

Isometric rigidity in codimension two

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Abstract

We show that among the Euclidean submanifolds with codimension two the ones of rank two that are parabolic but nonruled are isometrically rigid. This generalizes the result in [10] that these submanifolds are genuinely rigid. In addition, we give a parametric classifications of all parabolic submanifolds.

1 Introduction

In the local theory of submanifolds a fundamental but difficult problem is to describe the isometrically deformable isometric immersions $f: M^n \rightarrow \mathbb{R}^{n+p}$ into Euclidean space with low codimension p if compared to the dimension $n \geq 3$ of the Riemannian manifold. Moreover, one would like to understand the set of all possible isometric deformations.

Submanifolds in low codimension are generically rigid since the fundamental Gauss-Codazzi-Ricci system of equations is overdetermined. Rigidity means that there are no other isometric immersion up to rigid motion of the ambient space. As a consequence, it is quite easier to give a generic assumption implying rigidity than describing the submanifolds that are isometrically deformable. For instance, the results in [1] and [5] conclude rigidity provided the second fundamental is “complicated enough”.

The result stated by Beez [3] in 1876 but correctly proved by Killing [13] in 1885 says that any deformable hypersurfaces without flat points has two nonzero principal curvatures (rank two) at any point. For dimension three the deformation problem for hypersurfaces was first considered by Schur [16] as early as 1886 and then completely solved in 1905 by Bianchi [4]. The general case was solved by Sbrana [15] in 1909 and Cartan [6] in 1916; see [8] for additional information. From their result, we have that even hypersurfaces of rank two are generically rigid.

Outside the hypersurfaces case, the deformation question remains essentially unanswered to this day even for low codimension $p = 2$. From [5] or [9] any submanifold $f: M^n \rightarrow \mathbb{R}^{n+2}$ is rigid if at any point the index of relative nullity satisfies $\nu_f \geq n - 5$ and any shape operator has at least three nonzero principal curvatures. If only the relative nullity condition holds we know from [12]

that f is genuinely rigid. This means that given any other isometric immersion $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+2}$ there is an open dense subset of M^n such that restricted to any connected component $f|_U$ and $\tilde{f}|_U$ are either congruent or there are an isometric embedding $j: U \hookrightarrow N^{n+1}$ into a Riemannian manifold N^{n+1} and either flat or isometric Sbrana-Cartan hypersurfaces $F, \hat{F}: N^{n+1} \rightarrow \mathbb{R}^{n+2}$ such that $f|_U = F \circ j$ and $\tilde{f}|_U = \hat{F} \circ j$.

One may expect the nowhere flat submanifolds $f: M^n \rightarrow \mathbb{R}^{n+2}$ whose second fundamental form is “as simple as possible”, namely, with constant relative nullity index $\nu_f = n - 2$, to be easily deformable. However, we believe that, as part of a general trend, submanifolds in this class are generically rigid as happens in the hypersurface situation. These rank two submanifolds were divided into [10] into three classes: elliptic, parabolic and hyperbolic. It was shown there that the elliptic and the nonruled parabolic ones are genuinely rigid. The ruled parabolic submanifolds admit isometric immersions as hypersurfaces and have many isometric deformations. Examples of hyperbolic submanifolds that are not genuinely rigid were discussed in [10] but there are not general results for this class yet.

In this paper, we generalize [10] by showing that the nonruled parabolic submanifolds are not only genuinely rigid but are, in fact, isometrically rigid. In particular, this unexpected result provides the first known examples of locally rigid submanifolds of rank and codimension two. Observe that by our result any nonruled parabolic submanifold cannot be locally realized as a hypersurface of a Sbrana-Cartan hypersurface. This is certainly not the case for the elliptic submanifolds. The remaining of the paper is devoted to a local parametric classification of all parabolic submanifolds.

2 Preliminaries

Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ denote an isometric immersion with codimension two into Euclidean space of a Riemannian manifold of dimension $n \geq 3$. We denote its second fundamental with values in the normal bundle by

$$\alpha_f: TM \times TM \rightarrow T_f^\perp M.$$

The shape operator $A_\xi^f: TM \rightarrow TM$ for any $\xi \in T_f^\perp M$ is defined by

$$\langle A_\xi^f X, Y \rangle = \langle \alpha_f(X, Y), \xi \rangle.$$

We assume throughout the paper that f has constant rank 2. This condition is denoted by $\text{rank}_f = 2$ and means that the relative nullity subspaces $\Delta(x) \subset T_x M$ defined by

$$\Delta(x) = \{X \in T_x M : \alpha_f(X, Y) = 0; Y \in T_x M\}$$

form a tangent subbundle of codimension two. Equivalently, the index of relative nullity $\nu_f(x) = \dim \Delta(x)$ satisfies $\nu_f(x) = n - 2$. It is a standard fact that the

relative nullity distribution is integrable and that the $(n - 2)$ -dimensional leaves are totally geodesic submanifolds of the manifold and the ambient space.

