EXISTENCE OF PERIODIC ORBITS FOR SINGULAR-HYPERBOLIC ATTRACTORS

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Abstract. An *attractor* is a transitive set to which all nearby positive orbits converge. A non-trivial attractor for flows is *singular-hyperbolic* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction. We show that a singular-hyperbolic attractor of a C^1 flow on a compact 3-manifold has a periodic orbit. This result has the following corollaries. First every singular-hyperbolic attractor has topological dimension ≥ 2 (solving positively a question posed in [M2]). Second any of such attractors is the closure of the unstable manifold of a periodic orbit. Third every C^1 robust transitive set has a periodic orbit. Our result generalize well known properties of hyperbolic and geometric Lorenz attractors [PT].

Contents

1. Introduction

1.1. **Motivations.** The singular-hyperbolic sets were introduced in [MPP3] to classify C^1 robust transitive sets on 3-manifolds (see also [MPP2]). Since then several works exploring the similarity between hyperbolic and singular-hyperbolic sets have been appearing [BDV]. In this paper we further explore such similarities

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by studying the existence of periodic orbits on singular-hyperbolic sets. The motivation is the well known fact that all (non-trivial) isolated, hyperbolic sets have a periodic orbit. This follows from the Shadowing Lemma for flows [HK]. The natural question is whether (non-trivial) isolated, *singular-hyperbolic* sets have a periodic orbit. The answer is negative even if the isolated set is transitive [BDV], [M2]. Here we show positive answer *if the isolated set is an attractor.* More precisely, we show that a singular-hyperbolic attractor of a C^1 flow on a compact 3-manifold has a periodic orbit. In particular, all such attractors have topological dimension ≥ 2 (solving positively a question posed in [M2]) and are the closure of the unstable manifold of a periodic orbit. We also obtain the existence of periodic orbits for C^1 robust transitive sets on compact 3-manifolds. Our results generalize well known properties of hyperbolic and geometric Lorenz attractors [PT]. We also obtain an approach for a positive solution of the following conjecture stated in [M1].

Conjecture 1. *Singular-hyperbolic attractors for C^1 flow on compact 3-manifolds are homoclinic classes.*

1.2. **Basic definitions and the Main Theorem.** Hereafter M denotes a compact 3-manifold. A C^r flow $X = X_t$ on M is a C^r action $I\!R \times M \to M$, $r \geq 1$. We always assume that X is the integral solution of a C^r vector field still denoted by X. An *orbit* of X is the set $\mathcal{O} = \mathcal{O}_X(q) = \{X_t(q) : t \in I\!R\}$ for some $q \in M$. The *omega-limit set* of a point p is the set $\omega_X(p) = \{x \in M : x = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty\}$. A *singularity* of X is a point $\sigma \in M$ such that $X(\sigma) = 0$ (equivalently $\mathcal{O}_X(\sigma) = \{\sigma\}$). A *periodic orbit* of X is an orbit $\mathcal{O} = \mathcal{O}_X(p)$ such that $X_T(p) = p$ for some minimal $T > 0$ (equivalently $\mathcal{O}$ is compact and $\mathcal{O} \neq \{p\}$). A *closed orbit* of X is either a singularity or a periodic orbit of X.

A compact set $\Lambda \subset M$ is:

- *Invariant* if $X_t(\Lambda) = \Lambda$, $\forall t \in I\!R$;
- *Transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$;
- *Non-trivial* if Λ is not a closed orbit of X;
- *Isolated* if there is a compact neighborhood U of Λ such that
$$\Lambda = \cap_{t \in I\!R} X_t(U)$$
(U is called *isolating block*);
- *Attracting* if it is isolated and has a positively invariant isolating block U, i.e.,
$$X_t(U) \subset U, \quad \forall t \geq 0;$$
- *Attractor* if it is a transitive attracting set.

Attracting sets are isolated but not conversely. For example, a saddle-type singularity is isolated but not attracting. Many authors call attractors what we call attracting sets, see [Mi].

A compact invariant set H of X is *hyperbolic* if there is a continuous tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ over H such that E^s_H is contracting, E^u_H is expanding and E^X_H denotes the direction of X ([HK], [PT]). A closed orbit of X is hyperbolic if it is hyperbolic as a compact invariant set of X. Hereafter we denote by
$$m(A) = \inf_{v \neq 0} \frac{||Av||}{||v||}$$
the minimum norm of a linear operator A.

Definition 1. *Let Λ be a compact invariant set of X. A continuous invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ is* dominated *if there are positive constants K, λ such that*

$$\frac{\|DX_t(x)/_{E_x}\|}{m(DX_t(x)/_{F_x})} \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t \geq 0.$$

A compact invariant set Λ is partially hyperbolic *if it exhibits a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that E^s_Λ* is contracting, *i.e.*

$$\|DX_t(x)/_{E^s_x}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t \geq 0.$$

A compact invariant set Λ of X is singular-hyperbolic *if it has singularities (all hyperbolic) and is partially hyperbolic* with volume expanding central direction E^c_Λ, *i.e.*

$$\left|\det(DX_t(x)/_{E^c_x})\right| \geq K^{-1}e^{\lambda t}, \quad \forall x \in \Lambda, \forall t \geq 0.$$

A singular-hyperbolic attractor *is a non-trivial attractor which is also a singular-hyperbolic set.*

The most classical example of singular-hyperbolic attractors are the geometric Lorenz ones [GW]. Another examples different from Lorenz's are the ones in [MPu] or [M3]. All these examples have a periodic orbits. Our main result asserts that the existence of periodic orbits holds true for all singular-hyperbolic attractors. More precisely, we have the following result.

Theorem 1 (**Main Theorem**). *A singular-hyperbolic attractor of a C^1 flow on a compact 3-manifolds has a periodic orbit.*

1.3. **Corollaries.** Let us state some corollaries of the Main Theorem. Previously we recall some short definitions and facts. By the stable manifold theory [HPS], if $\mathcal{O}$ is a hyperbolic closed orbit with splitting $T_{\mathcal{O}}M = E^s_{\mathcal{O}} \oplus E^X_{\mathcal{O}} \oplus E^u_{\mathcal{O}}$ of a flow X on M, then its unstable set

$$W^u_X(\mathcal{O}) = \{q \in M : \operatorname{dist}(X_t(q), \mathcal{O}) \to 0, t \to -\infty\},$$

is an immersed submanifold tangent at $\mathcal{O}$ to the subbundle $E^X_{\mathcal{O}} \oplus E^u_{\mathcal{O}}$. An isolated set Λ is C^1*robust transitive* if there are a isolating block U of Λ and a neighborhood $\mathcal{U}$ of X (in the space of C^1 flows) such that $\cap_{t\geq 0} Y_t(U)$ is a non-trivial transitive set of Y, $\forall Y \in \mathcal{U}$. Denote by $Cl(B)$ the closure of $B \subset M$.

The following corollary was proved first in [M4] with different methods.

