Projective limit cycles [1]

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Abstract

In this article we study projective cycles in $\mathbb{P}^2_{\mathbb{R}}$. Our inspiring example is the Jouanolou foliation of odd degree which has a hyperbolic projective limit cycle. We prove that only odd degree foliations may have projective cycles and foliations with exactly one real simple singularity have a projective cycle. We also prove that after a perturbation of a generic Hamiltonian foliation with a projective cycle, we have a projective limit cycle if and only if the perturbation is not Hamiltonian.

1 Introduction

For a holomorphic foliation $\mathcal{F}_{\mathbb{C}}$ in $\mathbb{P}^2_{\mathbb{C}}$ and defined over $\mathbb{R}$ we have also the real foliation $\mathcal{F}_{\mathbb{R}}$ in $\mathbb{P}^2_{\mathbb{R}}$ for which the leaves are obtained by the intersection of the leaves of $\mathcal{F}_{\mathbb{C}}$ with $\mathbb{P}^2_{\mathbb{R}}$. In the complex context we have the minimal set problem (see [CLS89]) which asks the existence of a leaf of $\mathcal{F}_{\mathbb{C}}$ which does not accumulates on singularities. In the real context we have the celebrated Hilbert sixteen problem which asks for a uniform bound for the number of limit cycles of $\mathcal{F}_{\mathbb{R}}$. If the minimal set for a foliation defined over $\mathbb{R}$ exists and if it intersects the real projective space $\mathbb{P}^2_{\mathbb{R}}$ then the intersection may contain a union of closed cycles, among them at most one is projective, i.e. its neighborhood is isomorphic to the Möbius band. This projective cycle can exist only for odd degree foliations and it can be regarded as a counterpart of the minimal set in the real context. There is a huge literature on affine cycles and few is known about projective cycles. The objective of the present article is to describe in details such cycles, their properties, perturbations, degenerations and so on. The text is organized in the following way:

In §2 we introduce basic definitions related to projective and affine cycles. We show that the Jouanolou foliation of odd degree has always a projective limit cycle. In §3 we describe degenerations of projective cycles and in §4 we show that even degree foliations cannot posses a projective cycle. In §5 we show that foliations with only one simple real singularity in $\mathbb{P}^2_{\mathbb{R}}$ posses a projective cycle. §6 is dedicated to the study of foliations with a Hamiltonian first integral and with a projective cycle. In §7 we prove that the projective cycle of a generic Hamiltonian foliation after a perturbation turns into a projective limit cycle if and only if the deformed foliation is not Hamiltonian. The key idea of the proof is that the cycles which live over the projective cycle of a Hamiltonian foliation are vanishing cycles.

2 Projective limit cycles

In this section we present basic definitions and properties related to projective cycles.

Definition 1. Let δ be a connected component of a real smooth algebraic curve $C_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{R}}$. It is called an affine oval (resp. projective oval) if its complement has two (resp. one) connected components. In a similar way a closed leaf δ (without singularity) of a

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real algebraic foliation $\mathcal{F}_{\mathbb{R}}$ in $\mathbb{P}^2_{\mathbb{R}}$ is called an affine cycle (resp. projective cycle) if its complement has two components (resp. one component).

Let Σ be a transverse section to a foliation $\mathcal{F}$ in $\mathbb{P}^2_{\mathbb{R}}$ in a point p of the projective or affine cycle γ of $\mathcal{F}$. In a local coordinate $t : \Sigma \to \mathbb{R},\ t(p) = 0$ one can write the Taylor series of the Poincaré first return map (holonomy of γ):

$$h(t) = a_1 t + a_2 t^2 + \cdots,\ a_1, a_2, \ldots \in \mathbb{R}$$

It is easy to see that a_1 is independent of the choice of the coordinate t. It is called the multiplier of γ. If $a_1 = 1$ then the natural number m such that $a_2 = a_3 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$ is called the order of γ and it is also independent of the choice of the coordinate t. It is easy to see that

1. A projective cycle/oval intersects any projective cycle/oval.
2. The multiplier of a projective cycle (resp. affine cycle) is negative (positive).
3. A neighborhood of a projective cycle/oval is homeomorphic to the Möbius band.
4. A foliation in $\mathbb{P}^2_{\mathbb{R}}$ has at most one projective cycle.

Definition 2. *An affine (resp. a projective) cycle is called a limit cycle if its holonomy is not identity (resp. if twice iteration of its holonomy is not identity).*

It is useful to mention that we can obtain $\mathbb{P}^2_{\mathbb{R}}$ by gluing 3 quadrilaterals Q_1, Q_2 and Q_3 through its edges like Figure 1. We will use Q_1, Q_2 and Q_3 the squares $[-1,1] \times [-1,1]$ in the three canonical charts $\varphi_1, \varphi_3, \varphi_3 : \mathbb{R}^2 \to \mathbb{P}^2_{\mathbb{R}}$ of $\mathbb{P}^2_{\mathbb{R}}$:

$$\begin{array}{c} \varphi_1,\ \varphi_2, \varphi_3 :\ \mathbb{R}^2 \ \longrightarrow\ \mathbb{P}^2_{\mathbb{R}} \\ \varphi_1(x,y) = (x:y:1),\ \varphi_2(u,v) = (v:1:u),\ \varphi_3(s,t) = (1:s:t) \end{array} \tag{1}$$