The following fact was proved in [10].

Proposition 1. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion of $\text{rank}_f = 2$. Assume that M^n has no open flat subset. Then, given an isometric immersion $g: M^n \rightarrow \mathbb{R}^{n+2}$, there exists an open dense subset of M^n such that along each connected component V we have:*

- (i) $\text{rank}_g = 2$ and $\Delta_g = \Delta_f$, or
- (ii) $\text{rank}_g = 3$ and $g|_V = k \circ h: V \rightarrow \mathbb{R}^{n+2}$ is a composition of isometric immersions $h: V \rightarrow U$ and $k: U \rightarrow \mathbb{R}^{n+2}$ where $U \subset \mathbb{R}^{n+1}$ is open.

The simplest submanifolds $f: M^n \rightarrow \mathbb{R}^{n+2}$ of $\text{rank}_f = 2$ are called *surface-like*. This means that $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ with $L^2 \subset \mathbb{R}^4$ or $f(M) \subset CL^2 \times \mathbb{R}^{n-3}$ where $CL^2 \subset \mathbb{R}^5$ is a cone over a spherical surface $L^2 \subset \mathbb{S}^4$. The following characterization in terms of the splitting tensor is well-known; see [8] or [12]. Recall that associated to the relative nullity foliation the *splitting tensor* C is defined as follows: to each vector $T \in \Delta$ corresponds the endomorphism C_T of Δ^\perp given by

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

Proposition 2. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion of $\text{rank}_f = 2$. Assume that $C_T = \mu(T)I$ for any $T \in \Delta$. Then each point has a neighborhood where f is surface-like.*

3 Parabolic submanifolds

A submanifold $f: M^n \rightarrow \mathbb{R}^{n+2}$ of $\text{rank}_f = 2$ is called *parabolic* if we have:

- (i) At any $x \in M^n$, we have

$$T_{f(x)}^\perp M = \text{span}\{\alpha_f(X, Y); X, Y \in T_x M\}.$$

- (ii) There is a nonsingular (asymptotic) vector field $Z \in \Delta^\perp$ such that

$$\alpha_f(Z, Z) = 0.$$

Clearly, if f is parabolic there is a smooth orthonormal frame $\{\eta_1, \eta_2\}$ of $T_f^\perp M$ such that the shape operators have the form

$$A_{\eta_1}^f|_{\Delta^\perp} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A_{\eta_2}^f|_{\Delta^\perp} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

with $b, c \in C^\infty(M)$ nowhere vanishing. In particular, the asymptotic vector field Z is unique up to sign.

It is well-known (cf. [12] or [8]) that the differential equation

$$\nabla_T A_\xi^f|_{\Delta^\perp} = A_\xi^f|_{\Delta^\perp} \circ C_T + A_{\nabla_T \xi}|_{\Delta^\perp} \quad (2)$$

holds for any $T \in \Delta$ and $\xi \in T_f^\perp M$. In particular, $A_\xi^f|_{\Delta^\perp} \circ C_T$ is symmetric. From this and (ii) it follows easily that

$$C_T = \begin{bmatrix} m & 0 \\ n & m \end{bmatrix} \quad (3)$$

for any $T \in \Delta$. From Proposition 2, we conclude that f is surface-like if and only if $n = \langle C_T X, Z \rangle = 0$ for any $T \in \Delta$.

Recall that an isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+2}$ is ruled when there is a foliation by open subsets of $(n-1)$ -dimensional affine subspaces of \mathbb{R}^{n+2} . Observe that a ruled submanifold in codimension 2 does not have to be parabolic. In fact, generically we have $\text{rank}_f = 3$.

Ruled parabolic submanifolds are never locally isometrically rigid as seen in the converse part of the following result in [10]. In fact, the direct statement will be an important element in the proof of our main result.

Proposition 3. *If $f: M^n \rightarrow \mathbb{R}^{n+2}$ is a ruled parabolic submanifold. If M^n is simply connected then it admits an isometric immersion as a ruled hypersurface in \mathbb{R}^{n+1} .*

Conversely, let $g: M^n \rightarrow \mathbb{R}^{n+1}$ be a simply connected ruled hypersurface without flat points. Then, the family of ruled parabolic submanifolds $f: M^n \rightarrow \mathbb{R}^{n+2}$ is parametrized by the set of ternary smooth arbitrary functions in an interval.

4 The main result

The following result generalizes the one in [10] and provides the first known examples of locally rigid submanifolds of rank and codimension two.