Corollary 1. *A singular-hyperbolic attractor of a C^1 flow on a compact 3-manifold has topological dimension ≥ 2.*

Proof. Let Λ be a singular-hyperbolic attractor as in the statement. By the Main Theorem we have that Λ has a periodic orbit $\mathcal{O}$. Clearly $\mathcal{O}$ must be saddle-type i.e. $dim(E^s) = dim(E^u) = 1$. As noted before the unstable set $W^u_X(\mathcal{O})$ of $\mathcal{O}$ is a submanifold tangent at $\mathcal{O}$ to the direction $E^X_{\mathcal{O}} \oplus E^u_{\mathcal{O}}$. In particular, $dim(W^u_X(\mathcal{O})) = 2$. Then, $dim(\Lambda) \geq 2$ since $W^u_X(\mathcal{O}) \subset \Lambda$ as Λ is an attractor. □

Corollary 2. *If Λ is a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold, then $\Lambda = Cl(W^u_X(O))$ for some closed orbit O of X.*

Proof. If Λ is as in the statement, then Λ has a periodic orbit $\mathcal{O}$ by the Main Theorem. As before we have that $\mathcal{O}$ is saddle-type and so $dim(W^u_X(\mathcal{O})) = 2$. In addition, $W^u_X(\mathcal{O}) \subset \Lambda$ since Λ is an attractor. By using $dim(W^u_X(\mathcal{O})) = 2$, the dense orbit and the contracting direction of Λ we can prove that $W^u_X(\mathcal{O})$ is dense in Λ. Hence $\Lambda = Cl(W^u_X(\mathcal{O}))$ as desired. □

Corollary 3. *C^1 robust transitive sets for C^1 flows on compact 3-manifolds have a periodic orbit.*

Proof. By [MPP3], every C^1 robust transitive set is either a non-trivial isolated hyperbolic set or a singular-hyperbolic attractor (up to reversing the flow). Then the result follows from the Shadowing Lemma (in the first case) and the Main Theorem (in the second case). □

The Main Theorem provides a possible approach to prove Conjecture 1. Indeed, one could try to prove that the attractor is the homoclinic class of the periodic orbit obtained in the Main Theorem. However, this argument fails because a homoclinic class in a singular-hyperbolic attractor may be trivial [MPu].

The proof of the Main Theorem is based on the existence of periodic points for triangular maps and the techniques in [MPa]. The organization is as follows. In Section 2 we prove the existence of periodic orbits for triangular maps with some hyperbolicity (Theorem 2). Although this result is related to some works dealing with hyperbolic maps with singularities ([AP], [KS], [P]) we cannot use these works because our maps are not C^2 and have infinitely many discontinuity or no invariant measures (see for instance (H2) p. 125 in [P] or the proof of Theorem 11 in [P] p. 142). In Section 3 we prove the Main Theorem using Theorem 2. One of the links between these theorems is Lemma 6. This lemma is used to prove that if the flow has a periodic orbit in the attractor, then there is a return map close to the singularities satisfying the hypotheses of Theorem 2. We observe that Lemma 6 is true for attractor but not for isolated sets in general. We use the transitivity of the attractor only to prove Proposition 1. Consequently, a non-trivial singular-hyperbolic attracting set Λ satisfying the conclusion of this proposition has a periodic orbit. On the other hand, our arguments can be used also to study the existence of periodic orbits on a non-trivial singular-hyperbolic *Lyapunov stable* set. More precisely, we can prove that a non-trivial singular-hyperbolic Lyapunov stable set Λ satisfying the conclusion of Proposition 1 either has a periodic orbit or is accumulated by infinitely many periodic orbits.

2. Periodic orbits for triangular maps

We investigate the existence of periodic points for certain maps in $[-1,1]\times[-1,1]$ to be called triangular maps. In the literature the name "triangular maps" is reserved to continuous selft-maps of $[-1,1]\times[-1,1]$ which are skew produt, i.e. preserving the constant vertical foliation (see for instance [BGM], [JS]). In our context we have to consider discontinuous maps preserving a continuous (but not necessarily constant) vertical foliation. These maps will appear in Section 3 as return maps close to the singularities of a singular-hyperbolic attractor.

2.1. **Definition of triangular maps and Theorem 2.** Hereafter we denote by I the compact interval $[-1,1]$. We denote by Σ a disjoint union of k-copies $\{\Sigma_1,\cdots,\Sigma_k\}$ of the square I^2, i.e. $\Sigma_i = I_i^2$ where $I_i=[-1,1]$ is a copy of I. For all $i=1,\cdots,k$ we denote

$$L_{-i}=\{-1\}\times I_i;\quad L_{0i}=\{0\}\times I_i;\quad L_{+i}=\{1\}\times I_i.$$

We also denote

$$L_-=\bigcup_{i=1}^k L_{-i};\quad L_0=\bigcup_{i=1}^k L_{0i};\quad L_+=\bigcup_{i=1}^k L_{+i}.$$

Given a map F we denote by $Dom(F)$ the domain of F. The discontinuity set $D(F)$ of F is denoted by

$$D(F) = \{x \in Dom(F) : F \text{ is discontinuous in } x\}.$$

If F is a map, we say that $x \in Dom(F)$ is *periodic* if there is $n \geq 1$ such that $F^j(x) \in Dom(F)$ for all $0 \leq j \leq n-1$ and $F^n(x) = x$.

By a curve c in Σ we mean a one-to-one C^1 map $c : Dom(c) \subset I\!R \to \Sigma$ whose $Dom(c)$ is a compact interval. We shall identify c with its image in Σ. A curve L in Σ is called *vertical* if $Dom(L) = I$ and L is the graph $\{(g(y), y) : y \in I\}$ of some C^1 map $g : I \to I$.

A continuous foliation $\mathcal{F}_i$ (with leave's space I) in a component Σ_i of Σ is a *vertical foliation* of Σ_i if $\mathcal{F}_i$ is formed by vertical curves such that the curves L_{-i}, L_{0i}, L_{+i} are leaves of $\mathcal{F}_i$. A *vertical foliation* $\mathcal{F}$ of Σ is the union of vertical foliations $\mathcal{F}_i$ of Σ_i, $i = 1, \cdots, k$. Note that the leave's space SL of $\mathcal{F}$ is a disjoint union of k-copies $\{I_1, \cdots, I_k\}$ of I. If $\mathcal{F}$ is a vertical foliation of Σ, a subset $B \subset \Sigma$ is called *$\mathcal{F}$-invariant* if $L \in \mathcal{F}$ and $L \cap B \neq \emptyset$ implies $L \subset B$. Sometimes we use the notation $L \in \mathcal{F}$ to indicate that L is a leaf of $\mathcal{F}$. Next we state the definition of triangular maps to be used in the sequel.

Definition 2. *A* triangular map *is an injective map $F : Dom(F) \subset \Sigma \to \Sigma$ having an* invariant vertical foliation *$\mathcal{F}$, i.e. the following conditions hold:*

1. *$Dom(F)$ is $\mathcal{F}$-invariant.*
2. *If $L \in \mathcal{F}$ and $L \subset Dom(F)$, then there is $f(L) \in \mathcal{F}$ such that $F(L) \subset f(L)$.*
3. *$F/_L : L \to f(L)$ is continuous.*

If $F : Dom(F) \subset \Sigma \to \Sigma$ is a triangular map with associated foliation $\mathcal{F}$, then we have a one-dimensional map $f : Dom(f) \subset SL \to SL$. This map allows us to define the lateral limits

$$f(L_{**}+) = \lim_{L \subset Dom(F),\ L \to L_{**}^{+}} f(L) \quad \text{and} \quad f(L_{**}-) = \lim_{L \subset Dom(F),\ L \to L_{**}^{-}} f(L)$$

for all $L_{**} \subset Cl(Dom(F))$ when they exist.

We shall denote by $T\Sigma$ the tangent space of Σ. Given $x \in \Sigma$, $\alpha > 0$ and a one-dimensional subspace $V_x \subset T_x\Sigma$ we denote by $C_\alpha(x, V_x) \equiv C_\alpha(x)$ the cone in $T_x\Sigma$ with slope α around V_x, namely

$$C_\alpha(x) = \{v_x \in T_x\Sigma : \angle(v_x, V_x) \leq \alpha\},$$

where $\angle$ denotes angle. A *cone field* in Σ will be a continuous map $C_\alpha : x \in \Sigma \to C_\alpha(x) \subset T_x\Sigma$, where $C_\alpha(x)$ is a cone with slope α in $T_x\Sigma$. A cone field C_α is said to be *transverse* to a vertical foliation $\mathcal{F}$ in Σ if $v_x \notin T_xL$ for all leaf L of $\mathcal{F}$, $x \in L$ and all $v_x \in C_\alpha(x)$, i.e., if T_xL not is contained in $C_\alpha(x)$ for all $L \in \mathcal{F}$ and all $x \in L$.

