Monotone operators representable by l.s.c. convex functions

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August 6, 2004

Abstract

A theorem due to Fitzpatrick provides a representation of arbitrary maximal monotone operators by convex functions. This paper explores representability of arbitrary (non necessarily maximal) monotone operators by convex functions. In the finite dimensional case, we identify the class of monotone operators that admit a convex representation as the one consisting of intersections of maximal monotone operators and characterize the monotone operators that have a unique maximal monotone extension.

Key words: monotone operators, convex functions.
Mathematics Subject Classification (2000):47H05, 46B99, 47H17.

*This work has been partially supported by the Ministerio de Ciencia y Tecnología, project BEC2002-00642, and by the Departament d'Universitats, Recerca i Societat de la Informació, Direcció General de Recerca de la Generalitat de Catalunya, project 2001SGR-00162. This work has been completed during a visit of this author to IMPA, to which he is grateful for the support received.

†Partially supported by CNPq Grant 302748/2002-4 and by PRONEX-Optimization.

1 Introduction and Motivation

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c. from now on) proper convex function defined on a real Banach space $X \neq \{0\}$. Then its associated Fenchel subdifferential mapping $\partial f : X \rightrightarrows X^*$ is maximal monotone [14]. For $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$h(x, x^*) := f(x) + f^*(x^*), \tag{1}$$

f^* being the convex conjugate function of f, we have the Fenchel-Young inequality

$$h(x, x^*) \geq \langle x, x^* \rangle \, ,$$

$\langle \cdot, \cdot \rangle$ denoting the duality pairing between X and X^*. Moreover

$$x^* \in \partial f(x) \Leftrightarrow h(x, x^*) = \langle x, x^* \rangle \, . \tag{2}$$

Relation (2) shows that ∂f is fully determined by h.

Let A be an arbitrary maximal monotone subset of $X \times X^*$. Fitzpatrick [6] proved that the family of l.s.c. convex functions $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ satisfying

(i) $h(x, x^*) \geq \langle x, x^* \rangle$,

(ii) $(x, x^*) \in A \Leftrightarrow h(x, x^*) = \langle x, x^* \rangle$

is nonempty. His proof is constructive. He defined $\varphi_A : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi_A(x, x^*) := \sup_{(y, y^*) \in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \, . \tag{3}$$

and proved that this function has the above mentioned properties. Moreover, φ_A is the smallest function in the family. Note that φ_A characterizes (or represents) A. Indeed if A is maximal monotone, then

$$A = \{(x, x^*) \in X \times X^* \mid \varphi_A(x, x^*) = \langle x, x^* \rangle\} \, . \tag{4}$$

In a recent paper, Martínez-Legaz and Théra [8] rediscovered the Fitzpatrick function associated to maximal monotone operators and characterized the family

$$\{\varphi_A \mid A \subset X \times X^* \text{ is maximal monotone}\} \, .$$

In their study of enlargements of maximal monotone operators (see [4, 16]), Burachik and Svaiter [5] also rediscovered Fitzpatrick functions and studied the whole family of l.s.c. convex functions associated with a given maximal monotone operator A, i.e., those functions h satisfying (i) and (ii). They proved that this family is invariant under a suitable generalized conjugation operator and has a biggest element σ_A, which is characterized by

$$\sigma_A(x, x^*) = \text{clconv}(\pi + \delta_A)(x, x^*), \tag{5}$$

$\pi : X \times X^* \to \mathbb{R}$ and $\delta_A : X \times X^* \to \mathbb{R}\cup\{+\infty\}$ being the duality pairing (that is, $\pi(x, x^*) = \langle x, x^*\rangle$) and the indicator function of A (given by $\delta_A(x, x^*) = 0$ if $(x, x^*) \in A$, $+\infty$ if $(x, x^*) \notin A$), respectively, and clconv denoting l.s.c. convex envelope.

The purpose of this paper is to extend the representation of maximal monotone operators by l.s.c. convex functions to a larger class of monotone operators. We shall work in the framework of real Banach spaces. One of our main results is Theorem 31 (Section 5), which establishes that, in the finite dimensional case, the class of representable operators is the one consisting of the intersections of maximal monotone operators.

2 Basic Definitions and Results

Let Z be an arbitrary set. The *indicator function* of $Y \subset Z$, defined on Z, is $\delta_Y : Z \to \mathbb{R} \cup \{+\infty\}$,

$$\delta_Y(z) := \begin{cases} 0 & \text{if } z \in Y, \\ +\infty & \text{otherwise.} \end{cases}$$

Let W be another arbitrary set. Note that a multivalued operator $F : Z \rightrightarrows W$ is fully characterized by its *graph* $G(F)$,

$$G(F) := \{(z, w) \in Z \times W \mid w \in F(z)\}.$$

From now on, X is a real Banach space and X^* is its dual. The *ordered* duality product between X and X^* will be denoted by π,

$$\begin{array}{cccc} \pi & : & X \times X^* & \to & \mathbb{R}, \\ & & (x, x^*) & \mapsto & x^*(x). \end{array}$$

We shall use also the product notation $\langle \cdot, \cdot \rangle$,

$$\langle x, x^*\rangle := x^*(x).$$

A multivalued operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall x, y \in X, \;\; x^* \in T(x), \;\; y^* \in T(y).$$

The multivalued operator $T : X \rightrightarrows X^*$ is *maximal monotone* if it is monotone and, for any monotone $\widetilde{T} : X \rightrightarrows X^*$,

$$G(T) \subset G(\widetilde{T}) \Rightarrow T = \widetilde{T}.$$

An arbitrary $A \subset X \times X^*$ is *monotone* if

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall\, (x, x^*), (y, y^*) \in A.$$

The set A is *maximal monotone* if it is monotone and for any monotone $\widetilde{A} \subset X \times X^*$,

$$A \subset \widetilde{A} \Rightarrow A = \widetilde{A}.$$

Trivially, the operator $T : X \rightrightarrows X^*$ is monotone (resp. max. monotone) if and only if the set $G(T)$ is monotone (resp. max. monotone).

The Fitzpatrick function was defined originally for maximal monotone sets [6]. Its definition can be directly generalized for arbitrary subsets of $X \times X^*$.

Definition 1 *The Fitzpatrick function associated with* $A \subset X \times X^*$ *is* $\varphi_A : X \times X^* \to \overline{\mathbb{R}}$,

$$\begin{aligned}\varphi_A(x, x^*) &:= \sup_{(y,y^*)\in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \\ &= \sup_{(y,y^*)\in A} \Big(\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle \Big).\end{aligned}$$

Note that the Fitzpatrick function is convex and l.s.c.. An equivalent expression for $\varphi_A(x, x^*)$ is

$$\varphi_A(x, x^*) = \langle x, x^* \rangle - \inf_{(y,y^*)\in A} \langle x - y, x^* - y^* \rangle .$$

Given f, an extended real function on X, its Legendre conjugate, denoted by f^*, is the extended real function on X^* defined by

$$f^*(x^*) := \sup_{x\in X} \Big(\langle x, x^* \rangle - f(x) \Big).$$

The *effective domain* of f is the set where $f < +\infty$,

$$\mathrm{ed}(f) := \{x \in X \mid f(x) < +\infty\}.$$

The largest convex minorant of f will be denoted by $\operatorname{conv} f$:

$$\operatorname{conv} f := \sup_{\substack{h:X\to\bar{\mathbb{R}},\ h\le f \\ h \text{ convex}}} h.$$

The closed convex envelope of f, that is, the largest l.s.c. convex minorant of f, will be denoted $\operatorname{clconv} f$:

$$\operatorname{clconv} f := \sup_{\substack{h:X\to\bar{\mathbb{R}},\ h\le f \\ h \text{ convex, l.s.c.}}} h.$$

Given $A \subset X \times X^*$, define $s_A, \sigma_A : X \times X^* \to \bar{\mathbb{R}}$ by

$$s_A := \operatorname{conv}(\pi + \delta_A), \quad \sigma_A := \operatorname{clconv}(\pi + \delta_A). \tag{6}$$

These functions where defined and studied in [5] for A maximal monotone.