We will use transition maps from the (x,y) chart given by

$$\varphi_{12} : (x,y) \mapsto (u,v) = \left(\frac{1}{y}, \frac{x}{y}\right) \text{ and } \varphi_{13} : (x,y) \mapsto (s,t) = \left(\frac{y}{x}, \frac{1}{x}\right)$$

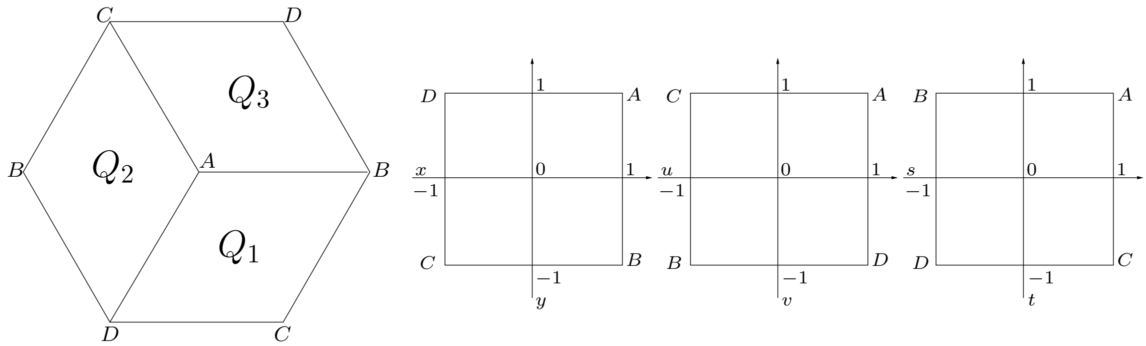

Figure 1: $\mathbb{P}^2_{\mathbb{R}}$ obtained by gluing 3 quadrilaterals

Example 1. Projective cycle in the Jouanolou foliation: The Jouanolou foliation of degree k in $\mathbb{P}^2_{\mathbb{R}}$, namely $\mathcal{J}^k_{\mathbb{R}}$, is defined in the affine chart (x, y), by the polynomial vector field

$$X : \begin{cases} \dot{x} &= y^k - x^{k+1} \\ \dot{y} &= 1 - yx^k \end{cases} .$$

In the affine charts (u, v) and (s, t) the vector fields $X_1(u, v) = u^{k-1}D\varphi_{12}(X)$ and $X_2(s, t) = t^{k-1}D\varphi_{13}(X)$, respectively, have the same expression like $X(x, y)$. Therefore, it is sufficient to know the behavior of $\mathcal{J}^k_{\mathbb{R}}$ inside the square $[-1, 1] \times [-1, 1]$ in the affine chart (x, y). We are going to study the vector field $\widetilde{X} = -X$ in the triangle $\triangle DCB$ (see Figure 2).

It is straightforward to see that all the orbits of $\widetilde{X}$ that meets DC and DB enter in the triangle $\triangle DCB$. Since $\widetilde{X}$ has not singularities in $\triangle DCB$ (the only singularity of $\widetilde{X}$ in $\mathbb{R}^2$ is the point $(1, 1)$), we conclude that all the orbits which begin in any point in DC necessarily leaves $\triangle DCB$ by some point in CB (see Figure 2). Therefore, we have an analytic function $f : [-1, 1] \to [-1, 1]$ which is the holonomy from DC to CB. We claim that there is a projective cycle of X crossing DC. By the observation that we made at the beginning, the projective cycle intersects CD is a point a with $f(a) = -a$. This point exists and it is unique because for $g(t) = t + f(t)$ we have $g'(t) = 1 + f'(t) > 0$ $\forall t \in (-1, 1)$, $g(-1) = -2$ and $g(1) = 1 + f(1) > 0$. It is not hard to see that the holonomy of the projective cycle δ of X is hyperbolic (see [Vi09]).