Theorem 4. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$, $n \geq 3$, be a parabolic submanifold neither ruled nor surface-like on any open subset of M^n . Then f is isometrically rigid.*

The following is a key ingredient of the proof of the theorem.

Proposition 5. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be a simply connected parabolic submanifold. If f is not of surface type in any open subset of M^n and admits an isometric immersion as a hypersurface of \mathbb{R}^{n+1} then f is ruled.*

Proof: Let $g: M^n \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion. We denote by N its Gauss map. Given $x \in M^n$, let $\beta: T_x M \times T_x M \rightarrow \mathbb{L}^2$ be the symmetric bilinear form

$$\beta(Y, V) = \langle A_m^f Y, V \rangle e_1 + \langle A_N^g Y, V \rangle e_2$$

where η_1 is as in (1) and $\{e_1, e_2\}$ is an orthonormal frame for the Lorentzian plane \mathbb{L}^2 such that $\|\epsilon_1\|^2 = 1 = -\|\epsilon_2\|^2$ and $\langle \epsilon_1, \epsilon_2 \rangle = 0$. Then β is flat since, from the Gauss equations for f and g , we easily see that

$$\langle \beta(X, Y), \beta(V, W) \rangle - \langle \beta(X, W), \beta(V, Y) \rangle = 0.$$

We claim that $\Delta_g(x) = \Delta_f(x)$. Since parabolic submanifolds have no flat points it follows that $\dim \Delta_g(x) \leq n - 2$. If $\Delta_g(x) \neq \Delta_f(x)$, it is easy to see that

$$S(\beta) = \text{span}\{\beta(Y, V) ; Y, V \in T_x M\}$$

satisfies $S(\beta) = \mathbb{L}^2$. From Corollary 1 in [14], we obtain that

$$N(\beta) = \{Y \in T_x M : \beta(Y, V) = 0 ; V \in T_x M\}$$

satisfies $\dim N(\beta) = n - 2$. Since $N(\beta) = \Delta_g \cap \Delta_f$, this is a contradiction and proves the claim.

Set

$$A_N^g|_{\Delta^\perp} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{bmatrix}.$$

Using (3) we conclude from the symmetry of

$$A_N^g \circ C_T = \begin{bmatrix} \bar{a}m + \bar{b}n & \bar{b}m \\ \bar{b}m + \bar{c}n & \bar{c}m \end{bmatrix}$$

that $\bar{c}n = 0$. It follows from Proposition 3 and our assumption that f is nowhere surface-like that $\bar{c} = 0$. In particular, we have from the Gauss equations for f and g that we can choose an orientation for g such that $\bar{b} = b$.

Taking the Z -component of the Codazzi equations for $A_{\eta_1}^f$ and A_N^g give

$$2b\langle \nabla_X X, Z \rangle - a\langle \nabla_Z X, Z \rangle - Z(b) = 0 \quad (4)$$

and

$$2b\langle \nabla_X X, Z \rangle - \bar{a}\langle \nabla_Z X, Z \rangle - Z(b) = 0.$$

Thus,

$$(a - \bar{a})\langle \nabla_Z Z, X \rangle = 0.$$

Suppose that $\langle \nabla_Z Z, X \rangle = 0$ in an open subset U of M^n . From (3) we have

$$\langle \nabla_Z T, X \rangle = -\langle C_T Z, X \rangle = 0. \quad (5)$$

Then taking the Z -component of the Codazzi equations for g applied to Z, T gives

$$\langle \nabla_T Z, X \rangle = 0.$$

It follows from the above that distribution $\text{span}\{Z\} \oplus \Delta$ is totally geodesic on U . But then f is ruled on U as we wanted.

Assume now that $a = \bar{a}$ on an open subset. Taking the X -component of the same Codazzi equations as above gives

$$X(b) - a\langle \nabla_X Z, X \rangle - Z(a) + 2b\langle \nabla_Z X, Z \rangle + c\langle \nabla_Z^\perp \eta_1, \eta_2 \rangle = 0 \quad (6)$$

and

$$X(b) - a\langle \nabla_X Z, X \rangle - Z(a) + 2b\langle \nabla_Z X, Z \rangle = 0.$$

Thus $\langle \nabla_Z^\perp \eta_1, \eta_2 \rangle = 0$. The Codazzi equation for $A_{\eta_2}^f$ applied to X, Z yields

$$c\langle \nabla_Z Z, X \rangle = b\langle \nabla_Z^\perp \eta_1, \eta_2 \rangle. \quad (7)$$

Again $\langle \nabla_Z Z, X \rangle = 0$, and the proof follows. ■

Proof of Theorem 4. Let $g: M^n \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion. Since f is nowhere ruled, Proposition 5 asserts that there is no local isometric immersion of M^n as a hypersurface in \mathbb{R}^{n+1} . From Proposition 1, we obtain $\text{rank}_g = 2$ and that $\Delta_g = \Delta_f$.