If h is an extended real function in $X \times X^*$, then h^* is defined in $(X \times X^*)^* = X^* \times X^{**}$. We will consider the canonical injection (if X is reflexive, canonical identification) of X in X^{**}, which assigns to $x \in X$ the functional

$$X^* \ni x^* \mapsto x^*(x).$$

With this convention, if we take $(x, x^*) \in X \times X^*$ and $A \subset X \times X^*$ then $(x^*, x) \in X^* \times X^{**}$ and

$$\begin{aligned}
(\pi + \delta_A)^*(x^*, x) &= \sup_{(y,y^*) \in X \times X^*} \Big(\langle y, x^* \rangle + \langle x, y^* \rangle - (\pi + \delta_A)(y, y^*) \Big) \\
&= \sup_{(y,y^*) \in A} \Big(\langle y, x^* \rangle + \langle x, y^* \rangle - \langle y, y^* \rangle \Big) \\
&= \sup_{(y,y^*) \in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle .
\end{aligned}$$

Hence, for any $(x, x^*) \in X \times X^*$,

$$\begin{aligned}
\varphi_A(x, x^*) &= (\pi + \delta_A)^*(x^*, x) \\
&= (s_A)^*(x^*, x) \\
&= (\sigma_A)^*(x^*, x).
\end{aligned} \tag{7}$$

Given $h : X \times X^* \to \overline{\mathbb{R}}$, define $\mathcal{J}h : X \times X^* \to \overline{\mathbb{R}}$,

$$\mathcal{J}h(x, x^*) := h^*(x^*, x). \tag{8}$$

Now, (7) can be synthetically rewritten as:

$$\varphi_A = \mathcal{J}(\pi + \delta_A) = \mathcal{J}(s_A) = \mathcal{J}(\sigma_A). \tag{9}$$

Trivially, from Definition 1,

$$\varphi_A(x, x^*) \geq \langle x, x^* \rangle , \quad \forall (x, x^*) \in A. \tag{10}$$

Moreover, monotonicity may be characterized by Fitzpatrick functions.

Proposition 2 *A set $A \subset X \times X^*$ is monotone if and only if*

$$\varphi_A(x, x^*) = \langle x, x^* \rangle , \quad \forall (x, x^*) \in A.$$

or equivalently,

$$\varphi_A \leq \pi + \delta_A.$$

Proof. Equivalence between the two last conditions follows from (10). Monotonicity of A means that, for any $(x, x^*) \in A$,

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in A.$$

This condition is trivially equivalent to

$$\varphi_A(x, x^*) \leq \langle x, x^* \rangle ,$$

for all $(x, x^*) \in A$. ■

Proposition 3 *A set $A \subset X \times X^*$ is monotone if and only if $\varphi_A \leq \sigma_A$.*

Proof. If A is monotone, then, by Proposition 2, $\varphi_A \leq \pi + \delta_A$. Since φ_A is convex and l.s.c., we conclude that $\varphi_A \leq \sigma_A$.

Assume now that $\varphi_A \leq \sigma_A$. Since $\sigma_A \leq \pi + \delta_A$, we get $\varphi_A \leq \pi + \delta_A$. Now, using Proposition 2 the conclusion follows. ■

Observe that, according to (9), the condition $\varphi_A \leq \sigma_A$ may be written as

$$\mathcal{J}(\sigma_A) \leq \sigma_A.$$

The preceding proposition suggests the introduction of the class of functions $h : X \times X^* \to \mathbb{R} \cup \{\infty\}$ which are convex, l.s.c., and satisfy the condition

$$\mathcal{J}\, h \leq h.$$

The above condition is an abbreviation for

$$h^*(x^*, x) \leq h(x, x^*), \quad \forall (x, x^*) \in X \times X^*,$$

which can be expressed more symmetrically as

$$\langle x, y^* \rangle + \langle y, x^* \rangle \leq h(x, x^*) + h(y, y^*) \qquad \forall (x, x^*), (y, y^*) \in X \times X^*.$$

We call this class $\Delta(X)$ or Δ. Formally we have:

$$\begin{aligned} \Delta \quad := \quad & \left\{ h : X \times X^* \to \mathbb{R} \cup \{+\infty\} \;\middle|\; \begin{array}{l} h \text{ is convex and l.s.c.}, \forall (x, x^*), (y, y^*) \\ \langle x, y^* \rangle + \langle y, x^* \rangle \leq h(x, x^*) + h(y, y^*) \end{array} \right\} \\ = \quad & \left\{ h : X \times X^* \to \mathbb{R} \cup \{+\infty\} \;\middle|\; \begin{array}{l} h \text{ is convex and l.s.c.}, \\ \mathcal{J} h \leq h \end{array} \right\}. \end{aligned}$$

If A is monotone, then (by Proposition 3) σ_A belong to this class. Note that if $h \in \Delta$, k is convex, l.s.c., and $k \geq h$ then $k \in \Delta$.

Take $h \in \Delta$. Setting $(x, x^*) = (y, y^*)$ here one deduces the inequality $h \geq \pi$, which, as we will see in the next section, is crucial for a convex function h to represent a monotone set.

We end this section with a result due to Fitzpatrick [6].

Theorem 4 *If $A \subset X \times X^*$ is maximal monotone then, for all $(x, x^*) \in X \times X^*$,*

$$\begin{array}{l} \varphi_A(x, x^*) \geq \langle x, x^* \rangle, \\ (x, x^*) \in A \Leftrightarrow \varphi_A(x, x^*) = \langle x, x^* \rangle. \end{array}$$

Moreover, φ_A is the smallest convex function with these properties.

3 Representable Monotone Operators

First we characterize monotone operators in terms of convex functions and the duality product.

Theorem 5 *Let $A \subset X \times X^*$. Then A is monotone if and only if there exists a convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that, for all $(x, x^*) \in X \times X^*$,*

$$\begin{aligned} &h(x, x^*) \geq \langle x, x^* \rangle \,, \\ &(x, x^*) \in A \Rightarrow h(x, x^*) = \langle x, x^* \rangle \,. \end{aligned} \tag{11}$$

Moreover, h may be taken l.s.c. also (closed convex).

Proof. Suppose that A is monotone. Using Zorn Lemma we conclude that there exists a maximal monotone $\widetilde{A} \subset X \times X^*$ such that $A \subset \widetilde{A}$. Taking $h = \varphi_{\widetilde{A}}$ and using Theorem 4 we conclude that (11) holds. By its definition, $\varphi_{\widetilde{A}}$ is l.s.c..

Assume now that there exists a convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that (11) holds. Let $(x, x^*), (y, y^*) \in A$. By the second condition in (11),

$$h(x, x^*) = \langle x, x^* \rangle \,, \quad h(y, y^*) = \langle y, y^* \rangle \,.$$

Using the convexity of h we get

$$h\left(\frac{1}{2}(x + y), \frac{1}{2}(x^* + y^*)\right) \leq \frac{1}{2} \langle x, x^* \rangle + \frac{1}{2} \langle y, y^* \rangle \,.$$

Now, using the first condition in (11) and the above inequality, we get

$$\frac{1}{4} \langle x + y, x^* + y^* \rangle \leq \frac{1}{2} \langle x, x^* \rangle + \frac{1}{2} \langle y, y^* \rangle \,,$$

which is equivalent to $\langle x - y, x^* - y^* \rangle \geq 0$. So, A is monotone. ■

Remark 6 *The function h may also be taken in Δ, by choosing $h = \sigma_A$. Indeed, σ_A is trivially convex and l.s.c.. Using Proposition 3 we conclude that if A is monotone then $\varphi_A \leq \sigma_A$, which by our choice of h and by (9) is equivalent to $\mathcal{J} h \leq h$, and so $h = \sigma_A \in \Delta$.*

Corollary 7 *For any set $A \subset X \times X^*$, the following conditions are equivalent:*

1. *A is monotone.*
2. *$\mathcal{J}(\sigma_A) \leq \sigma_A$.*
3. *σ_A majorizes π.*
4. *s_A majorizes π.*

Proof. By Proposition 3 and (9), items 1 and 2 are equivalent.

If A is monotone, then by the preceding theorem there exists a l.s.c. convex function h which coincides with π on A and majorizes π. In particular, $h \leq \pi + \delta_A$. Therefore, using also definition (6), we have $\pi \leq h \leq \sigma_A$. This proves that item 1 implies 3. The implication 3⇒4 is trivial. For 4⇒1, observe that item 4 together with (6) implies that s_A coincides with π on A, and then use the preceding theorem again. ■

A monotone set $A \subset X \times X^*$ is *representable* if there exists a l.s.c. convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $h \geq \pi$ and

$$A = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

We will use the notation $\mathcal{R}$ for the family of monotone representable subsets of $X \times X^*$. Given $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, define

$$L(h) := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\} \tag{12}$$

and

$$b\,(h) := \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^* \rangle\}.$$

Observe that $L(h) \subset b\,(h)$. If $h \geq \pi$, then $L(h) = b\,(h)$. Define also

$$\mathcal{F} := \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} \mid h \text{ is convex and l.s.c., } h \geq \pi\}. \tag{13}$$

Now we have a synthetic expression for $\mathcal{R}$:

$$\mathcal{R} = \{L(h) \mid h \in \mathcal{F}\}. \tag{14}$$

By Theorem 4, if $A \subset X \times X^*$ is maximal monotone then its Fitzpatrick representation φ_A belongs to $\mathcal{F}$ and one has $A = L(\varphi_A)$. Therefore all maximal monotone operators are representable. Monotone *representable* operators share some properties with maximal monotone operators:

Proposition 8 *Let A be a monotone representable set. Then*

1. *$A(x) := \{x^* \in X^* \mid (x, x^*) \in A\}$ is closed and convex, for all $x \in X$.*
2. *$A^{-1}(x^*) := \{x \in X \mid (x, x^*) \in A\}$ is closed and convex, for all $x^* \in X^*$.*
3. *A is closed. Moreover, let $\{(x_k, x_k^*)\}_{k \in \mathbb{N}}$ be a sequence in A.*

   a *If $x_k \xrightarrow{w} x$ and $x_k^* \to x^*$, then $(x, x^*) \in A$,*

   b *If $x_k \to x$ and $x_k^* \xrightarrow{w} x^*$, then $(x, x^*) \in A$.*

Proof. There exists some l.s.c. convex $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $h \geq \pi$ and $A = L(h)$. Then

$$A = \{(x, x^*) \in X \times X^* \mid h(x, x^*) - \langle x, x^* \rangle \leq 0\}.$$

Since the function $\tilde{h} : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{h}(x, x^*) = h(x, x^*) - \langle x, x^* \rangle$$

is l.s.c. and convex in x and x^* *separately*, items 1 and 2 hold.

To prove 3, observe that l.s.c. convex functions are weakly l.s.c.. If $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ is a sequence as in item 3.**a** or 3.**b**, then $\{x_k\}$, $\{x_k^*\}$ are bounded, $\langle x_k, x_k^*\rangle$ converges to $\langle x, x^*\rangle$ and

$$\begin{aligned} h(x,x^*) - \langle x, x^*\rangle &\leq \liminf_{k\to\infty} h(x_k, x_k^*) - \langle x, x^*\rangle \\ &= \liminf_{k\to\infty} h(x_k, x_k^*) + \lim_{k\to\infty}\Big(- \langle x_k, x_k^*\rangle\Big) \\ &= \liminf_{k\to\infty}\Big(h(x_k, x_k^*) - \langle x_k, x_k^*\rangle\Big) \\ &= 0. \end{aligned}$$

■

An *extension* of $A \subset X \times X^*$ is any $B \subset X \times X^*$ such that $A \subset B$. The family of representable monotone *extensions* of A will be called $\mathcal{R}(A)$. Formally,

$$\mathcal{R}(A) := \{R \in \mathcal{R} \mid A \subset R\}\,. \tag{15}$$

The family $\mathcal{R}(A)$ is determined by a subset of $\mathcal{F}$, namely, if we define

$$\mathcal{F}\,(A) := \{h \in \mathcal{F} \mid (x, x^*) \in A \Rightarrow h(x, x^*) = \langle x, x^*\rangle\}\,, \tag{16}$$

or, equivalently,

$$\mathcal{F}\,(A) = \{h \in \mathcal{F} \mid A \subset L(h)\}\,,$$

we have

$$\mathcal{R}(A) = \{L(h) \mid h \in \mathcal{F}\,(A)\}\,. \tag{17}$$

Note that $\mathcal{R}(\emptyset) = \mathcal{R}$ and $\mathcal{F}\,(\emptyset) = \mathcal{F}$. If A is *not* monotone, then $\mathcal{R}(A) = \emptyset$ and $\mathcal{F}\,(A) = \emptyset$. Some properties of the family $\mathcal{F}\,(A)$, in the case where A is maximal monotone, were studied in [5]. Even when A is monotone but *not* maximal monotone, some properties remain.

Proposition 9 *Let $A \subset X \times X^*$. Then A is monotone if and only if $\mathcal{F}\,(A) \neq \emptyset$. Moreover*

1. *For any $h_1, h_2 \in \mathcal{F}\,(A)$ and $t \in [0, 1]$,*

$$th_1 + (1-t)h_2 \in \mathcal{F}\,(A)\,.$$

2. *If $\{h_i\}_{i\in I} \subset \mathcal{F}\,(A)$, with $I \neq \emptyset$, then*

$$\sup_{i\in I} h_i \in \mathcal{F}\,(A)\,.$$

Proof. The first part of this proposition follows from Theorem 5. The proofs of items 1 and 2 are trivial. ■

Corollary 10 *Let $A \subset X \times X^*$. Then A is monotone if and only if $\mathcal{R}(A) \neq \emptyset$. If $\{R_i\}_{i\in I}$ is a nonempty family in $\mathcal{R}(A)$, then $\bigcap_{i\in I} R_i \in \mathcal{R}(A)$.*

Proof. The first assertion follows from Proposition 9 and (17). Use again (17) to conclude that, for each $i \in I$, there exists some $h_i \in \mathcal{F}(A)$ such that $R_i = L(h_i)$. Define

$$R := \bigcap_{i \in I} R_i, \qquad h := \sup_{i \in I} h_i.$$

Note that $h_i \in \mathcal{F}(R_i) \subset \mathcal{F}(R)$ for all $i \in I$. Hence, $h \in \mathcal{F}(R)$, that is, $R \subset L(h)$. If $(x, x^*) \in L(h)$ then $h_i(x, x^*) \leq \langle x, x^* \rangle$ for all $i \in I$. Since $h_i(x, x^*) \geq \langle x, x^* \rangle$, we conclude that, for all $i \in I$, $h_i(x, x^*) = \langle x, x^* \rangle$, so $(x, x^*) \in L(h_i) = R_i$. Hence $(x, x^*) \in R$. Altogether, $R = L(h)$ and, since $A \subset R$, $\mathcal{F}(R) \subset \mathcal{F}(A)$, so that $h \in \mathcal{F}(A)$; this shows that $R \in \mathcal{R}(A)$. ■

Let $A \subset X \times X^*$ be monotone. From Corollary 10 it follows that the intersection of all elements of the (nonempty) family $\mathcal{R}(A)$ still belongs to this family. This is the smallest representable extension of A and will be called the *monotone representable closure* of A, denoted $\mathrm{cl}_{\mathcal{R}}(A)$,

$$\mathrm{cl}_{\mathcal{R}}(A) := \bigcap_{R \in \mathcal{R}(A)} R. \tag{18}$$

Representable extensions of a monotone A can be seen as *outer* approximations of A by representable monotone sets. From this point of view, $\mathrm{cl}_{\mathcal{R}}(A)$ is the best (smallest) outer approximation of A by a representable monotone set.

Lemma 11 *Let $A \subset X \times X^*$ be monotone. If $A \subset B \subset \mathrm{cl}_{\mathcal{R}}(A)$ then*

$$\mathcal{R}(A) = \mathcal{R}(B), \qquad \mathcal{F}(A) = \mathcal{F}(B)$$

and

$$\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(B).$$

Some other properties of $\mathrm{cl}_{\mathcal{R}}$ are exposed below:

Proposition 12 *Let $A, B \subset X \times X^*$ be monotone. Then*

1. $A \subset \mathrm{cl}_{\mathcal{R}}(A)$.
2. $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{R}$.
3. $A \in \mathcal{R} \Leftrightarrow \mathrm{cl}_{\mathcal{R}}(A) = A$.
4. $\mathrm{cl}_{\mathcal{R}}(\mathrm{cl}_{\mathcal{R}}(A)) = \mathrm{cl}_{\mathcal{R}}(A)$.
5. $A \subset B \Rightarrow \mathrm{cl}_{\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(B)$.
6. $A \subset B, B \in \mathcal{R} \Rightarrow \mathrm{cl}_{\mathcal{R}}(A) \subset B$.

Let $\mathcal{M}$ be the family of monotone subsets of $X \times X^*$,

$$\mathcal{M} := \{A \subset X \times X^* \mid A \text{ is monotone}\}.$$

Then $\mathcal{R} \subset \mathcal{M}$ and, by Proposition 12, $\mathrm{cl}_{\mathcal{R}} : \mathcal{M} \to \mathcal{R}$ is a closure operator in $\mathcal{M}$ for which $\mathcal{R}$ is the family of "closed" sets.

Now we shall obtain another characterization of $\mathrm{cl}_{\mathcal{R}}(A)$.

Theorem 13 *For any set $A \subset X \times X^*$, the following conditions are equivalent:*

1. *A is monotone.*
2. *$\sigma_A \in \mathcal{F}(A)$.*
3. *$\sigma_A = \sup_{h \in \mathcal{F}(A)}$.*

Hence, if A is monotone then

$$\mathrm{cl}_{\mathcal{R}}(A) = L(\sigma_A).$$

Proof. By its definition, $\sigma_A \leq \pi + \delta_A$ (and is convex, l.s.c.). So, $\sigma_A \geq \pi$ if and only if $\sigma_A \in \mathcal{F}(A)$. Using this fact and the equivalence of items 1 and 3 in Corollary 7 we conclude that items 1 and 2 above are equivalent.