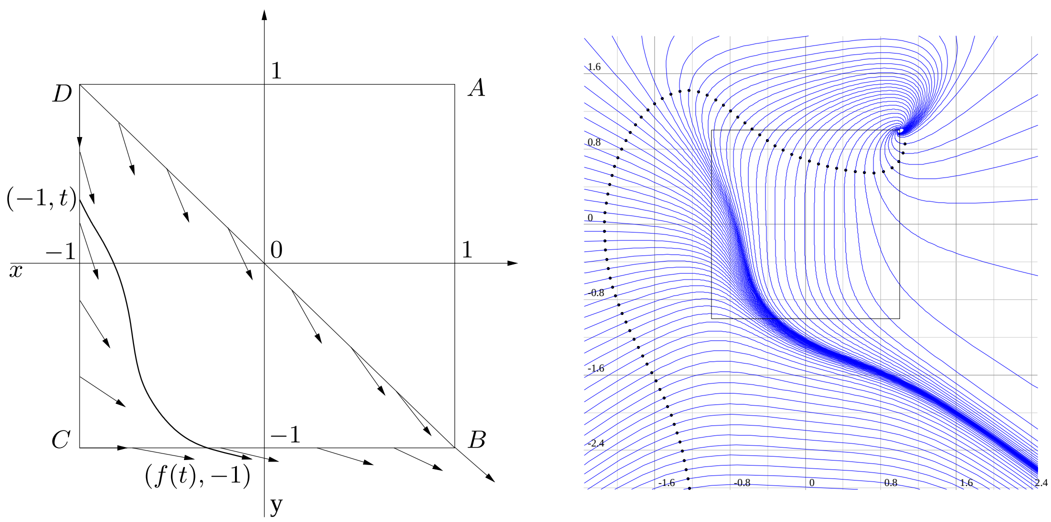

Figure 2: Projective limit cycle of the Jouanolou foliation

Remark 1. We always use the projective degree for a foliations $\mathcal{F}$ in $\mathbb{P}^2_{\mathbb{R}}$. It is the number of tangency points of a generic line in $\mathbb{P}^2_{\mathbb{C}}$ with $\mathcal{F}_{\mathbb{C}}$. In algebraic terms, a degree d foliation $\mathcal{F}$ in $\mathbb{P}^2_{\mathbb{R}}$ in an affine coordinates (x, y) is given by $\omega = p(x, y)dy + q(x, y)dx + g(x, y)(xdy - ydx)$, where $g, p, q \in \mathbb{R}[x, y]$ and either $g = 0$ and $\max\{\deg(p), \deg(q)\} = d$ or g is a non-zero homogeneous polynomial of degree d and $\deg(p), \deg(q) \leq d$.

3 Degeneration of projective limit cycles

A limit cycle with multiplier 1 and even order can disappear (go to the complex domain) by small perturbations of the foliation. However, the situation by a projective cycle is different because its multiplier is always negative and hence is never equal to 1.

Proposition 1. *Let γ be projective cycle of $\mathcal{F} = \mathcal{F}_{\mathbb{R}}$. For any small perturbation $\mathcal{F}_\varepsilon, \varepsilon \in (\mathbb{R}, 0)$ of $\mathcal{F}$, there is a projective cycle of $\mathcal{F}_\varepsilon$ near γ.*

Proof. Let $h_\varepsilon(z) = az + h.o.t.$ be the deformed holonomy of $\mathcal{F}_\varepsilon$ along γ. We know that $a < 0$ and so $h_\varepsilon(z) - z = (a-1)z + h.o.t.$ For $\varepsilon = 0$ this has a unique zero of multiplicity one and so its graph intersects the z axis transversely. This property is preserved for any ε near enough to 0. $\square$

Let $\mathcal{F}(Pdx + Qdy)$ be a germ of holomorphic foliation with an isolated singularity at $p \in \mathbb{C}^2$, i.e. $\{P = 0, Q = 0\} = \{p\}$. The multiplicity of $\mathcal{F}$ at the singularity p is defined to be the dimension of $\mathcal{O}_p/\langle P, Q\rangle_p$, where $\mathcal{O}_p$ is the germ of holomorphic functions in $p \in \mathbb{C}^2$ and $\langle P, Q\rangle_p$ is its ideal generated by P and Q in p. It can be verified that a singularity p is of multiplicity 1 if and only if the germ of varieties $P = 0$ and $Q = 0$ are smooth and interset each other transversely at p. If the multiplicity of a singularity p is m then we can obtain m singularities of multiplicity one after a generic perturbation of $\mathcal{F}$. The singularities with multiplicity bigger than one appear in a natural way in the study of real foliations in $\mathbb{P}^2_{\mathbb{R}}$ and their projective limit cycles.

Let $\mathcal{F}_\varepsilon, \varepsilon \in (\mathbb{R}^+, 0)$ be a family of foliations in $\mathbb{P}^2_{\mathbb{R}}$. Assume that for all non-zero ε, $\mathcal{F}_\varepsilon$ has a projective limit cycle γ_ε and $\mathcal{F}_0$ has not. Since for ε fixed, γ_ε is unique, it is natural to ask what happens to γ_ε when ε goes to zero. Since γ_ε is a continuous family of cycles, the only possibility is that pairs of singularities of $\mathcal{F}_\varepsilon$, which are complex conjugated, approach $\mathbb{P}^2_{\mathbb{R}}$ as ε goes to zero and for $\varepsilon = 0$ there appears at least one singularity of multiplicity bigger than one such that it has a separatrix which lies in the limit of γ_ε. In general the limit of γ_ε is a union of singularities and separatrices. Let us explain this phenomenon by two examples.

Example 2. Consider $\mathcal{F}_\varepsilon = \mathcal{F}((y^k - x^{k+1})dy - (\varepsilon - yx^k)dx)$ which is the Jouanolou of degree k foliation for $\varepsilon = 1$. As $\varepsilon \to 0^+$ the projective limit cycle approaches $y = 0$. Note that $y = 0$ is an algebraic solution of $\mathcal{F}_0$ and all the complex and real singularities of $\mathcal{F}_\varepsilon$ accumulate in $0 \in \mathbb{R}^2$ and so the singularity 0 of $\mathcal{F}_0$ has the maximum multiplicity $k^2 + k + 1$. One can also check this directly from the algebraic definition of multiplicity.

Example 3. Consider $\mathcal{F}_\varepsilon = \mathcal{F}((y^k - x^{k+1} - \varepsilon x)dy - (1 - yx^k)dx)$ which is the Jouanolou of degree k foliation for $\varepsilon = 0$. Using a computer one can see that there exists $\varepsilon_0 > 0$ such that $\mathcal{F}_{\varepsilon_0}$ has a singularity of multiplicity 2. This singularity destroys the projective limit cycle. Note that for $\varepsilon > \varepsilon_0$ this singularity separates itself in two real singularities, a sink and a saddle (see Figure 3 with $\varepsilon = 2 > \varepsilon_0$).