There are triangular maps $F : \Sigma \to \Sigma$ without periodic points. Indeed, put $k = 1$ and consider the natural coordinate system (x, y) in Σ. Then define $F(x, y) = ((x + \gamma \ (mod\ 2)) - 1, g(x, y))$ for suitable $g(x, y)$ with γ irrational. This map has no periodic orbits. On the other hand, the return map associated to the geometric Lorenz attractor [GW] is a triangular map with many periodic points. This example which suggests the existence of periodic points for triangular maps with some hyperbolicity. The definition is the following.

Definition 3. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a triangular map and $\lambda > 0$ be fixed. We say that F is* λ-hyperbolic *if there is a cone field C_α in Σ such that:*

1. *C_α is transverse to $\mathcal{F}$.*
2. *If $x \in Dom(F)$ and F is differentiable in x, then*
$$DF(x)(C_\alpha(x)) \subset Int(C_{\alpha/2}(F(x)))$$
and
3. $$\| DF(x) \cdot v_x \| \geq \lambda \cdot \| v_x \|,$$
for all $v_x \in C_\alpha(x)$.

To study λ-hyperbolic triangular maps we assume the following additional hypotheses:

(H1): If $L \in \mathcal{F}$, $L \subset Dom(F)$ and $F(L) \subset \Sigma \setminus (L_- \cup L_+)$, then F is C^1 in a neighborhood of L in Σ.

(H2): Let $L \in \cup_{i=1}^k \{L_{-i}, L_{+i}\}$ and $n \in I\!N$ be such that $F^j(L) \subset Dom(F) \cap (L_- \cup L_+)$ for all $0 \leq j \leq n-1$ and $F^n(L) \subset \Sigma \setminus (L_- \cup L_+)$. If $L_* \subset Dom(F) \cap F^{-1}(L)$ is a leaf of $\mathcal{F}$, then there is a neighborhood $S \subset Dom(F)$ of L_* in Σ such that $F(S \setminus L_*) \subset \Sigma \setminus (L_- \cup L_+)$. Moreover, the lateral limits $f(L_*+)$ and $f(L_*-)$ exist and belong to $\{L, f(L), \cdots, f^n(L)\}$.

Examples of λ-hyperbolic triangular maps F satisfying **(H1)**- **(H2)** with $\lambda > \sqrt{2}$ and $Dom(F) = I^2 \setminus L_0$ are the two-dimensional return maps associated to the geometric Lorenz attractors [ABS], [GW]. In these examples we have the existence of many periodic points. Such examples motivate the main result of this section.

Theorem 2. *Let F be a λ-hyperbolic triangular map satisfying* **(H1)-(H2)** *with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. Then F has a periodic point.*

See also [AP] and [P] for related results.

2.2. **Preliminary lemmas.** A *closed vertical band* in Σ will be the closed region $[L, L']$ in between two disjoint vertical curves L, L' in the same component Σ_i of Σ. The sets $(L, L') = [L, L'] \setminus (L \cup L')$, $(L, L'] = [L, L'] \setminus L$, $[L, L') = [L, L'] \setminus L'$ will be called vertical bands with (L, L') being called open vertical band. We say that $[L, L']$ is a (closed) vertical band around L_* if $L_* \subset (L, L')$.

Hereafter F is a λ-hyperbolic triangular map satisfying the hypotheses **(H1)-(H2)** with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. The foliation and cone field associated to F will be denoted by $\mathcal{F}$ and C_α respectively. We shall denote by $<$ the natural order in the leaves's space I of $\mathcal{F}_i$. If $B \subset \Sigma$ we denote by $\mathcal{F}_B$ the union of the leaves intersecting B. If $B = \{x\}$ reduces to a single x then $\mathcal{F}_x$ is precisely the leaf of $\mathcal{F}$ containing x. Obviously B is $\mathcal{F}$-invariant if and only if $B = \mathcal{F}_B$. If $S \subset \Sigma$ and $B \subset \Sigma$, we say that S *covers* B if S intersects all the leaves of $\mathcal{F}$ in $\mathcal{F}_B$, i.e., if $B \subset \mathcal{F}_S$.

The remark below is a direct consequence of the definition of F (F is an injective map, $Dom(F)$ is $\mathcal{F}-$invariant and $F/_L$ is continuous on each leaf L of the domain), the Hypothesis **(H1)** and the continuity of $\mathcal{F}$.

Remark 1. *$D(F)$ is $\mathcal{F}$-invariant and closed in $Dom(F)$. In addition we have $F(D(F)) \subset L_- \cup L_+$, $Dom(F) \setminus D(F)$ is open in Σ and $F/_{(Dom(F) \setminus D(F))}$ is a C^1 embedding.*

If c is a curve in Σ we denote by c_0, c_1 its end points and $Int(c) = c \setminus \{c_0, c_1\}$. A open curve is a curve without its end points. We say that c is *tangent* to C_α if $c'(t) \in C_\alpha(c(t))$ for all $t \in Dom(c)$. A C_α*-spine* of a vertical band $[L, L']$, (L, L'),

$[L, L')$ or $(L, L']$ will be a curve $c \subset [L, L']$ tangent to C_α such that $\{c_0, c_1\} \subset L \cup L'$ and $Int(c) \subset (L, L')$.

Definition 4. *Let B be a $\mathcal{F}$-invariant set. We say that B is* $\mathcal{F}$-discrete *if for every $\mathcal{F}$-invariant neighborhood U of L_0 in Σ the set $\{L \in \mathcal{F} : L \subset (B \setminus U)\}$ is finite.*

Lemma 1. *If F has no periodic points, then the following properties hold:*

1. *$D(F)$ is $\mathcal{F}$-discrete.*
2. *If $i \in \{1, \cdots, k\}$ and $D(F) \cap [L_{0i}, L_{+i}]$ consists of a finite number of leaves, then $f(L_{0i}+)$ exists.*
3. *If $i \in \{1, \cdots, k\}$ and $D(F) \cap [L_{-i}, L_{0i}]$ consists of a finite number of leaves, then $f(L_{0i}-)$ exists.*

Proof. Since F has no periodic points we have that there is $0 \leq n \leq 2k$ such that $F^n(L_*) \subset \Sigma \setminus (L_- \cup L_+)$ for all $L_* \in \cup_{i=1}^k \{L_{-i}, L_{+i}\}$. Suppose that there is a $\mathcal{F}$-invariant neighborhood U of L_0 in Σ such that $D(F) \setminus U$ has a infinite numbers of leaves. As $D(F)$ is closed in $Dom(F)$, then there is a sequence of leaves $L_n \subset D(F) \setminus U$ such that $L_n \to L \subset D(F)$. By **(H2)** there is a neighborhood $S \subset Dom(F)$ of L in Σ such that $F(S \setminus L) \subset \Sigma \setminus (L_- \cup L_+)$, and by **(H1)**, the leaf L cannot be accumulated by leaves in $D(F)$. This is a contradiction which proves the first part of Lemma 1.