To prove $2 \Longrightarrow 3$, it suffices to observe that every $h \in \mathcal{F}(A)$ is convex, l.s.c. and satisfies $h \leq \pi + \delta_A$, whence, by (6), $h \leq \sigma_A$. Combining this with item 2 we get item 3.

The implication $3 \Longrightarrow 2$ follows from Proposition 9.

Finally, if A is monotone then, using (18) and (17), we get

$$\begin{aligned} \mathrm{cl}_{\mathcal{R}}(A) &= \bigcap_{R \in \mathcal{R}(A)} R = \bigcap_{h \in \mathcal{F}(A)} L(h) = \bigcap_{h \in \mathcal{F}(A)} b(h) = b\left(\sup_{h \in \mathcal{F}(A)} h\right) = b(\sigma_A) \\ &= L(\sigma_A). \end{aligned}$$

■

Corollary 14 *Let $A \subset X \times X^*$ be monotone. Then $\sigma_A = \sigma_{\mathrm{cl}_{\mathcal{R}}(A)}$.*

Now we give another characterization of the functions belonging to $\mathcal{F}(A)$:

Proposition 15 *Let $A \subset X \times X^*$ be monotone and $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c. Then the following conditions are equivalent:*

1. *$h \in \mathcal{F}(A)$.*
2. *$\pi \leq h \leq \sigma_A$.*
3. *$\pi \leq h \leq \pi + \delta_A$.*

Lemma 16 *Let $A \subset X \times X^*$ be monotone. Then*

$$\mathrm{cl}_{\mathcal{R}}(A) \subset L(\varphi_A).$$

Proof. By Propositions 2 and 3 one has

$$\varphi_A(x, x^*) = \langle x, x^* \rangle, \forall (x, x^*) \in A$$

and

$$\sigma_A \geq \varphi_A, \tag{19}$$

respectively.

Let $(x, x^*) \in X \times X^*$. By Fenchel-Young inequality,

$$\sigma_A(x, x^*) + (\sigma_A)^*(x^*, x) \geq \langle (x, x^*), (x^*, x) \rangle.$$

Using now (7), we get

$$\sigma_A(x, x^*) + \varphi_A(x, x^*) \geq 2 \langle x, x^* \rangle .$$

By the preceding inequality and Theorem 13,

$$\varphi_A(x, x^*) < \langle x, x^* \rangle \Rightarrow \sigma_A(x, x^*) > \langle x, x^* \rangle \Rightarrow (x, x^*) \notin \mathrm{cl}_{\mathcal{R}}(A),$$

on the other hand, by (19) and Theorem 13,

$$\varphi_A(x, x^*) > \langle x, x^* \rangle \Rightarrow \sigma_A(x, x^*) > \langle x, x^* \rangle \Rightarrow (x, x^*) \notin \mathrm{cl}_{\mathcal{R}}(A).$$

Therefore $(x, x^*) \in \mathrm{cl}_{\mathcal{R}}(A) \Rightarrow \varphi_A(x, x^*) = \langle x, x^* \rangle$. ■

3.1 Translations

Given any $(x_0, x_0^*) \in X \times X^*$, define $\tau_{(x_0,x_0^*)} : X \times X^* \to X \times X^*$ by

$$\tau_{(x_0,x_0^*)}(x, x^*) := (x - x_0, x^* - x_0^*).$$

For any $A \subset X \times X^*$, one has

$$\begin{aligned} \tau_{(x_0,x_0^*)}(A) &= \{(x - x_0, x^* - x_0^*) \mid (x, x^*) \in A\} \\ &= A - \{(x_0, x_0^*)\} . \end{aligned}$$

Obviously, $\tau_{(x_0,x_0^*)}$ preserves monotonicity and maximal monotonicity and is continuous and invertible, with

$$\left(\tau_{(x_0,x_0^*)}\right)^{-1} = \tau_{-(x_0,x_0^*)}.$$

Given any $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, define $\mathcal{T}_{(x_0,x_0^*)}(h) : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{T}_{(x_0,x_0^*)}(h) := (h - \pi) \circ (\tau_{(x_0,x_0^*)})^{-1} + \pi. \tag{20}$$

Equivalently,

$$\mathcal{T}_{(x_0,x_0^*)}(h)(x, x^*) = h(x + x_0, x^* + x_0^*) - [\langle x, x_0^* \rangle + \langle x_0, x^* \rangle + \langle x_0, x_0^* \rangle] .$$

Hence, $\mathcal{T}$ maps convex (l.s.c.) functions into convex (resp., l.s.c.) functions. By (20),

$$\mathcal{T}_{(x_0,x_0^*)}(h) - \pi = (h - \pi) \circ (\tau_{(x_0,x_0^*)})^{-1},$$

and so we conclude that $\mathcal{T}_{(x_0,x_0^*)}$ is a bijection from $\mathcal{F}$ onto itself, which preserves the pointwise partial ordering. The above equality also proves, for any $A \subset X \times X^*$, the equivalence

$$h \in \mathcal{F}(A) \Leftrightarrow \mathcal{T}_{(x_0,x_0^*)} h \in \mathcal{F}(\tau_{(x_0,x_0^*)} A).$$

Direct calculation yields

$$\mathcal{T}_{(x_0,x_0^*)}\varphi_A = \varphi_{\tau_{(x_0,x_0^*)}(A)}.$$

As $\mathcal{T}_{(x_0,x_0^*)}$ is an order preserving bijection from $\mathcal{F}(A)$ onto $\mathcal{F}(\tau_{(x_0,x_0^*)}A)$, we also have

$$\mathcal{T}_{(x_0,x_0^*)}\sigma_A = \sigma_{\tau_{(x_0,x_0^*)}(A)}.$$

It is trivial to verify that, for any $h : X \times X^* \to \mathbb{R}\cup\{+\infty\}$,

$$\begin{aligned} L(\mathcal{T}_{(x_0,x_0^*)}h) &= \tau_{(x_0,x_0^*)}L(h) \\ b(\mathcal{T}_{(x_0,x_0^*)}h) &= \tau_{(x_0,x_0^*)}b(h). \end{aligned}$$

Proposition 17 *Let $A \subset X \times X^*$ be monotone. Then*

$$\mathrm{cl}_{\mathcal{R}}(\tau_{(x_0,x_0^*)}A) = \tau_{(x_0,x_0^*)}\left(\mathrm{cl}_{\mathcal{R}}(A)\right).$$

4 A Polarity Approach to Monotonicity

In this section we shall approach monotonicity from the point of view of the classical notion of polarity [1]. Let us consider the reflexive and symmetric binary relation μ on $X \times X^*$ given by

$$(x,x^*)\,\mu\,(y,y^*) \Longleftrightarrow \langle x-y, x^*-y^*\rangle \geq 0. \tag{21}$$

Definition 18 *The* monotone polar *of $A \subset X \times X^*$ is*

$$A^\mu := \{(x,x^*) \mid (x,x^*)\,\mu\,(y,y^*),\ \forall (y,y^*) \in A\}.$$

The following properties follow from the fact that the mapping $A \longmapsto A^\mu$ is a polarity [1]:

1. $(\cup_{i\in I}A_i)^\mu = \cap_{i\in I}A^\mu$, (with the convention $\cap_{i\in\emptyset}B_i = X \times X^*$).
2. $A \subset A^{\mu\mu}$.
3. $A^{\mu\mu\mu} = A^\mu$.
4. $A \subset B \Rightarrow B^\mu \subset A^\mu$.
5. $\emptyset^\mu = X \times X^*$.

Our next proposition, which has an obvious proof, gives a useful interpretation of the monotone polar of a monotone set:

Proposition 19 *Let A be monotone. Then $(x,x^*) \in A^\mu$ if and only if $A \cup \{(x,x^*)\}$ is monotone.*

From the preceding proposition it follows that, for this specific polarity, we have

$$(X \times X^*)^\mu = \emptyset.$$

The closure operator induced by the polarity μ is the mapping $A \longmapsto A^{\mu\mu}$. We call $A^{\mu\mu}$ the μ-closure of A and say that the set $A \subset X \times X^*$ is μ-closed if $A^{\mu\mu} = A$. It follows from item 3 above that the family of μ-closed sets is equal to the family of polars $\{B^\mu \mid B \subset X \times X^*\}$.

Remark 20 *For any $A \subset X \times X^*$, $A^{\mu\mu}$ is the smallest μ-closed set which contains A.*

Monotonicity and maximal monotonicity can be easily characterized in terms of the polarity $A \longmapsto A^\mu$:

Proposition 21 *Let $A \subset X \times X^*$. The following conditions are equivalent:*

1. *A is monotone.*
2. *$A \subset A^\mu$.*
3. *$A^{\mu\mu} \subset A^\mu$.*
4. *$A^{\mu\mu}$ is monotone.*

Moreover, A is maximal monotone if and only if $A = A^\mu$.