Note that in Example 2 (resp. Example 3), the foliation $\mathcal{F}_\varepsilon$ for all $0 < \varepsilon < 1$ (resp. $0 < \varepsilon < \varepsilon_0$) has a projective cycle (see Proposition 2).

4 Even degree foliations

In this section we prove the following:

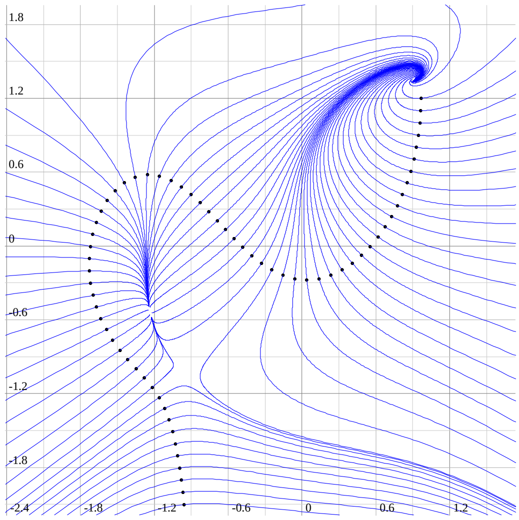

Figure 3: projective cycle destroyed

Theorem 1. *A foliation of even degree does not have any projective cycle.*

Proof. Let γ be a projective cycle of a foliation in $\mathbb{P}^2_{\mathbb{R}}$. We take the line at infinity $l_\infty \subset \mathbb{P}^2_{\mathbb{R}}$ in such a way that it intersects γ transversely.

Since $\pi_1(\mathbb{P}^2_{\mathbb{R}}) = \mathbb{Z}/2\mathbb{Z}$, γ is homotopic to l_∞. We claim that $\#\gamma \cap l_\infty$ is an odd number. To see this we take homotopy $\gamma_\varepsilon, \varepsilon \in [0,1]$ such that $\gamma_0 = l_\infty$ and $\gamma_1 = \gamma$. Further, we can assume that $\gamma_\varepsilon, \varepsilon \neq 0$ is either transverse to l_∞ or it has a unique tangency point of order two with l_∞. The integer valued function $\#\gamma_\varepsilon \cap l_\infty$ drops each time by two and near 0 it is identically one. This proves our affirmation.

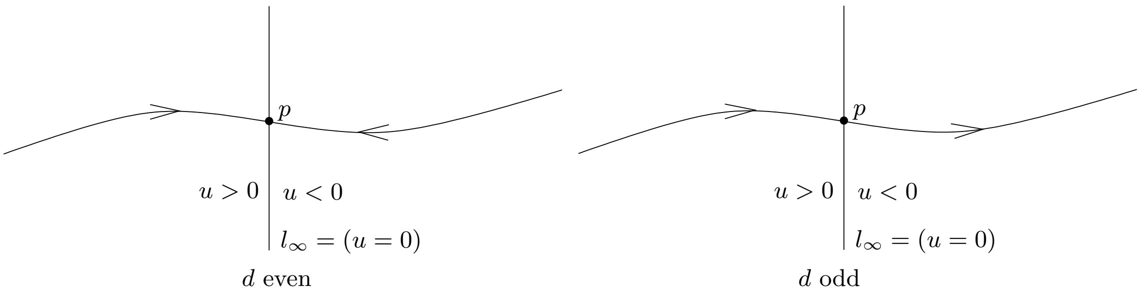

Figure 4:

Let us assume that the foliation $\mathcal{F}$ in the affine chart $\mathbb{R}^2 = \mathbb{P}^2_{\mathbb{R}} \backslash l_\infty$ is given by the vector field X. In this way each leaf of $\mathcal{F}$ in $\mathbb{R}^2$ is oriented. We consider the orientation induced by X. Let us look at this orientation at infinity. Let l be a leaf of $\mathcal{F}$ which intersects l_∞ transversely at p. We claim that if $d = \deg(\mathcal{F})$ is even then the orientation of L in both sides of l_∞ is opposite. To see this assume that p is in the chart (u, v) and take the change of coordinates φ_{12} as in (1). Now, the vector field $u^{d-1} D\varphi_{12}(X)$ is holomorphic and so it induces an orientation on l in this chart. Since $d-1$ is odd, this orientation coincides with the orientation induced by X only in the side $(u > 0)$ (see Figure 4).

Now, if $p_i, i = 1, 2, \ldots, k$ are the intersection points of γ with l_∞ and γ has opposite directions near each point p then k must be even which is a contradiction. $\square$

Remark 2. The Theorem 1 give us an important information. If the degree of the foliation is odd then a polynomial vector field which induces $\mathcal{F}$ induces an orientation to $\mathcal{F}$ too, so we can find a global C^∞ real vector field $\widetilde{X}$ in $\mathbb{P}^2_{\mathbb{R}}$ such that the orbits of $\widetilde{X}$ are the leaves of $\mathcal{F}$. We will use this later.

