. \tag{11}$$

Set $\Gamma = \mathbb{Z}\oplus\mathbb{Z}\xi_3$ in what follows. Because $\xi_3\Gamma = \Gamma$, multiplication by ξ_3 induces an automorphism of $E = E(\Gamma)$, of order 3 with two fixed points besides the origin:

$$p_+ = \frac{2+\xi_3}{3} + \Gamma \quad \text{ and } \quad p_- = \frac{1+2\,\xi_3}{3} + \Gamma.$$

The relation (11) implies that

$$\wp\big(p_\pm,\Gamma\big) = \tau^{-2}\wp\big(p_\pm,\Gamma\big)\,.$$

It follows that p_+ and p_- are two zeroes of $\wp(\cdot,\Gamma)$. Since $\wp(\cdot,\Gamma)$ has a unique pole of order two there are no other zeroes. The points $0, p_+, p_-$ form a subgroup T of the 3-torsion group $E(3)$ of E.

The 1-form $\omega = dx/\wp(x) + dy/\wp(y)$ is a logarithmic 1-form with polar set at $E_\pm = \{p_\pm\}\times E$ and $E^\pm = E\times\{p_\pm\}$. The residues of ω along E_- and E^- are equal and so are those along E_+ and E^+. Moreover the residue of ω along E_- is the opposite of its residue along E_+.

The singular set of the foliation $\mathcal{F} = [\omega]$ consists of five points: $p_{00} = (0,0)$ and

$$p_{--} = (p_-, p_-)\,, \quad p_{-+} = (p_-, p_+)\,, \quad p_{+-} = (p_+, p_-)\,, \quad p_{++} = (p_+, p_+)\,.$$

The inspection of the first non-zero jet of the closed 1-form ω at the singularity p_{00} reveals that $\mathcal{F}$ admits a local first integral analytically equivalent to $x^3 + y^3$ at this point. The two singularities p_{-+}, p_{+-} are radial ones with local meromorphic first integrals analytically equivalent to x/y. Finally the last two singularities p_{--} and p_{++} have local holomorphic first integrals analytically equivalent to xy.

If $\alpha\Gamma = \Gamma$ then (11) implies that

$$\varphi_{1,\alpha}^* \omega = \frac{\alpha\, dx}{\wp(\alpha x, \Gamma)} + \frac{dx}{\wp(x,\Gamma)} = \frac{\alpha^3 + 1}{\wp(x,\Gamma)}\, dx\,.$$

A simple consequence is that the three separatrices of $\mathcal{F}$ through p_{00} are the elliptic curves $E_{(1,-1)}, E_{(1,-\xi_3)}$ and $E_{(1,-\xi_3^2)}$.

Let $\pi : \widetilde{T_{\xi_3}} \to T_{\xi_3}$ be the blow-up of T_{ξ_3} at the radial singularities of $\mathcal{F}$ and denotes by $\widetilde{\mathcal{F}}$ the transformed foliation. If

$$D_1 = E_+ + E^+\,, \quad D_2 = E_- + E^-\,, \qquad D_3 = E_{(1,-1)} + E_{(1,-\xi_3)} + E_{(1,-\xi_3^2)}$$

and $\widetilde{D_1}, \widetilde{D_2}, \widetilde{D_3}$ designate their respective strict transforms then

$$\widetilde{D_1}^2 = \widetilde{D_2}^2 = \widetilde{D_3}^2 = 0\,.$$

The polar set of $\pi^*\omega$ has two connected components, one supported on $|\widetilde{D_1}|$ and the other on $|\widetilde{D_2}|$. The divisor $\widetilde{D_3}$ has connected support and is disjoint from the polar set of $\pi^*\omega$. It follows from the Hodge index Theorem that this divisor is numerically equivalent to multiples of $\widetilde{D_1}$ and $\widetilde{D_2}$. Of course this can be verified by a direct computation. Indeed,

$$3\,(E_- + E^-) \equiv 3\,(E_+ + E^+) \equiv E_{(1,-1)} + E_{(1,-\xi_3)} + E_{(1,-\xi_3^2)}$$

where $\equiv$ denotes numerical equivalence.

We can apply [46, Theorem 2.1] (see also [38, Theorem 2]) to conclude that the divisors $\widetilde{D_1}, \widetilde{D_2}$ and $\widetilde{D_3}$ are fibers of a fibration on $\widetilde{T}$. Consequently $\mathcal{F}$ admits a first integral $F : T_{\xi_3} \dashrightarrow \mathbb{P}^1$ satisfying $F^{-1}(0) = 3D_1$, $F^{-1}(\infty) = 3D_2$ and $F^{-1}(1) = D_3$. As before, F has generic fiber irreducible because $\widetilde{D_3}$ is connected and reduced. $\square$

To conclude we proceed as in Section 4.2. The proof of Propostion 4.2 shows that the linear equivalence class of $D_1 - D_2$ is a non-trivial 3-torsion point of $\mathrm{Pic}_0(T_{\xi_3})$. Therefore there exists a complex torus X, an étale covering $\rho : X \to T_{\xi_3}$ and a rational function $G : X \dashrightarrow \mathbb{P}^1$ with generic irreducible fiber fitting into the commutative diagram

(12)
$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, dashed, "G"'] & T_{\xi_3} \arrow[d, dashed, "F"] \\ \mathbb{P}^1 \arrow[r, "z\mapsto z^3"] & \mathbb{P}^1\,. \end{tikzcd}$$

Notice that G has 5 linear fibers (three of them over $F^{-1}(1)$) and, as in Section 4.2, Theorem 4.1 implies that $\mathcal{E}_6$ has maximal rank and therefore is exceptional.

4.5. **Explicit abelian relations for elliptic exceptional CDQL webs.** The results of the four preceding subsections give geometrical descriptions of the abelian relations of the webs $\mathcal{E}_5^\tau$ (for $\tau \in \mathbb{H}$), $\mathcal{E}_5$, $\mathcal{E}_6$ and $\mathcal{E}_\tau$. Closed explicit forms for the abelian relations of these elliptic exceptional CDQL webs can be deduced from theirs proofs.

4.5.1. *Explicit abelian relations for $\mathcal{E}_5^\tau$.* We recall the description of $\mathcal{A}(\mathcal{E}_5^\tau)$ that have been obtained in [42]. We fix $\tau \in \mathbb{H}$ and set $G = F_\tau^{1/2}$ (see (7)), $g_1 = x$, $g_2 = y$, $g_3 = x + y$ and $g_4 = x - y$. Then the following multiplicative abelian relations hold:

$$
\begin{aligned}
G &= \frac{\vartheta_1(g_1,\tau)\,\vartheta_1(g_2,\tau)}{\vartheta_4(g_1,\tau)\,\vartheta_4(g_2,\tau)}\\
1-G &= \frac{\vartheta_3\big(\frac{g_3}{2},\frac{\tau}{2}\big)\,\vartheta_4\big(\frac{g_4}{2},\frac{\tau}{2}\big)}{\vartheta_4(g_1,\tau)\,\vartheta_4(g_2,\tau)}\\
1+G &= \frac{\vartheta_4\big(\frac{g_3}{2},\frac{\tau}{2}\big)\,\vartheta_3\big(\frac{g_4}{2},\frac{\tau}{2}\big)}{\vartheta_4(g_1,\tau)\,\vartheta_4(g_2,\tau)}\ .
\end{aligned}
\tag{13}
$$

4.5.2. *Explicit abelian relations for $\mathcal{E}_7$.* We fix $\tau = 1 + i$ in this section and we note $H = F_\tau^{1/2} = F_{1+i}^{1/2}$. Let $g_1, \ldots, g_4$ designate the same functions than above and set $g_5 = i\,x + y$, $g_6 = x + i\,y$. The relations (13) of the subweb $\mathcal{E}_5^\tau$ are of course three abelian relations for $\mathcal{E}_7$. To obtain the last two, we just substitute ix to x in (13) and use the transformation formulas for thetas functions admiting complex multiplication (see Section §8 of [14, Chap.V] for instance) to get:

$$
\begin{aligned}
1-i\,H &= \frac{\vartheta_3\big(\frac{g_5}{2},\frac{\tau}{2}\big)\,\vartheta_4\big(i\,\frac{g_6}{2},\frac{\tau}{2}\big)}{\vartheta_4(i\,g_1,\tau)\,\vartheta_4(g_2,\tau)}\\
1+i\,H &= \frac{\vartheta_4\big(\frac{g_5}{2},\frac{\tau}{2}\big)\,\vartheta_3\big(i\,\frac{g_4}{2},\frac{\tau}{2}\big)}{\vartheta_4(i\,g_1,\tau)\,\vartheta_4(g_2,\tau)}\ .
\end{aligned}
$$

4.5.3. *Explicit abelian relations for $\mathcal{E}_5$.* To simplify the formulae, we shall abreviate ξ_3 by ξ, will write $\vartheta_i(z) = \vartheta_i(z,\xi)$ $(i = 1,\ldots,4)$ and will set $q = e^{i\pi\xi_3}$ in this subsection. We will also use the notations introduced in the proof of Proposition 4.1.

Let F be the rational function (9), that is $F = f_1/f_2$ with

$$
\begin{aligned}
&f_1(x,y) = \vartheta_1(x)\,\vartheta_1(y)\,\vartheta_1(x-y)\,\vartheta_1(x+\xi^2 y)\\
\text{and}\qquad &f_2(x,y) = \vartheta_2(x)\,\vartheta_3(y)\,\vartheta_4(x-y)\,\vartheta_3(x+\xi^2 y)\ .
\end{aligned}
$$

Since $f_1\big(x+\frac{\xi}{2}\,,\,y+\frac{1}{2}\big) = i\,q^{-1/2}e^{i\pi(y-2x)}\vartheta_4(x)\vartheta_2(y)\vartheta_3(x-y)\vartheta_2(x+\xi^2y)$ (see [14, p. 63-64]), the linear divisor $D_3 = L_{p_3}(D_1)$ on T_ξ is cut out by

$$
f_3(x,y) = \vartheta_4(x)\vartheta_2(y)\vartheta_3(x-y)\vartheta_2(x+\xi^2y)\ .
$$

One verifies that $f_3 \equiv a_3\,f_1 + b_3\,f_2$ where $a_3 = i\frac{\vartheta_2(0)\vartheta_4(0)}{\vartheta_3(0)}$ and $b_3 = \frac{\vartheta_2(0)}{\vartheta_3(0)}$. Consequently, D_3 is the linear fiber $F^{-1}(c_3)$ where $c_3 = -b_3/a_3 = i/\vartheta_4(0)$. According to (the proof of) Theorem 4.1, there is an associated logarithmic abelian relation. Explicitly, it is (in multiplicative form)

$$
a_3\,F + b_3 = \frac{\vartheta_4(x)\vartheta_2(y)\vartheta_3(x-y)\vartheta_2(x+\xi^2y)}{\vartheta_2(x)\,\vartheta_3(y)\,\vartheta_4(x-y)\,\vartheta_3(x+\xi^2\,y)}\ .
$$

In the same way, one proves that the linear divisor $D_4 = L_{p_4}(D_1)$ is cut out by

$$f_4(x,y) = \vartheta_3(x)\vartheta_4(y)\vartheta_2(x-y)\vartheta_4(x+\xi^2 y)\ .$$

One verifies that $f_4 \equiv a_4\, f_1 + b_4\, f_2$ where $a_4 = i\frac{\vartheta_2(0)}{\vartheta_3(0)}$ and $b_4 = \frac{\vartheta_4(0)}{\vartheta_3(0)}$. So $D_4 = F^{-1}(c_4)$ where $c_4 = i\vartheta_4(0)/\vartheta_2(0)$. The associated logarithmic abelian relation is

$$a_4\, F + b_4 = \frac{\vartheta_3(x)\vartheta_4(y)\vartheta_2(x-y)\vartheta_4(x+\xi^2 y)}{\vartheta_2(x)\,\vartheta_3(y)\,\vartheta_4(x-y)\,\vartheta_3(x+\xi^2\, y)}\ .$$

4.5.4. *Explicit abelian relations for $\mathcal{E}_6$.* We shall also abbreviate ξ_3 by ξ in this subsection and use the notations introduced in the proof of Proposition 4.2. Let $\wp(z)$ be the Weierstrass $\wp$-function (10) associated to the lattice $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\,\xi$. It satisfies the differential equation

$$\wp'(z)^2 = 4\,\wp(z)^3 - \left(\frac{\Gamma(1/3)^3}{2\,\pi}\right)^6 \tag{14}$$

and $(\wp) = (p_+) + (p_-) - 2(0)$ as divisors on the elliptic curve $E = \mathbb{C}/\Gamma$.

We want to make explicit the abelian relations of

$$\mathcal{E}_6 = \big[dx\,dy\,(dx+dy)\,(dx+\xi\,dy)\,(dx+\xi^2\,dy)\big] \boxtimes \big[dx/\wp(x) + dy/\wp(y)\big]$$

defined on E^2. Let f be the elliptic function defined by

$$f(x) = \frac{\wp'(x) - \wp'(p_+)}{\wp'(x) - \wp'(p_-)}\ .$$

Using (14), one verifies by a straight-forward computation that $F = f(x)f(y)$ is a first integral for the foliation $[dx/\wp(x) + dy/\wp(y)]$. We claim that this rational function corresponds exactly to the first integral deduced in the proof of Proposition 4.2 (also denoted by F there). One verifies that $(f) = 3\,(p_+) - 3\,(p_-)$.

Recall (from [14, Chap. IV] for instance) the definition of the Weierstrass sigma function associated to a lattice $\Lambda \subset \mathbb{C}$:

$$\sigma(z,\Lambda) = z \prod_{\lambda\in\Lambda\setminus\{0\}} \Big(1 - \frac{z}{\lambda}\Big)\, e^{\frac{z}{\lambda} + \frac{z^2}{2\,\lambda^2}}\ .$$

Lemma 4.1. *Let σ_1 be the Weierstrass sigma function associated to the lattice*

$$\Gamma_1 = (2+\xi)\,\Gamma = (2+\xi)\mathbb{Z} \oplus (1+2\,\xi)\mathbb{Z}\,.$$

If $E_1 = \mathbb{C}/\Gamma_1$ and

$$g(x) = -\,\frac{\sigma_1\big(x-p_+\big)\sigma_1\big(x-\xi\,p_+\big)\sigma_1\big(x-\xi^2\,p_+\big)}{\sigma_1\big(x-p_-\big)\sigma_1\big(x-\xi\,p_-\big)\sigma_1\big(x-\xi^2\,p_-\big)}\,.$$

then the product $G = g(x)g(y)$ is a function that makes commutative the diagram (12). More precisely, let $X = E_1^2$ and set $\rho = (\mu,\mu) : X \to E^2$ where $\mu : E_1 \to E$ denotes the isogeny of degree three induced by the natural inclusion $\Gamma_1 \subset \Gamma$. Then

1. *the functions g and G are rational functions on E_1 and X respectively;*
2. *they satisfy $g^3 = f \circ \mu$ and $G^3 = F \circ \rho$ on X.*

Proof. Item *(1)* follows at once from formulae (4). To establish item *(2)* one proceeds as usual by comparing the zeroes and the poles of g^3 and $f \circ \mu$ on E_1. $\square$

Using the function G one can give closed explicit formulae for the non-elementary abelian relations of

$$\big[dx\,dy\,(dx+dy)\,(dx+\xi\,dy)\,(dx+\xi^2\,dy)\big]\boxtimes\big[dG\big].$$

The simplest is certainly (in multiplicative form)

$$G = g(x)\,g(y)\,. \tag{15}$$

If we set $g_3 = x+y$, $g_4 = x+\xi\,y$ and $g_5 = x+\xi^2\,y$ then the other three are

$$1-G = \epsilon_0\,\frac{\sigma_1(g_3)\sigma_1(g_4)\sigma_1(g_5)}{\prod_{\ell=0}^2\sigma_1(x-\xi^\ell p_-)\sigma_1(y-\xi^\ell p_-)} \tag{16}$$

$$1-\xi\,G = \epsilon_1\,\frac{\sigma_1(g_3+\xi^2)\sigma_1(g_4+\xi)\sigma_1(g_5+1)}{\prod_{\ell=0}^2\sigma_1(x-\xi^\ell p_-)\sigma_1(y-\xi^\ell p_-)} \tag{17}$$

$$1-\xi^2\,G = \epsilon_2\,\frac{\sigma_1(g_3-\xi^2)\sigma_1(g_4-\xi)\sigma_1(g_5-1)}{\prod_{\ell=0}^2\sigma_1(x-\xi^\ell p_-)\sigma_1(y-\xi^\ell p_-)}\,. \tag{18}$$

where ϵ_0, ϵ_1 and ϵ_2 are complex constants. Notice that (17) and (18) can be obtained from (16) by using the relations $g(x+1)=g(x+\xi)=\xi\,g(x)$.

Remark 4.1. Since $1-F=(1-G)(1-\xi\,G)(1-\xi^2\,G)$, multiplying (16), (17) and (18) one get a multiplicative abelian relation of $\mathcal{E}_6$ involving $1-F$. After several simplifications (left to the reader), we find the relation

$$1-F = -\wp'(p_+)\sigma(p_-)^6\,\frac{\sigma(g_3)\sigma(g_4)\sigma(g_5)}{\prod_{\ell=0}^2\sigma(x-\xi^\ell p_-)\sigma(y-\xi^\ell p_-)} \tag{19}$$

where σ designates the Weierstrass sigma function associated to the lattice Λ.

Since $\wp'(x)-\wp'(p_\pm)=\frac{2}{\sigma(p_\pm)^3}\,\frac{\sigma(x-p_\pm)\sigma(x-\xi\,p_\pm)\sigma(x-\xi^2\,p_\pm)}{\sigma(x)^3}$ on E, one have also

$$1-F = \frac{\wp'(p_+)\sigma(p_-)^6}{2}\,\frac{\sigma(x)^3\sigma(y)^3\,\big(\wp'(x)+\wp'(y)\big)}{\prod_{\ell=0}^2\sigma(x-\xi^\ell p_-)\sigma(y-\xi^\ell p_-)}. \tag{20}$$

Comparing (19) and (20) yields the relation

$$-\frac{1}{2}\Big(\wp'(x)+\wp'(y)\Big) = \frac{\sigma(x+y)\sigma(x+\xi\,y)\sigma(x+\xi^2\,y)}{\sigma(x)^3\sigma(y)^3}\,.$$

This is the recently discovered addition formula (6.6) of [18].

5. The barycenter transform

5.1. The $[v]$-barycenter of a configuration. Let V be a two-dimensional vector space over $\mathbb{C}$ equipped with a non-zero alternating two-form $\sigma\in\bigwedge^2 V^*$. For a fixed $k\geq 1$ and $v\in V$ distinct from 0, consider the map

$$\begin{array}{rccl} \alpha_v & : & V^k & \longrightarrow & V \\ & & (v_1,\ldots,v_k) & \longmapsto & \sum_{i=1}^k \big(\prod_{j\neq i}\sigma(v,v_j)\big)\,v_i\,. \end{array} \tag{21}$$

These maps have the following properties:

1. $\alpha'_v=\lambda^{k-1}\alpha_v$ if $\sigma'=\lambda\,\sigma$ with $\lambda\in\mathbb{C}^*$;
2. $\alpha_{\lambda v}=\lambda^{k-1}\alpha_v$ for every $\lambda\in\mathbb{C}^*$;
3. α_v is symmetric;
4. $\alpha_v(v_1,\ldots,v_k)=0$ if and only if there exist i and j distinct such that v_i, v_j and v are multiples of each other or if one of the v_i's is zero.

The projectivization of α_v is a rational map $\beta_{[v]} : \mathbb{P}(V)^k \dashrightarrow \mathbb{P}(V)$ that admits a nice geometric interpretation: if $[v_i] \neq [v]$ for every $i \in \{1, \ldots, k\}$ then $\beta_{[v]}([v_1], \ldots, [v_k])$ is nothing but the barycenter of $[v_1], \ldots, [v_k]$ seen as points of the affine line $\mathbb{C} \cong \mathbb{P}(V) \setminus \{[v]\}$. Unlike α_v, $\beta_{[v]}$ does not depend on the choice of σ. The point $\beta_{[v]}([v_1], \ldots, [v_k])$ will be referred as the $[v]$-*barycenter of* $[v_1], \ldots, [v_k]$.

The naturalness of β_v is testified by its $\mathrm{PSL}(V)$-equivariance, that is, for every $g \in \mathrm{PSL}(V)$, $\beta_{gv}(gv_1, \ldots, gv_k) = g\beta_v(v_1, \ldots, v_k)$.

5.2. **Symmetric versions.** Since $\beta_{[v]}$ is a symmetric function it factors through the natural map $\mathbb{P}(V)^k \to \mathbb{P}(\mathrm{Sym}^k V)$. Still denoting by $\beta_{[v]}$ the resulting rational map from $\mathbb{P}(\mathrm{Sym}^k V)$ to $\mathbb{P}(V)$, it has been observed in [21] (see also [20]) that $\beta_{[v]}$ admits the affine expression

$$\beta_x\big(p(t)\big) = x - k\frac{p(x)}{p'(x)} \tag{22}$$

where $x \in \mathbb{C}$ and the roots of the degree k polynomial $p \in \mathbb{C}[t]$ correspond to k points in an affine chart $\mathbb{C} \subset \mathbb{P}(V)$.

There are also symmetrized versions of the above maps. Namely we can define

$$\begin{array}{rcccl} \alpha & : & \mathrm{Sym}^k V & \longrightarrow & \mathrm{Sym}^k V \\ & & v_1 \cdot v_2 \cdots v_k & \longmapsto & \displaystyle\prod_{i=1}^{k} \alpha_{v_i}(v_1, \ldots, \widehat{v_i}, \ldots, v_k)\,. \end{array}$$

Its projectivization

$$\beta : \mathbb{P}(\mathrm{Sym}^k V) \dashrightarrow \mathbb{P}(\mathrm{Sym}^k V)$$

is a $\mathrm{PSL}(V)$-equivariant rational map.

An affine expression for β is also presented in [21]. If all the k points belong to the same affine chart $\mathbb{C} \subset \mathbb{P}(V)$ then

$$\beta\left(p(t)\right) = \mathrm{Resultant}_z\big(p(z), (t-z)p''(z) + 2(k-1)p'(z)\big) \tag{23}$$

where $p \in \mathbb{C}[t]$ is a degree k polynomial whose roots correspond to k points in $\mathbb{C} \subset \mathbb{P}(V)$.

Remark 5.1. For $k = 2$, the rational map $\beta : \mathbb{P}(\mathrm{Sym}^k V) \dashrightarrow \mathbb{P}(\mathrm{Sym}^k V)$ is nothing more than the identity map. For $k = 3$ it is still rather simple: it is a birational involution of $\mathbb{P}^3$ with indeterminacy locus equal to a cubic rational normal curve. For $k = 4$ it is already more interesting from the dynamical point of view. Recall that for four unordered points of $\mathbb{P}^1$ there is a unique invariant, the so called j-invariant. It can be interpreted as a rational map $j : \mathbb{P}(\mathrm{Sym}^4 V) \dashrightarrow \mathbb{P}^1$ whose generic fiber contains an orbit of the natural $\mathrm{PSL}(V)$-action of $\mathbb{P}(\mathrm{Sym}^4 V)$ as an open and dense subset. Therefore there exists a rational map $\beta_* : \mathbb{P}^1 \to \mathbb{P}^1$ that fits in the commutative diagram below:

$$\begin{tikzcd} \mathbb{P}(\mathrm{Sym}^4 V) \arrow[r, dashed, "\beta"] \arrow[d, dashed, "j"'] & \mathbb{P}(\mathrm{Sym}^4 V) \arrow[d, dashed, "j"] \\ \mathbb{P}^1 \arrow[r, "\beta_*"] & \mathbb{P}^1\,. \end{tikzcd}$$

We learned from David Marín that there exists a choice of coordinates in $\mathbb{P}^1$ where

$$\beta_*(z) = \frac{z^2(z+540)^3}{(5z-216)^4}\,.$$

It can be immediately verified that β_* is a post-critically finite map. We do not know if a similar property holds for the map β when $k \geq 5$. For a more comprehensive discussion about the dynamic of β see [21].

5.3. **Multiplicities of the first iterate.** Notice that $\mathbb{P}(\mathrm{Sym}^k V)$ can be naturally identified with the set of degree k effective divisors on $\mathbb{P}(V)$. In particular, it makes sense to talk about the support of an element in $\mathbb{P}(\mathrm{Sym}^k V)$.

Lemma 5.1. *If $q_1, \ldots, q_k \in \mathbb{P}(V)$ are pairwise distinct points then every point in the support of $\beta(q_1 \cdots q_k)$ appears with multiplicity at most $k-2$.*

Proof. Let q be in S, where S stands for the support of $\beta(q_1, \ldots, q_k)$. Choose an affine coordinate system in $\mathbb{P}(V)$ where $q = 0$ and $q_i = t_i$ for $i = 1, \ldots, k$. If we set $p(t) = \prod(t - t_i)$ and $p_i(t) = p(t)/(t - t_i)$ then

$$p_i(t_i) = p'(t_i) \quad \text{and} \quad p_i'(t_i) = \frac{1}{2} p''(t_i)\,.$$

Thus, according to (22), the points in the support of S are of the form

$$t_i + 2\,(1-k)\,\frac{p'(t_i)}{p''(t_i)}\,.$$

If q appears in $\beta(q_1 \cdots q_k)$ with multiplicity at least $k-1$ then

$$t_i + 2\,(1-k)\,\frac{p'(t_i)}{p''(t_i)} = 0 \implies t_i\,p''(t_i) = 2\,(k-1)\,p'(t_i)$$

is verified for at least $(k-1)$ distinct values of i. Since both $p'(t)$ and $tp''(t)$ are polynomials of degree $k-1$ they must differ by a nonzero constant. Thus there exist $\lambda \in \mathbb{C}^*$ such that $y(t) = p'(t)$ satisfies the differential equation

$$y'(t) = \frac{\lambda}{t}\,y(t)\,.$$

Hence $p'(t) = C \cdot \exp\left(\frac{\lambda}{2} t^2\right)$ is not a polynomial. This contradiction proves the lemma. $\square$

6. The $\mathcal{F}$-barycenter of a web

If V is a two-dimensional vector space over an arbitrary field F of characteristic zero then it is still possible to define the $[v]$-barycenter of an element $\mathbb{P}(\mathrm{Sym}^k V)$. This can be inferred directly from equation (23) in Section 5.1.

More explicitly, one can specialize (22) to the $\mathcal{F}$-barycenter of a k-web $\mathcal{W}$ when there are at our disposal global rational coordinates x, y on S. Assume that $\mathcal{F} = [dx + a\,dy]$ with $a \in \mathbb{C}(S)$. If $\mathcal{W}$ is defined by an implicit differential equation $F(x, y, dy/dx) = 0$ where $F(x, y, p)$ is a polynomial of degree k in p with coefficients in $\mathbb{C}(S)$, then

$$\beta_{\mathcal{F}}(\mathcal{W}) = \left[dx + \left(a - k\,\frac{F(a)}{\frac{\partial F}{\partial p}(a)} \right) dy \right].$$

Note also that the $\mathrm{PSL}(V)$-equivariance of the barycenter transform yields

$$\beta_{\varphi^*\mathcal{F}}(\varphi^*\mathcal{W}) = \varphi^*\big(\beta_{\mathcal{F}}(\mathcal{W})\big)$$

for any $\varphi \in \mathrm{Diff}(S)$. Therefore the $\mathcal{F}$-barycenter of $\mathcal{W}$ is a foliation that is geometrically attached to the pair $(\mathcal{F}, \mathcal{W})$ and, as such, can be defined on an arbitrary surface by patching together over local coordinate charts the construction presented above.