Take $\{\eta_1, \eta_2\}$ as in (1) and recall that (3) holds for any $T \in \Delta$. Since $A_{\bar{\eta}}^g \circ C_T$ is symmetric for any $\bar{\eta} \in T_g^\perp M$, it follows as before that Z is asymptotic for g . Therefore, we may fix a orthonormal base $\{\bar{\eta}_1, \bar{\eta}_2\}$ of $T_g^\perp M$ such that

$$A_{\bar{\eta}_1}^g|_{\Delta^\perp} = \begin{bmatrix} \bar{a} & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A_{\bar{\eta}_2}^g|_{\Delta^\perp} = \begin{bmatrix} \bar{c} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since (4) holds for both immersions, we have

$$(a - \bar{a})\langle \nabla_Z Z, X \rangle = 0,$$

and conclude that $a = \bar{a}$. Similarly, we have from (6) that

$$c\langle \nabla_Z^\perp \eta_1, \eta_2 \rangle = \bar{c}\langle \nabla_Z^\perp \bar{\eta}_1, \bar{\eta}_2 \rangle.$$

Now computing the Z -component of the Codazzi equations for $A_{\eta_2}^f$ and $A_{\bar{\eta}_2}^g$ gives

$$\bar{c}\langle \nabla_Z^\perp \eta_1, \eta_2 \rangle = c\langle \nabla_Z^\perp \bar{\eta}_1, \bar{\eta}_2 \rangle.$$

It follows that

$$c = \bar{c} \quad \text{and} \quad \langle \nabla_Z^\perp \eta_1, \eta_2 \rangle = \langle \nabla_Z^\perp \bar{\eta}_1, \bar{\eta}_2 \rangle.$$

Then taking the X -components yields

$$\langle \nabla_X^\perp \eta_1, \eta_2 \rangle = \langle \nabla_X^\perp \bar{\eta}_1, \bar{\eta}_2 \rangle.$$

We conclude from the fundamental theorem of submanifolds (cf. [7]) that f and g are congruent by a rigid motion of the ambient space. ■

5 Ruled parabolic surfaces

In this section, we give a parametric description of all ruled submanifolds in \mathbb{R}^{n+2} that are parabolic.

Let $v: I \rightarrow \mathbb{R}^{n+2}$ be a smooth curve parametrized by arc length in an interval I in \mathbb{R} . Set $e_1 = dv/ds$ and let e_2, \dots, e_{n-1} be orthonormal normal vector fields along $v = v(s)$ parallel in the normal connection of v in \mathbb{R}^{n+2} . Thus,

$$\frac{de_j}{ds} = b_j e_1, \quad 2 \leq j \leq n-1, \quad (8)$$

where $b_j \in C^\infty(I)$. Set $\Delta = \text{span}\{e_2, \dots, e_{n-1}\}$ and let Δ^\perp be the orthogonal complement in the normal bundle of the curve. Take a smooth unit vector field $e_0 \in \Delta^\perp$ along v such that

$$P = \text{span}\{e_0, (de_1/ds)_{\Delta^\perp}\} \subset \Delta^\perp$$

satisfies:

$$\dim P = 2 \quad (9)$$

and P is nowhere parallel in Δ^\perp along v , that is,

$$\text{span}\{(de_0/ds)_{\Delta^\perp}, (d^2e_1/ds^2)_{\Delta^\perp}\} \not\subset P. \quad (10)$$

We parametrize a ruled submanifold M^n by

$$f(s, t_1, \dots, t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_j(s) \quad (11)$$

where $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $c(s)$ satisfies $dc/ds = e_0$. To see that f is parabolic, first observe that

$$TM = \text{span}\{f_s\} \oplus \text{span}\{e_1\} \oplus \Delta$$

where $f_s = e_0 + t_1 de_1/ds + \sum_{j \geq 2} t_j b_j e_1$. Consider the orthogonal decomposition

$$\left(\frac{de_1}{ds}\right)_{\Delta^\perp} = a_1 e_0 + \eta. \quad (12)$$

Thus $\eta(s) \neq 0$ for all $s \in I$ from (9). Hence,

$$TM = \text{span}\{e_0 + t_1(a_1 e_0 + \eta)\} \oplus \text{span}\{e_1\} \oplus \Delta, \quad (13)$$

and it follows easily that f is regular at any point.