Now let $i \in \{1, \cdots k\}$ be such that $D(F) \cap [L_{0i}, L_{+i}]$ consists of a finite number of leaves. Since $D(F) \cap [L_{0i}, L_{+i}]$ is the union of a discrete set of leaves and $Dom(F/_{\Sigma_i}) = \Sigma_i \setminus L_{0i}$, we can choose a leaf $B > L_{0i}$ in I_i such that $(L_{0i}, B] \subset Dom(F) \setminus D(F)$. It follows that f is continuous in $(L_{0i}, B]$. If $f/_{(L_{0i},B]}$ were not monotone then it would exist two different leaves $L, L' \subset (L_{0i}, B]$ with $L'' = f(L) = f(L')$. Choose a C_α-spine c of $[L, L']$. Then $c \subset Dom(F) \setminus D(F)$ and so $F(c)$ would be a curve transverse to $\mathcal{F}$ intersecting the common leaf L'' twice. This contradiction shows that $f/_{(L_{0i},B]}$ is monotone, and so, $f(L_{0i}+)$ exists proving the second part of the lemma. The third part can be proved with similar arguments. ☐

Lemma 2. *Let $c \subset Dom(F) \setminus D(F)$ be a open curve transverse to $\mathcal{F}$. If there are $n \geq 1$ and an open curve c^* with closure $Cl(c^*) \subset c$ such that $F^i(c^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq i \leq n-1$ and $F^n(c^*)$ covers c, then F has a periodic point.*

Proof. Since c is a open curve transverse to $\mathcal{F}$ we have that $\mathcal{F}_c$ is a vertical band (L, L'). Analogously $\mathcal{F}_{c^*}$ is a vertical band (L_*, L'_*) with closure $[L_*, L'_*]$ contained in (L, L'). Clearly we have $F^i((L_*, L'_*)) \subset Dom(F) \setminus D(F)$ for all $0 \leq i \leq n-1$, and so, the restricted map $f^n/_{(L_*,L'_*)}$ is continuous because $F^n/_{(L_*,L'_*)}$ is a C^1 embedding. On the other hand, if $W \in (L, L')$ then there is $x \in c^*$ such that $F^n(x) \in W$ because $F^n(c^*)$ covers c. But $F^n(x) \in f^n(\mathcal{F}_x)$. So $W = f^n(\mathcal{F}_x)$, i.e. $W \in f^n((L_*, L'_*))$. Henceforth $[L_*, L'_*] \subset (L, L') \subset f^n((L_*, L'_*))$. Since $f^n/_{(L_*,L'_*)}$ is continuous we conclude that f^n has a fixed point L_{**} in $[L_*, L'_*]$. Hence $F^n(L_{**}) \subset f^n(L_{**}) = L_{**}$ and so F^n has a fixed point in L_{**} by the Brower Theorem since $F^n/_{L_{**}}$ is continuous. This fixed point represents a periodic point of F and the proof follows. ☐

Lemma 3. *F carries a curve $c \subset Dom(F) \setminus D(F)$ tangent to C_α with length $\mid c \mid$ into a curve tangent to C_α with length $\geq \lambda \cdot \mid c \mid$.*

Proof. Let $c : Dom(c) \to Dom(F) \setminus D(F)$ be a curve tangent to C_α. If $t \in Dom(c)$ and $c'(t) \in C_\alpha(c(t))$, then $DF(c(t))c'(t) \in C_\alpha(F(c(t)))$, because

$$DF(c(t))(C_\alpha(c(t))) \subset Int(C_{\alpha/2}(F(c(t)))).$$

Also,

$$\mid F \circ c \mid = \int_{Dom(c)} \| DF(c(t))c'(t) \| \, dt \geq \int_{Dom(c)} \lambda \cdot \| c'(t) \| \, dt = \lambda \cdot \mid c \mid .$$

The poof follows. □

Observe that if F has no periodic leaves, then for each $i = 1, ..., k$ there are $n(-i), n(+i)$ with $0 \leq n(-i), n(+i) \leq 2k$ such that $F^j(L_{-i}), F^l(L_{+i}) \subset L_- \cup L_+$ for all $0 \leq j \leq n(-i) - 1$, $0 \leq l \leq n(+i) - 1$ and $F^{n(-i)}(L_{-i}), F^{n(+i)}(L_{+i}) \subset \Sigma \setminus (L_- \cup L_+)$.

Lemma 4. *Suppose that F has no periodic leaves. Let L, L' be two different leaves in $D(F)$ with (L, L') open vertical band in Σ and $(L, L') \subset Dom(F) \setminus D(F)$. If c is a C_α-spine of (L, L'), then $F(c)$ covers a vertical band of the form (W, W') with*

$$W, W' \in \bigcup_{i=1}^{k} \{L_{-i}, L_{+i}, f^{n(-i)}(L_{-i}), f^{n(+i)}(L_{+i})\}.$$

Proof. Note that $F/_{(L,L')}$ is a C^1 diffeomorphism since $(L, L') \subset Dom(F) \setminus D(F)$, and $F(L), F(L') \subset L_- \cup L_+$. Let c be a C_α-spine of (L, L') with $c(0) \in L$ and $c(1) \in L'$. Then $F(c)$ is a curve tangent to C_α with $Int(F(c))$ contained in the vertical band limited for$f(L+)$ and $f(L'-)$. If $L = L_{-i}$ for some $i = 1, ..., k$, then $f(L_{-i}+)$ exists and belongs to

$$\{f(L_{-i}), f^2(L_{-i}), \cdots, f^{n(-i)}(L_{-i})\}$$

because F has no periodic leaves. Analogously if $L' = L_{+i}$ for some $i = 1, ..., k$. Hence we can assume that $L \in \mathcal{F} \setminus \cup_{i=1}^{k}\{L_{-i}\}$. Then $F(L) \subset L_*$ with $L_* \in \cup_{i=1}^{k}\{L_{-i}, L_{+i}\}$. As F has no periodic leaves, there is $n(*)$, $0 \leq n(*) \leq 2k$ such that $F^j(L_*) \subset (L_- \cup L_+)$ for all $0 \leq j \leq n(*) - 1$, and $F^{n(*)}(L_*) \subset \Sigma \setminus (L_- \cup L_+)$. By **(H2)** $f(L+)$ exists and

$$f(L+) \in \{L_*, f(L_*), f^2(L_*), ..., f^{n(*)}(L_*)\}$$

Analogously if $L' \in \mathcal{F} \setminus \cup_{i=1}^{k}\{L_{+i}\}$ This finishes the proof. □

Define

$$\mathcal{L}_- = \{f(L_{0i}-) : f(L_{0i}-) \text{ exists}, \ i = 1, ..., k\}$$

and

$$\mathcal{L}_+ = \{f(L_{0i}+) : f(L_{0i}+) \text{ exists}, \ i = 1, ..., k\}.$$

Lemma 5. *Suppose that F has no periodic leaves. For all curve $c \subset Dom(F) \setminus D(F)$ tangent to C_α there are a smaller curve $c^* \subset c$ and $n'(c) > 0$ such that $F^j(c^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n'(c) - 1$ and $F^{n'(c)}(c^*)$ covers (W, W') where W, W' belong to*

$$\bigcup_{i=1}^{k} \left\{L_{-i}, L_{+i}, f^{n(-i)}(L_{-i}), f^{n(+i)}(L_{+i})\right\} \cup \mathcal{L}_- \cup \mathcal{L}_+.$$

Proof. For all curve $c \subset Dom(F) \setminus D(F)$ tangent to C_α we define

$$n(c) = Sup\left\{n \geq 1 : F^j(c) \subset Dom(F) \setminus D(F), \forall 0 \leq j \leq n-1\right\}.$$

Note that $1 \leq n(c) < \infty$ because $\lambda > 1$ and Σ has finite diameter. Moreover, $F^{n(c)}(c)$ is a curve tangent to C_α with $F^{n(c)}(c) \cap (D(F) \cup L_0) \neq \emptyset$.