5 The space of holomorphic foliations

Let $\overline{\mathcal{F}}_d$ denote the space of degree d foliations in $\mathbb{P}^2_{\mathbb{R}}$ and let $\mathcal{F}_d \subset \overline{\mathcal{F}}_d$ be its subset containing all foliations with non-degenerated singularities in $\mathbb{P}^2_{\mathbb{C}}$. By definition each real or complex singularity of $\mathcal{F} \in \mathcal{F}_d$ has multiplicity one and $\mathcal{F}_d$ is a dense open subset of $\overline{\mathcal{F}}_d$.

Proposition 2. *Any foliation $\mathcal{F}$ of odd degree in $\mathbb{P}^2_{\mathbb{R}}$ and with exactly one simple real singularity has a projective cycle.*

Proof. Let $\mathcal{F}$ be a foliation of odd degree in $\mathbb{P}^2_{\mathbb{R}}$ and with exactly one simple real singularity p. By topological arguments, this singularity is a sink, a source or a center, never a saddle. Suppose $\mathcal{F}$ has at least one affine cycle. Let us consider $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ the family of affine cycles of $\mathcal{F}$. Take $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ the family of discs in $\mathbb{P}^2_{\mathbb{R}}$ bounded by affine cycles in $\mathcal{F}$, where $\partial\Gamma_\lambda = \gamma_\lambda$. Each disc Γ_λ is invariant by $\mathcal{F}$ and so $\mathcal{F}$ has a singularity in it. Since $\mathcal{F}$ has only one singularity, the family $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ is ordered by inclusion.

Take the disc $\Gamma = \cup_{\lambda\in\Lambda}\Gamma_\lambda$. It is invariant by $\mathcal{F}$ and so $\gamma = \partial\Gamma$ is invariant by $\mathcal{F}$ too. Since γ has not a singular point of $\mathcal{F}$, it is a closed curve. If γ is a projective cycle, the proof is finished. Else, since d is odd, we can give an orientation to $\mathcal{F}$ (see the Remark after Theorem 1). Without loss of generality, we can suppose that γ is an ω-limit of an orbit δ external to Γ. Since $\mathbb{P}^2_{\mathbb{R}} - \Gamma$ has no singularities and α-limit and ω-limit of δ are disjoint sets (see [EL64]), the α-limit of δ is a closed curve ρ. Since $\rho \notin \{\Gamma_\lambda\}_{\lambda\in\Lambda}$, we conclude that ρ is a projective cycle. If $\mathcal{F}$ has not an affine cycle, the singularity of $\mathcal{F}$ is not a center, so by substituting γ by the singular point of $\mathcal{F}$ in the previous argument we have the same conclusion. $\square$

The space $\mathcal{F}_d$ has many connected components. For a generic element in each connected component, the type and number of singular points is fixed and we do not know whether it is possible to characterize each component by such numerical and topological data. If an element $\mathcal{F}$ of a component has a projective cycle then every element has also and this rise the question of classification of the components of $\mathcal{F}_d$ with projective cycles. In the next section we will study this question by looking at foliations with a first integral.

6 Deformation of foliations with a first integral

In this section we consider foliations in $\mathbb{P}^2_{\mathbb{R}}$ with a first integral and with a projective cycle. We investigate the perturbation of such foliations such that the projective cycle turns into a projective limit cycle. For simplicity, we will consider only first integrals of Hamiltonian type, however, the methods introduced in this section may be applied for other first integrals. For any algebraic object A we will use $A_{\mathbb{R}}$ (resp. $A_{\mathbb{C}}$) to denote the set of real (resp. complex) points of A.

Let us consider an odd number d and polynomials $f, l \in \mathbb{R}[x, y]$, $\deg(f) = d + 1$, $\deg(l) \leq 1$ such that the $\{f = 0\}_{\mathbb{R}}$ does not intersects the real line $\{l = 0\}_{\mathbb{R}}$ in

$\mathbb{P}^2_{\mathbb{R}}$. This can happen because d is an odd number. The foliation

$$\mathcal{F}_0: \ \omega_0 := ldf - (d+1)fdl = 0 \tag{2}$$

has the first integral $\frac{f}{l^{d+1}}$ and the projective cycle $\gamma_0 := \{l = 0\}_{\mathbb{R}}$. We may choose the coordinates so that $\{l = 0\}_{\mathbb{R}}$ is the line at infinity (take $l := 1$). Our assumption on f implies that the last homogeneous piece of f induces an empty variety in $\mathbb{P}^1_{\mathbb{R}}$. A typical example of this situation can be constructed by small perturbations of $g(x,y) := x^{d+1} + y^{d+1}$ by monomials of lower degree.

Remark 3. The foliation $\mathcal{F}_0$ has necessarily $d+1$ complex singularities in the line at infinity. Therefore, the maximum number of real singularties of $\mathcal{F}_0$ is d^2 and by this example we cannot produce projective cycles with number of real singularties between d^2 and d^2+d+1. In general if X is an algebraic projective oval in $\mathbb{P}^2_{\mathbb{R}}$ invariant by a foliation then the complex algebraic curve $X_{\mathbb{C}}$ may forcely contain complex singularities of the foliation. This arises the question: A foliation with an algebraic projective cycle has at most how many real singularities? The maximum number d^2+d+1 seems to be far from the reality.