Proof. The equivalence between items 1 and 2 is obvious. Since polarity reverses inclusions, by taking polars in item 2 one gets 3. As $A^{\mu\mu\mu} = A^{\mu}$, items 3 and 4 are nothing else than items 2 and 1, respectively, with A replaced by $A^{\mu\mu}$, and so they are equivalent. The implication $4 \Longrightarrow 1$ follows from the inclusion $A \subset A^{\mu\mu}$. The last statement follows from the equivalence between items 1 and 2 and Proposition 19. ■

From the above characterization, it trivially follows that maximal monotone operators are μ-closed.

Maximal monotone operators have many nice properties [2], [10], [15]. Any monotone $A \subset X \times X^*$ can be extended to a maximal monotone $B \subset X \times X^*$. For $A \subset X \times X^*$, let $\mathbf{M}(A)$ denote the set of maximal monotone extensions of A,

$$\mathbf{M}(A) := \{B \subset X \times X^* \mid B \text{ is maximal monotone, } A \subset B\}.$$

Obviously, $\mathbf{M}(A) \neq \emptyset$ if and only if A is monotone. Note that $\mathbf{M}(A) \subset \mathcal{R}(A)$.

Proposition 22 *Let $A \subset X \times X^*$ be monotone. Then*

1. *$A^\mu = \bigcup_{B \in \mathbf{M}(A)} B$.*
2. *$A^{\mu\mu} = \bigcap_{B \in \mathbf{M}(A)} B$.*
3. *$A^{\mu\mu} \in \mathcal{R}(A)$.*

Proof. Item 1 follows from Proposition 19 and the above mentioned fact that any monotone operator has a maximal monotone extension. Item 2 follows by taking monotone polars in item 1 and using the characterization of maximal operators given in Proposition 21. Finally, item 3 is a consequence of 2 and the inclusion $\mathbf{M}(A) \subset \mathcal{R}(A)$. ■

Corollary 23 *Let $A \subset X \times X^*$. Then A is monotone if and only if A^μ contains a maximal monotone set.*

Proof. The "only if" part is an immediate consequence of Proposition 22. Conversely, if there is a maximal monotone set $T \subset A^\mu$ then, using Proposition 21, we get $A \subset A^{\mu\mu} \subset T^\mu = T \subset A^\mu$; hence, again by Proposition 21, A is monotone. ■

As a consequence of item 3 of Proposition 22, if A is monotone then

$$\mathrm{cl}_{\mathcal{R}}(A) \subset A^{\mu\mu}.$$

In Theorem 31 (Section 5) we establish that, in finite dimensional spaces, the above inclusion holds as an equality. This is one of the main results of this paper.

Observe that the Fitzpatrick function is also linked to monotone polarity; indeed, for any $A \subset X \times X^*$ one has

$$b(\varphi_A) = A^\mu. \tag{22}$$

In particular,

$$b(\varphi_{A^\mu}) = A^{\mu\mu}. \tag{23}$$

For a nonmonotone set A, φ_A cannot be the Fitzpatrick function of a monotone set, as the next proposition states.

Proposition 24 *Let $A \subset X \times X^*$. If there exists a monotone set $B \subset X \times X^*$ such that $\varphi_A = \varphi_B$ then A is monotone.*

Proof. By (22) we have $A^\mu = b(\varphi_A) = b(\varphi_B) = B^\mu$; hence, using Proposition 21, we get $A \subset A^{\mu\mu} = B^{\mu\mu} \subset B^\mu = A^\mu$. Again by Proposition 21, A is monotone. ■

For a monotone but not maximal monotone A, one may have $\varphi_A \notin \mathcal{F}(A)$. A trivial example is $A = \{(0,0)\}$. In the last section we will characterize, in the finite dimensional case, those monotone operators A that satisfy $\varphi_A \in \mathcal{F}(A)$.

Proposition 25 *Let $A \subset X \times X^*$ be monotone. Then*

$$\varphi_{A^\mu} \in \mathcal{F}(A^{\mu\mu}), \quad L(\ \varphi_{A^\mu}) = A^{\mu\mu}.$$

Proof. Notice that, for any $B \subset X \times X^*$, φ_B majorizes π on B. Hence, as item 3 of Proposition 21, Definition 1, and (23) imply that $\varphi_{A^\mu} \geq \varphi_{A^{\mu\mu}} = \varphi_{b(\varphi_{A^\mu})}$, it follows that φ_{A^μ} majorizes π on $b(\varphi_{A^\mu})$. Since φ_{A^μ} strictly majorizes π on the complement of this set, we conclude that $\varphi_{A^\mu} \geq \pi$, so $\varphi_{A^\mu} \in \mathcal{F}$.

Therefore, $L(\varphi_{A^\mu}) = b(\varphi_{A^\mu})$. Using this equality, the fact that $A \subset A^{\mu\mu}$ and (23) again the conclusions follow. ■

Let $A \subset X \times X^*$ be monotone. Since, in this case, $A^\mu = \cup_{B\in\mathbf{M}(A)} B$, we also have

$$\varphi_{A^\mu} = \sup_{B\in\mathbf{M}(A)} \varphi_B.$$

The monotone relation (21) is preserved by translations, that is, for any $(x, x^*), (y, y^*), (z, z^*) \in X \times X^*$ one has

$$(y, y^*)\, \mu\, (z, z^*) \iff \tau_{(x,x^*)}(y, y^*)\ \mu\ \tau_{(x,x^*)}(z, z^*).$$

Therefore, for any $A \subset X \times X^*$ one has

$$\left(\tau_{(x,x^*)}(A)\right)^\mu = \tau_{(x,x^*)}(A^\mu)$$

and

$$\left(\tau_{(x,x^*)}(A)\right)^{\mu\mu} = \tau_{(x,x^*)}(A^{\mu\mu}).$$

5 A Characterization of Representable Monotone Operators in the Finite Dimensional Case

The aim of this section is to prove that, in the finite dimensional case, the representable closure of a *monotone* set coincides with its μ-closure. This result will be useful in Section 6, where we shall characterize monotone sets that admit only one maximal monotone extension.

To simplify the proof, we shall first study the origin of $X \times X^*$ and then extend the results to arbitrary points.

Let N be the set of points in $X \times X^*$ that are *not* in monotone relation with the origin $(0, 0)$:

$$N := \{(x, x^*) \in X \times X^* \mid \langle x, x^*\rangle < 0\}.$$

Then, for $A \subset X \times X^*$,

$$(0, 0) \in A^{\mu\mu} \iff N \cap A^\mu = \emptyset.$$

Proposition 26 *Let $A \subset X \times X^*$. If $\varphi_A(0, 0) < 0$ and*

$$N \cap \mathrm{ed}(\varphi_A) \neq \emptyset,$$

then $(0, 0) \notin A^{\mu\mu}$.

Proof. We shall produce a point in $N \cup A^\mu$ using the proposition's assumptions.

Take $(x, x^*) \in N \cap \mathrm{ed}(\varphi_A)$ and define, for $t \in \mathbb{R}$,

$$\begin{aligned}(x_t, x_t^*) &:= t\,(x, x^*) + (1-t)\,(0, 0)\\ &= (tx, tx^*).\end{aligned}$$

Clearly, for $t > 0$,

$$\langle x_t, x_t^* \rangle = t^2 \langle x, x^* \rangle < 0.$$

So, $(x_t, x_{t^*}) \in N$ for any $t > 0$. As φ_A is convex, for $t \in [0,1]$ one has

$$\varphi_A(x_t, x_t^*) \le t\,\varphi_A(x, x^*) + (1-t)\,\varphi_A(0,0).$$

Since $\varphi_A(0,0) < 0$ and $\varphi_A(x, x^*) < +\infty$, for $\bar{t} \in (0,1]$ small enough, one has

$$\bar{t}\,\varphi_A(x, x^*) + (1-\bar{t})\,\varphi_A(0,0) < \bar{t}^{\,2} \langle x, x^* \rangle\,.$$

Altogether,

$$\varphi_A(x_{\bar{t}}, x_{\bar{t}}^*) < \left\langle x_{\bar{t}}, x_{\bar{t}}^* \right\rangle\,.$$

Using (22) and the above inequality one has $(x_{\bar{t}}, x_{\bar{t}}^*) \in A^{\mu}$. So, $(x_{\bar{t}}, x_{\bar{t}}^*) \in N \cap A^{\mu}$ and hence $(0,0) \notin A^{\mu\mu}$. ■

Lemma 27 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. If $\varphi_A(0,0) < 0$, then $(0,0) \notin A^{\mu\mu}$.*