Define $\beta = (1/2) \cdot \lambda > 1$. Fix an open curve $c_1 \subset Dom(F) \setminus D(F)$ and define $n_1 = n(c_1)$. If $F^{n_1}(c_1)$ intersects $D(F) \cup L_0$ in a single leaf L_1, then $F^{n_1}(c_1) \cap L_1$ reduces to a single point p_1. In such a case we define $c_2^* =$ largest connected component of $F^{n_1}(c_1) \setminus \{p_1\}$ and $c_2 = F^{-n_1}(c_2^*)$. Then, we have the following properties:

1) $c_2 \subset c_1$ is an open curve tangent to C_α.
2) $F^j(c_2) \subset Dom(F) \setminus D(F)$, for all $0 \leq j \leq n_1$.
3) $\mid F^{n_1}(c_2) \mid \geq \beta \cdot \mid c_1 \mid$.

Indeed, the first property holds since $F^{-n_1}/_{\mathcal{F}_{c_2^*}}$ is a C^1 embedding. The second property follows from the definition of $n_1 = n(c_1)$ and the fact that $c_2^* = F^{n_1}(c_2)$ does not intersect the leaves of $D(F)$. The third property follows from Lemma 3 because

$$\mid F^{n_1}(c_2) \mid = \mid c_2^* \mid \geq (1/2) \cdot \mid F^{n_1}(c_1) \mid \geq (1/2) \cdot \lambda^{n_1} \mid c_1 \mid \geq (1/2) \cdot \lambda \mid c_1 \mid = \beta \cdot \mid c_1 \mid$$

since $\lambda > 2$ and $n_1 \geq 1$.

Now we define $n_2 = n(c_2)$. The second property implies $n_2 > n_1$. As before, if $F^{n_2}(c_2)$ intersects $D(F) \cup L_0$ in a single leaf L_2, then $F^{n_2}(c_2) \cap L_2$ reduces to a single point p_2. In such a case we define $c_3^* =$ largest connected component of $F^{n_2}(c_2) \setminus \{p_2\}$ and $c_3 = F^{-n_2}(c_3^*)$. As before we have

$$\mid F^{n_3}(c_3) \mid = \mid c_3^* \mid \geq (1/2) \cdot \mid F^{n_2}(c_2) \mid \geq (1/2) \cdot \lambda^{n_2 - n_1} \mid F^{n_1}(c_2) \mid \geq \beta 2 \mid c_1 \mid$$

by the third property. We have the following properties:

1) $c_3 \subset c_2$ is an open curve tangent to C_α.
2) $F^j(c_3) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n_2$.
3) $\mid F^{n_2}(c_3) \mid \geq \beta^2 \cdot \mid c_1 \mid$.

In this way we obtain a sequence $n_1, n_2, n_3, \cdots$ of positive integers with $n_l > n_{l-1}$; and a sequence $c_1, c_2, c_3, \cdots$ of curves such that the following properties hold for all $l \geq 1$

1) $c_{l+1} \subset c_l$ is an open curve tangent to C_α.
2) $F^j(c_{l+1}) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n_l$.
3) $\mid F^{n_l}(c_{l+1}) \mid \geq \beta^l \cdot \mid c_1 \mid$.

The sequences n_l and c_l have to be finite by the third property since Σ have finite diameter and $F^{n_l}(c_{l+1})$ is tangent to C_α. So, *there is a first l_0 such that $F^{n_{l_0}}(c_{l_0})$ intersects two different leaves L, L' of $\mathcal{F}$ in $D(F) \cup L_0$.*

If $L, L' \subset D(F)$ then we can assume that $(L, L') \subset Dom(F) \setminus D(F)$ since $D(F)$ is discrete by the first part of Lemma 1 (recall that F has no periodic points by assumption). It follows that $F^{n_{l_0}}(c_{l_0})$ contains a C_α-spine $c_{l_0+1}^*$ of (L, L'). In this case $c = c_1$, $c^* = F^{-n_{l_0}}(c_{l_0+1}^*)$ and $n'(c) = n_{l_0} + 1$ satisfy the conclusion of Lemma 5.

Now we assume that $L = L_{0i}$ for some $i = 1, \cdots, k$ and that $L_{0i} < L'$ (the argument when $L' < L_{0i}$ is similar). If $(L_{0i}, L') \cap D(F) \neq \emptyset$, then we can find a leaf L'' in $D(F)$ such that $(L'', L') \subset Dom(F) \setminus D(F)$ and $F^{n_{l_0}}(c_{l_0})$ contains a spine of (L'', L'). In this case we conclude as before. Hence we can assume that

$(L_{0i}, L') \cap D(F) = \emptyset$. Then, $D(F) \cap [L_{0i}, L_{+i}]$ consists of a finite number of leaves of $\mathcal{F}$. On the other hand, $f(L_{0i}+) = f(L+)$ exists by Lemma 1. Then, $F^{n_{l_0}}(c_{l_0})$ contains a C_α-spine $C^*_{l_0+1}$ of (L, L'). By choosing $c = c_1$, $c^* = F^{-n_{l_0}}(c^*_{l_0+1})$ and $n'(c) = n_{l_0} + 1$ we obtain the result. This proves the result. $\square$

2.3. **Proof of Theorem 2.** Let F be a λ-hyperbolic triangular map satisfying **(H1)-(H2)** with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. We assume by contradiction that F has no periodic points. By the first part of Lemma 1 we have that $D(F)$ is $\mathcal{F}$-discrete, and so, $Dom(F) \setminus D(F) = \Sigma \setminus (D(F) \cup L_0)$ is open-dense in Σ.

Let $\mathcal{B}$ the family of vertical bands in Σ of the form (W, W') where W, W' belong to

$$\bigcup_{i=1}^{k} \left\{ L_{-i}, L_{+i}, f^{n(-i)}(L_{-i}), f^{n(+i)}(L_{+i}) \right\} \cup \mathcal{L}_- \cup \mathcal{L}_+.$$

Clearly $\mathcal{B} = \{B_1, \cdots, B_m\}$ is a finite set. As $Dom(F) \setminus D(F)$ is open-dense in Σ we can choose a curve $c^{\cdot}_1 \subset B_1$ with closure $Cl(c^{\cdot}_1) \subset Dom(F) \setminus D(F)$.

By Lemma 5 applied to $c = c^{\cdot}_1$ there are a curve $c^{\cdot}_1{}^* \subset c^{\cdot}_1$ and $n'(c^{\cdot}_1) > 0$ such that $F^j(c^{\cdot}_1{}^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n'(c^{\cdot}_1) - 1$ and $F^{n'(c^{\cdot}_1)}(c^{\cdot}_1{}^*)$ covers B_{j_1} for some $j_1 \in \{1, \cdots, m\}$. Define $c_1 = c^{\cdot}_1{}^*$ and $n_1 = n'(c^{\cdot}_1)$. Note that $j_1 \neq 1$ for, otherwise, $F^{n_1}(c_1)$ covers $c^{\cdot}_1$ and then F would have a periodic point by Lemma 2 a contradiction. Then we have the following properties:

1) $j_1 \notin \{1\}$.
2) $c_1 \subset c^{\cdot}_1$ is a curve.
3) $F^j(c_1) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n_1 - 1$.
4) $F^{n_1}(c_1)$ covers B_{j_1}.