Since $f_{st_j} = b_j e_1 \in TM$, $2 \leq j \leq n-1$, we have that $\Delta \subset \Delta_f$. It follows easily from (12), (13) and $\eta(s) \neq 0$ that

$$f_{st_1} = \frac{de_1}{ds} \notin TM.$$

It is easy to see that $f_{ss} \notin \text{span}\{f_{st_1}\} \oplus TM$, i.e., $\dim N_1^f = 2$, is equivalent to

$$\left(\frac{de_0}{ds}\right)_{\Delta^\perp} + t_1 \left(\frac{d^2e_1}{ds^2}\right)_{\Delta^\perp} \notin P.$$

It follows easily that $\Delta = \Delta_f$ and that f is parabolic in at least an open dense subset of M^n .

Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be a ruled parabolic submanifold and $\{e_2, \dots, e_{n-1}\}$ an orthonormal frame for Δ_f along a integral curve $c = c(s)$, $s \in I$, of the unit vector field X orthogonal to the rulings. Without loss of generality (see Lemma 2.2 in [2]) we may assume that

$$\frac{de_j}{ds} \perp \Delta_f, \quad 2 \leq j \leq n-1.$$

Now parametrize f by (11), where $e_0 = X$ and $e_1 = Z$. That $f_{st_j} \in TM$ implies

$$\frac{de_j}{ds} \in \text{span}\{e_1, f_s\}, \quad 2 \leq j \leq n-1. \quad (14)$$

Taking $t_1 = 0$, we obtain that

$$\frac{de_j}{ds} = a_j e_0 + b_j e_1, \quad 2 \leq j \leq n-1, \quad (15)$$

where $a_j, b_j \in C^\infty(I)$. Since $\dim N_1^f = 2$, we have

$$\frac{de_1}{ds} = a_1 e_0 + (de_1/ds)_\Delta + \eta$$

where $\eta \perp \text{span}\{e_0, e_1\} \oplus \Delta$ satisfies $\eta(s) \neq 0$. Thus (14) reduces to

$$a_j e_0 \in \text{span}\{(1 + t_1 a_1 + \dots + t_{n-1} a_{n-1})e_0 + t_1 \eta\}, \quad 2 \leq j \leq n-1.$$

Therefore $a_j = 0$. From (15) we have $de_j/ds = b_j e_1$ for $2 \leq j \leq n-1$.

We have proved the following result.

Proposition 6. *Given a smooth curve $c: I \subset \mathbb{R} \rightarrow \mathbb{R}^{n+2}$ consider orthonormal fields $e_0 = dc/ds, e_1(s), \dots, e_{n-1}(s)$ satisfying (8), (9) and (10) at any point. Then, the ruled submanifold parametrized by*

$$f(s, t_1, \dots, t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_j(s) \quad (16)$$

with $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ is parabolic in an open dense subset of M^n . Conversely, any ruled parabolic submanifold can be parametrized as in (16).

6 Nonruled parabolic surfaces

In this section, we provide a parametrically description of all nonruled Euclidean parabolic submanifolds.

Let L^2 be a Riemannian manifold endowed with a global coordinate system (x, z) . Let $g: L^2 \rightarrow \mathbb{R}^N$, $N \geq 4$, be a surface whose coordinate functions are linearly independent solutions of the parabolic equation

$$\frac{\partial^2 u}{\partial z^2} + W(u) = 0 \quad (17)$$

where $W \in TL$. In terms of the Euclidean connection, we have

$$\tilde{\nabla}_Z g_* Z + g_* W = 0$$

where $Z = \partial/\partial z$. Thus, the coordinate field Z is asymptotic, i.e., the second fundamental form of g satisfies

$$\alpha_g(Z, Z) = 0, \quad (18)$$

and also $W = -\nabla_Z Z$. In particular, the coordinate functions satisfy

$$\text{Hess}_u(Z, Z) = 0. \quad (19)$$

Conversely, if $f: L^2 \rightarrow \mathbb{R}^N$ is a surface with a coordinate system (x, z) such that $\partial/\partial z = Z$ satisfies (18), then all coordinate functions of f satisfy (17) with $W = -\nabla_Z Z$.

Given a surface $g: L^2 \rightarrow \mathbb{R}^N$, $N \geq 4$, we denote

$$N_1^g(x) = \text{span}\{\alpha_g(X, Y); X, Y \in T_x L\}.$$

We call g a *parabolic* surface if the following conditions hold.

- (i) $\dim N_1^g(x) = 2$ at any $x \in L^2$.
- (ii) There is a nonsingular vector field $Z \in TL$ such that $\alpha_g(Z, Z) = 0$.