Remark 6.1. In [34], Nakai defines the *dual* 3*-line configuration* of a configuration $L = L_1 \cup L_2 \cup L_3$ of three concurrent lines in the plane: it is "the unique invariant 3-line configuration distinct from L invariant by the group generated by three involutions respecting the line L_i and L." The *dual* 3*-web* $\mathcal{W}^*$ of a 3-web $\mathcal{W}$ is then defined as the one obtained by integrating the dual 3-line configuration of the tangent 3-line fields of $\mathcal{W}$.

It turns out that $\mathcal{W}^*$ is nothing more than the barycenter transform of $\mathcal{W}$) in our terminology. Since β is an involution in the case of 3-points, it follows that $(\mathcal{W}^*)^* = \mathcal{W}$, a fact already noted by Nakai. Moreover he also observed that $K(\mathcal{W}) = K(\beta(\mathcal{W}))$, see [34, Theorem 4.1]. In particular, a 3-web $\mathcal{W}$ is flat if and only if $\beta(\mathcal{W})$ is flat [34, Corollary 4.1].

We have verified, with the help of a computer algebra system, that the identity $K(\mathcal{W}) = K(\beta(\mathcal{W}))$ also holds for 4-webs as soon as the four defining foliations of $\beta(\mathcal{W})$ are distinct. For 5-webs the situation is different: the barycenter transforms of most algebraic 5-webs do not have zero curvature. These blind constatations are crying for geometric interpretations.

6.1. **Barycenters of completely decomposable linear webs.** Let $p_0, \ldots, p_k$ be $(k+1)$ pairwise distinct points in $\mathbb{P}^2$. For any $i = 0, \ldots, k$, let $\mathcal{L}_i$ denotes the foliation of $\mathbb{P}^2$ tangent to the pencil of lines through p_i. In what follows, we give a description of the foliation $\beta_{\mathcal{F}}(\mathcal{W})$ when $\mathcal{F} = \mathcal{L}_0$ and $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$.

In the simplest case the points $p_0, \ldots, p_k$ are aligned. If one chooses an affine coordinate system where all the p_i's belong to the line at infinity then the foliations $\mathcal{L}_i$ are induced by constant 1-forms and so is the $\mathcal{F}$-barycenter of $\mathcal{W}$. The corresponding foliation $\beta_{\mathcal{F}}(\mathcal{W})$ is tangent to the pencil of lines through the point $\beta_{p_0}(p_1, \ldots, p_k)$ on the line at infinity.

If we think of $\mathbb{C}^2$ as the universal covering of a two-dimensional complex torus T then, if $p_0, \ldots, p_k$ are at the line at infinity, the foliations $\mathcal{L}_i$ are pull-backs of linear foliations on T under the covering map. In this geometric picture the line at infinity is identified with $\mathbb{P}H^0(T, \Omega^1_T)$ and the the linear foliations $\mathcal{L}_i$ are defined by points p_i in $\mathbb{P}H^0(T, \Omega^1_T)$. The $\mathcal{F}$-barycenter of $\mathcal{W}$ is the linear foliation on T determined by $\beta_{p_0}(p_1, \ldots, p_k) \in \mathbb{P}H^0(T, \Omega^1_T)$.

In the next simplest case $p_1, \ldots, p_k$ are on the same line while p_0 is not. In an affine coordinate system where $p_1, \ldots, p_k$ are on the line at infinity and p_0 is at the origin, the $\mathcal{F}$-barycenter will be induced by the 1-form $\sum_{i=1}^k d\log L_i$, where L_i is a linear polynomial vanishing on the line $\overline{p_0 p_i}$. In particular, the product

$$\prod_{i=1}^{k} L_i \tag{24}$$

is a first integral of the foliation $\beta_{\mathcal{F}}(\mathcal{W})$.

In order to describe the $\mathcal{F}$-barycenter of $\mathcal{W}$ without further restrictions on the points $p_0, \ldots, p_k$, let $\Pi : (S, E) \to (\mathbb{P}^2, p_0)$ be the blow-up of p_0; $\pi : S \to \mathbb{P}^1$ be the fibration on S induced by the lines through p_0; $\mathcal{G}$ be the foliation $\Pi^* \beta_{\mathcal{F}}(\mathcal{W})$; and ℓ_i be the strict transform of the line $\overline{p_0 p_i}$ under Π for $i = 1, \ldots, k$.

Lemma 6.1. *If the points $\{p_0, \ldots, p_k\}$ are not aligned then the foliation $\mathcal{G}$ is a Riccati foliation with respect to π, that is, $\mathcal{G}$ has no tangencies with the generic fiber of π. Moreover $\mathcal{G}$ has the following properties:*

(1) the exceptional divisor E of Π is $\mathcal{G}$-invariant;
(2) the only fibers of π that are $\mathcal{G}$-invariant are the lines ℓ_i, for $i = 1, \ldots, k$;
(3) the singular set of $\mathcal{G}$ is contained in the lines ℓ_i, for $i = 1, \ldots, k$;
(4) over each line ℓ_i there are two singularities of $\mathcal{G}$. One is a complex saddle at the intersection of ℓ_i with E, the other is a complex node at the p_0-barycenter of $\{p_1, \ldots, p_k\} \cap \ell_i$. Moreover, if r_i is the cardinality of $\{p_1, \ldots, p_k\} \cap \ell_i$ then the quotient of eigenvalues of the saddle (resp. node) over ℓ_i is $-r_i/k$ (resp. r_i/k);
(5) the monodromy of $\mathcal{G}$ around ℓ_i is finite of order $k/\gcd(k, r_i)$;
(6) the only separatrices of $\beta_{\mathcal{F}}(\mathcal{W})$ through p_0 are the lines $\overline{p_0 p_i}$, $i = 1, \ldots, k$.

Proof. Let $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ be an affine coordinate system where $\mathcal{F} = \mathcal{L}_0 = [dx]$ (that is $p_0 = [0 : 1 : 0]$) and $\mathcal{L}_i = \big[(x - x_i)\, dy - (y - y_i)\, dx\big]$ (that is $p_i = [x_i : y_i : 1]$) for $i = 1, \ldots, k$. It is convenient to assume also that $y_i \neq 0$ for $i = 1, \ldots, k$.

By definition $\beta_{\mathcal{F}}(\mathcal{W})$ is

$$\beta_{\mathcal{F}}(\mathcal{W}) = \left[k\, dy - \Big(\sum_{i=1}^{k} \frac{y - y_i}{x - x_i} \Big) dx \right]. \tag{25}$$

Since $\Pi : (S, E) \to (\mathbb{P}^2, p_0)$ is the blow-up of a point at infinity, the coordinates (x, y) still define affine coordinates on an affine chart of S. Notice that the fibration $\pi : S \to \mathbb{P}^1$ is nothing more than $\pi(x, y) = x$ in these new coordinates.

If we set $z = 1/y$ then $(x, z) \in \mathbb{C}^2$ is another affine chart of S. The intersection of the exceptional divisor $E = \pi^{-1}(p_0)$ with this chart is equal to $\{z = 0\}$. Notice that in the new coordinates (x, z) we have

$$\mathcal{G} = \left[k\, dz + z \Big(\sum_{i=1}^{k} \frac{1 - z y_i}{x - x_i} \Big)\, dx \right]. \tag{26}$$

It is clear from the equations (25) and (26) that: $\mathcal{G}$ has no tangencies with the generic fiber of π, that is, $\mathcal{G}$ is a Ricatti foliation; *(1)* the exceptional divisor E is $\mathcal{G}$-invariant; *(2)* the only $\mathcal{G}$-invariant fibers of π are the lines ℓ_i; and *(3)* the singularities of $\mathcal{G}$ are contained in the lines ℓ_i.

To prove items *(4)* and *(5)* suppose, without loss of generality, that $\ell_1 \cap \{p_1, \ldots, p_k\} = \{p_1, \ldots, p_{r_1}\}$ and that $x_1 = 0$. In particular $x_i \neq 0$ for $i > r_1$. Therefore, in the open set $U = \{(x, z) \in \mathbb{C}^2 \mid |x| \ll 1\}$ we can write

$$\mathcal{G} = \left[kxu(x)\, dz + \left(z \left(r - \sum_{i=1}^{r_1} z y_i \right) + zxv(x, z) \right) dx \right],$$

where u is an unity in $\mathcal{O}_U$ that does not depends on z and $v \in \mathcal{O}_U$ is a regular function. It follows that the singularities of $\mathcal{G}$ over $\ell_1 = \overline{\{x_1 = 0\}}$ are $(0, 0)$ and

$$\left(0\, ,\, \frac{r}{\sum_{i=1}^{r_1} y_i} \right).$$

Notice that this last point is the p_0-barycenter of $\{p_1, \ldots, p_{r_1}\}$ on ℓ_1.

The local expression for $\mathcal{G}$ over U also shows that $\mathcal{G}$ is induced by a vector field X with linear part at $(0,0)$ equal to

$$kx\frac{\partial}{\partial x} - r_1 z\frac{\partial}{\partial z}\,.$$

Clearly the quotient of eigenvalues in the direction of ℓ_1 is $-r_1/k$. Since the points $\{p_0,\ldots,p_k\}$ are not aligned $r_1 < k$ and, consequently, $-r_1/k \in \mathbb{Q}\setminus\mathbb{Z}$. Since ℓ_1 has zero self-intersection it follows from Camacho-Sad index Theorem that the quotient of eigenvalues (in the direction of the fiber of π) of the other singularity of $\mathcal{G}$ on ℓ_1 is r_1/k. Since this number is not an integer it follows (see [9, page 52]) that this singularity is a complex node. Moreover the monodromy around ℓ_1 is analytically conjugated to $z \mapsto \exp(2\pi\, i\, r_1/k)z$. This concludes the proof of *(4)* and *(5)*.

Finally, to settle *(6)* notice that the singular points of $\mathcal{G}$ contained in E are complex saddles. A classical result by Briot and Bouquet says that these singularities admit exactly two separatrices. In our setup one separatrix corresponds to E and the other corresponds to one of the lines ℓ_i. Thus *(6)* follows and so does the lemma. □

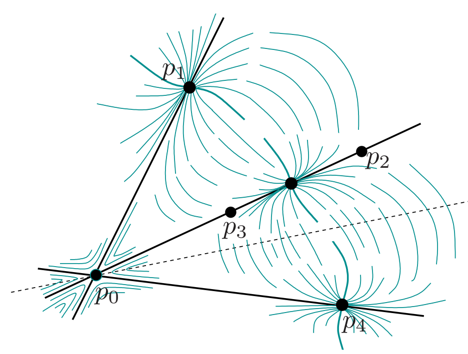

FIGURE 2. The $\mathcal{L}_{p_0}$-barycenter of the linear web $\mathcal{L}_{p_1} \boxtimes \cdots \boxtimes \mathcal{L}_{p_4}$

It is interesting to notice that the generic leaf of $\beta_{\mathcal{F}}(\mathcal{W})$ is transcendental in general. Indeed, the cases when there are more algebraic leaves than the obvious ones (the lines $\overline{p_0p_i}$) are conveniently characterized by the following proposition.

Proposition 6.1. *The foliation $\beta_{\mathcal{F}}(\mathcal{W})$ has an algebraic leaf distinct from the lines $\overline{p_0p_i}$ if and only if all its singularities distinct from p_0 are aligned. Moreover in this case all its leaves are algebraic.*

Proof. Since the Riccati foliation $\mathcal{G}$ leaves the exceptional divisor E invariant, it has affine monodromy. It follows from Lemma 6.1 item *(5)* that its monodromy group is generated by elements of finite order.

Suppose that $\mathcal{G}$ has an algebraic leaf L distinct from E and the lines ℓ_i. The existence of such leaf implies that the monodromy group $G \subset \mathrm{Aut}(\mathbb{P}^1)$ of $\mathcal{G}$ must have a periodic point corresponding to the intersections of L with a generic fiber of π. Since G already has a fixed point (thanks to the $\mathcal{G}$-invariance of E) it follows from Lemma 6.1 item *(5)* that G is conjugated to a finite subgroup of $\mathbb{C}^* \subset \mathrm{Aff}(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{P}^1)$. This is sufficient to show that $\mathcal{G}$ admits a holomorphic first integral defined on the complement of the $\mathcal{G}$-invariant fibers of π. Lemma 6.1 item *(4)* implies that

$\mathcal{G}$ is conjugated to $[r_i y dx - k x dy]$ in a neighborhood of ℓ_i and, consequently, the restriction of $\mathcal{G}$ to this neighborhood has a local meromorphic first integral. Putting all together it follows that $\mathcal{G}$ has a global rational first integral.

Notice that $\mathcal{G}$ admits two distinguished leaves that correspond to the two fixed points of the monodromy. One of these is the exceptional divisor E and the other is an algebraic curve C invariant by $\mathcal{G}$ such that $\pi_{|C} : C \to \mathbb{P}^1$ is a one to one covering.

For every $i = 1, \ldots, k$, the distinguished leaf C must intersect the line ℓ_i at a singularity of $\mathcal{G}$ away from E (by Lemma 6.1 item *(6)*). In a neighborhood of these singularities $\mathcal{G}$ has a meromorphic first integral of the form $y^k x^{-r_i}$ where r_i is the cardinality of $\{p_1, \ldots, p_k\} \cap \ell_i$ and the local coordinates (x, y) are such that $[dx]$ defines the reference fibration. The restriction of the projection $(x, y) \mapsto x$ to any local leaf not contained in $\{x\, y = 0\}$ is a $\frac{k}{\gcd(k, r_i)}$ to 1 covering of $\mathbb{D}^*$. Therefore, in these local coordinates around ℓ_i, the distinguished leaf C must be contained in $\{y = 0\}$. Notice that the Camacho-Sad index of the leaf $\{y = 0\}$ is $\frac{r_i}{k}$. Summing over the lines ℓ_i we obtain from the Camacho-Sad index Theorem that $C^2 = 1$. Since C does not intersects E (Lemma 6.1 item (6)) it follows that $\Pi(C)$ has self-intersection one. Thus $\Pi(C)$ is a line containing all the singularities of $\beta_{\mathcal{F}}(\mathcal{W})$ different from p_0. The proposition follows. □

Corollary 6.1. *If the foliation $\beta_{\mathcal{F}}(\mathcal{W})$ has an irreducible algebraic leaf C distinct from the lines $\overline{p_0 p_i}$ then C is a line or*

$$\deg C = \frac{\sum_{i=1}^m r_i}{\gcd(r_1, \ldots, r_m)} \tag{27}$$

where $\{\ell_1, \ldots, \ell_m\} = \cup_{i=1}^k \overline{p_0 p_i}$ and r_i is the cardinality of $\ell_i \cap \{p_1, \ldots, p_k\}$ for $i = 1, \ldots, m$. In particular the degree of C is bounded from below by m.

Proof. It follows from Proposition 6.1 that the singularities of $\beta_{\mathcal{F}}(\mathcal{W})$ distinct from p_0 are all contained in an invariant line ℓ. We can assume that ℓ is the line at infinity in an affine chart $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$. We can also assume that $p_0 = (0, 0)$ and, as a by product, that the m lines $\overline{p_0 p_i}$ are cut out by homogeneous linear polynomials $L_1, \ldots, L_m$. It can easily verified that the polynomial $P = L_1^{r_1} \cdots L_m^{r_m}$ is a first integral for $\beta_{\mathcal{F}}(\mathcal{W})$. Of course, if $s_i = r_i / \gcd(r_1, ..., r_m)$ then $(L_1^{s_1} \cdots L_m^{s_m})^{\gcd(r_1, \ldots, r_m)} = P$ and therefore $Q = L_1^{s_1} \cdots L_m^{s_m}$ is also a polynomial first integral for $\beta_{\mathcal{F}}(\mathcal{W})$. To conclude one has just to observe that $Q - c$ is irreducible when $c \neq 0$. Indeed the curve $\overline{\{Q = c\}}$ is smooth on $\mathbb{C}^2$ and has exactly one irreducible branch at each of its points of intersection with the line at infinity. □

7. Curvature

To settle the notation we recall the definition of curvature for a completely decomposable $(k+1)$-web $\mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. We start by considering 1-forms ω_i with isolated singularities such that $\mathcal{F}_i = [\omega_i]$. For every triple (r, s, t) with $0 \leq r < s < t \leq k$ we define

$$\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$$

as the unique meromorphic 1-form such that

$$\begin{cases} d(\delta_{st}\,\omega_r) &=& \eta_{rst}\wedge\delta_{st}\,\omega_r\\ d(\delta_{tr}\,\omega_s) &=& \eta_{rst}\wedge\delta_{tr}\,\omega_s\\ d(\delta_{rs}\,\omega_t) &=& \eta_{rst}\wedge\delta_{rs}\,\omega_t\end{cases}$$

where $\delta_{ij}=\sigma(\omega_i,\omega_j)$ and σ is the alternating two-form characterized by

$$\omega_i\wedge\omega_j=\sigma(\omega_i,\omega_j)\,dx\wedge dy.$$

Notice that the 1-forms ω_i are not uniquely defined but any two differ by an invertible function. Therefore, although dependent on the choice of the ω_i's, the 1-forms η_{rst} are well-defined modulo the addition of a closed holomorphic 1-form. The *curvature* of the web $\mathcal{W}=\mathcal{F}_0\boxtimes\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_k$ is thus defined by the formula

$$K(\mathcal{W})=K(\mathcal{F}_0\boxtimes\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_k)=d\,\eta(\mathcal{W})$$

where

$$\eta(\mathcal{W})=\eta(\mathcal{F}_0\boxtimes\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_k)=\sum_{0\le r<s<t\le k}\eta_{rst}\,.$$

Clearly, $K(\mathcal{W})$ is a meromorphic 2-form intrinsically attached to $\mathcal{W}$. More precisely for any invertible holomorphic map φ, one has

$$K(\varphi^*\mathcal{W})=\varphi^*\big(K(\mathcal{W})\big)\,.$$

We will say that a k-web $\mathcal{W}$ is *flat* if its curvature $K(\mathcal{W})$ vanishes identically. This extends to every $k\ge 3$ a classical terminology used for a long time for 3-webs.

7.1. **On the regularity of the curvature.** Our main motivation to introduce the $\mathcal{F}$-barycenter of a web $\mathcal{W}$ steams from an attempt to characterize the absence of poles of $K(\mathcal{W})$ over a generic point of an irreducible component of $\Delta(\mathcal{W})$.

In order to state concisely our result in this direction we introduce the following notation. If $\mathcal{F}$ is one of the defining foliations of a $(k+1)$-web $\mathcal{W}$, then we define the k-web $\mathcal{W}-\mathcal{F}$ by the relation

$$\mathcal{W}=(\mathcal{W}-\mathcal{F})\boxtimes\mathcal{F}\,.$$

We will also profit from the usual definition of the tangency between two foliations: if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two distinct holomorphic foliations then $\mathrm{tang}(\mathcal{F}_1,\mathcal{F}_2)$ is the divisor locally defined by the vanishing of

$$\omega_1\wedge\omega_2=0$$

where ω_i are holomorphic 1-forms with isolated zeros locally defining $\mathcal{F}_i$ for $i=1,2$.

Theorem 7.1. *Let $\mathcal{F}$ be a foliation and $\mathcal{W}=\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\cdots\boxtimes\mathcal{F}_k$ be a completely decomposable k-web, $k\ge 2$, both defined on the same domain $U\subset\mathbb{C}^2$. Suppose that C is an irreducible component of $\mathrm{tang}(\mathcal{F},\mathcal{F}_1)$ that is not contained in $\Delta(\mathcal{W})$. The curvature $K(\mathcal{F}\boxtimes\mathcal{W})$ is holomorphic over a generic point of C if and only if the curve C is invariant by $\mathcal{F}_1$ or by $\beta_{\mathcal{F}_1}(\mathcal{W}-\mathcal{F}_1)$.*

We will need the following lemma.

Lemma 7.1. *If C is an irreducible component of $\mathrm{tang}(\mathcal{F},\mathcal{F}_1)$ that is not contained in $\Delta(\mathcal{W})$ then $\eta(\mathcal{F}\boxtimes\mathcal{W})$ is holomorphic[6] over the generic point of C if and only if C is $\beta_{\mathcal{F}_1}(\mathcal{W}-\mathcal{F}_1)$-invariant.*

[6]Recall that $\eta(\mathcal{F}\boxtimes\mathcal{W})$ is defined up to the addition of a closed holomorphic 1-form. Thus the holomorphy of $\eta(\mathcal{F}\boxtimes\mathcal{W})$ is well-defined.

Proof. From the hypothesis we can choose a local coordinate system over a generic point of C such that

$$\begin{aligned} \mathcal{F} &= \{\omega_0 = dx + b\,dy = 0\}\,, \\ \mathcal{F}_1 &= \{\omega_1 = dx = 0\} \\ \text{and} \quad \mathcal{F}_i &= \{\omega_i = a_i\,dx + dy = 0\} \quad \text{for } \ i = 2, \ldots, k\,. \end{aligned}$$

A straight-forward computation shows that for every i:

$$\eta_{01i} = \frac{\frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y} - b\left(a_i \frac{\partial b}{\partial x} + \frac{\partial a_i}{\partial y}\right)}{b\,(a_i\,b - 1)}\;dx - \frac{a_i\,b\,\frac{\partial b}{\partial y} + \frac{\partial a_i}{\partial y}}{a_i\,b - 1}\;dy\,.$$

Over a generic point of C we have that C coincides with the zero locus of b. Thus C is not contained in the polar set of $\sum_{i=2}^k \eta_{01i}$ if and only if the expression

$$\sum_{i=2}^{k} \frac{\frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y}}{(a_i\,b - 1)}$$

is divisible by b. But

$$b \;\text{ divides }\; \sum_{i=2}^{k} \frac{\frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y}}{(a_i b - 1)} \iff b \;\text{ divides }\; \sum_{i=2}^{k} \Big(\frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y}\Big)\,.$$

The right hand side above is equivalent to

$$b \;\text{ divides }\; \left(\Big(\sum_{i=2}^{k} a_i\Big)\,dx + (k-1)\,dy\right) \wedge db.$$

From the very definition of the barycenter (see equation (21)) it follows that

$$\beta_{\mathcal{F}_1}(\mathcal{W} - \mathcal{F}_1) = \left[\sum_{i=2}^{k} \Big(\prod_{\substack{j=2 \\ j\neq i}}^{k} \delta_{1i}\Big)\,\omega_i\right] = \left[\Big(\sum_{i=2}^{k} a_i\Big)dx + (k-1)\,dy\right].$$

Notice that the 1-form $(\sum_{i=2}^k a_i)dx + (k-1)dy$ has no singularities. Thus $\sum_{i=2}^k \eta_{01i}$ is holomorphic on C if and only if C is invariant by $\beta_{\mathcal{F}_1}(\mathcal{W} - \mathcal{F}_1)$.

Since C is not contained in $\Delta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_r)$ for every set $\{r, s, t\}$ that does not contain $\{0, 1\}$ it follows that η_{rst} is holomorphic on C. The Lemma follows. □

Proof of Theorem 7.1. In the notation of the proof of Lemma 7.1

$$d\omega_0 = \frac{1}{k-1}\left(\sum_{i=2}^{k} (\eta_{01i} - d\log\delta_{1i})\right) \wedge \omega_0\,. \tag{28}$$

The definition of $\eta(\mathcal{W})$ laid down in the beginning of this section implies that

$$\sum_{i=2}^{k} \eta_{01i} = \eta(\mathcal{F} \boxtimes \mathcal{W}) - \eta(\mathcal{W})\,.$$

Because C is not contained in $\Delta(\mathcal{W})$ both $\eta(\mathcal{W})$ and $\sum_{i=2}^k \eta_{01i}$ are holomorphic at the generic point of C.

Suppose first that $K(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic over the generic point of C. If C is $\mathcal{F}$-invariant then there is nothing to prove. Thus assume that C is not $\mathcal{F}$-invariant.

If p is a generic point of C and α is a holomorphic primitive of $d\eta(\mathcal{F} \boxtimes \mathcal{W})$ on a neighborhood of p then

$$\eta(\mathcal{F} \boxtimes \mathcal{W}) - \alpha = \frac{df(b)}{b^n} + dg$$

where f and g are holomorphic functions on a neighborhood of p and n is a positive integer. Therefore (28) implies

$$d\omega_0 = \frac{1}{k-1}\left(\frac{df(b)}{b^k} + \alpha'\right) \wedge \omega_0 \,,$$

for some holomorphic 1-form α'. Since $d\omega_0$ is holomorphic and, by assumption, $\{b = 0\}$ is not $\mathcal{F}$-invariant the only possibility is that $f \equiv 0$. Therefore $\eta(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic along C. It follows from Lemma 7.1 that C is $\beta_{\mathcal{F}_1}(\mathcal{W} - \mathcal{F}_1)$-invariant.

Suppose now that C is left invariant by $\mathcal{F}$ or $\beta_{\mathcal{F}_1}(\mathcal{W} - \mathcal{F}_1)$. In the latter case the result follows from Lemma 7.1. In the former case we can assume, for a fixed $i \in \{2, \ldots, k\}$, that $C = \{x = 0\}$, $\omega_0 = dx + x^n u dy$, $\omega_1 = dx$ and $\omega_i = dy$ where u does not vanish identically on C. A straight-forward computation shows that

$$d\eta_{01i} = \frac{u\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x}}{u^2} \,.$$

Thus the 2-forms $d\eta_{01i}$ are holomorphic for every $i = 2, \ldots, k$. Because

$$K(\mathcal{F} \boxtimes \mathcal{W}) = \sum_{i=2}^{k} d\eta_{01i} + d\eta(\mathcal{W}) \,,$$

and the righthand side is a sum of holomorphic 2-forms, the curvature $K(\mathcal{F} \boxtimes \mathcal{W})$ is also holomorphic and the theorem follows. □

7.2. **Specialization to CDQL webs on complex tori.** Theorem 7.1 completely characterizes in geometric terms the flat CDQL webs on two-dimensional complex tori.

Theorem 7.2. *Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear k-web and $\mathcal{F}$ be a non-linear foliation on a complex torus T. If $k \geq 2$ then $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ if and only if any irreducible component of* $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ *is invariant by $\mathcal{F}$ or by $\beta_{\mathcal{L}_i}(\mathcal{W} - \mathcal{L}_i)$ for every $i = 1, \ldots, k$.*

Proof. Notice that the discriminant of $\mathcal{W}$ is empty. Therefore the hypotheses of Theorem 7.1 are all satisfied.

If every irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ is invariant by $\mathcal{F}$ or $\beta_{\mathcal{L}_i}(\mathcal{W} - \mathcal{L}_i)$ for every $i = 1, \ldots, k$ then Theorem 7.1 implies that $K(\mathcal{W})$ is a holomorphic 2-form. Since every foliation on T is induced by a global meromorphic 1-form, one can proceed as in the beginning of this Section to define a global meromorphic 1-form η on T such that $K(\mathcal{W}) = d\eta$. The result follows from the next proposition. □

Proposition 7.1. *Let ω be a meromorphic 1-form on a compact Kähler manifold M. If $d\omega$ is holomorphic then ω is closed.*

Proof. We learned the following proof from Marco Brunella. Notice that although ω is not closed a priori, the holomorphicity of $\Omega = d\omega$ ensures that its residues along the irreducible components Z_i of its polar set are well-defined complex numbers. If S is a real subvariety of M of real dimension 2, then Stoke's Theorem implies that

$$\int_S \Omega = \sum_{i=1}^{m} \mathrm{res}_{Z_i}(\omega) \cdot \big(S \cdot Z_i\big)$$

where $S \cdot Z_i$ stands for the topological intersection number of S with Z_i. It follows that the class of Ω, seen as a current, lies in $H^{1,1}(M, \mathbb{C})$.