Again, since $Dom(F) \setminus D(F)$ is open-dense in Σ, we choose a curve $c^{\cdot}_2 \subset F^{n_1}(c_1) \cap B_{j_1}$ with closure $Cl(c^{\cdot}_2) \subset Dom(F) \setminus D(F)$. By Lemma 5 applied to $c = c^{\cdot}_2$ there are a curve $c^{\cdot}_2{}^* \subset c^{\cdot}_2$ and $n'(c^{\cdot}_2) > 0$ such that $F^j(c^{\cdot}_2{}^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n'(c^{\cdot}_2) - 1$ and $F^{n'(c^{\cdot}_2)}(c^{\cdot}_2{}^*)$ covers B_{j_2} for some $j_2 \in \{1, \cdots, m\}$. Define $c_2 = F^{-n'(c^{\cdot}_1)}(c^{\cdot}_2{}^*)$ and $n_2 = n_1 + n'(c^{\cdot}_2)$. Note that $j_2 \notin \{1, j_1\}$. Indeed, if $j_2 = 1$, then $F^{n_2}(c_2)$ covers covers $c^{\cdot}_1$ and then F would have a periodic point by Lemma 2 a contradiction. If $j_2 = j_1$ then $F^{n'(c^{\cdot}_2)}(c^{\cdot}_2{}^*)$ covers $c^{\cdot}_2$ and so F would have a periodic point by Lemma 2 a contradiction. Then we have the following properties:

1) $j_2 \notin \{1, j_1\}$.
2) $c_2 \subset c_1$ is a curve.
3) $F^j(c_2) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n_2 - 1$.
4) $F^{n_2}(c_2)$ covers B_{j_2}.

In this way we can construct sequences $j_1, j_2, j_3, \cdots \in \{1, 2, \cdots, m\}$; $c_1, c_2, c_3, \cdots$; and $n_1, n_2, n_3, \cdots$ such that the following properties hold for all $s \geq 1$ with $c_0 = c^{\cdot}_1$ and $j_0 = 1$:

1) $j_s \notin \{1, j_1, j_2, j_3, \cdots, j_{s-1}\}$.
2) $c_s \subset c_{s-1}$ is a curve.
3) $F^j(c_s) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n_s - 1$.
4) $F^{n_s}(c_s)$ covers B_{j_s}.

However, the first property cannot be satisfied $\forall s \geq 1$ because $\{1, \cdots, m\}$ is a finite set. This is a contradiction which proves the result. $\square$

3. Proof of the Main Theorem

We investigate the existence of periodic orbits for singular hyperbolic attractors. In particular, we prove the Main Theorem using Theorem 2. We shall use some constructions some of which can be found in sections 5 and 6 in [MPa].

3.1. Singular cross-section.

Hereafter X will denote a C^1 flow on a compact 3-manifold M.

Definition 5. *A singularity σ of X is* Lorenz-like *if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy the following eigenvalue relation (up to some order):*

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

We consider Lorenz-like singularities by the following proposition.

Proposition 1. *Let Λ be a singular-hyperbolic attractor of X. Then, the following properties hold:*

1. *If $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ denotes the singular-hyperbolic splitting of Λ, then $X(x) \in E^c_x$, $dim(E^s_x) = 1$ and $dim(E^c_x) = 2$ for all $x \in \Lambda$.*
2. *If σ is a singularity of X in Λ, then σ is Lorenz-like and satisfies the following equality:*

$$W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}.$$

Proof. Since Λ is transitive we can fix $q \in \Lambda$ with dense orbit, i.e. $\Lambda = \omega_X(q)$. First we claim that $X(q) \notin E^s_q$. Indeed, suppose the contrary namely $X(q) \in E^s_q$. Then, $X(X_t(q)) = DX_t(q)(X(q)) \in E^s_{X_t(q)}$ for all $t > 0$ by invariance. Since E^s is contracting we have that $\lim_{n\to\infty} |X(X_{t_n}(q))| \to 0$ as $n \to \infty$ for all sequence $t_n \to \infty$. This implies $X(x) = 0$ for every $x \in \Lambda$ since the orbit of q is dense. We contradict the non-triviality of Λ and the result follows. Using the claim and the dominating splitting we obtain $X(x) \in E^c_x$ for all $x \in \Lambda$. On the other hand, the dimension $dim(E^c_x)$ is constant for $x \in \Lambda$ since Λ is connected. If such a dimension is 1, then we have that E^c_Λ is expanding (i.e. contracting for the reversed flow). As $X(x) \in E^c_x$ for all $x \in \Lambda$ (and M is compact) we obtain a contradiction unless $X(x) = 0$ for all $x \in \Lambda$. This would imply that Λ is a singularity which is absurd as Λ is non-trivial. We conclude that $dim(E^c_x) = 0, 2$ or 3 for all $x \in \Lambda$. Clearly such a dimension is neither 0 (for otherwise Λ would be an attracting closed orbit) nor 3 (for Λ is an attractor). This implies that $dim(E^c_x) = 2$ for all $x \in \Lambda$. It follows that $dim(E^s_x) = 1$ for all $x \in \Lambda$ proving (1). (2) follows from (1) and the arguments in [MPP1]. □

Every Lorenz-like singularity σ of X is hyperbolic. Hence the stable and unstable manifolds $W^s_X(\sigma)$, $W^u_X(\sigma)$ exist and are tangent at σ to the eigenspace associated to the set of eingenvalues $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$ respectively [HPS]. In particular, $dim(W^s_X(\sigma)) = 2$ and $dim(W^u_X(\sigma)) = 1$. A further invariant manifold, the strong stable manifold $W^{ss}_X(\sigma)$, is well defined and tangent at σ to the eigenspace associated the $\{\lambda_2\}$. Consequently $dim(W^{ss}_X(\sigma)) = 1$.

The classical Grobman-Hartman Theorem [dMP] gives the description of the flow nearby a Lorenz-like singularity σ of X. This is done at Figure 1. Note that $W^{ss}_X(\sigma)$ separates $W^s_X(\sigma)$ in two connected components: the top and the botton ones. In the top component we consider a cross-section $S^t = S^t_\sigma$ of X together with a curve $l^t = l^t_\sigma$ as in Figure 1. Similarly we consider a cross-section $S^b = S^b_\sigma$ and a curve $l^b = l^b_\sigma$ in the botton component. S^* is diffeomorphic to $[-1, 1] \times [-1, 1]$ and

l^* is contained in $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$ for $* = t, b$. The positive flow lines of X starting at $S^t \cup S^b \setminus (l^t \cup l^b)$ exit a small neighborhood of σ passing through the cusp region as indicated in Figure 1. The positive orbits starting at $l^t \cup l^b$ goes directly to σ. We note that the boundary of S^* is formed by four curves, two of them transverse to l^* and two of them parallel to l^*. The union of the curves in the boundary of S^* which are parallel (resp. transverse) to l^* is denoted by $\partial^v S^*$ (resp. $\partial^h S^*$).

Definition 6. *The cross-sections S^t, S^b above are called* singular cross-sections associated to σ. *The curves l^t, l^b are called* singular curves *of S^t, S^b respectively.*

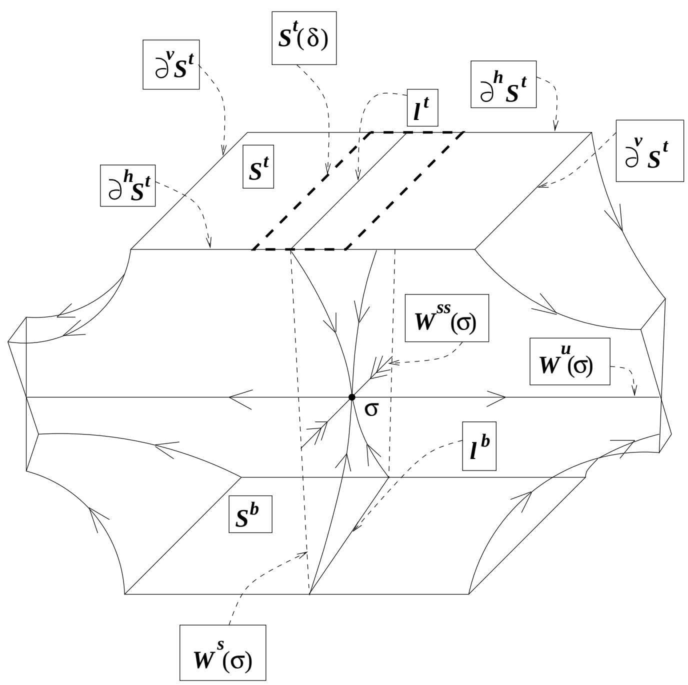

FIGURE 1. Singular cross-section.