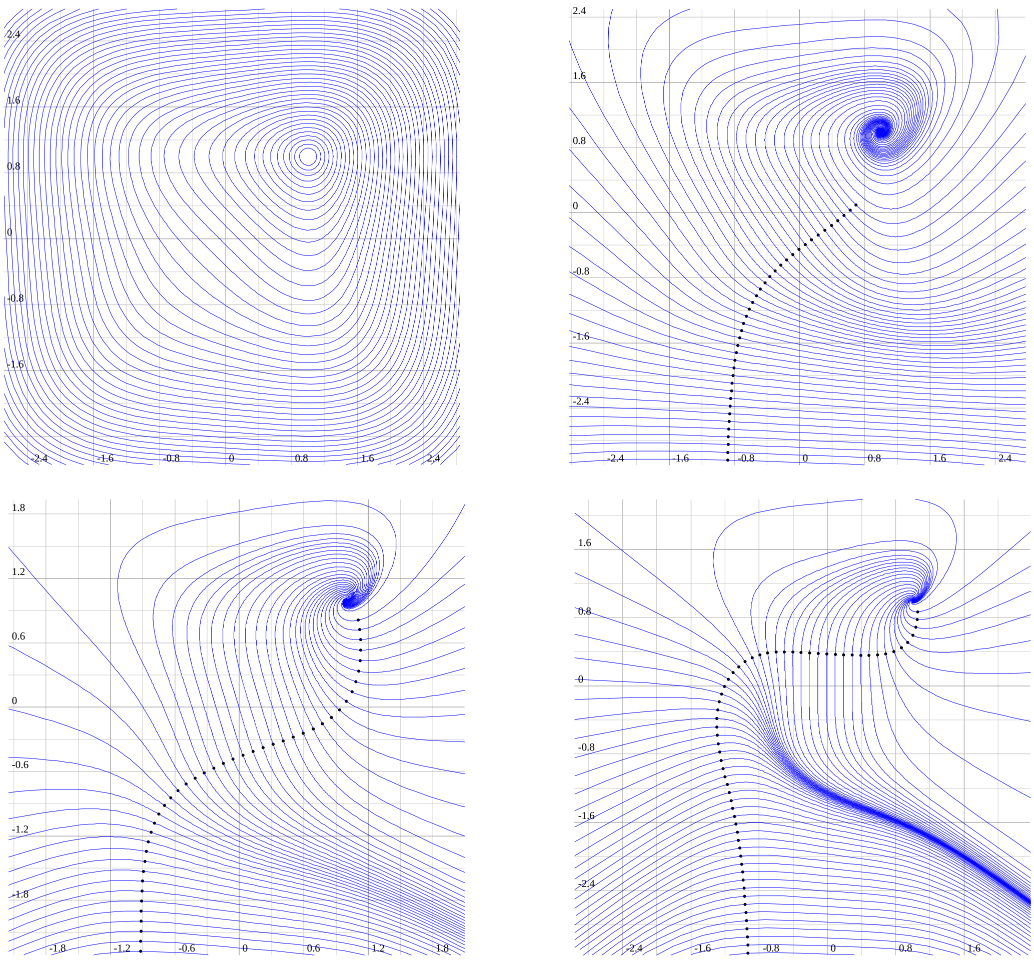

Figure 5: Deformations for $t=0$, $t=0.3$, $t=0.7$ and $t=1$ respectively

Example 4. We consider the example $f = \dfrac{x^4}{4} - x + \dfrac{y^4}{4} - y$ and $l = 1$. It is interesting to mention that both $\mathcal{J}^3_{\mathbb{R}}$ (Jouanolou's foliation of degree 3) and $\mathcal{F}(df)$ have a unique simple real singularity and, we can obtain $\mathcal{J}^3_{\mathbb{R}}$ from perturbations of $\mathcal{F}(df)$ without destroying the projective cycle. To see this take

$$\mathcal{F}_t := \mathcal{F}((y^3 - 1 + t - tx^4)dy - (1 - (1-t)x^3 - tyx^3)dx), \ t \in [0,1]$$

We have $\mathcal{F}_0 = \mathcal{F}(df)$ and $\mathcal{F}_1 = \mathcal{J}^3_{\mathbb{R}}$. It is not difficult to see that for all $t \in [0,1]$ $\mathcal{F}_t$ has a unique simple real singularity. We conclude that $\mathcal{J}^3_{\mathbb{R}}$ and $\mathcal{F}(df)$ are in the same connected component of $\mathcal{F}_3$. Therefore, by Proposition 2 the foliation $\mathcal{F}_t$ has a projective cycle for all $t \in [0,1]$ (see Figure 5).

Example 5. Consider $f(x,y) = p(x) + q(y)$, where $\deg(f)$ is even and $\deg(p) = \deg(q)$, and assume that the real zeros of $p'(x)$ and $q'(y)$ are isolated. In this example the critical points are

$$\{p'(x) = 0\} \times \{q'(y) = 0\}.$$

The foliation induced by $df = 0$ leaves the line at infinity l_∞ invariant, and $l_\infty \cap Sing(\mathcal{F}_{\mathbb{R}}) = \emptyset$. Using this model, we can construct foliations of degree d with a projective cycle and $\#\{Sing(\mathcal{F}_{\mathbb{R}})\}$ being any value from 1 to d^2, except the primes between d and d^2. In Figure 6 we can see the level curves of the function $f = x^4 - 2x^2 + y^4 - 2y^2$ which has 9 real singularities.