On the other hand, Ω being a closed holomorphic 2-form, its class lies also in $H^{2,0}(M, \mathbb{C})$. But $H^{1,1}(M, \mathbb{C}) \cap H^{2,0}(M, \mathbb{C}) = 0$ since M is Kähler. This implies that Ω is zero and consequently ω is closed. □

Theorem 7.2 admits the following consequence.

Corollary 7.1. *Let $\mathcal{W}$ be a linear k-web and $\mathcal{F}$ be a foliation both defined on the same complex torus T. Suppose that $\mathcal{W}$ decomposes as $\mathcal{W}_1 \boxtimes \mathcal{W}_2$ in such a way that $\mathcal{W}_1$ and $\mathcal{W}_2$ are not foliations. Suppose also that for every defining foliation $\mathcal{L}$ of $\mathcal{W}_i$, $i = 1, 2$, we have*

$$\beta_{\mathcal{L}}(\mathcal{W}_i - \mathcal{L}) = \beta_{\mathcal{L}}(\mathcal{W} - \mathcal{L}) \,.$$

Then $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ if and only if $K(\mathcal{W}_i \boxtimes \mathcal{F}) = 0$ for $i = 1, 2$.

Example 7.1. Consider the linear 4-web

$$\mathcal{W} = \underbrace{[dxdy]}_{\mathcal{W}_1} \boxtimes \underbrace{[(dx - dy)(dx + dy)]}_{\mathcal{W}_2}$$

on a two-dimensional complex torus T. Notice that

$$\beta_{[dx]}(\mathcal{W}) = [dy] = \beta_{[dx]}(\mathcal{W}_1) \quad \text{and} \quad \beta_{[dy]}(\mathcal{W}) = [dx] = \beta_{[dy]}(\mathcal{W}_1) \,.$$

Similarly $\beta_{[dx \pm dy]}(\mathcal{W}) = [dx \mp dy] = \beta_{[dx \pm dy]}(\mathcal{W}_2)$.

In [42], germs of exceptional CDQL 5-webs on $(\mathbb{C}^2, 0)$ of the form

$$[dxdy(dx - dy)(dx + dy)] \boxtimes \mathcal{F}$$

are classified under the additional assumption that $K([dxdy] \boxtimes \mathcal{F}) = 0$. Mihaileanu's criterion combined with the Corollary 7.1 above yields that the additional assumption is superfluous if $\mathcal{F}$ is supposed to be globally defined on a complex torus T. Translating the classification of [42] to our setup, we obtain that every flat and global 5-web on complex tori of the form $[dxdy(dx-dy)(dx+dy)] \boxtimes \mathcal{F}$ is isogeneous to one of the 5-webs $\mathcal{E}_\tau$ ($\tau \in \mathbb{H}$) presented in the Introduction. In particular the torus T has to be isogeneous to the square of an elliptic curve.

7.3. **Specialization to CDQL webs on the projective plane.** It would be interesting to extend Theorem 7.1 in order to deal with more degenerated discriminants. We do not know how to do it in general. Nevertheless under the assumption that $\mathcal{W}$ is a product of linear foliations on the projective plane we have the following weaker result.

Theorem 7.3. *Let $\mathcal{F}$ be a foliation and $\mathcal{W} = \mathcal{L}_1 \boxtimes \mathcal{L}_2 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a totally decomposable linear k-web, $k \geq 2$, both globally defined on $\mathbb{P}^2$. Suppose that C is an irreducible component of $tang(\mathcal{F}, \mathcal{L}_1)$. If $K(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic over a generic point of C then the curve C is invariant by $\mathcal{L}_1$ or by $\beta_{\mathcal{L}_1}(\mathcal{W} - \mathcal{L}_1)$.*

Proof. If C is not contained in $\Delta(\mathcal{W})$ then the result follows from Theorem 7.1. Thus, assume that $C \subset \Delta(\mathcal{W})$. The tangency of two linear foliations on $\mathbb{P}^2$ is a line invariant by both and, therefore, C must be a line invariant by at least two of the defining foliations of $\mathcal{W}$.

If C is $\mathcal{L}_1$-invariant then there is nothing to prove. Thus assume that this is not the case. Because $C \subset \mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$, we are also assuming that C is not $\mathcal{F}$-invariant.

First remark that Theorem 7.1 implies that $K(\mathcal{F} \boxtimes \mathcal{L}_i \boxtimes \mathcal{L}_j)$ is holomorphic over the generic point of C for every choice of distinct $i, j \in \{2, \ldots, k\}$. Indeed, on the one hand if $C \subset \mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ then $\mathcal{L}_i$ and $\mathcal{L}_1$ have to be tangent along C. Thus C is $\mathcal{L}_1$-invariant contrary to our assumptions. On the other hand if $C \subset \mathrm{tang}(\mathcal{L}_i, \mathcal{L}_j)$ then C is invariant by both $\mathcal{L}_i$ and $\mathcal{L}_j$ and the triple $(\mathcal{F}, \mathcal{F}_1, \mathcal{W}) = (\mathcal{L}_i, \mathcal{L}_j, \mathcal{F} \boxtimes \mathcal{L}_j)$ satisfies the hypotheses of Theorem 7.1. Thus $K(\mathcal{F} \boxtimes \mathcal{L}_i \boxtimes \mathcal{L}_j)$ is indeed holomorphic over the generic point of C.

Similarly, Theorem 7.1 implies that $K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_i)$ is holomorphic along C whenever C is $\mathcal{L}_i$-invariant.

If we write $\mathcal{W} = \mathcal{L}_1 \boxtimes \mathcal{W}_0 \boxtimes \mathcal{W}_1$ with $\mathcal{W}_1$ being the product of foliations in $\mathcal{W}$ leaving C invariant and $\mathcal{W}_0$ being the product of foliations in $\mathcal{W} - \mathcal{L}_1$ not leaving C invariant then $K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{W}_0)$ is holomorphic over the generic point of C.

Because C is not contained in $\Delta(\mathcal{L}_1 \boxtimes \mathcal{W}_0)$, Theorem 7.1 implies that C is $\beta_{\mathcal{L}_1}(\mathcal{W}_0)$-invariant. From the definition of the $\mathcal{L}_1$-barycenter it follows that C is also invariant by $\beta_{\mathcal{L}_1}(\mathcal{W}_0 \boxtimes \mathcal{W}_1) = \beta_{\mathcal{L}_1}(\mathcal{W} - \mathcal{L}_1)$. □

Notice that in Theorem 7.3, unlikely in Theorem 7.1, the invariance condition imposed on $C \subset \mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ is no longer a necessary and sufficient condition for the regularity of the curvature: it is just necessary. In fact, the converse to Theorem 7.3 does not hold in general. For instance, if $\mathcal{F} = [ydx + dy]$, $\mathcal{L}_1 = [dy]$ and $\mathcal{L}_2 = [ydx - xdy]$, then the line $L = \{y = 0\}$ is invariant by $\mathcal{F}$, $\mathcal{L}_1$ and $\beta_{\mathcal{L}_1}(\mathcal{L}_2) = \mathcal{L}_2$ but $K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_2)$ is not holomorphic over L since

$$K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_2) = \frac{dx \wedge dy}{y\,(x+1)^2}\,.$$

8. Constraints on flat CDQL webs

In this section we start the classification of exceptional CDQL webs on the projective plane. As already mentioned in the Introduction the starting point is Mihăileanu criterion: *If $\mathcal{W}$ is a web of maximal rank then $K(\mathcal{W}) = 0$.*

We will combine this criterion with Theorem 7.3 in order to restrict the possibilities for the pairs $(\mathcal{F}, \mathcal{P})$. For instance, Theorem 8.1 below shows that the degree of $\mathcal{F}$ is bounded by four when $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ is flat.

Here, as usual, the degree of a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2$ is the number of tangencies with a generic line $\ell \subset \mathbb{P}^2$. Concretely, in affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, a foliation $\mathcal{F}$ has degree d if and only if $\mathcal{F}$ is defined by a polynomial 1-form ω with isolated zeros that can be written in the form

$$\omega = a(x, y)dx + b(x, y)dy + h(x, y)(xdy - ydx)$$

where h is a homogeneous polynomial of degree d; a and b are polynomials of degree at most d and; when h is the zero polynomial the polynomial $xa + yb$ has degree exactly $d + 1$.

We point out that h vanishes identically if and only if the line at infinity is $\mathcal{F}$-invariant. In this case the zeros of the homogenous component of degree $d+1$ of the polynomial $xa+yb$ correspond to the singularities of $\mathcal{F}$ on the line at infinity. If h is non-zero then the points at infinity determined by h are in one to one correspondence with the tangencies of $\mathcal{F}$ with the line at infinity.

8.1. **Notations.** The notations below will be used in the proof of the classification of exceptional CDQL webs on the projective plane.

$\mathcal{P}$	finite set of points in $\mathbb{P}^2$;
k	the cardinality of $\mathcal{P}$;
$p_1,\ldots,p_k$	the points of $\mathcal{P}$;
$\mathcal{P}_i$	$\mathcal{P}\setminus\{p_i\}$;
$\mathcal{L}_i$	the linear foliation determined by p_i;
$\mathcal{W}(\mathcal{P})$	$\mathcal{L}_1\boxtimes\cdots\boxtimes\mathcal{L}_k$;
$\mathcal{W}(\mathcal{P}_i)$	$\mathcal{W}(\mathcal{P})-\mathcal{L}_i$;
$\widehat{\mathcal{L}_i}$	the $\mathcal{L}_i$-barycenter of $\mathcal{W}(\mathcal{P}_i)$ that is, $\beta_{\mathcal{L}_i}\big(\mathcal{W}(\mathcal{P}_i)\big)$;
$\mathcal{L}_p$	the pencil of lines through a point $p\in\mathbb{P}^2$;
$\widehat{\mathcal{L}_p}$	in case $p\in\mathcal{P}$, the $\mathcal{L}_p$-barycenter of $\mathcal{W}(\mathcal{P}\setminus\{p\})$;
ℓ	a line on $\mathbb{P}^2$;
$\mathcal{P}_\ell$	$\mathcal{P}\cap\ell$;
k_ℓ	the cardinality of $\mathcal{P}_\ell$;
$q_1,\ldots,q_{k_\ell}$	the points of $\mathcal{P}_\ell$;
$\widehat{q_i}$	the q_i-barycenter of $\mathcal{P}_\ell\setminus\{q_i\}$ in ℓ.

A set $\mathcal{P}$ is in *p_i-barycentric general position* if the only algebraic leaves of $\widehat{\mathcal{L}_i}$ are the lines $\overline{p_ip_j}$ (compare with Proposition 6.1). We will write $b(\mathcal{P})$ for the cardinality of

$$\mathcal{B}(\mathcal{P})=\big\{p\in\mathcal{P}\,|\,\mathcal{P}\text{ is in }p\text{-barycentric general position}\big\}\,.$$

8.2. **Configurations of points in barycentric general position.** As an immediate consequence of Theorem 7.1 it follows that a completely decomposable 3-web $\mathcal{W}=\mathcal{F}\boxtimes\mathcal{L}_1\boxtimes\mathcal{L}_2$ on $\mathbb{P}^2$ induced by two pencils of lines and a foliation has curvature zero if and only if it is projectively equivalent to a web of the form

$$\big[a(y)dx+b(x)dy\big]\boxtimes[dx]\boxtimes[dy]$$

where a and b are rational functions.

In the same vein, the next result combines Proposition 6.1 with Theorem 7.3 to show how *generic* configurations of points impose strong restrictions on a foliation $\mathcal{F}$ when $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})$ has curvature zero.

Proposition 8.1. *Let $\mathcal{W}(\mathcal{P})$ be the k-web naturally associated to a collection $\mathcal{P}$ of k distinct points in $\mathbb{P}^2$. If $\mathcal{F}$ is a non-linear foliation on $\mathbb{P}^2$ such that $K\big(\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})\big)=0$ then $b(\mathcal{P})$ is at most 4. Moreover, there exist affine coordinates x,y such that*

(a) if $b(\mathcal{P})=1$ then $\mathcal{F}=[a(y)dx+b(x,y)\,dy]$ for some $a\in\mathbb{C}[y]$, $b\in\mathbb{C}[x,y]$;

(b) if $b(\mathcal{P})=2$ then $\mathcal{F}=\big[a(y)\,dx+b(x)\,dy\big]$ for some $a,b\in\mathbb{C}[t]$;

(c) if $b(\mathcal{P})=3$ then the points in $\mathcal{B}(\mathcal{P})$ are not aligned and

$$\mathcal{F}=\big[y\,(y^{d-1}-\epsilon_1)\,dx-x\,(x^{d-1}-\epsilon_2)\,dy\big]$$

for some integer $d \geq 2$ and $\epsilon_1, \epsilon_2 \in \{0,1\}$ or

$$\mathcal{F} = \big[y\,dx - \lambda\,x\,dy\big]$$

for some constant $\lambda \in \mathbb{C} \setminus \{0,1\}$;

(d) if $b(\mathcal{P}) = 4$ then the points in $\mathcal{B}(\mathcal{P})$ are in general position and $\mathcal{F}$ is the pencil of conics through them.

Proof. Suppose that $\mathcal{P}$ is in p_1-barycentric general position and assume that $p_1 = [0:1:0]$. If $K\big(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})\big) = 0$ then Theorem 7.3 implies that the tangency between $\mathcal{F}$ and $\mathcal{L}_1$ is a union of lines through p_1. In the affine coordinates $(x,y) = [x:y:1]$, $\mathcal{L}_1 = [dy]$ and the lines through p_1 correspond to vertical lines. Therefore $\mathcal{F} = [a(y)dx + b(x,y)dy]$ for some polynomials a, b.

If $\mathcal{P}$ is also in p_2-barycentric general position and $p_2 = [1:0:0]$, the same argument shows that $\mathcal{F} = [a(y)dx + b(x)dy]$ for some polynomials a, b.

Notice that no point $p \in \mathcal{B}(\mathcal{P}) \setminus \{p_1, p_2\}$ can be aligned with p_1 and p_2. Indeed, suppose the contrary. One can assume that $p = p_3 = [1:1:0]$, or equivalently $\mathcal{L}_3 = [dx - dy]$. Then the tangency of $\mathcal{F}$ and $\mathcal{L}_3$ is given by vanishing of

$$(dx - dy) \wedge \big(a(y)dx + b(x)dy\big) = \big(b(x) + a(y)\big)dx \wedge dy\,.$$

Because $K\big(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})\big) = 0$, $\{b(x) + a(y) = 0\}$ must be a union of lines through p_3. Explicitly, up to a multiplicative constant,

$$b(x) + a(y) = \prod_{j=1}^{m} (x - y - c_j)$$

for suitable constants $c_1, \ldots, c_m$. Such identity is possible if and only if the homogenous component of higher order of $a(y)dx + b(x)dy$ is a constant multiple of $x\,dy - y\,dx$. Therefore $\mathcal{F}$ has degree zero and consequently is a linear foliation. This contradicts our assumptions on $\mathcal{F}$.

Suppose now that $\mathcal{P}$ is also in p_3-barycentric general position with $p_3 \notin \overline{p_1 p_2}$. It is harmless to assume that $p_3 = [0:0:1]$. Since the tangency of $\mathcal{F}$ and $\mathcal{L}_3$ is a union lines through $p_3 = (0,0) \in \mathbb{C}^2$ then the polynomial $xa(y) + yb(x)$ must be homogeneous. Thus for a certain $d \in \mathbb{N}^*$ and suitable $c_0, c_1, c_2 \in \mathbb{C}$

$$(a(y), b(x)) = (c_1 y^d + c_0 y, c_2 x^d - c_0 x)\,.$$

It is a simple matter to show that we are in one of the two cases displayed in part *(c)* of the statement, the first when $d \geq 2$ and the second when $d = 1$.

Finally, suppose that $b(\mathcal{P}) \geq 4$. Since no three points in $\mathcal{B}(\mathcal{P})$ are aligned we can assume that p_1, p_2, p_3 are as above and $p_4 = [1:1:1]$. Applying again the above argument to $\mathcal{L}_4$ and discarding the solutions corresponding to degree zero foliations, we prove that

$$\mathcal{F} = \big[a(y)dx + b(x)dy\big] = \big[y(y-1)\,dx - x(x-1)\,dy\big]\,.$$

Notice that the rational function $\frac{x(y-1)}{y(x-1)}$ is a first integral of $\mathcal{F}$, that is, $\mathcal{F}$ is a pencil of conics through the four points $p_1, \ldots, p_4$. Notice also that $\mathcal{F}$ leaves invariant exactly six lines: the line at infinity and the five affine lines cut out by the polynomial $xy(x-1)(y-1)(x-y)$. If $\mathrm{tang}(\mathcal{L}_p, \mathcal{F})$ is a union of lines through p then p must belong to three of the six $\mathcal{F}$-invariant lines. Since there are only four such points (p_1, p_2, p_3 and p_4) $b(\mathcal{P})$ has at most four elements. This concludes the proof. $\square$

Corollary 8.1. *Assume that the cardinality of $\mathcal{P}$ is at least 4. If it exists a non-linear foliation $\mathcal{F}$ such that $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then one of the following two situations occurs:*

(1) there are three aligned points in $\mathcal{P}$;

(2) $\mathcal{P}$ is the union of of 4 points in general position and $\mathcal{F}$ is the pencil of conics through them.

Proof. Assume that we are not in case *(1)*: any line contains at most two points of $\mathcal{P}$. Lemma 6.1 item *(4)* implies that the singularities of $\widehat{\mathcal{L}_p}$ coincide with $\mathcal{P} \setminus \{p\}$ for any $p \in \mathcal{P}$. By assumption, the set of points $\mathcal{P} \setminus \{p\}$ is not aligned and, according to Proposition 6.1, $\mathcal{P}$ is in p-barycentric general position. Thus $\mathcal{P} = \mathcal{B}(\mathcal{P})$ and Proposition 8.1 implies the result. □

8.3. **Aligned points versus invariant lines.** Non-generic configurations of points also impose non-trivial conditions on non-linear foliations $\mathcal{F}$ such that the curvature of $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ vanishes identically.

Proposition 8.2. *Let $\mathcal{P} \subset \mathbb{P}^2$ be a set of k points and $\mathcal{F}$ be a non-linear foliation on $\mathbb{P}^2$ such that $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$. If ℓ is a line that contains at least three points of $\mathcal{P}$ then ℓ is $\mathcal{F}$-invariant.*

Proof. Remind that $k_\ell = \mathrm{Card}(\mathcal{P}_\ell)$ with $\mathcal{P}_\ell = \mathcal{P} \cap \ell = \{q_1, \ldots, q_{k_\ell}\}$. By hypothesis, $k_\ell \geq 3$. If ℓ is not invariant by $\mathcal{F}$ then

$$|\mathrm{tang}(\mathcal{F}, \ell)| \subset |\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)| \cap \ell \tag{29}$$

for every $i = 1, \ldots, k_\ell$, since ℓ is invariant by $\mathcal{L}_i = \mathcal{L}_{q_i}$.

Notice that for every i ranging from 1 to k_ℓ, the Riccati foliation $\widehat{\mathcal{L}_i}$ leaves ℓ invariant and its singularities on ℓ are q_i and $\widehat{q_i}$ according to Lemma 6.1 items *(2)* and *(4)*.

If ℓ is not $\mathcal{F}$-invariant, Theorem 7.3 implies that each irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ is invariant by $\mathcal{L}_i$ or $\widehat{\mathcal{L}_i}$. Since the leaves of $\mathcal{L}_i$ are lines through q_i and because the algebraic curves invariant by $\widehat{\mathcal{L}_i}$ must intersect ℓ on $\mathrm{sing}(\widehat{\mathcal{L}_i}) \cap \ell = \{q_i, \widehat{q_i}\}$ (according to Lemma 6.1), it follows from (29) that

$$|\mathrm{tang}(\mathcal{F}, \ell)| \subset \bigcap_{i=1}^{k_\ell} \{q_i, \widehat{q_i}\}\,.$$

The only possibilities after an eventual reindexing are

(a) $\mathrm{tang}(\mathcal{F}, \ell) = \emptyset$ or

(b) $\mathrm{tang}(\mathcal{F}, \ell) = \{q_1\}$ and $\widehat{q_2} = \cdots = \widehat{q_{k_\ell}} = q_1$.

We aim at a contradiction. On the one hand (a) implies that $\mathcal{F}$ is everywhere transversal to ℓ. Therefore $\mathcal{F}$ is of degree zero what is not the case according to our hypothesis. On the other hand (b) implies that the support of $\beta(q_1, \ldots, q_{k_\ell})$ has a point with multiplicity at least $k_\ell - 1$, contradicting Lemma 5.1. □

Proposition 8.3. *Let $\mathcal{F}$ be a non-linear foliation on $\mathbb{P}^2$. Assume that ℓ is a line that contains at least three points of a set $\mathcal{P}$ of k points in $\mathbb{P}^2$. If $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ has curvature zero then the rational map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ induced by the linear system $\{\mathrm{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\}_{p \in \ell}$ does not contract ℓ.*

Proof. First of all, a rephrasing of Proposition 8.2, yields that ℓ is a fixed component of the pencil $\{\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)\}_{p\in\ell}$. Thus $\{\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)-\ell\}_{p\in\ell}$ is indeed a linear system.

Concretely, working with affine coordinates (x, y) such that ℓ is the line at infinity and $\mathcal{F}$ is induced by a polynomial 1-form $\omega = a(x, y)\, dx + b(x, y)\, dy$ with isolated zeros then $F(x : y : z) = (B(x, y, z) : -A(x, y, z))$, where A and B are homogenizations of a and b of degree $\max\{\deg(a), \deg(b)\} = \deg(\mathcal{F})$.

Assume that F contracts ℓ. It means that there exists a point $p \in \ell$ such that $2\,\ell \le \mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$. In other words the polynomials $A(x, y, 0)$ and $B(x, y, 0)$ are linearly dependent over $\mathbb{C}$. Therefore

$$|\mathrm{tang}(\mathcal{F}, \mathcal{L}_q) - \ell| \cap \ell = |\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q'}) - \ell| \cap \ell \subsetneq \ell \tag{30}$$

for every $q, q' \in \ell \setminus \{p\}$.

For any $i \in \{1, \ldots, k_\ell\}$, if C denotes an irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ distinct from ℓ, then Theorem 7.3 implies that C necessarily is $\mathcal{L}_i$-invariant or $\widehat{\mathcal{L}_i}$-invariant. Therefore, arguing as in the proof of Proposition 8.2, it follows from (30) that

$$|\mathrm{tang}(\mathcal{F}, \mathcal{L}_q) - \ell| \subset \bigcap_{i=1, q_i\neq p}^{k_\ell} \{q_i, \widehat{q_i}\}$$

for every $q \in \ell \setminus \{p\}$, in particular for every $q_i \in \mathcal{P}_\ell \setminus \{p\}$.

After an eventual reindexing we have four possibilities:

(a) $p \notin \{q_1, \ldots, q_{k_\ell}\}$
- (a.1) $\widehat{q_1} = \widehat{q_2} = \cdots = \widehat{q_{k_\ell}}$;
- (a.2) $q_1 = \widehat{q_2} = \cdots = \widehat{q_{k_\ell}}$;

(b) $p = q_1$
- (b.1) $\widehat{q_2} = \widehat{q_3} = \cdots = \widehat{q_{k_\ell}}$;
- (b.2) $q_2 = \widehat{q_3} = \cdots = \widehat{q_{k_\ell}}$.

Lemma 5.1 excludes the cases (a.1), (a.2) and (b.1). To deal with the case (b.2) we will choose an identification $\ell = \mathbb{P}(\mathbb{C}x \oplus \mathbb{C}y)$ where $q_1 = [x]$, $q_2 = [y]$ and $q_i = [x + \lambda_i\, y]$ with $\lambda_i \neq 0$ for $i = 3, \ldots, k_\ell$.

A straight-forward computation shows that

$$\begin{aligned}
\widehat{q_1} &= \left(\sum_{i=3}^{k_\ell} \frac{1}{\lambda_i}\right) x &&+ \quad (k_\ell - 1)\, y\,, \\
\widehat{q_2} &= \big(k_\ell - 1\big)\, x &&+ \quad \left(\sum_{i=3}^{k_\ell} \lambda_i\right) y\,, \\
\widehat{q_j} &= \left(-\frac{1}{\lambda_j} + \sum_{i=3, i\neq j}^{k_\ell} \frac{1}{\lambda_i - \lambda_j}\right) x &&+ \quad \left(1 + \sum_{i=3, i\neq j}^{k_\ell} \frac{\lambda_i}{\lambda_i - \lambda_j}\right) y
\end{aligned}$$

for j ranging from 3 to k_ℓ. Now $\widehat{q_j} = q_2$ for any such j implies that the coefficient of x in $\widehat{q_j}$ is zero. Summing up these coefficients for $j = 3, \ldots, k_\ell$, one obtains

$$\sum_{j=3}^{k_\ell} \frac{1}{\lambda_j} = 0\,.$$

Therefore $\widehat{q_1} = q_2$. Applying Lemma 5.1 once again we conclude that (b.2) is also impossible. This concludes the proof. $\square$

8.4. **A bound for the degree of $\mathcal{F}$.** Combining Propositions 8.2 and 8.3 with Riemman-Hurwitz formula we are able to bound the degree of $\mathcal{F}$.

Theorem 8.1. *Let $\mathcal{P} \subset \mathbb{P}^2$ be a set of $k \geq 4$ points and $\mathcal{F}$ be a a non-linear foliation on $\mathbb{P}^2$. If $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then $\deg(\mathcal{F}) \leq 4$. Moreover, if $\deg(\mathcal{F}) \geq 2$, and ℓ is a line containing k_ℓ points of $\mathcal{P}$ then $k_\ell \leq 7 - \deg(\mathcal{F})$.*

Proof. Assume that there is no line that contains at least three points of $\mathcal{P}$. Then Corollary 8.1 implies that $\mathcal{P}$ has cardinality four and that $\mathcal{F}$ is the degree two foliation tangent to the pencil of conics through $\mathcal{P}$.

From now on, we assume that there exists a line ℓ containing k_ℓ points of $\mathcal{P}$, with $k_\ell \geq 3$. Identifying ℓ with $\mathbb{P}^1$, let us note $f : \mathbb{P}^1 \to \mathbb{P}^1$ the restriction to ℓ of the rational map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ induced by the linear system $\{\mathrm{tang}(\mathcal{F}, \mathcal{L}_p) - \ell \,|\, p \in \ell\}$. Proposition 8.3 ensures that f is a non-constant map.

The map f is characterized by the following equalities between divisors on ℓ

$$f^{-1}(p) = \Big(\mathrm{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\Big)_{|\ell},$$

with $p \in \ell$ arbitrary.

Let d be the degree of $\mathcal{F}$. Recall from the proof of Proposition 8.3 that f is defined by degree d polynomials, that is, $\deg(f) = d$. Theorem 7.3 implies, for any $i = 1, \ldots, k_\ell$,

$$f^{-1}(q_i) = e_i\, q_i + (d - e_i)\, \widehat{q_i} \tag{31}$$

where e_i is an integer satisfying $0 \leq e_i \leq d$. Notice that the contribution of each of these fibers in Riemann-Hurwitz formula is at least $d-2$. Therefore

$$\chi(\mathbb{P}^1) = d\,\chi(\mathbb{P}^1) - (d-2)\, k_\ell - r$$

for some non-negative integer r. If $d > 2$ then

$$k_\ell \leq \frac{2d-2}{d-2}\,.$$

If we keep in mind that $k_\ell \geq 3$ and $d \geq 1$ then we end up with the following possibilities

$$d = 4 \;\text{ and }\; k_\ell = 3, \quad \text{or} \quad d = 3 \;\text{ and }\; k_\ell \leq 4, \quad \text{or} \quad 1 \leq d \leq 2 \;\text{ and }\; k_\ell \geq 3\,.$$

If one realizes that for $d = 2$ the map f will have at most three fixed points and two totally ramified points then one sees that in this case $k_\ell \leq 5$. □

The map $f : \mathbb{P}^1 \to \mathbb{P}^1$ used in the proof of Theorem 8.1 codifies a lot of information about the foliation $\mathcal{F}$. From now on we will refer to f as the *ℓ-polar map of $\mathcal{F}$*.