Remark 2. *The following holds by Proposition 1-(2). Let Λ a singular-hyperbolic attractor and σ be a singularity in Λ. Then, there are singular cross-sections S^t, S^b associated to σ which are arbitrarily close to σ and satisfy*

$$\Lambda \cap (\partial^h S^t_\sigma \cup \partial^h S^b_\sigma) = \emptyset.$$

For simplicity we denote by $Sing_X(B)$ the set of singularities of X in a subset $B \subset M$. If

$$S = \{S^t_\sigma, S^b_\sigma : \sigma \in Sing_X(\Lambda)\}$$

is a collection formed by singular cross-sections S^t_σ, S^b_σ, $\sigma \in Sing_X(\Lambda)$, then we define

$$\partial^h S = \bigcup_{\sigma \in Sing_X(\Lambda)} (\partial^h S^t_\sigma \cup \partial^h S^b_\sigma).$$

Definition 7. *Let Λ be a singular-hyperbolic attractor of X. A* singular cross-section of Λ *is a disjoint collection*

$$S = \{S^t_\sigma, S^b_\sigma : \sigma \in Sing_X(\Lambda)\},$$

formed by singular cross-sections S^t_σ, S^b_σ, $\sigma \in Sing_X(\Lambda)$, *such that*

$$\Lambda \cap \partial^h S = \emptyset.$$

We still denote by S *the union of the elements of* S. *The* singular curve *of* S *is the associated collection of singular curves*

$$l = \{l^t_\sigma, l^b_\sigma : \sigma \in Sin_X(\Lambda)\}.$$

3.2. **Return map for singular cross-sections.** Associated to any singular cross-section S of a singular-hyperbolic attractor Λ of X we have a return map

$$\Pi = \Pi_S : Dom(\Pi) \subset S \to S,$$

given by

$$\Pi(x) = X_{T(x)}(x),$$

where $Dom(\Pi)$ and $T(x)$ denotes respectively the domain of Π and the first (positive) return time of x respectively. The following lemma describes $Dom(\Pi)$ *when* X *has no periodic orbits in* Λ.

Lemma 6. *Let* Λ *be a singular-hyperbolic attractor of a* C^1 *flow* X *in* M. *Then, there is a positively invariant isolating block* U_0 *of* Λ *arbitrarily close to* Λ *with the following property: If* X *has no periodic orbits in* Λ *and* $S \subset U_0$ *is a singular cross-section of* Λ, *then*

$$Dom(\Pi) = S \setminus l.$$

Proof. Since Λ is an attractor we can find a positively invariant isolating block U_0 arbitrarily close to Λ (this is a well known exercise in topological dynamics). Now, let $S \subset U_0$ be a singular cross-section of Λ. Clearly $l \cap Dom(\Pi) = \emptyset$ and so $Dom(\Pi) \subset S \setminus l$. On the other hand, pick $x \in S \setminus l$. If $x \notin Dom(\Pi)$, then $\omega_X(x) \subset \Lambda$ and $\omega_X(x)$ has no singularities. It follows that $\omega_X(x)$ is hyperbolic by [MPP1]. By applying the Shadowing Lemma for flows [HK] we can find a periodic orbit of X close to $\omega_X(x)$. It follows that such a periodic orbit belongs to Λ which is absurd. This proves $S \setminus l \subset Dom(\Pi)$ and the result follows. □

3.3. **Induced foliations on singular cross-sections.** Hereafter Λ denotes a singular-hyperbolic attractor of X and by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the singular-hyperbolic splitting of Λ. The contracting direction E^s is one-dimensional by Proposition 1-(1). So, E^s_Λ can be extended to an invariant contracting splitting $E^s_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ. The standard Invariant Manifold Theory [HPS] implies that $E^s_{U(\Lambda)}$ is integrable, i.e. tangent to an invariant continuous one-dimensional contracting foliation $\mathcal{F}^{ss}$ on $U(\Lambda)$. If $x \in U(\Lambda)$ we denote by $\mathcal{F}^{ss}_x$ the leaf of $\mathcal{F}^{ss}$ containing x.

Now, we obtain a foliation $\mathcal{F}$ on S by *projecting* $\mathcal{F}^{ss}$ *onto* S in the following way. Let S be a singular cross-section contained in $U(\Lambda)$. If $I \subset \mathbb{R}$ and $B \subset M$ we define

$$X_I(B) = \{X_t(x) : (t,x) \in I \times B\}.$$

For every $\epsilon > 0$ and $x \in U(\Lambda)$ we define

$$\mathcal{F}^s_{x,\epsilon} = X_{[-\epsilon,\epsilon]}(\mathcal{F}^{ss}_x).$$

If x is a regular point of x (i.e. $X(x) \neq 0$) then $\mathcal{F}^s_{x,\epsilon}$ is a two-dimensional submanifold of M (this remark applies for all $x \in S$). If $x \in S$ we define

$$\mathcal{F}_{x,\epsilon} = \mathcal{F}^{ss}_{x,\epsilon} \cap S.$$

Since S is compact (and formed by regular points) we can find $\epsilon > 0$ such that if $\mathcal{F}_x = \mathcal{F}_{x,\epsilon}$, then the family

$$\mathcal{F} = \{\mathcal{F}_x : x \in S\}$$

defines a continuous one-dimensional foliation of S having l as union of leaves.

3.4. **Refinement of singular cross-sections.** We use the foliation $\mathcal{F}$ in Subsection 3.3 to *refine* a singular cross-section $S \subset U(\Lambda)$ in the following way. Let S be a singular-cross section of Λ. By definition S is a disjoint collection $S = \{S^t_\sigma, S^b_\sigma : \sigma \in Sing_X(\Lambda)\}$ of singular cross-sections S^t_σ, S^b_σ, $\sigma \in Sing_X(\Lambda)$. Recall that $l = \{l^t_\sigma, l^b_\sigma : \sigma \in Sin_X(\Lambda)\}$ is the singular curve of S. By construction l^*_σ divides S^*_σ in two connected components $S^{*,+}_\sigma, S^{*,-}_\sigma$ $(* = t, b)$. For $\delta > 0$ small we choose two points $x^+_\delta, x^-_\delta \in S^{*,\pm}_\sigma$ whose distance to l^*_σ is δ. Define $S^*_\sigma(\delta)$ as the singular cross-sections of σ satisfying the following property:

$$\partial^v S^*_\sigma(\delta) = \mathcal{F}_{x^-_\delta} \cup \mathcal{F}_{x^+_\delta}.$$

We have depicted $S^t(\delta) = S^t_\sigma(\delta)$ in Figure 1. It follows from the definition that S^*_σ and $S^*_\sigma(\delta)$ have the same singular curve l^*_σ. In addition, $\partial^h S^*_\sigma(\delta) \subset \partial^h S^*_\sigma$ and $S^*_\sigma(\delta)$ is invariant for the foliation $\mathcal{F}$ in S. Since S is a singular cross section of Λ we conclude that the set

$$S(\delta) = \{S^t_\sigma(\delta), S^b_\sigma(\delta) : \sigma \in Sing_X(\Lambda)\}$$

is also a singular cross-section of Λ. Note that S and $S(\delta)$ have the same singular curve l. For simplicity we denote by $\Pi_\delta = \Pi_{S(\delta)}$ the return map associated to $S(\delta)$ and by $T_\delta(x)$ the return time of $x \in Dom(\Pi_\delta)$. Clearly $S(\delta) \subset S$ and so $S(\delta) \subset U(\Lambda)$ for all δ. A simple observation is that *the return time T_δ is uniformly large as $\delta \to 0^+$*, namely

$$\lim_{\delta \to 0^+} \inf_{x \in S(\delta)} T_\delta(x) = \infty.$$

Lemma 7. *Let Λ be a singular-hyperbolic attractor of a C^1 flow X in M. Then, there is a positively invariant isolating block $U_1 \subset U(\Lambda)$ of Λ arbitrarily close to Λ with the following property: If X has no periodic orbits in Λ and $S \subset U_1$ is a singular cross-section of Λ, then for all $\delta > 0$ small one has:*