Proof. If $A = \emptyset$ the lemma holds trivially. So, we assume $A \neq \emptyset$.
Take any $(x_0, x_0^*) \in A$. Since $\varphi_A(0,0) < 0$,

$$\langle x_0, x_0^* \rangle \ge -\varphi_A(0,0) > 0\,.$$

In particular, $x_0 \neq 0$. Define

$$\begin{aligned} C_0 \quad &:= \quad \{(\lambda x_0, x^*) \in X \times X^* \mid \lambda > 0, \langle x_0, x^* \rangle < 0\} \\ &= \quad \{\lambda x_0 \mid \lambda > 0\} \times \{x^* \in X^* \mid \langle x_0, x^* \rangle < 0\}. \end{aligned}$$

Note that $C_0 \subset N$. If $C_0 \cap \mathrm{ed}(\varphi_A)$ is nonempty, then we may apply Proposition 26 to conclude that $(0,0) \notin A^{\mu\mu}$. So, assume that

$$C_0 \cap \mathrm{ed}(\varphi_A) = \emptyset.$$

As C_0 and $\mathrm{ed}(\varphi_A)$ are nonempty convex sets and X is finite dimensional, C_0 and $\mathrm{ed}(\varphi_A)$ can be *properly* separated by a hyperplane [13, Theorem 13.3]. This means that there exist $z^* \in X^*$, $z \in X$ and $\beta \in \mathbb{R}$ such that:

1. For any $(x, x^*) \in C_0$ and $(y, y^*) \in \mathrm{ed}(\varphi_A)$,

$$\langle x, z^* \rangle + \langle z, x^* \rangle \le \beta \le \langle y, z^* \rangle + \langle z, y^* \rangle. \tag{24}$$

2. For some $(x, x^*) \in C_0$ and $(y, y^*) \in \mathrm{ed}(\varphi_A)$,

$$\langle x, z^* \rangle + \langle z, x^* \rangle < \langle y, z^* \rangle + \langle z, y^* \rangle. \tag{25}$$

Since $(0,0) \in \mathrm{ed}(\varphi_A) \cap \overline{C_0}$, we have $\beta = 0$. By the structure of C_0 and (24), we conclude that

$$\langle x_0, z^* \rangle \leq 0 \tag{26}$$

and, for some $\alpha \geq 0$,

$$z = \alpha x_0. \tag{27}$$

By (24), we get

$$\langle y, z^* \rangle + \langle \alpha x_0, y^* \rangle \geq 0, \qquad \forall (y, y^*) \in \mathrm{ed}(\varphi_A). \tag{28}$$

Therefore, for any $(y, y^*) \in A \subset \mathrm{ed}(\varphi_A)$,

$$\langle y, -z^* \rangle + \langle -\alpha x_0, y^* \rangle - \langle y, y^* \rangle \leq - \langle y, y^* \rangle ,$$

which implies that

$$\varphi_A(-\alpha x_0, -z^*) \leq \varphi_A(0,0) < 0.$$

In particular,

$$(-\alpha x_0, -z^*) \in \mathrm{ed}(\varphi_A). \tag{29}$$

Since $\alpha \geq 0$, using (26) we get

$$\langle -\alpha x_0, -z^* \rangle = \alpha \langle x_0, z^* \rangle \leq 0.$$

If this inequality holds strictly then $(-\alpha x_0, -z^*) \in N \cap \mathrm{ed}(\varphi_A)$, and so we may use Lemma 26 to conclude that $(0,0) \notin A^{\mu\mu}$. Thus, we assume that

$$\alpha \langle x_0, z^* \rangle = 0. \tag{30}$$

Now we will discuss the cases $\alpha = 0$ and $\alpha > 0$ separately.

First assume that $\alpha = 0$, that is, $z = 0$. The inclusion $(x_0, x_0^*) \in A \subset \mathrm{ed}(\varphi_A)$ and (28) yield

$$\langle x_0, z^* \rangle \geq 0\,.$$

Using also (26) we get

$$\langle x_0, z^* \rangle = 0.$$

By (25), there exist $(\lambda_0 x_0, x^*) \in C_0$ and $(y, y^*) \in \mathrm{ed}(\varphi_A)$ such that

$$\lambda_0 \langle x_0, z^* \rangle + \langle z, x^* \rangle < \langle y, z^* \rangle + \langle z, y^* \rangle.$$

Since $z = 0$ and $\langle x_0, z^* \rangle = 0$, the above inequality reduces to

$$\langle y, z^* \rangle > 0. \tag{31}$$

Define, for $t \in [0,1]$,

$$\begin{aligned} (y_t, y_t^*) &:= t(y, y^*) + (1-t)(-\alpha x_0, -z^*) \\ &= t(y, y^*) + (1-t)(0, -z^*) \\ &= (ty, ty^* - (1-t)z^*). \end{aligned}$$

As φ_A is convex, $(y_t, y_t^*) \in ed(\varphi_A)$ for any $t \in [0,1]$. Moreover,

$$\langle y_t, y_t^* \rangle = t^2 \langle y, y^* \rangle - t(1-t) \langle y, z^* \rangle .$$

Using also (31) we conclude that, for $t \in (0,1]$ small enough, $\langle y_t, y_t^* \rangle < 0$, and hence $(y_t, y_t^*) \in N \cap \mathrm{ed}(\varphi_A)$. So we may apply Proportion 26 to conclude again that $(0,0) \notin A^{\mu\mu}$.

It remains to consider the case $\alpha > 0$. Then, using assumption (30) we obtain again that

$$\langle x_0, z^* \rangle = 0.$$

Define, for $t \in [0,1]$,

$$\begin{array}{rcl} (x_t, x_t^*) & := & t(x_0, x_0^*) + (1-t)(-\alpha x_0, -z^*) \\ & = & ((t-(1-t)\alpha)x_0, t x_0^* - (1-t)z^*). \end{array}$$

As φ_A is convex, $(x_t, x_t^*) \in ed(\varphi_A)$ for any $t \in [0,1]$. Moreover,

$$\begin{array}{rcl} \langle x_t, x_t^* \rangle & = & [t-(1-t)\alpha] \langle x_0, t x_0^* - (1-t) z^* \rangle \\ & = & t[t-(1-t)\alpha] \langle x_0, x_0^* \rangle . \end{array}$$

Since $\alpha > 0$ and $\langle x_0, x_0^* \rangle > 0$, we conclude that, for $t \in (0,1]$ small enough, $\langle x_t, x_t^* \rangle < 0$. So $(x_t, x_t^*) \in N \cap \mathrm{ed}(\varphi_A)$, and we may apply Proposition 26 for the last time to conclude again that $(0,0) \notin A^{\mu\mu}$. ■

Proposition 28 *Let $A \subset X \times X^*$ be monotone, $(x, x^*) \in X \times X^*$. If $\varphi_A(0,0) = 0$, $\langle x, x^* \rangle < 0$ and $\varphi_A(x, x^*) < 0$, then $(0,0) \notin A^{\mu\mu}$.*

Proof. Define, for $t \in [0,1]$,

$$\begin{array}{rcl} (x_t, x_t^*) & := & t(x, x^*) \\ & = & t(x, x^*) + (1-t)(0,0). \end{array}$$

Trivially,

$$\langle x_t, x_t^* \rangle = t^2 \langle x, x^* \rangle .$$

As φ_A is convex and $\varphi_A(0,0) = 0$, for $t \in [0,1]$,

$$\begin{array}{rcl} \varphi_A(x_t, x_t^*) & \leq & t\varphi_A(x, x^*) + (1-t)\varphi_A(0,0) \\ & = & t\varphi_A(x, x^*). \end{array}$$

Since $\varphi_A(x, x^*) < 0$, for some $\widehat{t} > 0$ small enough

$$\widehat{t}\varphi_A(x, x^*) < \widehat{t}^2 \langle x, x^* \rangle .$$

Hence, $\varphi_A(x_{\widehat{t}}, x_{\widehat{t}}^*) < \left\langle x_{\widehat{t}}, x_{\widehat{t}}^* \right\rangle$. So $(x_{\widehat{t}}, x_{\widehat{t}}^*) \in A^\mu$ which, together with $\left\langle x_{\widehat{t}} - 0, x_{\widehat{t}}^* - 0 \right\rangle = \widehat{t}^2 \langle x, x^* \rangle < 0$, implies $(0,0) \notin A^{\mu\mu}$. ■

Proposition 29 *Suppose that X is reflexive and let $A \subset X \times X^*$ be monotone. If $\sigma_A(0,0) > 0$ and $\varphi_A(0,0) = 0$, then $(0,0) \notin A^{\mu\mu}$.*

Proof. As $\varphi_A(0,0) = 0 > -\infty$, using Definition 1 we conclude that A is nonempty. By their definitions, φ_A and σ_A are *closed* (l.s.c.) *convex functions.* Moreover, as A is a nonempty monotone set we also conclude that φ_A and σ_A are proper ($\not\equiv +\infty$ and $> -\infty$).