Let $h: L^2 \rightarrow \mathbb{R}^N$ be a smooth map satisfying

$$h_*(TL) \subset T_g^\perp L. \quad (20)$$

Set $h = U + \delta$ where $U \in TL$ and $\delta \in T_g^\perp L$. Given $Y \in TL$, we have

$$h_*(Y) = \nabla_Y U - A_\delta^g(Y) + \alpha_g(Y, U) + \nabla_Y^\perp \delta.$$

It follows that (20) is equivalent to

$$\nabla_Y U = A_\delta^g Y, \quad \text{for any } Y \in TL. \quad (21)$$

In particular, the map $(Y, X) \mapsto \langle \nabla_Y U, X \rangle$ is symmetric, i.e., the one-form U^* is closed. Thus $U = \nabla\varphi$ for some function $\varphi \in C^\infty(L)$. We obtain from (21) that

$$\text{Hess}_\varphi = A_\delta^g$$

and hence φ satisfies (19). Let Λ denote the orthogonal complement of N_1^g in $T_g^\perp L$, that is,

$$T_g^\perp L = N_1^g \oplus \Lambda.$$

We have just proved the direct statement in the following result since the proof of the converse is immediate.

Proposition 7. *Let $g: L^2 \rightarrow \mathbb{R}^N$ be a parabolic surface. Then any smooth map $h: L^2 \rightarrow \mathbb{R}^N$ satisfying (20) has the form*

$$h_\varphi = g_* \nabla\varphi + \gamma_1 + \gamma_0 \tag{22}$$

where the function φ satisfies (17)(or (19)), the section $\gamma_1 \in N_1^g$ is unique such that $A_{\gamma_1}^g = \text{Hess}_\varphi$ and γ_0 is any section of Λ . Conversely, any function h_φ as in (22) satisfies (20).

Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be parabolic. For simplicity, we assume that f does not split a Euclidean factor. Since we are working locally, we assume that M^n is the saturation of a transversal section L^2 to Δ . In relation to the following definition recall that the normal bundle $T_f^\perp M$ is constant along Δ in \mathbb{R}^{n+2} .

Definition 8. Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be a parabolic submanifold and let L^2 be as above. We call a *polar surface* associated to f any immersion $g: L^2 \rightarrow \mathbb{R}^{n+2}$ satisfying $T_{g(x)}L = T_{f(x)}^\perp M$ up to parallel identification in \mathbb{R}^{n+2} .

Proposition 9. *Any parabolic submanifold f admits locally a polar surface. Moreover, any polar surface g associated to f is parabolic and substantial.*

Proof: Let $\eta_1, \eta_2 \in T_f^\perp M$ be orthonormal vector fields such that (1) holds. We claim that they are parallel along Δ in the normal connection. It follows from (2) that

$$\nabla_T A_\xi^f = A_\xi^f \circ C_T$$

is satisfied if $\xi \in T_f^\perp M$ is parallel along Δ . Given $x \in M^n$, let γ be a geodesic with $\gamma(0) = x$ contained in the leaf of relative nullity tangent to $\Delta(x)$. If δ_t is the parallel transport of η_x along γ , we have

$$\nabla_{\gamma'} A_{\delta_t}^f = A_{\delta_t}^f \circ C_{\gamma'}.$$

Hence, $A_{\delta_t}^f = A_{\eta_x}^f e^{\int_0^t C_{\gamma'} d\tau}$. Thus $A_{\delta_t}^f$ has constant rank along γ . It follows that A_ξ has constant rank along the leaves of relative nullity. Now observe that η_1

is the unique (up to sign) unit vector field in $T_f^\perp M$ such that the corresponding shape operator $A_{\eta_1}^f$ has rank one. Thus η_1 and hence η_2 are parallel as claimed.

Next, we claim that also the orthonormal frame $\{X, Z\}$ in Δ^\perp as in (1) is parallel along Δ . In fact, using (5) and the Codazzi equation we obtain

$$\nabla_T^\perp \alpha_f(Z, X) = \langle \nabla_T Z, X \rangle \alpha_f(X, X) - \langle \nabla_Z T, Z \rangle \alpha_f(Z, X).$$

Since η_1 is colinear with $\alpha_f(Z, X)$ and parallel along Δ , the claim follows easily.