1. *$Dom(\Pi_\delta) = S(\delta) \setminus l$.*
2. *$Dom(\Pi_\delta)$ is $\mathcal{F}$-invariant.*
3. *If $L \in \mathcal{F}$ and $L \subset Dom(\Pi_\delta)$, then there is $f(L) \in \mathcal{F}$ such that $\Pi_\delta(L) \subset f(L)$.*
4. *$\Pi_\delta/_L : L \to f(L)$ is continuous for all $L \in \mathcal{F}$ with $L \subset Dom(\Pi_\delta)$.*

Proof. Let U_0 be the positively invariant isolating block of Λ coming from Lemma 6. Let $U(\Lambda)$ be the domain of $\mathcal{F}^{ss}$. Since U_0 is arbitrarily close to Λ we can assume that $U_0 \subset U(\Lambda)$. Define $U_1 = U_0$. Let $S \subset U_1$ be a singular cross-section of Λ. Clearly $S(\delta) \subset S \subset U_0$ for all δ. Suppose that X has no periodic orbits in Λ. Then, Lemma 6 implies

$$Dom(\Pi_\delta) = S(\delta) \setminus l$$

proving (1). We obtain (2) from (1) because l is union of leaves of $\mathcal{F}$. On the other hand, T_δ is uniformly large for $\delta > 0$ small. In addition,

$$\partial^h S(\delta) \subset \partial^h S$$

for all δ. Pick $L = \mathcal{F}_x \in \mathcal{F}$ with $L \subset Dom(\Pi_\delta)$. We can assume that L is contained in $\mathcal{F}^{ss}_x$ by projecting along the flow. Since T_δ is uniformly large we have that the positive orbit of the elements of L stay uniformly close to the one of x. In addition, $\Pi_\delta(x)$ is close to $\Lambda \cap S(\delta)$. As $\Lambda \cap \partial^h S(\delta) = \emptyset$ we obtain the following: If $\Pi_\delta(x) \in S^*_\sigma(\delta)$ for some $\sigma, *$, then

$$X_{T_\delta(x)}(L) \subset \left(\mathcal{F}^s_{\Pi_\delta(x),\epsilon} \cap X_{[-\epsilon,\epsilon]}(S^*_\sigma(\delta))\right) \setminus X_{[-\epsilon,\epsilon]}(\partial^h S^*_\sigma(\delta)).$$

See Figure 2. By projecting along the flow we obtain

$$\Pi_\delta(L) \subset \mathcal{F}_{\Pi_\delta(x)}.$$

Setting $f(L) = \mathcal{F}_{\Pi_\delta(x)}$ we obtain (3). To finish we observe that (4) follows from the Tubular Flow Box Theorem [dMP]. ☐

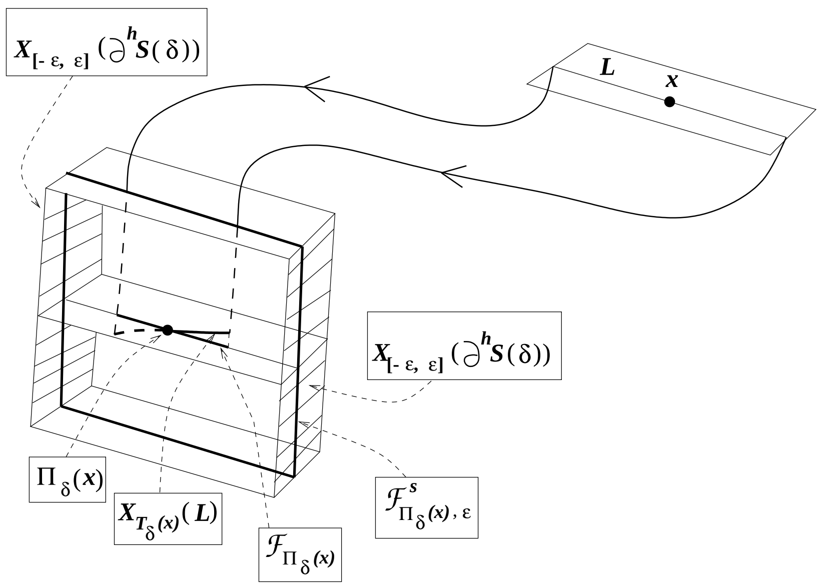

FIGURE 2. Proof of Lemma 7.

3.5. **Proof of the Main Theorem.** Let Λ be a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold M. *We assume by contradiction that X has no periodic orbits in Λ.* Let $U(\Lambda)$ be the domain of the foliation $\mathcal{F}^{ss}$ described in Subsection 3.3. Recall that the contracting direction E^s_Λ extend to a contracting direction $E^s_{U(\Lambda)}$ on $U(\Lambda)$. Analogously we can extend the central direction E^c_Λ to obtain a subbundle $E^c_{U(\Lambda)}$ but this extension is invariant only in Λ.

Let $U_1 \subset U(\Lambda)$ be the isolating block of Λ coming from Lemma 7. By Remark 2 we can fix a singular cross-section $S = \{S^t_\sigma, S^b_\sigma : \sigma \in Sing_X(\Lambda)\}$ of Λ contained in U_1. Let $S(\delta)$ be the refinement of S described in Subsection 3.4. We can consider $S(\delta) = \Sigma$ as a finite disjoint collection Σ of copies of $[-1,1] \times [-1,1]$ by identifying L_-, L_+ with the connected components of $\partial^v S(\delta)$; and L_0 with the singular curve l of $S(\delta)$ (see Subsection 2.1 for the corresponding definitions). As X has no periodic orbits in Λ, Lemma 7 implies that the return map $F = \Pi_\delta$ of Σ is a triangular map (with associated foliation $\mathcal{F}$) satisfying

$$Dom(F) = \Sigma \setminus L_0.$$

Fix $\lambda > 2$. We claim that there is $\delta > 0$ small such that F is a λ-hyperbolic triangular map satisfying **(H1)-(H2)** with $Dom(F) = \Sigma \setminus L_0$. We already obtained the last property holds. To prove that F is λ-hyperbolic we need to find a cone field C_α in Σ satisfying the properties (1), (2) and (3) in Definition 3. To define C_α we set $\alpha = 1/3$, and, for each $x \in \Sigma$ we define

$$C_\alpha(x) = \{v_x \in T_x\Sigma : \angle(v_x, V_x) \leq \alpha\},$$

where $V_x = E^c_x \cap T_x\Sigma$. These choices imply Definition 3-(1) because $\mathcal{F}$ is obtained by projecting $\mathcal{F}^{ss}$ to Σ. On the other hand, since the return time is large we obtain Definition 3-(2) because the splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is dominated. Analogously we obtain Definition 3-(3) because E^c_Λ is volume expanding (compare with the proof of Corollary 6.5, p. 1589 in [MPa]). To prove that F satisfies **(H1)**-**(H2)** we can use tubular flow-box arguments ([dMP]) because F is the return map induced by X. This proves the claim. It follows from the claim and Theorem 2 that F has a periodic point. This periodic point must belong to a periodic orbit of X in Λ, a contradiction. This contradiction proves the result. □

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