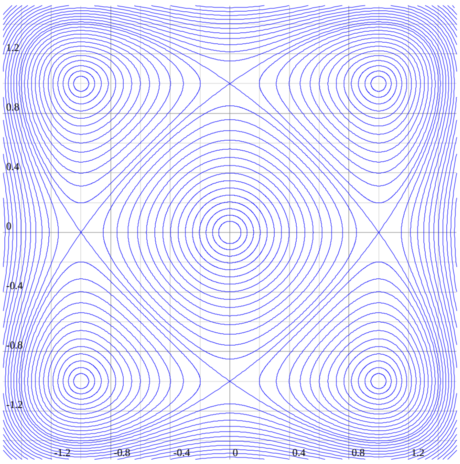

Figure 6: A foliation of degree 3 with 9 real simple singularities and a projective cycle

7 Deformations

Let us consider the foliation (2) with the first integral $\dfrac{f}{l^{d+1}}$ and with the projective cycle $\gamma_0 := \{l = 0\}_{\mathbb{R}}$. The objective of this section is to show that for generic f any perturbation of $\mathcal{F}_0$ which is not Hamiltonian has a projective limit cycle obtained from γ_0.

Let Σ be a transverse section to $\mathcal{F}_0$ at a point p_∞ of γ_0 oriented in an arbitrary way. Let also $h_0 : \Sigma \to \Sigma$ be the corresponding holonomy. By topological arguments one can see that h_0^2 is the identity map. Near γ_0 we have a continuous family of cycles

$\delta_t \subset f^{-1}(f(t)),\ t \in \Sigma - \{p_\infty\}$ such that δ_t is a double covering of γ_0. We call such a δ_t a cycle near infinity. In Figure 6 we have an example for which $\{l = 0\}_{\mathbb{R}}$ is the line at infinity.

Let us consider the perturbation

$$\mathcal{F}_\varepsilon : ldf - (d+1)fdl + \varepsilon\omega + \varepsilon^2(\cdots),\ \deg(\mathcal{F}_\varepsilon) \leq d \tag{3}$$

and the perturbed holonomy $h_\varepsilon : \Sigma \to \Sigma$. Note that $\deg(\cdot)$ is the projective degree. By Proposition 1 the holonomy h_ε has a unique fixed point and so it has a projective cycle γ_ε obtained by deformation of γ_0. We are interested to know when γ_ε is a limit cycle. To state our results we have to put some generic conditions on f as follows:

We say that the first integral $\frac{f}{l^{d+1}}$ is tame if $\{f = 0\}_{\mathbb{C}}$ intersects $\{l = 0\}_{\mathbb{C}}$ transversely in the complex domain $\mathbb{P}^2_{\mathbb{C}}$ and $\frac{f}{l^{d+1}}$ has only non-degenerated critical points with distinct images. The tame first integrals form an open dense subset of the set of first integrals of the form $\frac{f}{l^{d+1}}$.

Theorem 2. *Assume that* $\frac{f}{l^{d+1}}$ *is a tame first integral. If the deformed holonomy* h^2_ε *of the perturbation (3) is the identity map then* ω *is of the form*

$$\omega = ld\tilde{f} - (d+1)\tilde{f}dl + \tilde{l}df - (d+1)fd\tilde{l}$$

for some polynomials $\tilde{f},\ \tilde{l} \in \mathbb{R}[x,y]$ *with* $\deg(\tilde{f}) \leq d+1$ *and* $\deg(\tilde{l}) \leq 1$. *In particular, if the line at infinity is* $\mathcal{F}_\varepsilon$*-invariant then* $\tilde{l}$ *is a constant and so* ω *is exact.*

To prove the above theorem we need the following topological lemma.

Lemma 1. *Let* $\frac{f}{l^{d+1}}$ *be a tame first integral. Then cycles near infinity are vanishing cycles.*

For the definition of a vanishing cycle the reader is referred to [AGV88, La81].

Proof. In order to prove that δ_t is a vanishing cycle we proceed as follows: Without loss of generality we fix l to be the line at infinity and consider the the projectivization $\mathcal{P}_{d+1}$ of the space of polynomials $\mathbb{C}_{d+1}[x,y]$ of degree less than or equal to $d+1$. Its subset containing polynomials $\tilde{f}$ such that $\{\tilde{f} = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ is not smooth form a codimension one irreducible variety $\Delta_{d+1} \subset \mathcal{P}_{d+1}$ which we call it the discriminant variety (see [La81] 1.4.1). The fibration $\{f = t\},\ t \in \mathbb{C}$ corresponds to a line $G_{\mathbb{C}}$ in $\mathcal{P}_{d+1}$ which intersects the discriminant variety $\Delta_{d+1} \subset \mathcal{P}_{d+1}$ transversely in d^2 points $B = \{p_1, p_2, \ldots, p_{d^2}\}$. Let $t_0 \in \mathbb{R}$ be a big positive number and $a \in G_{\mathbb{R}}$ be the point corresponding to $f - t_0 \in (\mathcal{P}_{d+1})_{\mathbb{R}}$. Let also δ_a be the cycle near infinity in $\{f = t_0\}_{\mathbb{R}}$. We want to find a path in $G_{\mathbb{C}} - B$ starting from a and ending in one of the points of B such that δ_a vanishes along it. Since the discriminant variety Δ_{d+1} is irreducible, the map

$$\pi_1(G_{\mathbb{C}} - B, a) \to \pi_1(\mathcal{P}_{d+1} - \Delta_{d+1}, a)$$

induced by inclusion is surjective (see [La81], (7.3.5)). Therefore, it is enough to construct the vanishing path in $\mathcal{P}_{d+1} - \Delta_{d+1}$ and not necessarily in $G_{\mathbb{C}} - B$.