Since X is reflexive, the canonical injection of X into X^{**} becomes an identification. Moreover, in this setting

$$(X \times X^*)^* = X^* \times X^{**} = X^* \times X, \qquad (X \times X^*)^{**} = X \times X^*.$$

So, $X \times X^*$ is also reflexive. Therefore [3, Theorem 1.10]

$$\sigma_A = (\sigma_A)^{**}.$$

In particular,

$$\sigma_A(0,0) = \sup_{(y^*,y)\in X^*\times X} \Big(-(\sigma_A)^*(y^*,y)\Big) = -\inf_{(y^*,y)\in X^*\times X} (\sigma_A)^*(y^*,y).$$

Using now (7) we get

$$-\sigma_A(0,0) = \inf_{(y^*,y)\in X^*\times X} \quad \varphi_A(y,y^*).$$

Since $\sigma_A(0,0) > 0$, there exists some (y, y^*) such that

$$\varphi_A(y,y^*) < 0. \tag{32}$$

If $\langle y, y^*\rangle < 0$, then we may apply Proposition 28 to conclude that $(0,0) \notin A^{\mu\mu}$. So we assume that

$$\langle y, y^*\rangle \geq 0. \tag{33}$$

Define

$$t := \frac{(1/4)\,|\varphi_A(y,y^*)|}{\langle y,y^*\rangle + (1/2)\,|\varphi_A(y,y^*)|}. \tag{34}$$

Trivially, $0 < t \leq 1/2$. Since $\varphi_A(0,0) = 0$, there exist some $(x, x^*) \in A$ such that

$$0 \leq \langle x, x^*\rangle \leq (1/2)\, t\, |\varphi_A(y,y^*)|\,. \tag{35}$$

Since $(x, x^*) \in A$,

$$\langle x, y^*\rangle + \langle y, x^*\rangle - \langle x, x^*\rangle \leq \varphi_A(y,y^*) < 0. \tag{36}$$

Define

$$\begin{aligned} z &:= ty + (1-t)x, \\ z^* &:= ty^* + (1-t)x^*. \end{aligned}$$

Using (36) we get

$$\begin{aligned}
\langle z, z^* \rangle &= t^2 \langle y, y^* \rangle + t(1-t) \left[\langle y, x^* \rangle + \langle x, y^* \rangle \right] + (1-t)^2 \langle x, x^* \rangle \\
&= t^2 \langle y, y^* \rangle + t(1-t) \left[\langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle \right] + (1-t) \langle x, x^* \rangle \\
&\leq t^2 \langle y, y^* \rangle + (1-t) \left[t\varphi_A(y, y^*) + \langle x, x^* \rangle \right]. \qquad (37)
\end{aligned}$$

From (32) and (35),

$$t\varphi_A(y, y^*) + \langle x, x^* \rangle \leq (t/2)\varphi_A(y, y^*) < 0.$$

Therefore,

$$\begin{aligned}
\langle z, z^* \rangle &\leq t^2 \langle y, y^* \rangle + (1-t)(t/2)\varphi_A(y, y^*) \\
&= t^2 \langle y, y^* \rangle - (t^2/2)\varphi_A(y, y^*) + (t/2)\varphi_A(y, y^*) \\
&= t \left[t \left(\langle y, y^* \rangle + (1/2) \left| \varphi_A(y, y^*) \right| \right) + (1/2)\varphi_A(y, y^*) \right] \\
&\leq t \left[(1/4) \left| \varphi_A(y, y^*) \right| + (1/2)\varphi_A(y, y^*) \right] \\
&= t \left[(1/4)\varphi_A(y, y^*) \right] \\
&< 0. \qquad (38)
\end{aligned}$$

Since φ_A is convex, A is monotone and $(x, x^*) \in A$, we have $\varphi_A(x, x^*) = \langle x, x^* \rangle$ and

$$\begin{aligned}
\varphi_A(z, z^*) &\leq t\varphi_A(y, y^*) + (1-t)\varphi_A(x, x^*) \\
&= t\varphi_A(y, y^*) + (1-t) \langle x, x^* \rangle \\
&\leq t\varphi_A(y, y^*) + \langle x, x^* \rangle \qquad (39) \\
&< 0. \qquad (40)
\end{aligned}$$

Therefore $\varphi_A(z, z^*) < 0, \langle z, z^* \rangle < 0$ and we may apply again Proposition 28 to conclude that $(0, 0) \notin A^{\mu\mu}$. ■

Proposition 30 *Let $A \subset X \times X^*$ be monotone. If $\varphi_A(0, 0) > 0$, then $(0, 0) \notin A^{\mu\mu}$.*

Proof. If $\varphi_A(0, 0) > 0$, then $(0, 0) \notin A^\mu$ and, since $A^{\mu\mu} \subset A^\mu$, $(0, 0) \notin A^{\mu\mu}$. ■

Theorem 31 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then $\mathrm{cl}_{\mathcal{R}}(A) = A^{\mu\mu}$.*

Proof. First we claim that

$$(0, 0) \in \mathrm{cl}_{\mathcal{R}}(A) \Leftrightarrow (0, 0) \in A^{\mu\mu}. \qquad (41)$$

We already know that $\mathrm{cl}_{\mathcal{R}}(A) \subset A^{\mu\mu}$. So, we only need to prove that

$$(0, 0) \notin \mathrm{cl}_{\mathcal{R}}(A) \Rightarrow (0, 0) \notin A^{\mu\mu}.$$

Assume that $(0,0) \notin \mathrm{cl}_{\mathcal{R}}(A)$. This is equivalent to the inequality

$$\sigma_A(0,0) > 0.$$

Hence, if $\varphi_A(0,0) = 0$, by Proposition 29 one has $(0,0) \notin A^{\mu\mu}$. The cases $\varphi_A(0,0) < 0$ and $\varphi_A(0,0) > 0$ are covered by Lemma 27 and Proposition 30, respectively. This proves (41).

Now let (x_0, x_0^*) be an arbitrary point in $X \times X^*$. Obviously,

$$\tau_{(x_0,x_0^*)}\,(x_0, x_0^*) = (0,0).$$

Since

$$\tau_{(x_0,x_0^*)}\,(\mathrm{cl}_{\mathcal{R}}(A)) = \mathrm{cl}_{\mathcal{R}}(\tau_{(x_0,x_0^*)}A)$$

and

$$\tau_{(x_0,x_0^*)}\,(A^{\mu\mu}) = \left(\tau_{(x_0,x_0^*)}\,(A)\right)^{\mu\mu},$$

we conclude that $(x_0, x_0^*) \in \mathrm{cl}_{\mathcal{R}}(A)$ is equivalent to $(0,0) \in \mathrm{cl}_{\mathcal{R}}(\tau_{(x_0,x_0^*)}A)$. In the same way, $(x_0, x_0^*) \in A^{\mu\mu}$ is equivalent to $(0,0) \in \tau_{(x_0,x_0^*)}(A^{\mu\mu})$. Now we apply (41) ■

As trivial consequences of Theorem 31, we get the following results:

Corollary 32 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then A is representable if and only if A is μ-closed. Hence, in finite dimensional spaces, the monotone representable sets are the intersections of arbitrary families of maximal monotone sets.*

Corollary 33 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then*

$$\mathrm{cl}_{\mathcal{R}}(A) = \cap_{B \in \mathbf{M}(A)} B.$$

By Theorem 13, we know that if $A \in \mathcal{R}$ then σ_A is a convex representation of A. Now, in the finite dimensional case, from Theorem 31 and Proposition 25 we obtain another convex representation of the elements of $\mathcal{R}$, by means of Fitzpatrick functions:

Corollary 34 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then*

$$\mathrm{cl}_{\mathcal{R}}(A) = L(\varphi_{A^\mu}).$$

In particular, if $A \in \mathcal{R}$, then φ_{A^μ} is a representation of A.

Note that if A is maximal monotone, then $A = A^\mu$, and so in this case the representation provided by the preceding corollary reduces to the original Fitzpatrick representation φ_A.