8.5. **The polar map: properties and normal forms.** We use here the same notations than in the preceding section and keep the hypothesis of Theorem 8.1.

We first state two properties of the polar map that will be used in the sequel.

Lemma 8.1. *If the line ℓ is $\mathcal{F}$-invariant then the singularities of $\mathcal{F}$ on ℓ correspond to the fixed points of f.*

Proof. Let $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ be affine coordinates and assume that ℓ is the line at infinity. The foliation $\mathcal{F}$ is induced by a polynomial 1-form $\omega = a(x,y)\,dx + b(x,y)\,dy$ where $a(x,y)$ and $b(x,y)$ are relatively prime polynomials of degree d. If $a_d(x,y)$ and $b_d(x,y)$ are the homogeneous components of degree d of $a(x,y)$ and $b(x,y)$

(respectively) then, in the homogeneous coordinates $(x : y : 0) \in \ell$, the polar map f is

$$f(x : y) = \big[b_d(x, y) : -a_d(x, y) \big] \, .$$

On the other hand, one has

$$\mathrm{sing}(\mathcal{F}) \cap \ell = \Big\{ \, [x : y : 0] \in \mathbb{P}^2 \, \Big| \, x \, a_d(x, y) + y \, b_d(x, y) = 0 \, \Big\} \, .$$

Thus $[x : y : 0] \in \ell$ is a fixed point of f if and only if it belongs to $\mathrm{sing}(\mathcal{F})$. □

For $i = 1, \ldots, k_\ell$, let e_i be the non-negative integer appearing in (31).

Lemma 8.2. *There are exactly $e_i + 1$ lines invariant by $\mathcal{F}$ through q_i counted with the multiplicities that appear in* $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_i})$.

Proof. Let C be an irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_i})$ passing through q_i. According to Theorem 7.3, C is $\mathcal{L}_{q_i}$-invariant or $\widehat{\mathcal{L}_{q_i}}$-invariant. Since the only algebraic leaves of $\widehat{\mathcal{L}_{q_i}}$ trough q_i are lines (see Lemma 6.1 item *(6)*) C must be a line. This fact together with (31) proves the lemma. □

It turns out that the relations (31) determine f and $\mathcal{P}_\ell$ up to automorphism of $\mathbb{P}^1$ when $\deg(\mathcal{F}) \geq 2$. Indeed by routine elementary computations we arrive at the list presented in TABLE 1 below. For the sake of conciseness, we have chosen not to derive this list here in full generality but just to deal with a particular case in the lemma below. All the other cases follow from similar arguments.

Lemma 8.3. *Assume that $k_\ell = 3$ and $\deg(\mathcal{F}) = 4$. Then we are in one of the two cases* (c.1) *or* (c.2) *of TABLE 1.*

Proof. In what follows, $[a : b]$ designates the point $[a : b : 0]$ on the line ℓ that is supposed to be at infinity. Since $k_\ell = 3$, one can assume that $q_1 = [1 : 0]$, $q_2 = [0 : 1]$ and $q_3 = [1 : -1]$; so $\widehat{q_1} = [-1 : 2]$, $\widehat{q_2} = [2 : -1]$ and $\widehat{q_3} = [1 : 1]$. By hypothesis $\deg(\mathcal{F}) = 4$ so the polar map is $f(x : y) = (P(x, y) : Q(x, y))$ where P and Q are homogenous polynomials in x, y, of degree 4. According to (31), one have $(P)_0 = e_2 \, q_2 + (4 - e_2) \, \widehat{q_2}$ so P is of the form $\lambda \, x^{e_2} \, (2 \, y + x)^{4-e_2}$ for a certain $\lambda \in \mathbb{C}^*$ that can be supposed equal to 1. Similarly, $Q = \mu \, y^{e_1} \, (2 \, x + y)^{4-e_1}$ with $\mu \in \mathbb{C}^*$. Since $(P + Q)_0 = e_3 \, q_3 + (4 - e_3) \, \widehat{q_3}$, there exists $\nu \in \mathbb{C}^*$ such that

$$x^{e_2} \, (x + 2 \, y)^{4-e_2} + \mu \, y^{e_1} \, (y + 2 \, x)^{4-e_1} = \nu \, (x + y)^{e_3} (x - y)^{4-e_3} \, .$$

After straight-forward computations, it appears that such a relation in the space of homogeneous polynomials in two variables is only possible when (e_1, e_2, e_3) takes one of the two values: $(1, 1, 1)$ or $(3, 3, 3)$. These correspond respectively to the cases (c.1) and (c.2) in TABLE 1 below. □

8.6. Points of $\mathcal{P}$ versus singularities of $\mathcal{F}$.

We start with a simple observation.

Lemma 8.4. *Let $\mathcal{P}$ be a collection of points of $\mathbb{P}^2$. If $\mathcal{F}$ is a non-linear foliation on $\mathbb{P}^2$ such that $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then each point $p \in \mathcal{P}$ is contained in an $\mathcal{F}$-invariant line.*

Proof. The argument used to settle Lemma 8.2 implies that every irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$ containing p must be an $\mathcal{F}$-invariant line. □

TABLE 1 allows us to restrain the possibilities of $\mathcal{F}$ when $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ and $\deg(\mathcal{F}) > 1$. The next result shows that once $\mathcal{F}$ is known there are not many possibilities for $\mathcal{P}$.

$\mathbf{k}_\ell$	d	action	normal form for $f(x:y)$	label
3	2	$f^{-1}(q_i) = q_i + \widehat{q_i}$	$\big(x\,(2y+x) : -\,y\,(2x+y)\big)$	(a.1)
		$f^{-1}(q_1) = 2\,q_1$ $f^{-1}(q_2) = 2\,q_2$ $f^{-1}(q_3) = q_3 + \widehat{q_3}$	$\big(x^2 : -\,y^2\big)$	(a.2)
		$f^{-1}(q_1) = 2\,\widehat{q_1}$ $f^{-1}(q_2) = 2\,\widehat{q_2}$ $f^{-1}(q_3) = q_3 + \widehat{q_3}$	$\big((x+2y)^2 : -(2x+y)^2\,\big)$	(a.3)
	4	$f^{-1}(q_i) = q_i + 3\,\widehat{q_i}$	$\Big(x\,(2y+x)^3 : -\,y\,(2x+y)^3\Big)$	(c.1)
		$f^{-1}(q_i) = 3\,q_i + \widehat{q_i}$	$\Big(x^3(2y+x) : -\,y^3(2x+y)\Big)$	(c.2)
4	3	$f^{-1}(q_i) = q_i + 2\,\widehat{q_i}$	$\Big(3x\,\big(x+y\,(1-\xi_3^2)\big)^2 : -y\big(3x+y\,(1-\xi_3^2)\big)^2\Big)$	(b.1)
5	2	$f^{-1}(q_1) = 2\,\widehat{q_1}$ $f^{-1}(q_2) = 2\,\widehat{q_2}$ $f^{-1}(q_3) = q_3 + \widehat{q_3}$ $f^{-1}(q_4) = q_4 + \widehat{q_4}$ $f^{-1}(q_5) = q_5 + \widehat{q_5}$	$\big(y^2 : -x^2\big)$	(a.4)

TABLE 1. The ℓ-polar map of $\mathcal{F}$ assuming that $K\big(\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})\big)=0$. The integer k_ℓ stands for the cardinality of $\ell\cap\mathcal{P}$, d designates $\deg(\mathcal{F})$ and the points $q_1,q_2,q_3\in\ell$ are normalized as $q_1=[1:0:0]$, $q_2=[0:1:0]$ and $q_3=[1:-1:0]$.

Proposition 8.4. *Let $\mathcal{P}$ be a finite set of points of $\mathbb{P}^2$. Suppose there exists a line ℓ containing at least three points of $\mathcal{P}$. If $\mathcal{F}$ is a non-linear foliation on $\mathbb{P}^2$ such that the curvature of $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})$ vanishes identically then $\mathcal{P}\setminus\ell\subset\mathrm{sing}(\mathcal{F})$.*

Proof. Let $f:\mathbb{P}^1\to\mathbb{P}^1$ be the ℓ-polar map of $\mathcal{F}$. Recall that $\mathcal{P}_\ell=\mathcal{P}\cap\ell=\{q_1,\ldots,q_{k_\ell}\}$ where the q_i's are pairwise distinct.

For any distinct $i,j\in\{1,\ldots,k_\ell\}$,

$$\mathrm{sing}(\mathcal{F})\cap(\mathbb{P}^2\setminus\ell)=\big|\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_i})\big|\cap\big|\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_j})\big|\cap(\mathbb{P}^2\setminus\ell)\,. \tag{32}$$

Let p be a point in $\mathcal{P}\setminus\ell$. Assume that $p\notin\mathrm{sing}(\mathcal{F})$. After an eventual reordering, (32) implies that p does not belong to $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$ nor to $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_2})$.

Since $p\notin\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$, the line $\overline{pq_1}$ is not $\mathcal{F}$-invariant. Thus Proposition 8.2 implies that $\mathcal{P}\cap\overline{pq_1}=\{p,q_1\}$. Consequently $p\in\mathrm{sing}(\widehat{\mathcal{L}_{q_1}})$ thanks to Lemma 6.1 item *(4)*.

Let C be an irreducible component of $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$. If C is not $\mathcal{L}_{q_1}$-invariant then it must be $\widehat{\mathcal{L}_{q_1}}$-invariant by Theorem 7.3 and cannot contain q_1 by Lemma 6.1 item *(6)*. Thus C must intersect the $\widehat{\mathcal{L}_{q_1}}$-invariant line $\overline{pq_1}$ at p. Since $p\notin\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$ we deduce that every irreducible component of $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$ is $\mathcal{L}_{q_1}$-invariant. Lemma 8.2 implies that $f^{-1}(q_1)=\deg(\mathcal{F})\,q_1$. Similarly, $\mathcal{P}\cap\overline{pq_2}=\{p,q_2\}$ and $f^{-1}(q_2)=\deg(\mathcal{F})\,q_2$.

Every rational self-map of $\mathbb{P}^1$ has at most two totally ramified points (or at most two fixed points when the degree is one and the map is not the identity). Consequently p must belong to $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_i})$ for every $i\in\{3,\ldots,k_\ell\}$. The only possibility is that $k_\ell=3$ (otherwise p would be in $\mathrm{sing}(\mathcal{F})$ according to (32)).

Lemma 8.4 implies that there is a $\mathcal{F}$-invariant line ℓ_p through p. Since $\overline{pq_1}$ and $\overline{pq_2}$ are not $\mathcal{F}$-invariant, the line ℓ_p must be distinct from these. In particular, $\ell_p \cap \ell$ must be contained in $(\mathrm{sing}(\mathcal{F}) \cap \ell) \setminus \{q_1, q_2\}$. Therefore $\mathrm{sing}(\mathcal{F}) \cap \ell$ has cardinality at least three and consequently, the degree of $\mathcal{F}$ is at least two. After analyzing TABLE 1, one concludes that the map f must be as in case (a.2). Explicitly $f^{-1}(q_1) = 2\,q_1$, $f^{-1}(q_2) = 2\,q_2$ and $f^{-1}(q_3) = q_3 + \widehat{q_3}$. In particular $\mathcal{F}$ has degree two; admits exactly three singularities on ℓ, namely q_1, q_2 and q_3; and $\ell_p = \overline{pq_3}$ is the unique $\mathcal{F}$-invariant line through p.

Notice that $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$ is an effective divisor of degree three and its support contains both ℓ_p and the singular points of $\mathcal{F}$. Since q_1, q_2 and p are not aligned, there exists an irreducible component C of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$ distinct from ℓ_p and with degree at most two. According to Theorem 7.3 C is invariant by $\widehat{\mathcal{L}_p}$ or $\mathcal{L}_p$.

If C contains p then by Lemma 6.1 item *(6)* it must be a line and therefore is equal to ℓ_p. This is not possible due to our choice of C. Thus C does not contains p and must be $\widehat{\mathcal{L}_p}$-invariant.

Recall from above that $\overline{pq_i} \cap \mathcal{P} = \{p, q_i\}$ for $i = 1, 2$. Corollary 6.1 implies that the irreducible curves invariant by $\widehat{\mathcal{L}_p}$ that are not lines must have degree at least three. Thus we can assume that C is a line. Moreover

$$\mathrm{sing}\big(\widehat{\mathcal{L}_p}\big) \cap \overline{pq_i} = \{p, q_i\} \quad \text{for} \quad i = 1, 2$$

thanks to Lemma 6.1 item *(4)*. Because the intersections of C with $\overline{pq_1}$ and $\overline{pq_2}$ are singularities of $\widehat{\mathcal{L}_p}$ that are distinct from p, we conclude that $C = \ell$. However ℓ is $\mathcal{F}$-invariant but not $\mathcal{L}_p$-invariant and consequently cannot be in $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$. Thus the assumption $p \notin \mathrm{sing}(\mathcal{F})$ leads to a contradiction. The proposition follows. □

9. Exceptional CDQL webs of degree one on $\mathbb{P}^2$

The degree of a web $\mathcal{W}$ on $\mathbb{P}^2$ is, like in the case of foliations, the number of tangencies of $\mathcal{W}$ with a generic line. In particular the degree of a completely decomposable web is nothing more than the sum of the degrees of its defining foliations and the degree of an CDQL web is nothing more than the degree of its non-linear defining foliation.

9.1. **Infinitesimal automorphisms.**

Proposition 9.1. *Let $\mathcal{W} = \mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ be a CDQL $(k+1)$-webs of degree one with $k \geq 4$. If $K(\mathcal{W}) = 0$ then it exists a line ℓ containing at least $k-1$ points of $\mathcal{P}$. Moreover there is a system of affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ where ℓ is the line at infinity, $\mathcal{F}$ is induced by a homogeneous 1-form ω_0 of degree 1, and the radial vector field $R = x\partial_x + y\partial_y$ is an infinitesimal automorphism of $\mathcal{W}$.*

Proof. If $K(\mathcal{W}) = 0$ then Corollary 8.1 and Proposition 8.2 imply that there is a $\mathcal{F}$-invariant line ℓ that contains (at least) three points of $\mathcal{P}$. A classical result by Darboux says that a degree $d(=1)$ foliation on $\mathbb{P}^2$ has $d^2 + d + 1 (= 3)$ singularities counted with multiplicities. Since at least two of the three singularities of $\mathcal{F}$ necessarily lie on ℓ, it follows that $\mathrm{sing}(\mathcal{F}) \setminus \ell$ reduces to a point or is empty. Proposition 8.4 yields that at least $k-1$ points of $\mathcal{P}$ lie on ℓ. According to Proposition 8.3 the ℓ-polar map of $\mathcal{F}$ does not contract ℓ so one of the singularities of $\mathcal{F}$ is not contained in ℓ.

All that said we can choose affine coordinates where ℓ is the line at infinity and $\mathcal{F}$ is induced by a homogeneous linear 1-form ω_0 that vanishes only at the origin of $\mathbb{C}^2$. It is then clear that R is a infinitesimal automorphism of $\mathcal{W}$. $\square$

It is a simple exercise to show that after a linear change of coordinates the 1-form ω_0 that defines $\mathcal{F}$ on the system of affine coordinates given by Proposition 9.1 can be written as

$$\omega_0' = y\,dx - (x-y)\,dy \qquad \text{or} \qquad \omega_0^\kappa = y\,dx - \kappa\,x\,dy \quad \text{with} \quad \kappa \neq 0, 1\,.$$

The canonical first integral u_0 of $\mathcal{F}$ (see Section 2.1) with respect to the radial vector field R is then

$$u_0' = x/y + \log(y) \qquad \text{or} \qquad u_0^\kappa = \frac{1}{1-\kappa}\log\left(\frac{x}{y^\kappa}\right)\,. \tag{33}$$

Denote by $\mathcal{F}_R$ the foliation induced by the radial vector field. If $\mathcal{P}$ is not included in ℓ then $\mathcal{W}(\mathcal{P}) = \mathcal{F}_R \boxtimes \mathcal{W}(\mathcal{P}\cap\ell)$. It follows from Theorem 2.1 that $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})$ has maximal rank if and only if $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P}\cap\ell)$ does. Therefore we will restrict our attention to the case where $\mathcal{P}\subset\ell$. Until we say otherwise, $\mathcal{W}(\mathcal{P})$ is a web induced by $k\geq 3$ constant 1-forms $\omega_1,\ldots,\omega_k$ and $\mathcal{W} = \mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})$ is a CDQL web of degree one admitting R as an infinitesimal automorphism.

9.2. **The action on $\mathcal{A}(\mathcal{W})$ is semisimple.** Recall from Section 2.1 that L_R acts on the space of abelian relations of $\mathcal{W}$. In our particular setup this action is semisimple as the next proposition shows.

Proposition 9.2. *The linear map $L_R : \mathcal{A}(\mathcal{W}) \to \mathcal{A}(\mathcal{W})$ is diagonalizable and its eigenvalues are non-negative integers. Moreover the zero eigenspace of L_R, if not trivial, has dimension one.*

Proof. For $i = 1,\ldots,k$, the canonical first integrals for the foliations $[\omega_i]$ (with respect to the radial vector field) are the functions

$$u_i = \int \frac{\omega_i}{\omega_i(R)}\,.$$

Clearly $u_i = \log h_i$ for suitable linear forms $h_i \in \mathbb{C}[x,y]$.

On a simply connected open set contained in the complement of $\Delta(\mathcal{W})$, let us consider an abelian relation of the form

$$\sum_{j=0}^{k} P_j(u_j)\,e^{\lambda\,u_j}\,du_i = 0 \tag{34}$$

corresponding to an eigenvalue λ of L_R (see Proposition 2.2).

Analytic continuation of (34) along closed paths homotopic to zero in $\mathbb{C}^2 \setminus \{h_2\cdots h_k = 0\}$ but not homotopic to zero in $\mathbb{C}^2\setminus\{h_1=0\}$ implies that

$$P_1(z+2\pi i)\,e^{2\pi i\lambda} = P_1(z)$$

for every $z\in\mathbb{C}$. Therefore P_1 must be constant and λ an integer. In the same way, one proves that $P_2,\ldots,P_k$ are constant polynomials. Thus $P_0(u_0)\,e^{\lambda\,u_0}$ must be a rational function. Taking into account (33), one deduces that P_0 is also constant. According to the last part of Proposition 2.2, the linear operator L_R is diagonalizable.

Assume that $\lambda \leq 0$. Equation (34) takes the form

$$c_0\, e^{\lambda\, u_0} du_0 + \sum_{j=1}^{k} c_j\, {h_j}^{\lambda-1} dh_j = 0 \tag{35}$$

for certain constants $c_0, c_1, \ldots, c_k \in \mathbb{C}$. It follows that the line $h_j = 0$ is invariant by $\mathcal{F}$ as soon as $c_j \neq 0$. But $\mathcal{F}$ admits at most two invariant lines through the origin. Thus if (35) is not trivial, one can assume that it has the form

$$e^{\lambda\, u_0} du_0 + c_1\, x^{\lambda-1} dx + c_2\, y^{\lambda-1} dy = 0\,.$$

Since the curvature of $[dx\, dy] \boxtimes [ydx - (x-y)dy]$ is non-zero, an identity of the form (34) holds only when $\omega_0 = \omega_0^{\kappa}$ for a certain κ and the eigenvalue λ is zero. $\square$

Corollary 9.1. *If $rk(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) > rk(\mathcal{W}(\mathcal{P})) + 1$ then $\mathcal{F}$ admits a polynomial first integral of the form $x^p y^q$ where p and q are relatively prime positive integers.*

Proof. It follows from Proposition 9.2 that there is at least one abelian relation of the form (35) with $c_0 = 1$ and $\lambda \in \mathbb{N}^*$. Integrating these, it follows that

$$\int e^{\lambda u_0} du_0 = \frac{1}{\lambda} \sum_{j=1}^{k} c_j\, h_j^{\lambda}$$

is a homogeneous polynomial first integral of $\mathcal{F}$. Then $\omega_0 = \omega_0^{p/q}$ where p and q are relatively prime positive integers. $\square$

9.3. **Characterization of $\mathcal{F}$.** Let $\delta \in \{0, 1, 2\}$ be the number of lines through the origin of $\mathbb{C}^2$ that are invariant by $\mathcal{F}$.

Lemma 9.1. *If $\mathcal{F}$ has a first integral of the form $x^p y^q$ where $p, q \in \mathbb{N}$ are relatively prime then*

$$rk\big(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})\big) - rk\big(\mathcal{W}(\mathcal{P})\big) \leq \frac{2k-2}{p+q}\,.$$

Moreover if the equality holds then $\delta \neq 1$.

Proof. The abelian relations involving $\mathcal{F}$ and corresponding to strict positive eigenvalues can be written in integrated form as

$$(x^p y^q)^r = \sum_{i=1}^{k} \mu_i\, h_i^{(p+q)r} \qquad (\mu_1, \ldots, \mu_k \in \mathbb{C})\,.$$

Let $\alpha \in \mathbb{C}^*$ be sufficiently general and set $\varphi(x, y) = (x, \alpha\, y)$. Then

$$0 = (x^p y^q)^r - \alpha^{-qr} \varphi^*\big((x^p y^q)^r\big) = \sum_{i=1}^{k} \mu_i\, h_i^{(p+q)r} - \alpha^{-qr} \sum_{i=1}^{k} \mu_i\, \varphi^*\big(h_i^{(p+q)r}\big)\,.$$

where the right-hand side involves at most $2k - \delta$ distinct linear forms.

It is very convenient to interpret geometrically this equality in terms of the rational normal curve Γ in $\mathbb{P}^{(p+q)r}$ of degree $(p+q)r$. It says that there are $2k - \delta$ distinct points on Γ that are not in general position. It is classical result that m distinct points on the rational normal curve of degree l are in general position if $m \leq l + 1$. Therefore $(p+q)r + 2 \leq 2k - \delta$. Hence

$$r \leq \frac{2k - \delta - 2}{p+q}\,.$$

Recall from the proof of Proposition 9.2 that when $\delta \neq 2$, all the abelian relations are polynomials identities and when $\delta = 2$ there is exactly one extra logarithmic abelian relation. It follows that

$$\dim \frac{\mathcal{A}\big(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})\big)}{\mathcal{A}\big(\mathcal{W}(\mathcal{P})\big)} \leq \begin{cases} \dfrac{2k-\delta-2}{p+q} & \text{when} \quad \delta \in \{0,1\}, \\ \dfrac{2k-2}{p+q} & \text{when} \quad \delta = 2. \end{cases}$$

□

9.4. **The classification.** If a web $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ of degree 1 is of maximal rank it follows from Lemma 9.1 that δ must be 0 or 2 and $p = q = 1$. It turns out that there exist examples of $(k+1)$-webs of maximal rank and of degree 1 for any $\delta \in \{0,2\}$ and any $k \geq 3$, namely the webs $\mathcal{A}_I^k$ and $\mathcal{A}_{III}^{k-2}$ of the Introduction.

To complete the classification of exceptional CDQL webs of degree one on $\mathbb{P}^2$, it suffices to show that these examples are the only ones up to projective automorphisms.

In what follows, $\mathcal{P}$ and $\mathcal{Q}$ designate two sets of k points on the line at infinity, disjoint of $[1:0:0]$ and $[0:1:0]$, with $k \geq 3$ if $\delta = 0$ and $k \geq 1$ if $\delta = 2$. Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{C}^*$ be such that

$$\mathcal{W}(\mathcal{P}) = \big[d(x+a_1\,y)\cdots d(x+a_k\,y)\big] \quad \text{and} \quad \mathcal{W}(\mathcal{Q}) = \big[d(x+b_1\,y)\cdots d(x+b_k\,y)\big]\,.$$

In particular, $a_i \neq a_j$ and $b_i \neq b_j$ for all i,j such that $1 \leq i < j \leq k$.

Proposition 9.3. *If the two CDQL $(k+1)$-webs $[d(xy)]\boxtimes\mathcal{W}(\mathcal{P})$ and $[d(xy)]\boxtimes\mathcal{W}(\mathcal{Q})$ are both of maximal rank, then they are projectively equivalent.*

Proof. From the proof of Lemma 9.1, it follows that

$$\begin{aligned} \big(xy\big)^{k-1} &= \sum_{i=1}^{k} \lambda_i \,(x+a_i\,y)^{2\,k-2} = \\ &= \sum_{i=1}^{k} \mu_i \,(x+b_i\,y)^{2\,k-2} \end{aligned}$$

for suitable complex numbers λ_i, μ_i. Since, for any $\nu \in \mathbb{C}^*$, the automorphisms $(x,y) \mapsto (x, \nu\, y)$ preserve the foliation $[d(xy)]$, one can assume that $a_1 = b_1$ with no loss of generality. Subtracting the two summations yields

$$0 = (\lambda_1 - \mu_1)\,(x+a_1\,y)^{2\,k-2} + \sum_{i=2}^{k} \lambda_i \,(x+b_i\,y)^{2\,k-2} - \sum_{i=2}^{k} \mu_i \,(x+b_i\,y)^{2\,k-2}\,. \tag{36}$$

One can interpret this relation geometrically as in the proof of Lemma 9.1 by considering the powers $(x+a_i\,y)^{2\,k-2}$ and $(x+b_i\,y)^{2k-2}$ (for $i=1,\ldots,k$) as points on the rational normal curve of degree $2k-2$. Notice that the number of these points is at most $2k-1$. Since m distinct points on the rational normal curve of degree $2k-2$ are inevitably in general position when $m \leq 2k-1$, the relation (36) implies that the sets $\{a_1,\ldots,a_k\}$ and $\{b_1,\ldots,b_k\}$ coincide. The proposition follows. □

In the same way, one proves the

Proposition 9.4. *If the two CDQL webs $[d(xy)\,dx\,dy]\boxtimes\mathcal{W}(\mathcal{P})$ and $[d(xy)\,dx\,dy]\boxtimes\mathcal{W}(\mathcal{Q})$ are both of maximal rank, then they are projectively equivalent.*

At this point we have concluded the classification of exceptional CDQL webs of degree one on $\mathbb{P}^2$. They are all projectively equivalent to one the webs in the families $\mathcal{A}_k^*$. We point out that we have made a heavy use of the structure of the space of abelian relations of these webs. It would be interesting to find an alternative approach more focused on the curvature. For instance one could try to classify all the flat CDQL webs of degree 1 on $\mathbb{P}^2$.

10. Flat CDQL webs on $\mathbb{P}^2$ of degree at least two

Based on the results of Section 8 we will derive a complete list of flat CDQL $(k+1)$-webs on $\mathbb{P}^2$ of degree at least two when $k \geq 4$. Up to automorphisms of $\mathbb{P}^2$, there are exactly sixteen examples — nine of degree 2, three of degree 3 and four of degree 4.

10.1. **Flat CDQL webs of degree two.** In the present and in the next two subsections, we will treat independently the three possibilities for the degree of $\mathcal{F}$. We start by considering flat CDQL webs of degree two.