Let $U, V \in TL$ be such that

$$Z = U + \delta_1, \quad X = V + \delta_2$$

where $\delta_1, \delta_2 \in \Delta$. Now let (u, v) be a coordinate system in L^2 such that

$$\partial u =: \partial/\partial u = \lambda_1 U, \quad \partial v =: \partial/\partial v = \lambda_2 V$$

where $\lambda_1, \lambda_2 \in C^\infty(L)$. We show next that there exist linearly independent one-forms θ_1, θ_2 such that the differential equation

$$dg = \theta_1 \eta_1 + \theta_2 \eta_2 \tag{23}$$

is integrable. Set

$$\theta_1 = c\lambda_2 \phi dv \quad \text{and} \quad \theta_2 = b\lambda_1 \phi du + \sigma dv \tag{24}$$

where $\phi, \sigma \in C^\infty(L)$ and b, c denote the restriction to L^2 of the functions in M^n defined in (1). The integrability condition of (23) is

$$\begin{aligned} 0 &= d\theta_1 \eta_1 + d\theta_2 \eta_2 + \theta_1 \wedge d\eta_1 + \theta_2 \wedge d\eta_2 \\ &= d\theta_1 \eta_1 + d\theta_2 \eta_2 - (c\lambda_2 \phi \partial \eta_1 / \partial u - b\lambda_1 \phi \partial \eta_2 / \partial v + \sigma \partial \eta_2 / \partial u) dV \end{aligned}$$

where dV is the volume element of L^2 . We have

$$\tilde{\nabla}_{\partial u} \eta_j = \tilde{\nabla}_{\lambda_1(Z-\delta_1)} \eta_j = \lambda_1 \tilde{\nabla}_Z \eta_j \quad \text{and} \quad \tilde{\nabla}_{\partial v} \eta_j = \tilde{\nabla}_{\lambda_2(X-\delta_2)} \eta_j = \lambda_2 \tilde{\nabla}_X \eta_j.$$

Thus, we obtain from (1) that

$$(c\lambda_2 \phi \partial \eta_1 / \partial u - b\lambda_1 \phi \partial \eta_2 / \partial v + \sigma \partial \eta_2 / \partial u)_{TM} = 0.$$

Then,

$$c\lambda_2 \phi \partial \eta_1 / \partial u - b\lambda_1 \phi \partial \eta_2 / \partial v + \sigma \partial \eta_2 / \partial u = e \eta_1 + \ell \eta_2$$

where $e, \ell \in C^\infty(L)$. To conclude, we observe that the integrability condition now follows from the integrability of the system

$$\begin{cases} (c\lambda_2 \phi)_u &= e \\ \sigma_u - (b\lambda_1 \phi)_v &= \ell. \end{cases}$$

It remains to see that g is parabolic. It is clear that $N_1^g = \Delta^\perp$, and hence $\dim N_1^g = 2$. It follows from (24) that $g_*(\partial u) = b\lambda_1 \phi \eta_2$. Then, we obtain from (1) that $\tilde{\nabla}_{\partial u} g_*(\partial u) \in TL$ as we wished. ■

Theorem 10. *Let $g: L^2 \rightarrow \mathbb{R}^{n+2}$ be a parabolic surface and let $\Psi: \Lambda \rightarrow \mathbb{R}^{n+2}$ be the map defined by*

$$\Psi(\delta) = h(x) + \delta, \quad \delta \in \Lambda(x), \quad (25)$$

where $h: L^2 \rightarrow \mathbb{R}^{n+2}$ satisfies (20). Then, $M^n = \Psi(\Lambda)$ is, at regular points, a parabolic submanifold with polar surface g . Moreover, Ψ is nonruled if and only if g is nonruled.

Conversely, any parabolic submanifold $f: M^n \rightarrow \mathbb{R}^{n+2}$ admits a parametrization (25) locally where g is a polar surface of f .

Proof: Since g is parabolic there is an orthonormal tangent frame $\{X, Z\}$ of TL and an orthonormal normal frame of $\{\eta_1, \eta_2\}$ of N_1^g such that

$$A_{\eta_1}^f = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A_{\eta_2}^f = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$$

with $b, c \in C^\infty(M)$ nowhere vanishing.

It follows from the symmetry of the tensor (see [Sp])

$$\alpha^2(V, Y, W) = (\nabla_V^\perp \alpha_g(Y, W))_{(N_1^g)^\perp}$$

that

$$(\nabla_Z^\perp \alpha_g(X, Z))_{(N_1^g)^\perp} = 0, \quad (26)$$

that is,

$$(\nabla_Z^\perp \eta_1)_{(N_1^g)^\perp} = 0.$$

Next we show that

$$(\tilde{\nabla}_Z h_*(Z))_{TL} = 0. \quad (27)$$

In fact, we have

$$\langle \tilde{\nabla}_Z h_*(Z), W \rangle = \langle \tilde{\nabla}_Z h, \alpha_g(W, Z) \rangle = \langle \tilde{\nabla}_W \tilde{\nabla}_Z h, Z \rangle = \langle \tilde{\nabla}_Z \tilde{\nabla}_W h, Z \rangle = 0$$

since Z is asymptotic.