6 Pre-maximal Monotone Operators

In many cases, one needs to obtain a maximal monotone operator out of the sum of two maximal monotone operators T_1, T_2. Under some regularity assumptions, $T_1 + T_2$ is maximal monotone. If these regularity assumptions are lacking, then one can try to obtain a *regularized* sum S, which should coincide with the usual sum in the case when $T_1 + T_2$ is maximal monotone [7, 9, 11, 12]. Evidently, the operator $T_1 + T_2$ is monotone. The regularized sum of T_1 and T_2, whichever definition of regularization may be considered, should contain $T_1 + T_2$. If it happens that $T_1 + T_2$ admits a unique maximal monotone extension, then we are in a comfortable position, because any consistent regularized sum should yield such extension. This motivates the following definition:

Definition 35 *A monotone set $A \subset X \times X^*$ is* pre-maximal monotone *if it has a unique maximal monotone extension.*

It is natural to ask under which conditions an operator is pre-maximal monotone:

Proposition 36 *Let A be monotone. Then the following conditions are equivalent:*

1. *A is pre-maximal monotone.*
2. *$A^\mu = A^{\mu\mu}$.*
3. *$\varphi_{A^\mu} = \varphi_{A^{\mu\mu}}$.*
4. *A^μ is monotone.*
5. *A^μ is maximal monotone.*
6. *$A^{\mu\mu}$ is maximal monotone.*

Moreover, if these conditions holds, A^μ (or equivalently, $A^{\mu\mu}$) is the unique maximal monotone extension of A.

Proof. Suppose that item 1 holds. Let T be the unique maximal monotone extension of A. Them $\mathbf{M}(A) = \{T\}$. Hence, by items 1 and 2 of Proposition 22, item 2 holds.

Item 2 trivially implies item 3. From item 3 of the observation that follows Definition 18, using (22) we get $A^\mu = b(\varphi_{A^{\mu\mu}})$; from (23), $A^{\mu\mu} = b(\varphi_{A^\mu})$. So, item 3 implies 2 and these items are equivalent.

Item 2 obviously implies item 4.

Assume item 4 and take a maximal monotone extension B of A. By item 1 of Proposition 22, $B \subset A^\mu$. As A^μ is monotone, we conclude that it is maximal monotone. This proves the implication $4 \Longrightarrow 5$.

Assume that item 5 holds. Let B be a maximal monotone extension of A. By item 1 of Proposition 22, $B \subset A^\mu$. So, $B = A^\mu$, and we conclude that A^μ

is the unique maximal monotone extension of A. This proves the implication 5 $\implies$ 1.

To end the proof, applying Proposition 21 to $A^{\mu\mu}$ and using again the equality $A^{\mu\mu\mu} = A^{\mu}$, we conclude that item 6 is equivalent to item 2. ■

Next we give some sufficient conditions for an operator to be pre-maximal monotone.

Lemma 37 *Let $A \subset X \times X^*$ be monotone. If $\mathrm{cl}_{\mathcal{R}}(A)$ is maximal monotone, then A is pre-maximal monotone and $\mathrm{cl}_{\mathcal{R}}(A)$ is the unique maximal monotone extension of A.*

Proof. Let B be a maximal monotone extension of A. Since maximal monotone operators are representable, it follows that $\mathrm{cl}_{\mathcal{R}}(A) \subset B$. Now, since $\mathrm{cl}_{\mathcal{R}}(A)$ is maximal monotone, we conclude that $B = \mathrm{cl}_{\mathcal{R}}(A)$. ■

Lemma 38 *Let $A \subset X \times X^*$ be monotone. The following conditions are equivalent:*

1. $L(\varphi_A) = b(\varphi_A)$.
2. $\varphi_A \geq \pi$
3. $\varphi_A \in \mathcal{F}(A)$.
4. $\varphi_A = \varphi_{A^{\mu}}$.

Moreover, if the above conditions hold, then A is pre-maximal monotone.

Proof. Items 1 and 2 are trivially equivalent. Since A is monotone, by Proposition 2,

$$\varphi_A \leq \pi + \delta_A.$$

Therefore, items 2 and 3 are equivalent also.

Assume now that item 4 holds. By (10), $\varphi_{A^{\mu}}(x, x^*) \geq \langle x, x^* \rangle$, for all $(x, x^*) \in A^{\mu}$. Therefore,

$$\varphi_A(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in A^{\mu}.$$

By (22),

$$\varphi_A(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \notin A^{\mu}.$$

Combining these inequalities, we conclude that $\varphi_A \geq \pi$. We just proved that 4 $\implies$ 2.

Now assume that item 2 holds. By Theorem 5 we conclude that L^{φ_A} is monotone. Since items 1, 2 and 3 are equivalent, using also (22), we conclude that, in this case

$$A^{\mu} = b(\varphi_A) = L(\varphi_A).$$

Therefore, A^{μ} is monotone. Using now Proposition 36 we conclude that A^{μ} is maximal monotone and that A is pre-maximal monotone. As A is monotone $A \subset A^{\mu}$. Therefore,

$$\pi \leq \varphi_A \leq \varphi_{A^{\mu}},$$

and $\varphi_A \in \mathcal{F}(A^\mu)$. Since A^μ is maximal monotone, this inclusion, together with Theorem 4 yields

$$\varphi_{A^\mu} \leq \varphi_A.$$

Combining the above inequalities, we conclude that $\varphi_A = \varphi_{A^\mu}$. So we proved that $2 \Longrightarrow 4$.

Therefore, items 2 and 4 are also equivalent. In proving the implication $2 \Longrightarrow 4$ we also proved that item 2 implies that A is pre-maximal monotone. ■

Is is natural to ask what are the relations between the conditions of Lemma 37 and Lemma 38.

Proposition 39 *Let $A \subset X \times X^*$ be monotone. If $\mathrm{cl}_{\mathcal{R}}(A)$ is maximal monotone, then $\varphi_A \geq \pi$. If X is reflexive, the converse also holds.*

Proof. By Corollary 14, $\sigma_A = \sigma_{\mathrm{cl}_{\mathcal{R}}(A)}$. Using also (7) we get

$$\varphi_A = \varphi_{\mathrm{cl}_{\mathcal{R}}(A)}.$$

If $\mathrm{cl}_{\mathcal{R}}(A)$ is maximal monotone, then, by Theorem 4, $\varphi_{\mathrm{cl}_{\mathcal{R}}(A)} \geq \pi$. So, this implies $\varphi_A \geq \pi$.

Assume now that X is reflexive and $\varphi_A \geq \pi$. Using Lemma 38 and Proposition 36 we conclude that A is pre-maximal monotone, A^μ is maximal monotone and

$$\varphi_A = \varphi_{A^\mu}.$$

Using now (7) for φ_A and φ_{A^μ} we obtain

$$(\sigma_A)^*(x^*, x) = (\sigma_{A^\mu})^*(x^*, x), \quad \forall (x^*, x) \in X^* \times X.$$

As X is reflexive, $X^* \times X = (X \times X^*)^*$ and $(\sigma_A)^* = (\sigma_{A^\mu})^*$. Now, applying Fenchel-Moreau Theorem [3, Theorem 1.10] we get

$$\sigma_A = (\sigma_A)^{**} = [(\sigma_A)^*]^* = [(\sigma_{A^\mu})^*]^* = (\sigma_{A^\mu})^{**} = \sigma_{A^\mu}.$$

Using also Theorem 13 we conclude that

$$\mathrm{cl}_{\mathcal{R}}(A) = L(\sigma_A) = L(\sigma_{A^\mu}) = \mathrm{cl}_{\mathcal{R}}(A^\mu).$$

Since A^μ is maximal monotone, $\mathrm{cl}_{\mathcal{R}}(A^\mu)$ is equal to A^μ, and so, $\mathrm{cl}_{\mathcal{R}}(A) = A^\mu$. ■

In finite dimensional spaces we have stronger results.

Proposition 40 *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then the following conditions are equivalent:*

1. *A is pre-maximal monotone.*
2. *$A^\mu = A^{\mu\mu}$.*
3. *$\varphi_{A^\mu} = \varphi_{A^{\mu\mu}}$.*

4. A^{μ} *is monotone.*

5. A^{μ} *is maximal monotone.*

6. $A^{\mu\mu}$ *is maximal monotone.*

7. $\mathrm{cl}_{\mathcal{R}}(A)$ *is maximal monotone.*

8. $L(\varphi_A) = b(\varphi_A)$.

9. $\varphi_A \geq \pi$

10. $\varphi_A \in \mathcal{F}(A)$.

11. $\varphi_A = \varphi_{A^{\mu}}$.

Under these conditions, one has:
a) $L(\varphi_A)$ *is maximal monotone.*
b) $\varphi_{L(\varphi_A)} = \varphi_A$.

Proof. By Proposition 36, items 1, 2, 3, 4, 5, 6, are equivalent.

Since finite dimensional spaces are reflexive, using Proposition 39 and Lemma 38 we conclude that items 7, 8, 9, 10, 11 are equivalent.

Equivalence between item 6 and item 7 follows from Theorem 31. Therefore conditions 1-11 are equivalent.

To prove *a),* use item 8 and (22) to deduce that $L(\varphi_A) = A^{\mu}$ and then use item 5. To prove *b),* use the equality $L(\varphi_A) = A^{\mu}$ and item 11. ∎

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