Proposition 10.1. *Let $\mathcal{F}$ be a foliation of degree 2 and $\mathcal{P} \subset \mathbb{P}^2$ be a finite set of at least four points. If $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then $\mathcal{F}$ is projectively equivalent to one of the following foliations:*

(a.1.h) $\mathcal{F} = \big[d\,(xy(x+y))\,\big]$;

(a.2.h) $\mathcal{F} = \Big[d\big(\frac{xy}{x+y}\big)\Big]$;

(a.3.h) $\mathcal{F} = \big[d\,\big((4y^2+xy+4x^2)^3(x+y)\big)\,\big]$;

(a.4.h) $\mathcal{F} = \big[d\,\big(x^3+y^3\big)\,\big]$;

(a.2) $\mathcal{F} = \Big[d\left(\frac{y^2-1}{x^2-1}\right)\Big]$.

Moreover in the cases (a.1.h), (a.2.h) and (a.3.h), $\mathcal{P}$ has cardinality four and is equal to the singular set of $\mathcal{F}$. In the case (a.4.h) there are two possibilities for $\mathcal{P}$. Either $\mathcal{P}$ is equal to $\mathrm{sing}(\mathcal{F}) \cup \{[0:1:0],[1:0:0]\}$ *or to* $(\mathrm{sing}(\mathcal{F}) \cup \{[0:1:0],[1:0:0]\}) \setminus \{[0:0:1]\}$. *Finally, in case (a.2) the set $\mathcal{P}$ is any of the subsets of* $\mathrm{sing}(\mathcal{F})$ *containing the four base points of the pencil* $< x^2 - z^2, y^2 - z^2 >$. *Up to the automorphism group of $\mathcal{F}$ there are only four possibilities.*

Proof. If the points in $\mathcal{P}$ are in general position then, according to Corollary 8.1, $\mathcal{F}$ is the pencil generated by two reduced conics intersecting transversally and $\mathcal{P}$ is the set of base points of this pencil. So $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ is Bol's web and we are in case *(a.2)*.

From now on we will assume that there is a line $\ell \subset \mathbb{P}^2$ that contains at least three points of $\mathcal{P}$. Up to the end of the proof we work with affine coordinates $[x:y:1]$ on $\mathbb{C}^2 \subset \mathbb{P}^2$, for which $\ell = \{z=0\}$ is the line at infinity. We will also assume that $\mathcal{P} \cap \ell$ contains $q_1 = [1:0:0]$, $q_2 = [0:1:0]$ and $q_3 = [1:-1:0]$.

We will deal separately with each one of the four possibilities given by TABLE 1 for the ℓ-polar map f of $\mathcal{F}$.

Case (a.1). In this case $k_\ell = 3$ and $f^{-1}(q_i) = q_i + \widehat{q}_i$ for $i = 1,2,3$. Notice that $f^{-1}(q_1) = q_1 + \widehat{q}_1$ implies that $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_1})$ is the union of three lines: the line at infinity ℓ together with two other lines, one intersecting ℓ at q_1 and the other at $\widehat{q_1}$. A similar situation occurs for $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_2})$ and $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_3})$.

Therefore $f^{-1}(q_1) = q_1 + \widehat{q_1}$ and $f^{-1}(q_2) = q_2 + \widehat{q_2}$ imply that $\mathcal{F}$ is induced by a 1-form like

$$\omega = (y + c_1)(2x + y + c2)\, dx + (x + c_3)(2y + x + c_4)\, dy\,,$$

where c_1, c_2, c_3 and c_4 are complex constants. After composing with a translation we can assume that $c_1 = c_3 = 0$.

It remains to consider the conditions imposed by $f^{-1}(q_3) = q_3 + \widehat{q_3}$. Notice that $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_3})$ is cut out by

$$y(2x + y + c2) - x(2y + x + c_4) = y^2 - x^2 + c_2 y - c_4 x\,.$$

This latter expression is a product of lines if and only if $c_2 = \pm c_4$. When $c_2 = c_4 = 0$ we arrive at the homogeneous foliation

$$\mathcal{F} = \big[d\big(xy(x + y)\big)\big]\,.$$

We are in case *(a.1.h)*. Because the cardinality of the singular set of $\mathcal{F}$ is four, there is just one possible choice for $\mathcal{P}$: $\mathcal{P} = \mathrm{sing}(\mathcal{F})$.

If $c_2 \neq 0$ then after applying a homothety we can assume that $c_2 = 1$. We arrive at two possibilities for ω, namely

$$\omega_\pm = y\,(2x + y + 1)\, dx + x\,(2y + x \pm 1)\, dy\,.$$

Let $\mathcal{F}_\pm$ are the corresponding foliations. By hypothesis, $k_\ell = 3$ and $\mathcal{P}_\ell = \{q_1, q_2, q_3\}$. If $\mathcal{F}_\pm \boxtimes \mathcal{W}(\mathcal{P})$ is assumed to be flat then Proposition 8.4 implies that $\mathcal{P} \setminus \ell$ is included in the support of $\mathrm{sing}(\mathcal{F}_\pm) \cap \mathbb{C}^2$. In particular there are only a finite number of possibilities for $\mathcal{P}$. Lengthy, but straight-forward, computations shows that $K(\mathcal{F}_\pm \boxtimes \mathcal{W}(\mathcal{Q} \cup \{q_1, q_2, q_3\})) \neq 0$ for any non-empty subset $\mathcal{Q} \subset \mathrm{sing}(\mathcal{F}_\pm) \cap \mathbb{C}^2$. Therefore the foliations $\mathcal{F} = \mathcal{F}_\pm$ are not among the defining foliations of any flat CDQL webs of order at least five.

Case (a.2). In this case $k_\ell = 3$, $f^{-1}(q_i) = 2q_i$ for $i = 1, 2$ and $f^{-1}(q_3) = q_3 + \widehat{q_3}$. Arguing as in the paragraph above, we conclude that $\mathcal{F}$ is induced by

$$\omega = y\,(y - 1)\, dx + x\,(x - 1)\, dy \qquad \text{or} \qquad \omega' = y^2 dx + x^2 dy\,. \tag{37}$$

Recall that $\mathcal{P} \setminus \ell$ is included in $\mathrm{sing}(\mathcal{F}) \cap \mathbb{C}^2$ (according to Proposition 8.4). If $\mathcal{F}$ is induced by ω', only one possibility can happen, namely $\mathcal{P} = \{q_1, q_2, q_3, p_4\}$ where $p_4 = [0 : 0 : 1]$ (since $\mathrm{sing}(\mathcal{F}) = \{q_1, q_2, q_3, p_4\}$). By a direct computation, one verifies that the 5-web defined by $\mathcal{P}$ and ω' is indeed flat.

Let us now consider the case when $\mathcal{F}$ is the foliation induced by ω. If we set $p_5 = [1 : 1 : 1 :]$, $p_6 = [0 : 1 : 1]$ and $p_7 = [1 : 0 : 1]$ then

$$\mathrm{sing}(\mathcal{F}) = \{q_1, q_2, q_3, p_4, p_5, p_6, p_7\}\,.$$

A direct computation shows that there are exactly four subsets $\mathcal{P}$ of $\mathrm{sing}(\mathcal{F})$ that strictly contain$\mathcal{P}_\ell = \{q_1, q_2, q_3\}$ and that verify $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$, namely

$$\mathcal{P} = \left\{ \begin{array}{lcl} \mathcal{P}_\ell & \cup & \{p_4, p_5\}\,, \\ \mathcal{P}_\ell & \cup & \{p_4, p_5, p_6\}\,, \\ \mathcal{P}_\ell & \cup & \{p_4, p_5, p_7\}\,, \\ \mathcal{P}_\ell & \cup & \{p_4, p_5, p_6, p_7\}\,. \end{array} \right.$$

Notice that $\mathcal{F}\boxtimes\mathcal{W}(\{q_1,q_2,p_4,p_5\})$ is nothing more than Bol's exceptional 5-web. The second and the third possibilities for $\mathcal{P}$ are equivalent since they are interchanged by the $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P}_\ell)$-automorphism $(x,y)\mapsto(y,x)$. All the cases above lead to exceptional webs. Indeed they are the webs $\mathcal{B}_6,\mathcal{B}_7$ and $\mathcal{B}_8$ of the Introduction that have been previously studied in [44, 40].

Case (a.3). Here $k_\ell=3$, $f^{-1}(q_i)=2\,\widehat{q_i}$ for $i=1,2$ and $f^{-1}(q_3)=q_3+\widehat{q_3}$. Theorem 7.3 tell us that every irreducible component C of $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})$ is invariant by $\mathcal{L}_{q_1}$ or $\widehat{\mathcal{L}_{q_1}}$. Because $f^{-1}(q_1)=2\widehat{q_1}$, there exists such C invariant by $\widehat{\mathcal{L}_{q_1}}$ and distinct from ℓ. The divisor $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_1})-\ell$ is effective and of degree 2. Consequently the degree of C is at most two. If it is two then Corollary 6.1 implies that for every point $p\in\mathcal{P}\setminus\{q_1,q_2,q_3\}$ the line $\overline{q_1p}$ contains a third point of $\mathcal{P}$. Proposition 8.2 implies the $\mathcal{F}$ invariance of $\overline{q_1p}$ contradicting $f^{-1}(q_1)=2\widehat{q_1}$. This proves that for $i=1,2$ every irreducible component of $\mathrm{tang}(\mathcal{F},\mathcal{L}_{q_i})-\ell$ must be a $\widehat{\mathcal{L}_{q_i}}$-invariant line through $\widehat{q_i}$. Because $\mathrm{Card}(\mathcal{P})\geq 4$ and $\mathcal{P}\not\subseteq\ell$, for $i=1,2$, the foliation $\widehat{\mathcal{L}_{q_i}}$ has only one invariant line through $\widehat{q_i}$ distinct from ℓ. Therefore there exists constants c_1 and c_2 for which $\mathcal{F}=[(2x+y+c_1)^2dx+(x+2y+c_2)^2dy]$. Modulo a translation, we can assume that $c_1=c_2=0$. Thus

$$\mathcal{F}=\left[(2x+y)^2dx+(x+2y)^2dy\right]=\left[d\left((x+y)(4y^2+xy+4x^2)^3\right)\right].$$

We are in case *(a.3.h)* and necessarily $\mathcal{P}=\mathrm{sing}(\mathcal{F})$ since $\mathrm{Card}(\mathrm{sing}(\mathcal{F}))=4$.

Case (a.4). We finally arrive at the last case of TABLE 1 where $k_\ell=5$, $f^{-1}(q_i)=2\,\widehat{q_i}$ for $i=1,2$ and $f^{-1}(q_j)=q_j+\,\widehat{q_j}$ for $j=3,4,5$.

Arguing exactly as in case (a.3) one can show that in this case $\mathcal{F}$ is also homogeneous. Therefore $\mathcal{F}=[x^2dx+y^2dy]=[d(x^3+y^3)]$ and, as it was shown in Section 2, any of the two possibilities for $\mathcal{P}$, namely

$$\mathcal{P}=\{q_1,\ldots,q_5,[0:0:1]\}=\mathrm{sing}(\mathcal{F})\cup\{[0:1:0],[1:0:0]\}$$

or

$$\mathcal{P}=\{q_1,\ldots,q_5\}=(\mathrm{sing}(\mathcal{F})\cup\{[0:1:0],[1:0:0]\})\setminus\{[0:0:1]\}$$

leads to exceptional, and in particular flat, webs. $\square$

10.2. **Flat CDQL webs of degree three.** The classification of flat CDQL webs of degree three is given by the following proposition.

Proposition 10.2. *Let $\mathcal{F}$ be a foliation of degree three and $\mathcal{P}\subset\mathbb{P}^2$ be a finite set of at least four points. If $K(\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P}))=0$ then $\mathcal{F}$ is projectively equivalent to one of the following foliations:*

(a) $\mathcal{F}=\left[\left(x^3+y^3+1+6\,xy^2\right)dx-\left(x^3+y^3+1+6\,x^2y\right)dy\right]$;

(b) $\mathcal{F}=\left[d\big(x(x^3+y^3)\big)\right]$.

Moreover $\mathcal{P}=\mathrm{sing}(\mathcal{F})\cap\{x-y=0\}$ in case (a) and $\mathcal{P}=\mathrm{sing}(\mathcal{F})$ or $\mathcal{P}=\mathrm{sing}(\mathcal{F})\setminus\{[0:0:1]\}$ in case (b).

Proof. Corollary 8.1 implies that there exists a line ℓ containing at least three points of $\mathcal{P}$. According to TABLE 1, ℓ must contain indeed four points of $\mathcal{P}$, say $q_1,\ldots,q_4$, and the ℓ-polar map f of $\mathcal{F}$ is completely determined. It satisfies

$$f^{-1}(q_i)=q_i+2\,\widehat{q_i}\quad\text{for}\quad i=1,\ldots,4. \tag{38}$$

Recall from [9] that a foliation of degree 3 has at most four singularities on an invariant line. Therefore $\mathrm{sing}(\mathcal{F}) \cap \ell = \{q_1, \ldots, q_4\}$. Lemma 8.2 implies that through each q_i there is a $\mathcal{F}$-invariant line ℓ_i distinct from ℓ.

From (38) one deduces that

$$\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_i}) = \ell + \ell_i + C_i$$

where C_i is a conic (not necessarily reduced nor irreducible) intersecting ℓ at $\widehat{q_i}$ with multiplicity two. Theorem 7.3 implies moreover that C_i is $\widehat{\mathcal{L}_{q_i}}$-invariant.

Claim 10.1. *None of the conics C_i is reduced and irreducible.*

Proof. Aiming at a contradiction, suppose that C_1 is reduced and irreducible. Then C_1 is a $\widehat{\mathcal{L}_{q_1}}$-invariant curve of degree two. Corollary 6.1 implies that $\mathcal{P}$ is contained in the union of ℓ and ℓ_1 and that $\mathcal{P} \cap \ell_1$ must have the same cardinality of $\mathcal{P} \cap \ell$, that is $\mathrm{Card}(\mathcal{P} \cap \ell_1) = 4$. Recall from above that ℓ_1 is $\mathcal{F}$-invariant and $\ell \cap \ell_1 = q_1$. Let p_5, p_6 and p_7 be the points of $\mathcal{P}$ in ℓ_1 distinct from q_1, see Figure 3.

A simple computation shows that $\widehat{q_2} \neq q_1$ and (38) implies that q_2 is contained in at most one $\mathcal{F}$-invariant line different from ℓ. Therefore at least two of the three lines $\overline{q_2p_5}, \overline{q_2p_6}$ and $\overline{q_2p_7}$ are not $\mathcal{F}$-invariant. Proposition 8.2 combined with item *(4)* of Lemma 6.1 imply that two of the three points p_5, p_6 and p_7 are singularities of $\widehat{\mathcal{L}_{q_2}}$. Therefore the singularities of $\widehat{\mathcal{L}_{q_2}}$ are not aligned. Proposition 6.1 tell us that the only algebraic leaves of $\widehat{\mathcal{L}_{q_2}}$ are lines through q_2. Theorem 7.3 implies that $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_2})$ is constituted of four lines passing trough q_2. Consequently $f^{-1}(q_2) = 3\, q_2$ by Lemma 8.2 contradicting (38). □

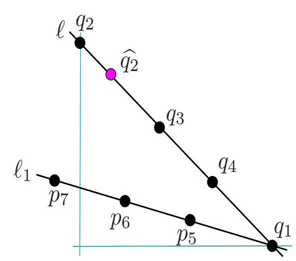

FIGURE 3. The singularities of $\widehat{\mathcal{L}_{q_2}}$ are not aligned.

If each C_i is a union of two distinct lines then the linear system of cubics

$$\left\{ \mathrm{tang}(\mathcal{F}, \mathcal{L}_p) - \ell \right\}_{p \in \ell} \tag{39}$$

contains four totally decomposable fibers. These are triangles (three lines in general position) with one of the vertices on ℓ. This is sufficient (see [45, Section 4.4]) to ensure that (39) is the Hesse pencil and that ℓ is one of its nine harmonic lines. Recall from [3] that the Hesse pencil is classically presented as the one generated by the cubic forms $x^3 + y^3 + z^3$ and xyz. In these coordinates the harmonic lines are

$$\begin{array}{lll} \{x - y = 0\} & \{x - \epsilon y = 0\} & \{x - \epsilon^2 y = 0\} \\ \{x - z = 0\} & \{x - \epsilon z = 0\} & \{x - \epsilon^2 z = 0\} \\ \{y - z = 0\} & \{y - \epsilon z = 0\} & \{y - \epsilon^2 z = 0\} . \end{array}$$

The subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ that preserves the Hesse Pencil is the Hessian group G_{216} isomorphic to $\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2 \rtimes \mathrm{SL}(2,\mathbb{F}_3)$. The projective transformations

$$a : (x:y:z) \mapsto (y:z:x) \quad \text{and} \quad b : (x:y:z) \mapsto (x:\epsilon y:\epsilon^2 z)$$

generates a subgroup Γ of G_{216} isomorphic to $\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2$ acting transitively on the set of harmonic lines. Thus we loose no generality by assuming that $\ell = \{x-y=0\}$.

Notice that the singular set of $\mathcal{F}$ contains the base points of the linear system (39). Thus the singular set of $\mathcal{F}$ contains the nine base points of the Hesse pencil together with the four fixed points of f on ℓ. Since $\mathcal{F}$ has degree 3, it has at most $3^2+3+1 = 13$ singular points. Therefore the singular set of $\mathcal{F}$ has been completely determined and each of its points has multiplicity one. In other words the singular scheme of $\mathcal{F}$ is everywhere reduced.

The main Theorem of [12] says that a foliation on $\mathbb{P}^2$ of degree greater than one is completely determined by its singular scheme. Therefore $\mathcal{F}$ is determined and it is equal to the foliation induced by the 1-form

$$\omega = \left(x^3+y^3+1+6\,xy^2\right)dx - \left(x^3+y^3+1+6\,x^2y\right)dy. \tag{40}$$

Therefore $\mathcal{F}$ is in case *(a)* of the statement. Concerning the set of points $\mathcal{P}$ it must be equal to $\{q_1,\ldots,q_4\}$. Otherwise Corollary 6.1 would imply that there would exist just one $\widehat{\mathcal{L}_{q_i}}$-invariant line through $\widehat{q_i}$ contrary to our assumptions on C_i. A direct computation shows that $K(\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P}))=0$.

If at least one of the conics C_i is non-reduced then [39, Proposition 3.1] implies that all the C_i's are double lines. Therefore $\mathcal{F}$ is a homogeneous foliation on the affine chart where ℓ is the line at infinity and the singularity of $\mathcal{F}$ corresponding to the unique base point of $\{\mathrm{tang}(\mathcal{F},\mathcal{L}_p)-\ell\}_{p\in\ell}$ is the origin. Thus $\mathcal{F}$ is defined by a homogenous 1-form with coefficients equal to the coefficients of ℓ-polar map, that is

$$\mathcal{F} = \Big[y\big(3x+y\,(1-\xi_3^2)\big)^2dx + 3x\left(x+y\,(1-\xi_3^2)\right)^2dy\Big] = \big[d\big(xy(x+y)(x-\xi_3 y)\big)\big]\ .$$

A linear change of coordinates envoys $\mathcal{F}$ to the form presented in case *(b)* of the statement. Finally, it follows from Proposition 8.4 that there are only two possibilities for $\mathcal{P}$: those mentioned in the statement of the proposition. Both cases are exceptional, and in particular flat, as we have shown in Section 2. □

10.3. **Flat CDQL webs of degree four.** Finally we turn our attention to the flat CQDL webs $\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})$ when $\deg(\mathcal{F})=4$ and the cardinality of $\mathcal{P}$ is at least four. Corollary 8.1 implies that $\mathcal{P}$ cannot be in general position and Theorem 8.1 shows that no four points in $\mathcal{P}$ are aligned. Therefore there exists a line ℓ such that $\mathcal{P}\cap\ell=\{q_1,q_2,q_3\}$.

According to TABLE 1 there are only two possibilities for the ℓ-polar map f of $\mathcal{F}$. In both cases f has 5 distinct fixed points that are cut out by the polynomial $xy(x+y)(x^2+xy+y^2)$. In particular, $\mathcal{F}$ has exactly 5 singular points on ℓ according to Lemma 8.1. Notice that $\mathrm{sing}(\mathcal{F})\cap\ell$ does not intersect $\{\widehat{q_1},\widehat{q_2},\widehat{q_3}\}$.

Lemma 10.1. *For $i=1,2,3$, the tangency of $\mathcal{F}$ and $\mathcal{L}_{q_i}$ is a union of lines.*

Proof. Let's first consider case (c.2) of TABLE 1, that is $f^{-1}(q_i)=3q_i+\widehat{q_i}$ for every $i=1,2,3$. By Theorem 7.3, any irreducible component C of the tangency between $\mathcal{F}$ and $\mathcal{L}_{q_i}$ is invariant by $\mathcal{L}_{q_i}$ or $\widehat{\mathcal{L}_{q_i}}$. In the former case C has to be a line as all the irreducible curves left invariant by $\mathcal{L}_{q_i}$. In the latter case, C is also a line. This follows from Lemma 6.1 item *(2)* when C passes through q_i and from $f^{-1}(q_i) = 3\,q_i + \widehat{q_i}$ when C passes through $\widehat{q_i}$.

We will now deal with case (c.1) of TABLE 1, that is $f^{-1}(q_i) = q_i + 3\widehat{q_i}$ for every $i = 1, 2, 3$. We can assume that $q_1 = p_1 = [0 : 1 : 0]$, $q_2 = p_2 = [1 : 0 : 0]$, $q_3 = p_3 = [1 : -1 : 0]$ and $p_4 = [0 : 0 : 1] \notin \ell$.

We will deal separately two cases: (a) the cardinality of $\mathcal{P}$ is four, and (b) the cardinality of $\mathcal{P}$ is at least five.

Case (a): $\mathbf{k} = \mathbf{Card}(\mathcal{P}) = 4$. In this case we will work in the affine coordinates $(x, y) = [x : y : 1]$. Notice that

$$\widehat{\mathcal{L}_1} = \left[d\Big(\frac{(x+2y)^3}{x} \Big) \right] , \quad \widehat{\mathcal{L}_2} = \left[d\Big(\frac{(2x+y)^3}{y} \Big) \right] \quad \text{and} \quad \widehat{\mathcal{L}_4} = \Big[d\big((xy(x+y) \big) \Big] .$$

If we write $\mathcal{F} = [a(x, y)dx + b(x, y)dy]$, where a and b are relatively prime polynomials, then $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_1})$ is defined by the vanishing of $a(x, y)$. Similarly $\mathrm{tang}(\mathcal{F}, \mathcal{L}_2)$ is defined by the vanishing of $b(x, y)$. Theorem 7.3 implies that

$$\mathcal{F} = \big[(y - \lambda_1) \left((2x+y)^3 - \mu_1 y \right) dx + (x - \lambda_2) \left((x+2y)^3 - \mu_2 x \right) dy \big] \tag{41}$$

for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$.

On the one hand, $\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)$ contains the singular points of $\mathcal{F}$ on ℓ. Theorem 7.3 implies that its irreducible components must be irreducible cubics in the pencil $< z^3, xy(x + y) >$ or lines connecting p_4 to one of the 5 singularities of $\mathcal{F}$ at ℓ (corresponding to the 5 fixed points of the ℓ-polar map of $\mathcal{F}$). Thus,

$$\mathrm{tang}(\mathcal{F}, \mathcal{L}_4) = \big\{ (x^2 + xy + y^2)(xy(x+y) - \lambda_3) = 0 \big\} \tag{42}$$

for a certain $\lambda_3 \in \mathbb{C}$.

On the other hand, the tangency between $\mathcal{F}$ and $\mathcal{L}_4$ is defined by the vanishing of the contraction of the 1-form in (41) with $x\partial_x + y\partial_y$. Explicitly

$$\mathrm{tang}(\mathcal{F}, \mathcal{L}_4) = \big\{ x(y - \lambda_1) \left((2x+y)^3 - \mu_1 y \right) + y(x - \lambda_2) \left((x+2y)^3 - \mu_2 x \right) = 0 \big\} .$$

Comparing the homogeneous components of degree two of the two presentations of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)$, one concludes that $\lambda_3 = \lambda_1\mu_1 = \lambda_2\mu_2 = 0$. Plugging $\lambda_3 = 0$ into (42) shows that all the five lines cut out by $xy(x + y)(x^2 + xy + y^2)$ are $\mathcal{F}$-invariant. The $\mathcal{F}$-invariance of $\{x = 0\}$ and $\{y = 0\}$ ensures that $\lambda_1 = \lambda_2 = 0$. Finally, since the homogeneous component of degree three of (42) is zero, $\mu_1 = \mu_2 = 0$. It is then clear that the expression of $\mathcal{F}$ in (41) is homogeneous. Consequently $\mathrm{tang}(\mathcal{F}, \mathcal{L}_q)$ is a union of lines for every $q \in \ell$.

Case (b): $\mathbf{k} = \mathbf{Card}(\mathcal{P}) \geq \mathbf{5}$. Notice that $\mathcal{P}$ is not in barycentric general position with respect to none of the points q_1, q_2, q_3 because $f^{-1}(q_i) \neq 4q_i$ for $i = 1, 2, 3$. Proposition 6.1 implies that all the leaves of $\widehat{\mathcal{L}_i}$ are algebraic. From the proof of Corollary 6.1, one deduces that the leaves of $\widehat{\mathcal{L}_1}$ (for instance) are irreducible components of elements of a pencil of the form $\mathcal{H} =< (x+2y+\lambda z)^{\deg(R)}, R(x, z) >$, where $\lambda \in \mathbb{C}$ and R is a homogeneous polynomial of degree $k - 1$. The irreducible factors of R correspond to the lines $\overline{q_1 p}$ where p ranges in $\mathcal{P}_1 = \mathcal{P} \setminus \{q_1\}$ and their multiplicities correspond to number of points of $\mathcal{P}_1$ contained in the respective lines.

If $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_1})$ has an non-linear irreducible component C then its degree is at most three and is an irreducible component of an element of the pencil $\mathcal{H}$. But $(x + 2y + \lambda z)^{\deg(R)} - \mu R(x, z)$ admits an irreducible factor of degree at most three for some $\mu \in \mathbb{C}^*$ only when R is a square. Indeed, on the one hand the square of each linear factor of R must divide R otherwise Corollary 6.1 would imply that C has degree $k-1 \geq 4$. On the other hand the third power of any linear factor of R cannot divide R, otherwise it would exist four points in $\mathcal{P}$ on the same line contradicting Theorem 8.1.

Since R is a square it must exist a third point $p_5 \in \mathcal{P}$ contained in the the line $\overline{q_1 p_4}$. From the fact that $\mathcal{P}$ is not in q_2-barycentric general position it follows that $\mathrm{sing}(\widehat{\mathcal{L}_{q_2}}) - \{q_2\}$ is contained in a line. Using that $\widehat{q_2} \neq q_1$ one deduces that it must exist a point $p_6 \in \overline{q_2 p_4} \cap \mathcal{P}$. Since $R(x,z)$ is a square, the line $\overline{q_1 p_6}$ must contain another point of $\mathcal{P}$ (noted p_7 in Figure 4 below).

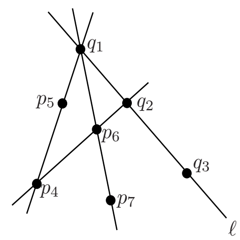

FIGURE 4. Seven points of $\mathcal{P}$.

Proposition 8.2 tell us that any line containing at least 3 points of $\mathcal{P}$ must be $\mathcal{F}$-invariant. Thus there are at least three $\mathcal{F}$-invariant lines through q_1. This contradicts $f^{-1}(q_1) = q_1 + 3\,\widehat{q_1}$ and ends the proof of the lemma. $\square$

We will also need a classical result of Darboux about the degree of foliations induced by pencil of curves. We state it below as a lemma.