Since h satisfies (20) it follows easily from the regularity assumption that $T_{\Psi(\delta_x)} M = T_g^\perp L(x)$. Moreover, we have that $\Delta_{\Psi(\delta_x)} = \Lambda(x)$. Let us see that Ψ is parabolic. Since

$$\Psi_*(Z) = h_*(Z) + \tilde{\nabla}_Z \delta \in T_{g(x)}^\perp L = T_{\Psi(\delta_x)} M,$$

we have using (26) and (27) that

$$\begin{aligned} \langle \tilde{\nabla}_Z \Psi_*(Z), W \rangle &= \langle h_*(Z) + \tilde{\nabla}_Z \delta, \alpha_g(W, Z) \rangle \\ &= \langle h_*(Z), \alpha_g(W, Z) \rangle + \langle \delta, \nabla_Z^\perp \alpha_g(W, Z) \rangle \\ &= 0. \end{aligned} \quad (28)$$

Since $\eta_1 \in T_g^\perp L = T_\Psi M$, and because $(\tilde{\nabla}_X \eta_1)_{TL}, (\tilde{\nabla}_Z \eta_1)_{TL}$ are linearly independent and belong to $T_\Psi^\perp M$, then $\dim N_1^\Psi = 2$. Thus Ψ is parabolic.

Let us see that Ψ is not ruled. We have from (28) that the asymptotic direction of Ψ is normal to η_1 , thus colinear with η_2 . Since Ψ is ruled if and only if $\langle \tilde{\nabla}_Z \eta_2, \eta_1 \rangle = 0$, we have from (7) that g is ruled.

For the converse, consider a polar surface $g: L^2 \rightarrow \mathbb{R}^{n+2}$ of f . Since f has no Euclidean factor then g is substantial. It is now easy to conclude that $\Delta_f = \Lambda$ and $TM = T_g^\perp L$ along L^2 . Thus, $h = f|_L$ satisfies (20). ■

References

- [1] Allendoerfer, C., *Rigidity for spaces of class greater than one*. Amer. J. Math. **61** (1939), 633–644.
- [2] Barbosa, J. L., Dajczer, M. and Jorge, L., *Minimal Ruled Submanifolds in Spaces of Constant Curvature*. Indiana Math. J. **33** (1984), 531–547.
- [3] Beez, R., *Zur Theorie des Krümmungsmasses von Mannigfaltigkeiten höhere Ordnung*. Zeit. für Math. und Physik **21** (1876), 373–401
- [4] Bianchi, L., *Sulle varietà a tre dimensioni deformabili entro lo spazio euclideo a quattro dimensioni*. Memorie di Matematica e di Fisica della Società Italiana delle Scienze, serie III, t. XIII (1905), 261–323.
- [5] do Carmo, M. and Dajczer M., *Conformal Rigidity*. Amer. J. Math. **109** (1987), 963–985.
- [6] Cartan, E., *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions*. Bull. Soc. Math. France **44** (1916), 65–99.
- [7] Dajczer, M. et al., *Submanifolds and isometric immersions*, Mathematics. Lecture Ser. 13, Publish or Perish, Houston, 1990.
- [8] Dajczer, M., Florit, L. and Tojeiro, R., *On deformable hypersurfaces in space forms*. Ann. Mat. Pura Appl. **147** (1998), 361–390.
- [9] Dajczer, M. and Florit, L., *Genuine deformations of submanifolds*. Comm. Anal. Geom. **5** (2004), 1105–1121.
- [10] Dajczer, M. and Florit, L., *Genuine rigidity of Euclidean submanifolds in codimension two*. Geom. Dedicata **106** (2004), 195–210.
- [11] Dajczer, M. and Gromoll, D., *Rigidity of complete Euclidean hypersurfaces*. J. Diff. Geometry **31** (1990), 401–416.

- [12] Dajczer, M. and Gromoll, ., *Isometric deformations of compact euclidean submanifolds in codimension 2*. Duke Math. J. **79** (1995), 605–618.
- [13] Killing, W., *Die nicht-Euklidischen Raumformen in Analytische Behandlung*. Teubner, Leipzig, 1885
- [14] Moore, J. D., *Submanifolds of Constant positive curvature I*. Duke Math. J. **44** (1977), 449–484.
- [15] Sbrana, V., *Sulla varietà ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni*, Rend. Circ. Mat. Palermo 27 (1909), 1–45.
- [16] Schur, F., *Ueber die Deformation eines dreidimensionalen Raumes in einem ebenen vierdimensionalen Raume*. Math. Ann. **28** (1886), 343–353.
- [Sp] M. Spivak, *A comprehensive introduction to differential geometry*, vol IV, Publish or Perish Inc. Berkeley, 1979.

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