We connect by a path $\gamma_1(s),\ s \in [0,1]$ the point a to the point $b \in (\mathcal{P}_{d+1})_{\mathbb{R}}$ corresponding to the polynomial $f_1 = x^{d+1} + y^{d+1} - x - y$ in $(\mathcal{P}_{d+1})_{\mathbb{R}}$. Since $d+1$ is even, we can

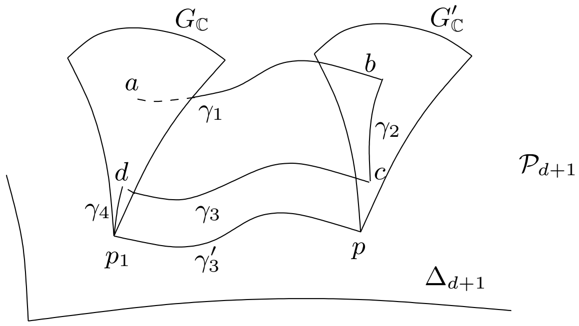

Figure 7:

construct $\gamma_1(s)$ such that, for all $s \in [0,1]$, the last homogeneous piece of the polynomial associated to $\gamma_1(s)$ has no real roots. Then, for all $s \in [0,1]$, the line at infinity is a projective cycle for $\mathcal{F}(df_s)$. Since $[0,1]$ is compact, there is a continuous function $c : [0,1] \to \mathbb{R}$, with $c(0) = c(1) = 0$, such that $\{f_s + c(s) = 0\}_{\mathbb{R}}$ has an oval (cycle) near infinity. We replace γ_1 with the one given by $f_s + c(s),\ s \in [0,1]$. Let $G'_{\mathbb{R}}$ be the line in $(\mathcal{P}_{d+1})_{\mathbb{R}}$ corresponding to the fibration $f_1 = t,\ t \in \mathbb{R}$. Since f_1 has a unique non-degenerated singularity at $\mathbb{R}^2$, a cycle near infinity δ_b for the point $b \in G'_{\mathbb{R}}$ vanishes automatically along a path γ_2 which connects b to a point c near $p \in (G'\Delta_{d+1})_{\mathbb{R}}$, where p is a point corresponding to the unique real non-degenerated critical value of f_1. Since Δ_{d+1} is connected, we can connect p to $p_1 \in B$ by a path γ'_3 in the smooth part of Δ_{d+1}. Let γ_3 be a path in $\mathcal{P}_{d+1} - \Delta_{d+1}$ which is near enough to γ'_3 and it connects a point c to a point $d \in G_{\mathbb{C}}$ near p_1. Now let γ_4 be a straight path connecting d to p_1. The path $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is the desired vanishing path. (see Figure 7)

$\square$

Proof of Theorem 2. The image t of the first integral $\frac{f}{l^{d+1}}$ gives us a coordinate system in $\Sigma - \{p_\infty\}$ (it is meromorphic in p_∞). We write the Taylor expansion of h_ε^2 in ε:

$$h_\varepsilon^2(t) : t + \varepsilon M_1(t) + \varepsilon^2(\cdots),\ t \in \Sigma - \{p_\infty\}$$

Similar to the context of affine projective cycles we call M_1 the first Melnikov function of the deformation (3). In this case we have also

$$M_1(t) = -\int_{\delta_t} \frac{\omega}{l^{d+2}},\ t \in \Sigma - \{p_\infty\}$$

(see for instance [Fr96] or [Mo04] Proposition 4.1). Note that in the context of projective cycles the integration makes sense only for cycles δ_t's which are affine and are double covering of γ_0 and it does not make sense for γ_0 itself. If h_ε^2 is the identity map then $M_1(t)$ is identically zero. Now by Lemma 1 δ_t is a vanishing and by a small modification of Ilyashenko's result in [Il69] (for instance see [Mo04] Proposition 3.3 and the proof of Proposition 6.1), we conclude that $\frac{\omega}{l^{d+2}}$ is a relatively exact 1-form with respect to the fibration $\{f = t\}_{\mathbb{C}},\ t \in \mathbb{C}$ and so ω is of the desired form. $\square$

By a slight modification of the Ilyashenko's argument in [Il69] (see [Mo04]) one can prove that the space of complex foliations of type (2) form an irreducible component of the space of foliations with a center singularity. Using this, we can replace the tameness condition in Theorem 2 by saying that f is in some open dense subset of $(\mathcal{P}_{d+1})_{\mathbb{R}}$ and conclude that the deformed foliation has still a first integral of the type $\frac{f}{l^{d+1}}$. This means that if a projective cycle after a perturbation persists to be a projective and not a limit cycle then the deformed foliation has a first integral.

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