Lemma 10.2. *If $F, G \in \mathbb{C}[x,y,z]$ are relatively prime homogeneous polynomials of degree e then*

$$FdG - GdF = \left(\prod_H H^{e(H)-1} \right) \cdot \omega$$

where ω is a homogenous polynomial 1-form with codimension two singular set; H runs over the irreducible components of the polynomials $\{sF + tG = 0\}_{(s:t)\in\mathbb{P}^1}$; and $e(H)$ denotes de maximum power of H that divides the member of the pencil that contains H. In particular if $\mathcal{F} = [d(F/G)]$ then

$$\deg(\mathcal{F}) = 2e - 2 - \sum_H \deg(H)(e(H)-1)\,.$$

Proof. See [27, Proposition 3.5.1, pages 110–111]. $\square$

Proposition 10.3. *Let $\mathcal{F}$ be a foliation of degree four and $\mathcal{P} \subset \mathbb{P}^2$ be a finite set of at least four points. If $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then $\mathcal{F}$ is projectively equivalent to one of the following foliations:*

(a) $\mathcal{F} = \Big[d\Big(xy\,(x+y)\,(x^2+xy+y^2)^3 \Big) \Big]$;

(b) $\mathcal{F} = \Big[d\Big(\dfrac{xy\,(x+y)}{x^2+xy+y^2} \Big) \Big]$;

(c) $\mathcal{F} = \left[d\Big(\dfrac{x^3+y^3+1}{xy}\Big)\right]$.

Moreover $\mathcal{P} = \{[1:-1:0],[1:0:0],[0:1:0],[0:0:1]\}$ *in cases (a) and (b). In case (c),* $\mathcal{P}$ *is equal to the nine base points of the pencil* $< xy, x^3+y^3+1 >$ *or is equal to the three base points of this pencil at the line at infinity union with* $[0:0:1]$.

Proof. We keep the notations from the beginning of this section. According to TABLE 1 there two possibilities for the ℓ-polar map of $\mathcal{F}$: (c.1) and (c.2). We will deal with them separately.

Case (c.1). We are assuming that the ℓ-polar map of $\mathcal{F}$ satisfies $f^{-1}(q_i) = q_i + 3\,\widehat{q_i}$ for $i = 1,\dots,3$. According to Lemma 10.1, the tangency between $\mathcal{F}$ and $\mathcal{L}_{q_i}$ is a union of lines. Since $\mathcal{P}$ has cardinality at least four, there exists $p_4 \in \mathcal{P}\setminus\ell$. Moreover $\mathcal{P}$ is not in q_i-barycentric general position. Proposition 6.1 implies that the foliation $\widehat{\mathcal{L}_{q_i}}$ admits exactly one invariant line $\widehat{\ell_i}$ through $\widehat{q_i}$. Moreover $f(q_i) = q_i + 3\,\widehat{q_i}$ implies that it exists exactly one $\mathcal{L}_{q_i}$-invariant line ℓ_i through q_i distinct from ℓ. Thus

$$\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i}) = \ell + \ell_i + 3\,\widehat{\ell_i} \quad \text{for } \ i = 1,\dots,3.$$

If $\mathcal{G}$ is the foliation induced by the pencil $\{(\operatorname{tang}(\mathcal{F},\mathcal{L}_q) - \ell)\}_{q\in\ell}$ then Lemma 10.2 implies that $\mathcal{G}$ has degree at most $2\cdot4-2-3\cdot(3-1) = 0$. In an affine coordinate system where ℓ is the line at infinity and the origin belongs to $\operatorname{sing}(\mathcal{G})$, the foliation $\mathcal{F}$ is induced by a polynomial 1-form with homogeneous components. Therefore it is completely determined by its ℓ-polar map and can be explicitly presented as

$$\mathcal{F} = \left[y\,(2x+y)^3 dx + x\,(2y+x)^3 dy \right].$$

A simple computation shows that $xy\,(x+y)(x^2+xy+y^2)^3$ is a first integral of $\mathcal{F}$. Since the singular set of $\mathcal{F}$ has cardinality four it has to be equal to $\mathcal{P}$. A direct computation shows that $K(\mathcal{F}\boxtimes\mathcal{W}(\mathcal{P})) = 0$. This example corresponds to case *(a)* of the statement.

Case (c.2). Suppose now that the ℓ-polar map of $\mathcal{F}$ is in case (c.2) of TABLE 1. Lemma 10.1 implies (for any $i = 1,\dots,3$) that $\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i})$ is the union of five lines: ℓ, one line through $\widehat{q_i}$ and three lines (counted with multiplicities) through q_i. It follows from [39, Proposition 3.1] that the multiplicities appearing in $\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i})$ do not depend on the choice of $i \in \{1,2,3\}$. Therefore, if $\mathcal{G}$ denotes the foliation associated to the pencil $\{\operatorname{tang}(\mathcal{F},\mathcal{L}_p) - \ell\}_{p\in\ell}$ then Lemma 10.2 implies that the degree of $\mathcal{G}$ is at most:

(c.2.1) zero when there is one line with multiplicity 3 in $\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i})$;
(c.2.2) three when there is one line with multiplicity 2 in $\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i})$;
(c.2.3) six when all the lines in $\operatorname{tang}(\mathcal{F},\mathcal{L}_{q_i})$ have multiplicity one.

Case (c.2.1). If the degree of $\mathcal{G}$ is equal to zero then, as in case (c.1) above, $\mathcal{F}$ is completely determined by its ℓ-polar map. In a suitable affine coordinate system, the foliation $\mathcal{F}$ is induced by

$$\omega = y^3(2x+y)\,dx + x^3(x+2y)\,dy\,.$$

One can verify that ω admits $\frac{xy\,(x+y)}{x^2+xy+y^2}$ as a rational first integral and, again as in the case (c.1), $\mathcal{P} = \operatorname{sing}(\mathcal{F})$. This example corresponds to case *(b)* of the statement.

Case (c.2.2). If the degree of $\mathcal{G}$ is at most three and distinct from zero then $\mathcal{G}$ is tangent to a pencil of quartics with three completely decomposable fibers, each formed by three distinct lines with one of these lines with multiplicity two. Therefore $\mathcal{G}$ has at least nine invariant lines. Since a degree d foliation has at most $3d$ invariant lines (see [37]) it follows that the degree of $\mathcal{G}$ is exactly 3.

It is not hard to show that, up to automorphisms of $\mathbb{P}^2$, there exists a unique foliation $\mathcal{G}$ as above. In suitable affine coordinates where ℓ is the line at infinity and $q_1 = [1:0:0]$, $q_2 = [0:1:0]$, $q_3 = [1:-1:0]$, the foliation $\mathcal{G}$ is defined by the rational function

$$\frac{x^2(x-1)(x+2y-1)}{y^2(y-1)(2x+y-1)}.$$

We leave the details to the reader.

It follows that

$$\mathcal{F} = \Big[y^2(y-1)(2x+y-1)dx + x^2(x-1)(x+2y-1)dy\Big].$$

By a direct computation, it can checked that the 4-web $\mathcal{F} \boxtimes \mathcal{W}(\{q_1, q_2, q_3\})$ has curvature zero. Nevertheless a lengthy computation shows that there is no set $\mathcal{P}$ verifying $\{q_1, q_2, q_3\} \subsetneq \mathcal{P} \subset \mathrm{sing}(\mathcal{F})$ such that $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$.

Case (c.2.3). We are now assuming that for each $i = 1, \ldots, 3$, $\mathrm{tang}(\mathcal{F}, \mathcal{L}_{q_i})$ consists of five distinct lines, four of them being $\mathcal{F}$-invariant. It implies that $\mathcal{F}$ has at least 10 invariant lines, ℓ plus nine others.

We will further divide this case in two subcases: (c.2.3.a) when $k = \mathrm{Card}(\mathcal{P}) = 4$, and (c.2.3.b) when $k = \mathrm{Card}(\mathcal{P}) \geq 5$.

Case (c.2.3.a). Assume that $\mathcal{P} = \{q_1, q_2, q_3, p\}$ with $p \notin \ell$. Notice that $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$ intersects ℓ at the five singular points of $\mathcal{F}$ on ℓ: q_1, q_2, q_3 and two other that we will call s_1 and s_2. Recall from Lemma 8.1 that these five points coincide with the fixed points of the ℓ-polar map of $\mathcal{F}$. With no loss of generality, one can assume that the points of $\mathcal{P}$ are normalized such that $q_1 = [1:-1:0]$, $q_2 = [1:-\xi_3:0]$, $q_3 = [1:-\xi_3^2:0]$ and $p = [0:0:1]$. Then, by Corollary 6.1, the foliation $\widehat{\mathcal{L}_p}$ admits $x^3 + y^3$ as a first integral. Consequently, any irreducible $\widehat{\mathcal{L}_p}$-invariant algebraic curve C is of degree less than 3 and satisfies $C \cap \ell \subset \{q_1, q_2, q_3\}$. Observe that $|\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)|$ must contain all the singularities of $\mathcal{F}$, in particular s_1 and s_2. Because none of the curves $\{x^3 + y^3 = cst.\}$ contains s_1 or s_2, Theorem 7.3 implies that the lines $\overline{ps_1}$ and $\overline{ps_2}$ are $\mathcal{F}$-invariant irreducible components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$. Therefore $\mathcal{F}$ has at least 12 invariant lines: ℓ, $\overline{ps_1}$, $\overline{ps_2}$, and the three linear components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ passing trough q_i for each $i \in \{1, 2, 3\}$. It is well-known that a degree d foliation of $\mathbb{P}^2$ has at most $3d$ invariant lines (see [37] for instance). Therefore $\mathcal{F}$ has exactly 12 invariant lines.

Because $\mathcal{F}$ has degree 4, over each $\mathcal{F}$-invariant lines there are at most 5 singularities of $\mathcal{F}$. Notice that over the $\mathcal{F}$-invariant line $\overline{ps_1}$ we know already two: p and s_1. The three $\mathcal{F}$-invariant lines through q_1 distinct from ℓ must intersect $\overline{ps_1}$ in three distinct singular points of $\mathcal{F}$, none of them equal to p or s_1 (see Figure 6 below).

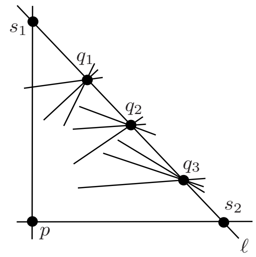

FIGURE 5. The twelve lines invariant by $\mathcal{F}$.

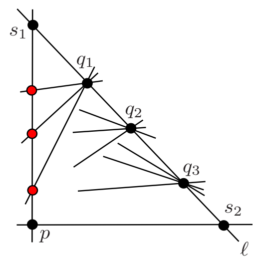

FIGURE 6.

The same being true for the $\mathcal{F}$-invariant lines through q_2 and q_3 it follows that on $\overline{ps_1}$ there are three singularities of $\mathcal{F}$ distinct from p and s_1 such that through each passes four $\mathcal{F}$-invariant lines. Of course the line $\overline{ps_2}$ has the same property. Thus we have a set $\mathcal{Q} \subset \mathbb{P}^2$ of cardinality 9 such that each of the points of $\mathcal{Q}$ is contained in four of the the twelve $\mathcal{F}$-invariant lines. It is a then a simple combinatorial exercise to show that these twelve lines support a $(4,3)$-net in the sense of Section 3. Therefore (see [45, Section 4.4]) the arrangement of twelve $\mathcal{F}$-invariant lines is projectively equivalent to the Hesse arrangement. Because the foliation determined by the Hesse pencil also has degree four and the tangency of two distinct foliations of degree four has degree nine it follows that $\mathcal{F}$ is the Hesse Pencil.

With the normalizations made above on the points q_1, q_2, q_3 and p, we obtain

$$\mathcal{F} = \left[d\Big(\frac{x^3 + y^3 + 1}{xy} \Big) \right].$$

This 5-web appeared in the introduction under the label $\mathcal{H}_5$. In Section 3 it is shown that it is an exceptional web and in particular has curvature zero.

Case (c.2.3.b). Suppose now that $\mathcal{P}$ has cardinality greater than four. As in case (c.2.3.a) we will denote by s_1 and s_2 the two other singularities of $\mathcal{F}$ on ℓ distinct from q_1, q_2 and q_3.

Claim 10.2. *There exists a pair of points $p, s \in \mathcal{P} \setminus \{q_1, q_2, q_3\}$ such that the line $\overline{ps}$ intersects ℓ in one of the points q_1, q_2, q_3.*

Proof. Suppose that the claim is not true and let p_4, p_5 be any two points in $\mathcal{P} \setminus \{q_1, q_2, q_3\}$. Proposition 8.2 combined with Theorem 8.1 implies that the line $\overline{p_4p_5}$ intersects $\mathcal{P}$ in at most three points. Thus there are only two possibilities for $\mathcal{P}$: (*i*) $\overline{p_4p_5} \cap \mathcal{P} = \{p_4, p_5\}$ or (*ii*) $\overline{p_4p_5} \cap \mathcal{P} = \{p_4, p_5, p_6\}$ for some point $p_6 \in \mathcal{P}$ distinct of p_4 and p_5.

If we are in case (i) then $\mathcal{P}$ is in p_4 and p_5-barycentric general position because, by assumption, the lines $\overline{p_4q_i}$ and $\overline{p_5q_i}$ (for $i = 1, 2, 3$) have only two elements of $\mathcal{P}$ each and the points q_1, q_2, q_3 are not aligned with p_4 nor with p_5. Theorem 7.3 ensures that $|\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)|$ is a union of five $\mathcal{F}$-invariant lines. Since $|\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)|$ contains p_4 and the singularities of $\mathcal{F}$, these lines have to be $\overline{p_4s_1}, \overline{p_4s_2}, \overline{p_4q_1}, \overline{p_4q_2}$ and $\overline{p_4q_3}$. Similarly the irreducible components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_p)$ are the $\mathcal{F}$-invariant lines $\overline{p_5q_i}$ for $i = 1, \ldots, 3$ and $\overline{p_5s_i}$ for $i = 1, 2$.

Through at least one of the points s_1, s_2, say s_1, passes three $\mathcal{F}$-invariant lines: $\overline{p_4s_1}$, $\overline{p_5s_1}$ and ℓ. This contradicts the behavior of the ℓ-polar map because s_1 appears in $f^{-1}(s_1)$ with multiplicity one as a simple computation shows.

Suppose now that we are in case (ii). Because the barycenter transform of three distinct points in $\mathbb{P}^1$ is still three distinct points, $\mathcal{P}$ is in barycentric general position with respect to at least two points in $\{p_4, p_5, p_6\}$. Exactly as before we arrive at a contradiction. The claim follows. □

By Claim 10.2 we can suppose that p_4, p_5 are two points in $\mathcal{P} \setminus \ell$ such that the line $\ell' = \overline{p_4p_5}$ intersect ℓ at q_1. Notice that ℓ' is $\mathcal{F}$-invariant (by Proposition 8.2) and that the ℓ'-polar map of $\mathcal{F}$ must also be in case (c.2) of TABLE 1. Therefore Lemma 8.2 implies that through each of the points p_4 and p_5 passes four $\mathcal{F}$-invariant lines. Since these intersect ℓ at $\mathrm{sing}(\mathcal{F})$, there will be one $\mathcal{F}$-invariant line through s_1 (say $\overline{p_4s_1}$) and one through s_2, say $\overline{p_5s_2}$. In the total $\mathcal{F}$ has the maximal number of invariant lines for a degree 4 foliation: twelve.

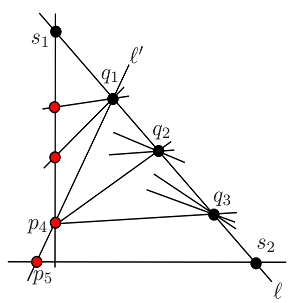

FIGURE 7. The twelve lines invariant by $\mathcal{F}$ in case **c.2.3.b**.

Consider the effective divisor $\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)$. It has degree 5, contains four lines through p_4 (namely $\overline{p_4q_2}$, $\overline{p_4q_3}$, $\overline{p_4s_1}$ and $\ell' = \overline{p_4q_1} = \overline{p_4p_5}$) and the point s_2. Since the four lines through p_4 do not contain s_2 there is a line $\ell'' \subset |\mathrm{tang}(\mathcal{F}, \mathcal{L}_4)|$ through s_2. By Theorem 7.3, ℓ'' must be $\widehat{\mathcal{L}_4}$ invariant and Lemma 6.1 item *(4)* implies that ℓ'' contains $\widehat{p_4}$: the p_4-barycenter of $\{p_5, q_1\}$ in ℓ'. In particular, $\ell'' = \overline{s_2\widehat{p_4}}$. Clearly $q_2 \notin \ell''$. Consequently Lemma 6.1 item *(4)* ensures the existence of an extra point in $\mathcal{P}$, say p_6, such that $p_6 \in \overline{p_4q_2}$ and the p_4-barycenter of $\{q_2, p_6\}$ in $\overline{p_4q_2}$ lies in ℓ''. Similarly, there exists another extra point $p_7 \in \mathcal{P}$ contained in $\overline{p_4q_3}$ such that the p_4-barycenter of $\{q_3, p_7\}$ in $\overline{p_4q_3}$ also lies in ℓ''.

Notice that the line $\overline{p_4q_2}$ contains three points of $\mathcal{P}$: q_2, p_4 and p_6. Therefore the $\overline{p_4q_2}$-polar map of $\mathcal{F}$ must be also in the case (c.2) of TABLE 1. Consequently through p_6 pass four $\mathcal{F}$-invariant lines. Remark that p_4,p_6 and s_1 are not aligned and that through s_1 pass just two $\mathcal{F}$-invariant lines (ℓ and $\overline{p_4s_1}$). Thus one of the four $\mathcal{F}$-invariant lines through p_6 must be the line $\overline{p_6 s_2}$. Similarly, through p_7 pass four $\mathcal{F}$-invariant lines and the line $\overline{p_7 s_2}$ is among these four lines. Since through s_2 passes just one $\mathcal{F}$-invariant distinct from ℓ it follows that $\overline{p_6 s_2} = \overline{p_7 s_2} = \overline{p_5 s_2}$.

Changing the role of p_4 and p_5 in the preceding argument it follows that there exist $p_8, p_9 \in \mathcal{P} \setminus \{q_1, q_2, q_3, p_4, \ldots, p_7\}$ in the lines $\overline{p_5 q_2}$ and $\overline{p_5 q_3}$ respectively. As before, through each of these points passes four $\mathcal{F}$-invariant lines.

Putting all together we have just proved that $\mathcal{F}$ leaves invariant an arrangement of twelve lines and $\mathcal{P}$ contains a subset of at least nine points such that each of these points is contained in four distinct lines of the arrangement. At this point it is clear that the arrangement is the Hesse arrangement (see [45]), that $\mathcal{F}$ is projectively equivalent to the Hesse pencil (it is the unique degree 4 foliation leaving the Hesse arrangement invariant because the tangency of two degree four foliations has degree nine) and that $\mathcal{P}$ contains the nine base points of it. It remains to show that $\mathcal{P}$ cannot be larger than the base points of the Hesse pencil. Indeed if there exists a point $p_{10} \in \mathcal{P}$ distinct from the nine base points it would exist a line in the arrangement containing four points of $\mathcal{P}$ contradicting Theorem 8.1. Therefore there exists only one flat CDQL $(k+1)$-web of degree four with $k \geq 5$: the 10-web $\mathcal{H}_{10}$ from the Introduction. □

10.4. **Proof of Theorem 2.** According to Section 9, the exceptional CDQL webs of degree one are projectively equivalent to one of the webs $\mathcal{A}^k_I, \mathcal{A}^k_{II}, \mathcal{A}^k_{III}, \mathcal{A}^k_{IV}$.

Propositions 10.1,10.2,10.3 putted together give a complete classification of flat CDQL $(k+1)$-webs of degree bigger than two, on the projective plane, when $k \geq 4$. There are only sixteen such webs (up to projective transformations). Thirteen of these have been presented in the Introduction and their exceptionality has been put in evidence in Sections 2 and 3.

It can be verified the that 5-web described in Proposition 10.1 case *(a.3.h)*, the 5-web described in Proposition 10.2 case *(a)* and the 5-web described in Proposition 10.3 case *(a)* are not exceptional. For this sake one can use, as we did, the criterion [42, Proposition 4.3] or Hénaut's curvature as indicated by Ripoll in [43, Theorem 5.1] or even Pantazi's criterion. Aiming at conciseness we decided not to reproduce the lengthy computations here. □

Remark 10.1. As already mentioned in the Introduction the non-linear defining foliation of all the exceptional CDQL webs on $\mathbb{P}^2$ admits a rational first integral. The non-linear defining foliations of the flat CDQL 5-webs of degrees two and four also admit rational first integrals. The situation is different for the flat CDQL 5-web of degree three.

Proposition 10.4. *The foliation* $\mathcal{F} = \left[y\,(2x+y)^3 dx + x\,(2y+x)^3 dy\right]$ *does not admit a rational first integral.*

Proof. Let $\omega = y\,(2x+y)^3 dx + x\,(2y+x)^3 dy$. If we set $X = x+y$ and $Y = x-y$ then $\omega = 2(X^3+1)dY + (3/2)(Y^3 - X^2 Y)dX$. It can be promptly verified that the algebraic function $Y^3\sqrt{X^3+1}$ is an integrating factor of ω, that is, $\omega/(Y^3\sqrt{X^3+1})$ is a closed 1-form. If we set $Z = \sqrt{X^3+1}$ then

$$\frac{\omega}{Y^3 Z} = -d\left(\frac{Z}{Y^2}\right) + \frac{dZ}{(Z^2-1)^{\frac{2}{3}}}\,.$$

Let $\wp(t)$ be a Weierstrass function satisfying $\wp'(t)^2 = 4\wp(t)^3 + 1$. Notice that $\wp''(t) = 6\wp(t)^2$. If we set $Z = \wp'(t)$ then, in the coordinates (t, Y),

$$\frac{\omega}{Y^3\wp'(t)} = -d\left(\frac{\wp'(t)}{Y^2}\right) + \frac{3}{2^{\frac{1}{3}}}dt\,.$$

Hence the generic leaf of $\mathcal{F}$ is defined by the equation

$$-\frac{\wp'(t)}{Y^2} + \frac{3}{2^{\frac{1}{3}}}t = \lambda \quad \Longrightarrow \quad Y = \sqrt{\frac{\wp'(t)}{\frac{3}{2^{\frac{1}{3}}}t + \lambda}}$$

where λ is a constant. Since $\wp'(t)$ is doubly-periodic while $(3/2^{\frac{1}{3}})t + \lambda$ is not, it follows that back in the coordinates (X, Y) the generic leaf of $\mathcal{F}$ cuts the lines with constant X coordinate in infinitely many distinct points. In particular they are non-algebraic and therefore $\mathcal{F}$ does not admit a rational first integral. □

11. From global to local...

11.1. **Degenerations.** Let $\mathcal{W}_t$ be a holomorphic family of webs in the sense that it is defined by an element

$$W(x, y, t) = \sum_{i+j=k} a_{ij}(x, y, t)dx^i dy^j$$

in $\mathrm{Sym}^k\Omega^1(\mathbb{C}^2)$ with coefficients in $\mathcal{O} = \mathbb{C}\{x, y, t\}$ (convergent power series) and such that $W(\cdot, \cdot, t)$ defines a (possibly singular) k-web on $(\mathbb{C}^2, 0)$ for every $t \in (\mathbb{C}, 0)$.

We do not claim originality on the next result. Indeed the first author, modulo memory betrayals, first heard about it in a talk delivered by Hénaut at CIRM in 2003. Anyway it follows almost immediately from the main result of [24]. Since it would take us too far afield to recall the notations and the results of [24], we include a sketchy proof below freely using them. We refer to this work for more precisions.

Theorem 11.1. *The set* $\{t \in (\mathbb{C}, 0) \,|\, \mathcal{W}_t$ *has maximal rank*$\}$ *is closed.*

Proof. The differential system $M_t(d)$ can be defined over $\mathcal{O}$ (with t considered as a constant of derivations) and the restriction of $M_t(d)$ to a parameter t_0 coincides with the definition of $M_{t_0}(d)$.

The prolongations p_k of the associated morphism are morphisms of $\mathcal{O}$-modules and the kernels R_k of the morphisms p_k are $\mathcal{O}$-modules locally free outside the discriminant. Notice that the discriminant is a hypersurface in $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ that does not contain any fiber of the projection $(x, y, t) \mapsto t$ by our definition of family of webs.

If $r_k = \dim R_k$ then Cartan's Theorem B implies the existence of r_k sections of R_k over a polydisk $D \subset (\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ that generates R_k on a Zariski open subset of $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$. Moreover this subset can be supposed to contain any given point on the complement of the discriminant. Therefore we can find a meromorphic inverse of the morphism $\overline{\pi}_{k-4}$ holomorphic at any given point in the complement of the discriminant.

Following [24], we can construct a holomorphic family of meromorphic connections Δ_t such that $\mathcal{W}_t$ has maximal rank if and only if $\Delta_t^2 = 0$. The theorem follows. □

11.2. **Singularities of certain exceptional webs.** Theorem 11.1 combined with the classification of CDQL exceptional webs in $\mathbb{P}^2$ yields the following result.

Corollary 11.1 (Corollary 2 of the Introduction)**.** *Let $\mathcal{W}$ be a smooth k-web, $k \geq 4$, and $\mathcal{F}$ be a singular holomorphic foliation, both on $(\mathbb{C}^2, 0)$, such that the $(k+1)$-web $\mathcal{W} \boxtimes \mathcal{F}$ has maximal rank. Then one of the following holds:*

(1) the foliation $\mathcal{F}$ is of the form $\big[H(x,y)(\alpha dx + \beta dy) + h.o.t.\big]$ where H is a non-zero homogeneous polynomial and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$;

(2) the foliation $\mathcal{F}$ is of the form $\big[H(x,y)(ydx - xdy) + h.o.t.\big]$ where H is a non-zero homogeneous polynomial;

(3) $\mathcal{W} \boxtimes \mathcal{F}$ is exceptional and its first non-zero jet is one of the following webs

$$\mathcal{A}_I^k,\ \mathcal{A}_{III}^{k-2},\ \mathcal{A}_5^d \ (\textit{only when } k=4) \ \textit{ and } \ \mathcal{A}_6^b \ (\textit{only when } k=5)\,.$$

Proof. Suppose that $\mathcal{W} = [\Omega]$ where Ω is a germ at the origin of a holomorphic k-symmetric 1-form. Consider the expansion of Ω in its homogeneous components:

$$\Omega = \sum_{i=0}^{\infty} \Omega_i$$

where Ω_i is a k-symmetric 1-form with homogenous coefficients of degree i. According to our assumptions $\Omega_0 \neq 0$ and $\Omega_0 = \prod_{i=1}^k dL_i$, where the L_i's are linear forms defining the tangent spaces of the leaves of $\mathcal{W}$ at the origin.

Similarly, suppose that $\mathcal{F} = [\omega]$ where ω is a germ of holomorphic 1-form with codimension two zero set. Let

$$\omega = \sum_{i=i_0}^{\infty} \omega_i\,, \quad \omega_{i_0} \neq 0$$

be the expansion of ω in its homogeneous components, with $i_0 > 0$ according to the hypothesis made on $\mathcal{F}$. If $\alpha_t(x,y) = (tx, ty)$ then

$$W(x,y,t) = \frac{\alpha_t^*(\Omega \cdot \omega)}{t^{k+i_0+2}} = \left(\sum_{i=0}^{\infty} t^i \Omega_i\right)\left(\sum_{i=i_0}^{\infty} t^{i-i_0}\omega_i\right) = \Omega_0 \cdot \omega_{i_0} + t(\cdots)$$

is an element of $\mathrm{Sym}^k \Omega^1(\mathbb{C}^2)$ with coefficients in $\mathcal{O} = \mathbb{C}\{x,y,t\}$. For every $t \neq 0$, the web $W_t = [W(\cdot,\cdot,t)]$ is isomorphic to $\mathcal{W} \boxtimes \mathcal{F}$.

If ω_{i_0} is a multiple of a constant 1-form (equivalently if $\mathcal{F}_0 = [\omega_{i_0}]$ is a smooth foliation) then $\mathcal{F}$ must be like in item (1) of the statement. Notice that when $W(x,y,0)$ does not define a $(k+1)$-web we are in this situation. Otherwise the foliation $\mathcal{F}_0 = [\omega_{i_0}]$ has a singularity at the origin and $\mathcal{W}(x,y,0)$ is a $(k+1)$-web. Since for every $t \neq 0$ the web W_t is of maximal rank, W_0 also has maximal rank thanks to Theorem 11.1. If $\mathcal{F}_0$ is linear then we are in case (2) of the statement. Otherwise $W_0 = [\Omega_0] \boxtimes \mathcal{F}_0$ is the product of a parallel k-web with a non-linear foliation. Since $k \geq 4$, Proposition 2.1 implies that W_0 is exceptional. Therefore it must be one of the thirteen sporadic exceptional CDQL webs or belong to one of the four infinite families of exceptional CDQL webs. The only ones that are the product of a parallel web with a non-linear foliation are listed in *(3)*. □

12. ...AND BACK: QUASI-LINEAR WEBS ON COMPLEX TORI

12.1. **First integrals of linear foliations on tori.** Let T be a two-dimensional complex torus. The set of linear foliations on T is naturally identified with the 1-dimensional projective space $\mathbb{P}H^0(T, \Omega^1_T)$. We are interested in the set $\mathcal{I}(T) \subset \mathbb{P}H^0(T, \Omega^1_T)$ corresponding to linear foliations which admit a holomorphic first integral.

Proposition 12.1. *The cardinality $i(T)$ of $\mathcal{I}(T)$ is 0, 1, 2 or ∞. Moreover*

(1) If $i(T) = 0$ then T is a simple complex torus;
(2) If $i(T) = 1$ then T is a non-algebraic complex torus;
(3) If $i(T) = 2$ then T is isogenous to the product of two non-isogenous elliptic curves;
(4) If $i(T) = \infty$ then T is isogeneous to the square of an elliptic curve E. Moreover if ω_1, ω_2 is a pair of linearly independent 1-forms on T admitting rational first integrals then

$$\left\{\lambda \in \mathbb{C} \,\middle|\, \omega_1 + \lambda\omega_2 \text{ has a holomorphic first integral}\right\} = \mathrm{End}(E) \otimes \mathbb{Q}.$$

Proof. Let $\mathcal{F}$ be a linear foliation on T. It is induced by a 1-form with constant coefficients $\omega = sdx + tdy$ on $\mathbb{C}^2$ viewed here as the universal covering of T.

Notice that ω is invariant by the action of T on itself. Therefore, since this action is transitive, $\mathcal{F}$ admits a compact leaf if and only if it has a compact leaf through 0. Notice also that a compact leaf is nothing more than a subtorus of T. Reciprocally if T contains a subtorus T' then translations of T' by elements in T form a linear foliation on T admiting a holomorphic first integral given by the quotient map $T \to T/T'$.

Therefore if $i(T)$ is equal to zero, T has no closed subgroups of dimension one that is, T is a simple complex torus. If $i(T)$ is equal to one then T admits exactly one closed subgroup of dimension one. It implies that T is non-algebraic otherwise T would be isogeneous to a product of two elliptic curves (according to Poincaré's reducibility Theorem) then would be such that $i(T) > 1$. If $i(T) = 2$ then T admits two closed subgroup T' and T'' of dimension one. The natural map

$$(x, y) \in T' \times T'' \longmapsto x + y \in T$$

has finite kernel equal to $T' \cap T''$ therefore is an isogeny between $T' \times T''$ and T. Notice that T' can't be isogenous to T'' otherwise $\mathcal{I}(T) = \mathcal{I}(T' \times T'') = \mathcal{I}(T' \times T')$ and the latter set has infinite cardinality since it is invariant under the induced action of $\mathrm{Aut}(T' \times T') \supseteq \mathrm{PSL}(2, \mathrm{End}(T')) \supseteq \mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{P}H^0(T, \Omega^1_T) \simeq \mathbb{P}^1$.

If $\mathcal{I}(T)$ has cardinality at least three then there exist three pairwise distinct subtorus T',T'' and T''' passing through the origin of T. As before one get that T is isogenous to $T' \times T''$. The existence of the natural projections $T''' \to T/T'$ and $T''' \to T/T''$ implies that all the three curves are isogenous. Moreover, up to an isogeny, T can be assumed to be $T' \times T'$ with T', T'' and T''' identified with the horizontal, vertical and diagonal subtori respectively. It follows that $\mathcal{I}(T)$ is an orbit of the natural action of $\mathrm{PGL}(2, \mathrm{End}(T'))$, hence $i(T) = \infty$. □

Remark 12.1. Item *(4)* of Proposition 12.1 can be traced back to Abel, see [2, §X]. According to Markushevich [30, p. 158], it is the first appearance of the so-called *complex multiplication* in the theory of elliptic functions.

Lemma 12.1. *Let T be a complex torus isogeneous to the square of an elliptic curve E. If $[\omega_1], \ldots, [\omega_4] \in \mathbb{P}H^0(T, \Omega^1_T)$ are linear foliations on T with holomorphic first integral then the cross-ratio $([\omega_1], [\omega_2] : [\omega_3], [\omega_4])$ belongs to* $\mathrm{End}(E) \otimes \mathbb{Q}$.

Proof. According to the proof of Proposition 12.1 we can assume that $T = E \times E, \omega_2 = dx - dy, \omega_3 = dy$ and $\omega_4 = dx$. Since the leaves of ω_1 are algebraic they must be translates of $E_{\alpha,\beta}$ (defined by (8) in Section 4.2) for suitable $\alpha, \beta \in \mathrm{End}(E)$. Thus $\omega_1 = [\beta dx - \alpha dy]$. Therefore

$$([\omega_1], [\omega_2] : [\omega_3], [\omega_4]) = \frac{\beta}{\alpha}.$$

The lemma follows. □

12.2. **Flat CDQL webs on complex tori.** Let $\mathcal{W}$ be a linear k-web on T. Clearly it is a completely decomposable web. Thus we can write $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ where the $\mathcal{L}_i$'s are linear foliations. For $i = 1, \ldots, k$, set $\widehat{\mathcal{L}_i} = \beta_{\mathcal{L}_i}(\mathcal{W} - \mathcal{L}_i)$. and define the *polar map* of a foliation $\mathcal{F}$ on T as the rational map $P_{\mathcal{F}} : T \dashrightarrow \mathbb{P}H^0(T, \Omega^1_T)$ characterized by the property

$$P_{\mathcal{F}}^{-1}(\mathcal{L}) = \mathrm{tang}(\mathcal{F}, \mathcal{L})$$

for every $\mathcal{L} \in \mathbb{P}H^0(T, \Omega^1_T)$.

Recall from the Introduction that a fiber of a rational map from a two-dimensional complex torus onto a curve is linear if it is set-theoretically equal to a union of subtori.

Lemma 12.2. *Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear k-web on T, with $k \geq 2$. If $\mathcal{F}$ is a non-linear foliation on T such that $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ then the rational map $P_{\mathcal{F}}$ has at least k linear fibers, one for each $\mathcal{L}_i$. Moreover, if $k \geq 3$ then each of the fibers $P_{\mathcal{F}}^{-1}(\mathcal{L}_i)$ contains at least one elliptic curve invariant by $\mathcal{L}_i$ and at least one invariant by $\widehat{\mathcal{L}_i}$.*

Proof. By Theorem 7.2, any irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ is $\mathcal{L}_i$ or $\widehat{\mathcal{L}_i}$-invariant. Since $\mathcal{L}_i$ and $\widehat{\mathcal{L}_i}$ are linear foliations, it follows that the fibers $P_{\mathcal{F}}^{-1}(\mathcal{L}_i)$ are linear for $i = 1, \ldots, k$. This proves the first part of the lemma.

Suppose now that $k \geq 3$. Aiming at a contradiction, assume that all the irreducible components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ are $\widehat{\mathcal{L}_1}$-invariant. Proposition 12.1 implies that $\widehat{\mathcal{L}_1}$ is tangent to an elliptic fibration.

Since both K_T and $N\mathcal{L}_i$ are trivial, $\mathcal{O}_T(\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)) = K_T \otimes N\mathcal{F} \otimes N\mathcal{L}_i = N\mathcal{F}$ for every $i = 1, \ldots, k$. Taking $i = 1$, we get that $N\mathcal{F}$ is linearly equivalent to a divisor supported on some fibers of the fibration $\widehat{\mathcal{L}_1}$. Taking $i = 2, \ldots, k$, we see that the divisors $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ are linearly equivalent to $N\mathcal{F}$ and consequently to $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$. Therefore, being all of them effective, they also have to be supported on elliptic curves invariant by $\widehat{\mathcal{L}_1}$.

Since two distinct linear foliations on T are everywhere transversal, Theorem 7.2 implies that for every $i = 2, \ldots, k$, $\mathcal{L}_i$ or $\widehat{\mathcal{L}_i}$ is equal to $\widehat{\mathcal{L}_1}$. By hypothesis the linear foliations $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are pairwise distinct. Therefore at least $k-1$ of the foliations $\widehat{\mathcal{L}_i}$ $(i = 1, \ldots, k)$ coincide. This contradicts Lemma 5.1.

If one assumes that all the irreducible components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ are invariant by $\mathcal{L}_1$ then the same argument with minor modifications also leads to a contradiction. The lemma follows. □

Proposition 12.2. *Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear k-web on T, with $k \geq 3$. If $\mathcal{F}$ is a non-linear foliation on T such that $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ then*

(1) *T is isogenous to the square of an elliptic curve. In particular T is an abelian surface;*
(2) *the foliations $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are tangent to elliptic fibrations;*
(3) *the foliations $\widehat{\mathcal{L}_1}, \ldots, \widehat{\mathcal{L}_k}$ are tangent to elliptic fibrations;*
(4) *$\mathcal{P}_{\mathcal{F}}$ has k linear fibers.*

Proof. The points *(2)*, *(3)* and *(4)* follow from Lemma 12.2 since a linear foliation on T is tangent to an elliptic fibration if and only if it leaves an elliptic curve invariant. Since $k \geq 3$, Proposition 12.1 implies *(1)*. □

12.3. **On the number of linear fibers of a pencil on a complex torus.** Let $F : T \dashrightarrow \mathbb{P}^1$ be a meromorphic map on a two-dimensional complex torus T. We are interested in the number k of linear fibers of F.

Theorem 12.1 (Theorem 4 of the Introduction)**.** *If k is finite then $k \leq 6$. Moreover, if $k = 6$ then every fiber of F is reduced.*

Proof. If x, y are homogeneous coordinates on $\mathbb{P}^1$ then $xdy - ydx \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(2))$. Therefore $\omega = F^*(xdy - ydx) \in H^0(T, \Omega^1_T \otimes \mathcal{N}^{\otimes 2})$ with $\mathcal{N} = F^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Let also $X \in H^0(T, T_T \otimes (\mathcal{N}^*)^{\otimes 2})$ be dual to ω, that is $\omega = i_X \Omega$ where Ω is a non-zero global holomorphic 2-form on T. The twisted vector field X can be represented by a covering of $\mathcal{U} = \{U_i\}$ of T and holomorphic vector fields $X_i \in T_T(U_i)$ subjected to the conditions

$$X_i = g_{ij} X_j$$

on any non-empty $U_i \cap U_j$, where $\{g_{ij}\}$ is a cocycle in $H^1(\mathcal{U}, \mathcal{O}_T^*)$ representing $\mathcal{N}^{\otimes 2}$.

If $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ form a basis of $H^0(T, T_T)$ then $X_i = A_i \frac{\partial}{\partial z} + B_i \frac{\partial}{\partial w}$ for suitable holomorphic functions $A_i, B_i \in \mathcal{O}(U_i)$. Consider the divisor Δ locally cut out by

$$\det \begin{pmatrix} A_i & B_i \\ X_i(A_i) & X_i(B_i) \end{pmatrix}.$$

Clearly these local expressions patch together to form an element of $H^0(T, \mathcal{N}^{\otimes 6})$.

Any divisor corresponding to a fiber of F is defined by the vanishing of a non-zero element of $H^0(T, F^*\mathcal{O}_{\mathbb{P}^1}(1)) = H^0(T, \mathcal{N})$. By the very definition of X, (the closures of) its 1-dimensional orbits are irreducible components of fibers of F. Outside the zero locus of X_i, the divisor $\Delta_{|U_i}$ corresponds to the inflection points of the orbits of X_i. Indeed, if $\gamma : (\mathbb{C}, 0) \to U_i$ is an orbit of X_i, that is if $X_i(\gamma(t)) = \gamma'(t)$ for $t \in (\mathbb{C}, 0)$, then (with an obvious abuse of notation)

$$\det \begin{pmatrix} A_i & B_i \\ X_i(A_i) & X_i(B_i) \end{pmatrix} (\gamma) \equiv \gamma' \wedge \gamma''.$$

Let L be a linear irreducible components of a fiber of F. Its generic point belongs to Δ since it is an inflection point of L relatively to X (see [37, §6]). It follows that $L \leq \Delta$.

From the preceding discussion, it follows that to prove the theorem it suffices to show that any effective divisor $D_1, \ldots, D_k$ corresponding to a linear fiber of F is smaller than Δ. Indeed, the support of distinct fibers of F do not share irreducible components in common and consequently

$$\sum_{i=1}^{k} D_i \leq \Delta \, .$$

Since $\sum_{i=1}^k D_i$ is defined by the vanishing of an element in $H^0(T, \mathcal{N}^{\otimes k})$ while Δ is defined by an element in $H^0(T, \mathcal{N}^{\otimes 6})$, it would follow that $k \leq 6$ as wanted. It remains to show that $D_i \leq \Delta$ for any $i = 1, \ldots, k$.

The divisorial components of the zero locus of X_i correspond to multiple components of the fibers of F just like in Darboux's Lemma 10.2. If there is a fiber of F containing an irreducible component with multiplicity $a \geq 2$ and locally cut out over U_i by a reduced holomorphic function f then we can write $X_i = f^{a-1}\widetilde{X_i}$ with $\widetilde{X_i} = \widetilde{A_i}\frac{\partial}{\partial z} + \widetilde{B_i}\frac{\partial}{\partial w}$ holomorphic. Therefore Δ is locally defined by

$$\det \begin{pmatrix} f^{a-1}\widetilde{A_i} & f^{a-1}\widetilde{B_i} \\ f^{a-1}\widetilde{X_i}(f^{a-1}\widetilde{A_i}) & f^{a-1}\widetilde{X_i}(f^{a-1}\widetilde{B_i}) \end{pmatrix} = f^{3a-3} \det \begin{pmatrix} \widetilde{A_i} & \widetilde{B_i} \\ \widetilde{X_i}(\widetilde{A_i}) & \widetilde{X_i}(\widetilde{B_i}) \end{pmatrix} .$$

Since $3a - 3 > a$ when $a \geq 2$ it follows that every linear fiber of F is smaller than Δ as wanted. Moreover if $k = 6$ then F cannot have non-reduced fibers. □

Theorem 12.1 combined with Proposition 12.2 yields the following corollary.

Corollary 12.1. *Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear k-web on T. If $\mathcal{F}$ is a non-linear foliation on T such that $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ then T is isogenous to the square of an elliptic curve and $k \leq 6$.*

12.4. **Constraints on the linear web.** Let $\mathcal{W} \boxtimes \mathcal{F}$ be a flat CDQL $(k+1)$-web on a complex torus T. If $\mathcal{P}_{\mathcal{F}}$ denotes the polar map of $\mathcal{F}$ and if $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ then the fibers $\mathcal{P}_{\mathcal{F}}^{-1}(\mathcal{L}_i)$ are all linear and supported on a union of elliptic curves invariant by $\mathcal{L}_i$ or by $\widehat{\mathcal{L}_i}$ according to Proposition 12.2. From the very definition of $\mathcal{P}_{\mathcal{F}}$ it is clear that the singular set of $\mathcal{F}$ coincides with the indeterminacy set of $\mathcal{P}_{\mathcal{F}}$.

In order to determine the linear web $\mathcal{W}$ under the assumption that $\mathcal{W} \boxtimes \mathcal{F}$ has maximal rank we will take a closer look at the singularities of $\mathcal{F}$. It will be convenient to consider the natural affine coordinates (x, y) on the universal covering $\mathbb{C}^2 \to T$.

Lemma 12.3. *Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear k-web, with $k \geq 3$, and $\mathcal{F}$ be a non-linear foliation, both defined on T. Suppose that $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$. If $p \in \mathrm{sing}(\mathcal{F})$ is the origin in the affine coordinate system (x, y) then one of the following two alternatives holds:*

1. *the foliation $\mathcal{F}$ is locally given by $[xdy - ydx + h.o.t.]$. In this case, for each $i = 1, \ldots, k$, the divisor $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ has multiplicity one at p and there exists an elliptic curve through p invariant by $\mathcal{L}_i$ and by $\mathcal{F}$.*
2. *the foliation $\mathcal{F}$ is locally given by $[\omega_d + h.o.t.]$ where ω_d is a non-zero homogeneous 1-form of degree $d \geq 1$ in the coordinates x, y with singular set reduced to $(0,0)$ and not proportional to $xdy - ydx$. In particular the foliation $[\omega_d]$ is non-linear.*

Proof. According to the proof of Proposition 12.1, one can assume that $\mathcal{L}_1 = [dx]$, $\mathcal{L}_2 = [dy]$ and $\mathcal{L}_3 = [dx - dy]$. If $\mathcal{F}$ is locally given by $[a(x,y)dx + b(x,y)dy]$ where a and b designate holomorphic functions without common factors, then $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1) =$

$\{b=0\}$, $\mathrm{tang}(\mathcal{F},\mathcal{L}_2)=\{a=0\}$ and $\mathrm{tang}(\mathcal{F},\mathcal{L}_3)=\{a+b=0\}$. Notice that the assumption $p\in\mathrm{sing}(\mathcal{F})$ implies that $a(0,0)=b(0,0)=0$.

Recall from Proposition 12.2 that $\mathrm{tang}(\mathcal{F},\mathcal{L}_1)$ is supported on a union of elliptic curves. Therefore the first non-zero jet of b will be a constant multiple of $x^k\cdot h(x,y)^l$, $k,l\in\mathbb{N}$, where h is a linear form vanishing defining the tangent space of $\widehat{s\mathcal{L}_1}$ at zero. Similarly for a and $a+b$.

The first non-zero jet of a,b and $a+b$ have the same degree and are pairwise without common factor. Otherwise the supports of $\mathrm{tang}(\mathcal{F},\mathcal{L}_i)$ and $\mathrm{tang}(\mathcal{F},\mathcal{L}_j)$ would share an irreducible component in common for some pair (i,j) satisfying $1\le i<j\le 3$. But this is impossible since $\mathrm{tang}(\mathcal{L}_i,\mathcal{L}_j)$ is empty as soon as $i\neq j$.

Thus we can write $\omega=\omega_d+h.o.t.$ where ω_d is homogeneous and with singular set equal to the origin. We are in the first case of the statement when ω_d is proportional to $xdy-ydx$ and in the second case otherwise. $\square$

We are now in position to use Corollary 11.1 to restrict the possibilities of the maximal linear subweb of an exceptional CDQL web on complex tori.

Proposition 12.3. *Let $\mathcal{W}=\mathcal{L}_1\boxtimes\cdots\boxtimes\mathcal{L}_k$ be a linear k-web, with $k\ge 4$, and $\mathcal{F}$ be a non-linear foliation on T. If $\mathcal{W}\boxtimes\mathcal{F}$ has maximal rank then, up to isogenies, one of the following alternatives holds:*

1. *The torus T is the square of an elliptic curve, $k=4$ and $\mathcal{W}=[dxdy(dx^2-dy^2)]$;*
2. *The torus T is E_i^2, $k=6$ and $\mathcal{W}=[dxdy(dx^2-dy^2)(dx^2+dy^2)]$;*
3. *The torus T is $E_{\xi_3}^2$, $k=5$ and $\mathcal{W}=[dxdy(dx^3+dy^3)]$;*
4. *The torus T is $E_{\xi_3}^2$, $k=4$ and $\mathcal{W}=[dxdy(dx+dy)(dx-\xi_3\,dy)]$.*

Proof. Corollary 12.1 tell us that T is isogeneous to the square of an elliptic curve E and that $k\le 6$. Lemma 5.1 implies that we can assume, after an eventual reordering, that $\widehat{\mathcal{L}_1}\neq\widehat{\mathcal{L}_2}$. For $i=1,2$, let $\widehat{E_i}$ be an elliptic curve contained in $\mathrm{tang}(\mathcal{F},\mathcal{L}_i)$ that is $\widehat{\mathcal{L}_i}$-invariant. Notice that the existence of these curves is ensured by Lemma 12.2.

Since $\widehat{\mathcal{L}_1}\neq\widehat{\mathcal{L}_2}$ there exists $p\in\widehat{E_1}\cap\widehat{E_2}$. Notice that p belongs to $\mathrm{sing}(\mathcal{F})$. Moreover our choice of p implies that it fits in the second alternative of Lemma 12.3. Therefore we can apply Corollary 11.1 to conclude that the first non-zero jet of $\mathcal{W}\boxtimes\mathcal{L}$ at p is equivalent, under a linear change of the affine coordinates (x,y), to one of the following webs:

$$\mathcal{A}_I^4,\mathcal{A}_I^5,\mathcal{A}_I^6,\,\mathcal{A}_{III}^2,\mathcal{A}_{III}^3,\mathcal{A}_{III}^4,\,\mathcal{A}_5^d,\mathcal{A}_6^b. \tag{43}$$

To prove the proposition we will analyze the constraints imposed on the torus T by the above *local models*.

Notice that the 5-web $\mathcal{A}_I^4=[(dx^4-dy^4)]\boxtimes[d(xy)]$ is isomorphic (via a linear map) to $[dxdy(dx^2-dy^2)]\boxtimes[d(x^2+y^2)]$. All the defining foliations of $[dxdy(dx^2-dy^2)]$ are tangent to elliptic fibrations on the square of an arbitrary elliptic curve E. Therefore these local models do not impose restrictions on the curve E. Similarly the 5-web $\mathcal{A}_{III}^2=[dxdy(dx^2-dy^2)]\boxtimes[d(xy)]$ also does not impose restrictions on E. Indeed these two local models coexist in distinct singular points of the exceptional CDQL 5-webs $\mathcal{E}_\tau$.

The 6-webs $\mathcal{A}_{III}^3=[dxdy(dx^3-dy^3)]\boxtimes[d(xy)]$ and $\mathcal{A}_6^b=[dxdy(dx^3+dy^3)]\boxtimes[d(x^3+y^3)]$ share the same linear 5-web (after the change of coordinates $(x,y)\mapsto$

$(x,-y)$ on $\mathcal{A}_6^b$). On the one hand Proposition 12.2 implies that all the defining foliations of the linear 5-web $[dxdy(dx^3-dy^3)]$ must be tangent to elliptic fibrations. On the other hand Lemma 12.1 implies that $\xi_3 \in \mathrm{End}(E)\otimes\mathbb{Q}$. Therefore T must be isogenous to $E^2_{\xi_3}$. Notice that both local models coexist in distinct singular points of the exceptional CDQL 6-web $\mathcal{E}_6$.

The same argument shows that the 5-web $\mathcal{A}_5^d = [dxdy(dx+dy)(dx-\xi_3 dy)] \boxtimes [d(xy(x+y)(x-\xi_3 y))]$ can only be a local model for an exceptional CDQL web when T is isogenous to $E^2_{\xi_3}$. Similarly the 7-web $\mathcal{A}^4_{III} = [dxdy(dx^4-dy^4)]\boxtimes[d(xy)]$ can only be a local model for an exceptional CDQL web when T is isogeneous to E_i^2.

To conclude the proof of the Proposition it suffices to show that the two remaining possibilities in the list (43) (namely $\mathcal{A}_I^5$ and $\mathcal{A}_I^6$) cannot appear as local models for exceptional CDQL webs on a torus.

We will first deal with the 6-web $\mathcal{A}_I^5 = [(dx^5-dy^5)]\boxtimes[d(xy)]$. If ξ_5 is a primitive 5th root of the unity then the cross-ratio $(1,\xi_5 : \xi_5^2, \xi_5^3)$ is a root of the polynomial $p(x) = x^2 - x - 1$. Notice that the roots of $p(x)$ are the golden-ratio and its conjugate: $1/2 \pm \sqrt{5}/2$. In particular they are irrational real numbers and, as such, cannot induce an endomorphism on any elliptic curve E. Lemma 12.1 implies that does not exist a two-dimensional complex torus T where all the defining foliations of $[(dx^5-dy^5)]$ are tangent to elliptic fibrations. Proposition 12.2 implies that $\mathcal{A}_I^5$ cannot appear as a local model of an exceptional CDQL web on a torus.

We also claim that the 7-web $\mathcal{A}_I^6 = [(dx^6-dy^6)]\boxtimes[d(xy)]$ cannot appear as a local model of an exceptional CDQL web on a torus T. Using Lemma 12.1 it is a simple matter to show that T is isogenous to $E^2_{\xi_3}$. Assume now that $\mathcal{L}_1$ and $\mathcal{L}_2$ are such that $\mathcal{L}_1 \neq \widehat{\mathcal{L}_2}$. Lemma 12.2 ensures that there are: an elliptic curve E_1 in $\mathrm{tang}(\mathcal{F},\mathcal{L}_1)$ invariant by $\mathcal{L}_1$ and an elliptic curve $\widehat{E_2}$ in $\mathrm{tang}(\mathcal{F},\mathcal{L}_2)$ invariant by $\widehat{\mathcal{L}_2}$. Since $\mathcal{L}_1 \neq \widehat{\mathcal{L}_2}$ there exits $p \in E_1 \cap \widehat{E_2}$. Since $p \in |\mathrm{tang}(\mathcal{F},\mathcal{L}_1)| \cap |\mathrm{tang}(\mathcal{F},\mathcal{L}_2)|$, it is a singular point of $\mathcal{F}$.

Notice that our choice of p implies that the first non-zero jet of $\mathcal{F}$ at p is non-linear, see Lemma 12.3. Since E_1 is also $\mathcal{F}$-invariant, the linear polynomial defining it on the affine coordinates (x,y) will be also invariant by the first jet of $\mathcal{F}$. But for the 7-web $\mathcal{A}_I^6$ none of the invariant lines through 0 of the non-linear foliation is invariant by any of the linear foliations. Therefore the local model at p must be the only other 7-web appearing in the list (43): $\mathcal{A}^4_{III} = [dxdy(dx^4-dy^4)]\boxtimes[d(xy)]$. But this implies that T is isogenous to E_i^2. Since E_i^2 is not isogenous to $E^2_{\xi_3}$ the claim follows and so does the proposition. □

12.5. **The classification of exceptional CDQL webs on tori.** To obtain the classification of exceptional CDQL webs on tori we will analyze in Sections 12.5.1, 12.5.2, 12.5.3 and 12.5.4 the respective alternatives *(1)*,*(2)*,*(3)* and *(4)* provided by Proposition 12.3.

12.5.1. **The continuous family of exceptional CDQL 5-webs**. In case *(1)* of Proposition 12.3, the torus T is isogenous to the square of an elliptic curve, $k=4$ and the linear web is $\mathcal{W} = [dxdy(dx^2-dy^2)]$. As we have proved in Example 7.1 every flat (in particular exceptional) CDQL 5-web of the form $\mathcal{W}\boxtimes\mathcal{F}$ must be isogenous to one of the 5-webs $\mathcal{E}_\tau$ (with $\tau \in \mathbb{H}$) presented in the Introduction.

12.5.2. **The exceptional CDQL 7-web on E_i^2.** In the second alternative of Proposition 12.3, the torus T is isogenous to E_i^2, $k=6$ and the linear web is $\mathcal{W}=\mathcal{W}_1\boxtimes\mathcal{W}_2$ where $\mathcal{W}_1=[dxdy(dx^2-dy^2)]$ and $\mathcal{W}_2=[dx^2+dy^2]$. This decomposition of $\mathcal{W}$ satisfies the hypothesis of Corollary 7.1. Therefore a non-linear foliation $\mathcal{F}$ satisfies $K(\mathcal{F}\boxtimes\mathcal{W})=0$ if and only if $K(\mathcal{F}\boxtimes\mathcal{W}_1)=K(\mathcal{F}\boxtimes\mathcal{W}_2)=0$. Thus the subweb $\mathcal{F}\boxtimes\mathcal{W}_1$ is isogenous to a web of the continuous family $\mathcal{E}_\tau$. We loose no generality by assuming that $\mathcal{F}\boxtimes\mathcal{W}_1=\mathcal{E}_\tau$ for some $\tau\in\mathbb{H}$. It remains to determine τ. Since T is isogenous to E_i^2 we know that $\tau=\alpha+\beta i$ for suitable rational numbers α,β. Set $\Gamma=\mathbb{Z}\oplus(\alpha+\beta i)\mathbb{Z}$.

Recall from Section 4.2 that the non-linear foliation $\mathcal{F}$ is equal to $[dF_\tau]$ where

$$F_\tau(x,y)=\left(\frac{\vartheta_1(x,\tau)\vartheta_1(y,\tau)}{\vartheta_4(x,\tau)\vartheta_4(y,\tau)}\right)^2.$$

Recall also that $\operatorname{Indet}(F_\tau)=\big\{(\tau/2,0),(0,\tau/2)\big\}$ and that these indeterminacy points correspond to radial singularities of $\mathcal{F}$.

The tangency of $\mathcal{F}$ with the linear foliation $[dx+idy]$ at $(0,\tau/2)$ has first non-zero jet equal to $(x+i\,y)$ since $(x\,dy-y\,dx)\wedge(dx+i\,dy)=-(x+i\,y)\,dx\wedge dy$. Therefore, Theorem 7.2 implies that there exists an elliptic curve C through $(0,\tau/2)$ invariant by $\mathcal{F}$ and by $[dx+idy]$. Notice that C is the image of the entire map

$$\begin{aligned}\varphi:\mathbb{C}\;&\longrightarrow\; E_\tau^2=(\mathbb{C}/\Gamma)^2\\ z\;&\longmapsto\;(-i\,z,z+\tau/2)\,.\end{aligned}$$

Thus $C\cap E_{0,1}=\varphi(i\Gamma)$. The curve $E_{0,1}$ is also $\mathcal{F}$ invariant (but do not coincide with C) so the set $C\cap E_{0,1}$ is contained in $\operatorname{sing}(\mathcal{F})$. But the singularities of $\mathcal{F}$ over $E_{0,1}$ are $(0,0)$ and $(0,\tau/2)$. Moreover the singularity at $(0,0)$ has only two separatrices, namely $E_{1,0}$ and $E_{0,1}$. It follows that $C\cap E_{0,1}=\varphi(i\Gamma)$ is equal to the radial singularity $(0,\tau/2)$ of $\mathcal{F}$ on $E_{0,1}$. Therefore $i\,\Gamma+\tau/2\subset\Gamma+\tau/2$. Consequently $i\Gamma\subset\Gamma$ and $-\Gamma\subset i\,\Gamma$. Thus $i\,\Gamma=\Gamma$, that is $i\in\operatorname{Aut}(E_\tau)$. This is sufficient to show that the elliptic curve E_τ is isomorphic to E_i.

Recall that

$$\Gamma_0(2)=\left\{\left(\begin{array}{cc} a & b\\ c & b\end{array}\right)\in\mathrm{SL}(2,\mathbb{Z})\;\middle|\; b\equiv 0 \mod 2\right\}.$$

Thus, modulo the action of $\Gamma_0(2)$ we can assume that $\tau\in\{i,1+i,(1+i)/2\}$. Moreover the $\mathbb{Z}_2$-extension of $\Gamma(2)$ by the transformation $z\mapsto -2/z$ identifies $1+i$ with $(1+i)/2$ because $-2((1+i)/2)^{-1}=-2+2i$. Therefore we can assume that $\mathcal{F}\boxtimes\mathcal{W}_1$ is equal to $\mathcal{E}_{1+i}$ or to $\mathcal{E}_i$. If $\tau=i$ then $(i/2,0)$ is a radial singularity of $\mathcal{F}$ and, as above, the curve $L_{(i/2,0)}E_{1,i}$ invariant by $[dx+idy]$ is also $\mathcal{F}$-invariant. But this curve intersects the $\mathcal{F}$-invariant curve $E_{0,1}$ at $(0,1/2)$ which is not a singularity of $\mathcal{F}$. This contradiction implies that, up to isogenies, $\mathcal{E}_7=[dx^2+dy^2]\boxtimes\mathcal{E}_{1+i}$ is the unique exceptional CDQL 7-web on complex tori.

12.5.3. **The exceptional CDQL 6-web on $E_{\xi_3}^2$.** In the third alternative of Proposition 12.3 the torus T is isogenous to $E_{\xi_3}^2$, $k=5$ and the linear web is $\mathcal{W}=\mathcal{W}_1\boxtimes\mathcal{W}_2$ with $\mathcal{W}_1=[dxdy]$ and $\mathcal{W}_2=[(dx^3+dy^3)]$. As in the previous case this decomposition satisfies the hypothesis of Corollary 7.1. Therefore $\mathcal{F}$ is a non-linear foliation on T satisfying $K(\mathcal{W}\boxtimes\mathcal{F})=0$ if and only if $K(\mathcal{W}_1\boxtimes\mathcal{F})=K(\mathcal{W}_2\boxtimes\mathcal{F})=0$.

If $K(\mathcal{F}\boxtimes[dxdy])=0$ then Theorem 7.2 (see also [42]) implies that $\mathcal{F}=[a(x)dx+b(y)dy]$ for suitable rational functions $a,b\in\mathbb{C}(E_{\xi_3})$. Moreover, according to item

(3) of Corollary 11.1, we can assume that the singularity of $\mathcal{W}\boxtimes\mathcal{F}$ at $(0,0)$ has first non-zero jet equivalent to $\mathcal{A}_6^b = [dxdy(dx^3+dy^3)]\boxtimes[x^2dx+y^2dy]$. In particular, interpreting x,y as coordinates on the universal covering of T, we can assume that the meromorphic functions a,b satisfy $a(x)=x^2+O(x^3)$ and $b(y)=y^2+O(y^3)$. In particular, $a(0)=a'(0)=b(0)=b'(0)=0$.

A tedious (but trivial) computation shows that $K(\mathcal{F}\boxtimes[dx^3+dy^3])$ is equal to

$$6\frac{ba^3\left(aa''-2a'^2\right)-ab^3\left(bb''-2b'^2\right)+a^4\left(b'^2+bb''\right)-b^4\left(a'^2+aa''\right)}{(a^3-b^3)^2}\,dx\wedge dy.$$

After deriving twice the numerator with respect to y, one obtains

$$a^3b''\left(2(a')^2-a\,a''-3\,b''a\right)-4\,a^4b'b'''+bR$$

where R is a polynomial in $a(x),b(y)$ and theirs derivatives up to order four. Evaluation of this expression at $y=0$ yields the following second order differential equation identically satisfied by a:

$$a^3\big(2\,(a')^2-aa''-6\,a\big)=0. \tag{44}$$

Lemma 12.4. *If $a:(\mathbb{C},0)\to\mathbb{C}$ is a germ of solution of (44) satisfying the boundary conditions $a(0)=a'(0)=0$ and $a''(0)=2$ then*

$$a(x)=x^2 \qquad \text{or} \qquad a(x)=\frac{\lambda^2}{\wp(\lambda^{-1}x,\xi_3)}$$

for a suitable $\lambda\in\mathbb{C}^*$.

Proof. Notice that the 6-web $[dxdy(dx^3+dy^3)]\boxtimes[a(x)dx+a(y)dy]$ with $a(x)=x^2$ is the 6-web $\mathcal{A}_6^b$ from the introduction. Similarly when $a(x)=\lambda^2/\wp(\lambda^{-1}x,\xi_3)$ then the 6-web $[dxdy(dx^3+dy^3)]\boxtimes[a(x)dx+a(y)dy]$ can be obtained from $\mathcal{E}_6$ by the change of coordinates $(x,y)\mapsto(\lambda x,\lambda y)$. Since both $\mathcal{E}_6$ and $\mathcal{A}_6^b$ are exceptional, the corresponding a's are solutions of (44). Clearly they all satisfy the boundary conditions. To prove the lemma it suffices to verify that they are the only solutions.

If $a(x)$ is a solution of (44) satisfying the boundary conditions then it is indeed a solution of $2(a')^2-aa''-6a=0$. Therefore $\gamma(t)=(a(t),a'(t))$ is an orbit of the following vector field

$$Z(x,y)=y\frac{\partial}{\partial x}+\frac{2y^2-6x}{x}\frac{\partial}{\partial y}\,,$$

that is $Z(\gamma(t))=\gamma'(t)$.

Notice that Z admits as a rational first integral the function $\frac{y^2-4x}{x^4}$. Therefore every solution $a(x)$ of (44) satisfying $a(0)=a'(0)=0$ and $a''(0)=2$ must parameterize (through the map $t\mapsto(a(t),a'(t))$) a branch of one of the curves $y^2-4x+\mu x^4$ for some $\mu\in\mathbb{C}$. When $\mu=0$, the corresponding curve is parameterized by $a(x)=x^2$. For $\mu\neq0$ it is parameterized by $a(x)=\lambda^2/\wp(\lambda^{-1}x,\xi_3)$ with λ satisfying $\mu\lambda^6=1$. Notice that the different choices for λ leads to the same function a. Indeed, the symmetry $-\xi_3(\mathbb{Z}\oplus\xi_3\mathbb{Z})=\mathbb{Z}\oplus\xi_3\mathbb{Z}$ combined with (11) implies that

$$\frac{(-\xi_3)^2}{\wp\big((-\xi_3)^{-1}x,\xi_3\big)}=\frac{1}{\wp(x,\xi_3)}. \tag{45}$$

Since each of the curves $\{y^2-4x-\mu x^4=0\}$ admits only one parametrization of the form $t\mapsto(a(t),a'(t))$ with $a''(0)=2$, the lemma follows. $\square$

Keeping in mind that the coefficients of the defining 1-form of $\mathcal{F}$ must be doubly-periodic functions and the symmetry of our setup, so far we have proved that $K(\mathcal{F}\boxtimes\mathcal{W})=0$ implies that, up to homotetheies,

$$\mathcal{F}=\left[\frac{dx}{\wp(x,\xi_3)}+\frac{\lambda^2 dy}{\wp(\lambda^{-1}y,\xi_3)}\right]$$

for a suitable $\lambda\in\mathbb{C}^*$. Computing again $K(\mathcal{F}\boxtimes[dx^3+dy^3])$ yields

$$\frac{9\lambda^2(\lambda^6-1)\wp(x,\xi_3)^2\wp(y/\lambda,\xi_3)^2}{\lambda^{12}\wp(x,\xi_3)^6-2\lambda^6\wp(y/\lambda,\xi_3)^3\wp(x,\xi_3)^3+\wp(y/\lambda,\xi_3)^6}\,dx\wedge dy.$$

The vanishing of the curvature, taking into account (45), implies that

$$\mathcal{F}=\left[\frac{dx}{\wp(x,\xi_3)}+\frac{dy}{\wp(y,\xi_3)}\right].$$

It follows that the 6-web $\mathcal{F}\boxtimes\mathcal{W}$ is isogenous to the 6-web $\mathcal{E}_6$ from the Introduction.

12.5.4. **The exceptional CDQL 5-web on $E^2_{\xi_3}$. Combinatorial patchwork**. In the last case of Proposition 12.3 (transformed via the change of coordinates $(x,y)\mapsto(y,-x)$), the complex torus T is isogenous to $E^2_{\xi_3}$, $k=4$ and the linear web $\mathcal{W}$ is $[dxdy(dx-dy)(\xi_3\,dx+dy)]$. Unlikely in the previous case the web $\mathcal{W}$ does not admit a decomposition satisfying the hypothesis of Corollary 7.1. We have not succeeded in dealing with this case using analytic methods as in the previous section and in [42]. We were lead to adopt a geometrical/combinatorial approach.

If $\mathcal{L}_1=[dx]$, $\mathcal{L}_2=[dy]$, $\mathcal{L}_3=[dx-dy]$ and $\mathcal{L}_4=[\xi_3\,dx+dy]$ then straightforward computations using formula (21) show that

$$\widehat{\mathcal{L}_1}=\big[dx+(\xi_3^2-1)\,dy\big]\qquad \widehat{\mathcal{L}_3}=\big[dx-\xi_3\,dy\big]$$
$$\widehat{\mathcal{L}_2}=\big[(\xi_3-1)\,dx+dy\big]\qquad \widehat{\mathcal{L}_4}=\big[dx+\xi_3\,dy\big].\tag{46}$$

For $i=1,2,3,4$, the leaves of $\widehat{\mathcal{L}_i}$ are translates of the elliptic curve $\widehat{E_i}$ where

$$\widehat{E_1}=E_{1-\xi_3^2,1},\quad \widehat{E_2}=E_{1,1-\xi_3},\quad \widehat{E_3}=E_{\xi_3,1}\quad\text{and}\quad \widehat{E_4}=E_{\xi_3,-1}\,.$$

Suppose that $\mathcal{F}$ is a non-linear foliation on T such that $\mathcal{W}\boxtimes\mathcal{F}$ has maximal rank. According to Corollary 11.1 and taking into account the change of coordinates $(x,y)\mapsto(y,-x)$, there are only two possibilities for a singularity p of $\mathcal{F}$: either p is a radial singularity or the first non-zero jet of $\mathcal{F}\boxtimes\mathcal{W}$ at p is equivalent to

$$\big[dx\,dy\,(dx-dy)\,(\xi_3\,dx+dy)\big]\boxtimes\big[d\big(xy(x-y)(\xi_3\,x+y)\big)\big].$$

We will say that the former singularities are of type A whereas the latter are of type B. We will write $\mathrm{sing}^A(\mathcal{F})$ (resp. $\mathrm{sing}^B(\mathcal{F})$) for the set of singularities of type A (resp. of type B).

By the very definition, the first non-zero jet of $\mathcal{F}$ at a point $p\in\mathrm{sing}^B(\mathcal{F})$ is

$$\mathcal{F}_0=\Big[d\big(xy(x-y)(\xi_3\,x+y)\big)\Big].$$

Simple computations show that

$$\mathrm{tang}(\mathcal{F}_0,\mathcal{L}_i)=\begin{cases}\big\{\,x\left(x+(\xi_3^2-1)y\right)^2=0\,\big\} & \text{when}\quad i=1\\ \big\{\,y\left((\xi_3-1)x+y\right)^2=0\,\big\} & \text{when}\quad i=2\\ \big\{\,(x-y)\,(x-\xi_3\,y)^2=0\,\big\} & \text{when}\quad i=3\\ \big\{\,(\xi_3\,x+y)\,(x+\xi_3\,y)^2=0\,\big\} & \text{when}\quad i=4\,.\end{cases}\tag{47}$$

Being aware of the first nonzero jets of the singularities of $\mathcal{F}$, we are able to describe the first non-zero jets of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$. This is the content of the two following lemmata.

Lemma 12.5. *Let $p \in \mathrm{sing}^A(\mathcal{F})$. For every $i \in \{1, \ldots, 4\}$, there is an unique irreducible component of the divisor $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ passing through p: it is an irreducible curve C invariant by $\mathcal{L}_i$. In particular, there is no $\widehat{\mathcal{L}_i}$-invariant curve in $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ passing through p.*

Proof. Since the first non-zero jet of $\mathrm{tang}(\mathcal{F},\mathcal{L}_i)$ at p coincides with $\mathrm{tang}([xdy - ydx], \mathcal{L}_i)$ the lemma follows from Theorem 7.3. $\square$

Lemma 12.6. *Let $p \in \mathrm{sing}^B(\mathcal{F})$. For every $i \in \{1, \ldots, 4\}$, the divisor $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ contains in its support two distinct irreducible curves C_i and $\widehat{C}_i$ both containing p. Moreover, C_i (resp. $\widehat{C}_i$) is invariant by $\mathcal{L}_i$ (resp. by $\widehat{\mathcal{L}_i}$).*

Proof. Since the first non-zero jet of $\mathrm{tang}(\mathcal{F},\mathcal{L}_i)$ at p coincides with $\mathrm{tang}(\mathcal{F}_0, \mathcal{L}_i)$ the lemma follows from Theorem 7.3 combined with (46) and (47). $\square$

The core of our argument to characterize $\mathcal{E}_5$ is contained in the next lemma.

Lemma 12.7. *Let $\mathcal{F}$ be a non-linear foliation on the torus $T = E^2_{\xi_3}$. Suppose that the 5-web $\mathcal{F} \boxtimes [dxdy(dx - dy)(\xi_3\, dx + dy)]$ has maximal rank. If $0 \in \mathrm{sing}^B(\mathcal{F})$ then*

- *(a) $(0, y) \in \mathrm{sing}(\mathcal{F})$ if and only if $(y, 0) \in \mathrm{sing}(\mathcal{F})$;*
- *(b) If $(y, 0) \in \mathrm{sing}(\mathcal{F})$ then $(2y, 0) \in \mathrm{sing}^B(\mathcal{F})$;*
- *(c) If $(y, 0) \in \mathrm{sing}(\mathcal{F})$ then $(-\xi_3^2\, y, 0) \in \mathrm{sing}(\mathcal{F})$;*
- *(d) Both $\mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ and $\mathrm{sing}^B(\mathcal{F}) \cap E_{1,0}$ are subgroups of $E_{1,0}$. Similarly $\mathrm{sing}(\mathcal{F}) \cap E_{0,1}$ and $\mathrm{sing}^B(\mathcal{F}) \cap E_{0,1}$ are subgroups of $E_{0,1}$.*

Proof. To prove *(a)*, we will use that the curves $E_{0,1}$, $E_{1,0}$ and $E_{1,1}$ passing through $(0, 0)$ are $\mathcal{F}$-invariant (what is ensured by Lemma 12.7). If $(0, y) \in \mathrm{sing}(\mathcal{F})$ then Lemma 12.5 implies that $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$-invariant. Therefore $L_{(0,y)}E_{1,0} \cap E_{1,1} = (y, y)$ is the intersection of two distinct leaves of $\mathcal{F}$. It follows that $(y, y) \in \mathrm{sing}(\mathcal{F})$. Consequently $L_{(y,y)}E_{0,1}$ is also $\mathcal{F}$-invariant. Since $(y, 0) = L_{(y,y)}E_{0,1} \cap E_{1,0}$, item *(a)* follows.

To prove *(b)*, start by noticing that $(0, y) \in \mathrm{sing}(\mathcal{F})$ by *(a)*. Therefore $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$-invariant according to Lemma 12.5. By hypothesis $(0, 0) \in \mathrm{sing}^B(\mathcal{F})$ thus Lemma 12.6 ensures that the elliptic curve $\widehat{E_1} = E_{1-\xi_3^2,1}$ belongs to $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$. The curve $L_{(0,y)}E_{1,0}$ being invariant by $\mathcal{L}_2$ and $\mathcal{F}$ (since $(0, y) \in \mathrm{sing}(\mathcal{F})$), it is necessarily an irreducible component of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_2)$. As a consequence, the intersection $\widehat{E_1} \cap L_{(0,y)}E_{1,0}$ is included in $\mathrm{sing}^B(\mathcal{F})$. In particular, the point $p = ((1 - \xi_3^2)y, y)$ belongs to $\mathrm{sing}^B(\mathcal{F})$. Considering now the $\widehat{\mathcal{L}_3}$-invariant curve through p, that is $L_pE_{\xi_3,1}$, we see that it intersects $E_{0,1}$ at $(2y, 0)$. Thus $(2y, 0) \in \mathrm{sing}^B(\mathcal{F})$ proving item *(b)*.

To prove item *(c)*, recall from the previous paragraph that $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$-invariant. The curve $E_{1,-\xi_3}$ intersects $L_{(0,y)}E_{1,0}$ at $p = (-\xi_3^2\, y, y)$. Since $E_{1,-\xi_3}$ is $\mathcal{F}$-invariant (by Lemma 12.5) it follows $p \in \mathrm{sing}(\mathcal{F})$. Consequently $L_pE_{0,1}$ is $\mathcal{F}$-invariant (again by Lemma 12.5) and $(-\xi_3^2\, y, 0) = L_pE_{0,1} \cap E_{1,0} \in \mathrm{sing}(\mathcal{F})$. Item *(c)* follows.

It remains to prove item *(d)*. We will first prove that $S = \mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ is a subgroup of $E_{1,0}$. From item *(c)* it follows $(y, 0) \in S$ if and only if $(-y, 0) \in S$. Thus it suffices to show that, given two elements $(y_1, 0)$ and $(y_2, 0)$ of S, their sum $(y_1 + y_2, 0)$ is also in S. Item *(a)* implies that $(0, y_2) \in \mathrm{sing}(\mathcal{F})$ and consequently the curve $L_{(0,y_2)}E_{1,0}$ is $\mathcal{F}$-invariant (by Lemma 12.5). For the same reason the curve $L_{(y_1,0)}E_{1,1}$ is also $\mathcal{F}$-invariant thus the point $p = (y_2 + y_1, y_2) \in L_{(0,y_2)}E_{1,0} \cap L_{(y_1,0)}E_{1,1}$ belongs to $\mathrm{sing}(\mathcal{F})$. Since $L_pE_{0,1}$ intersects $E_{0,1}$ at $(y_1 + y_2, 0)$ and because these two curves are $\mathcal{F}$-invariant, it follows that $(y_1 + y_2, 0) \in S$. Therefore $\mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ is a subgroup of $E_{1,0}$.

Consider now the group homomorphism

$$\begin{array}{ccc} S & \longrightarrow & S \\ x & \longmapsto & x + x\,. \end{array}$$

Item *(b)* implies that its image is $\mathrm{sing}^B(\mathcal{F}) \cap E_{1,0}$. Therefore $\mathrm{sing}^B(\mathcal{F}) \cap E_{1,0}$ is also a subgroup of $E_{1,0}$.

Mutatis mutandis we obtain the same statements for $\mathrm{sing}(\mathcal{F}) \cap E_{0,1}$ and $\mathrm{sing}^B(\mathcal{F}) \cap E_{0,1}$: both are subgroups of $E_{0,1}$. □

Theorem 12.2. *Let $\mathcal{F}$ be a non-linear foliation on $T = E^2_{\xi_3}$. If the 5-web $[dxdy(dx - dy)(\xi_3\, dx + dy)] \boxtimes \mathcal{F}$ has maximal rank then it is isogenous to $\mathcal{E}_5$.*

Proof. Let us denotes by $\equiv$ the numerical equivalence of divisors on T. Since $\mathcal{O}_T(\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)) = N\mathcal{F}$ for $i = 1, \ldots, 4$, all the divisors $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$ are pairwise linearly equivalent. Moreover, Theorem 7.2 implies that

$$\mathrm{tang}(\mathcal{F}, \mathcal{L}_i) \equiv a_i E_i + b_i \widehat{E_i}$$

for $i = 1, \ldots, 4$, where E_i and $\widehat{E_i}$ are elliptic curves in T invariant by $\mathcal{L}_i$ and $\widehat{\mathcal{L}_i}$ respectively and a_i, b_i are non-negative integers. Indeed Lemma 12.2 implies that a_i, b_i are positive integers. In particular we obtain that

$$a_1 E_{0,1} + b_1 E_{1-\xi_3^2,1} \equiv a_2 E_{1,0} + b_2 E_{1,1-\xi_3}\,.$$

Intersecting both members with $E_{0,1}$, $E_{1,0}$ and $E_{1,1}$ we obtain respectively

$$3b_1 = a_2 + b_2\,, \quad a_1 + b_1 = 3b_2 \quad \text{and} \quad a_1 + b_1 = a_2 + b_2\,.$$

Thus $a_1/b_1 = a_2/b_2 = 2$.

Assume, without loss of generality, that $0 \in T$ is point in $\mathrm{sing}^B(\mathcal{F})$. Notice that $E_{1,0}$ is $\mathcal{F}$-invariant and $\mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ is equal to the set of intersection points of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ with $E_{1,0}$. Moreover $\mathrm{sing}^B(\mathcal{F})$ corresponds to the intersection with $E_{1,0}$ of the irreducible components of $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ that are invariant by $\widehat{\mathcal{L}_1}$. Equation (47) implies that each of the $\widehat{\mathcal{L}_1}$-invariant curves in $\mathrm{tang}(\mathcal{F}, \mathcal{L}_1)$ appears with multiplicity two. From $a_1/b_2 = 2$ it follows that the cardinality of $\mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ is four times the cardinality of $\mathrm{sing}^B(\mathcal{F}) \cap E_{1,0}$. Recall from Lemma 12.7 item *(d)* that $S = \mathrm{sing}(\mathcal{F}) \cap E_{1,0}$ and $S^B = \mathrm{sing}^B(\mathcal{F}) \cap E_{1,0}$ are subgroups of $E_{1,0}$. It is now clear that the kernel of the map $S \to S^B$ given by multiplication by two is the subgroup of two-torsion points of $E_{1,0}$.

Notice that we can reconstruct the divisors $\mathrm{tang}(\mathcal{F}, \mathcal{L}_i)$, for $i = 2, 3, 4$, from the subgroups S and S^B. Indeed

$$\mathrm{tang}(\mathcal{F}, \mathcal{L}_i) = \sum_{p \in S} L_p E_i + 2 \left(\sum_{p \in S^B} L_p \widehat{E_i} \right) .$$

It follows that the foliation $\mathcal{F}$ is invariant by the natural action of $S^B \subset E_{1,0}$ in T, that is,

$$\begin{array}{ccc} S^B \times T & \longrightarrow & T \\ (g, 0), (x, y) & \mapsto & (x + g, y) . \end{array}$$

Indeed, due to the symmetry of our setup, $\mathcal{F}$ is left invariant by the following action of $(S^B)^2$,

$$\begin{array}{ccc} (S^B)^2 \times T & \longrightarrow & T \\ \big((g, 0), (h, 0), (x, y)\big) & \mapsto & (x + g, y + h) . \end{array}$$

The quotient of $\mathcal{F} \boxtimes \mathcal{W}$ by this action is a CDQL 5-web on $E^2_{\xi_3}$ of the form $\mathcal{G} \boxtimes \mathcal{W}$. If $E_{0,1}(2)$ denotes the two-torsion points on $E_{0,1}$ then, by construction,

$$\mathrm{tang}(\mathcal{G}, \mathcal{L}_i) = 2\, \widehat{E_i} + \sum_{p \in E_{0,1}(2)} L_p E_i .$$

for $i = 2, 3, 4$. This is sufficient to show that $\mathcal{G} \boxtimes \mathcal{W}$ is the 5-web $\mathcal{E}_5$ of the Introduction. $\square$

With Theorem 12.2 we complete the classification of exceptional CDQL webs on complex tori and, consequently, on compact complex surfaces.

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