A Geometric Approach to the Weighting Method Scalarization in Vector Optimization

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Abstract

We consider the weighting method for constrained (finite dimensional) vector optimization. First we show that the closest points in the objective's image from certain hyperplanes are weakly efficient; this approach allows us to give a geometrical interpretation of the method. We also give some conditions on the the existence of weakly efficient optima, based on the connection between the recession cone of the convex hull of the objective's image and the negative of the ordering cone.

Keywords: Vector optimization, weak efficiency, scalarization, weighting method, recession cone.

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1 Introduction

In this work we consider the problem of finding weakly efficient points (or weak Pareto minimal elements) of a constrained vector optimization problem. Our setting will be a finite dimensional linear space, say $\mathbb{R}^m$, with the canonical inner product $\langle \cdot, \cdot \rangle$, and a preference order induced by $K \subset \mathbb{R}^m$, a closed convex and pointed (i.e. $-K \cap K = \{0\}$) cone with nonempty interior $\mathrm{int}(K)$ (see Fig.1(a)). Our objective function, defined on a subset of another finite dimensional space, will take its values in $\mathbb{R}^m$. To be more precise, the space $\mathbb{R}^m$ is endowed with the following partial order

$$u \preceq v \, (v \succeq u) \text{ for } u, v \in \mathbb{R}^m \text{ iff } v - u \in K \text{ (see Fig.1}(b)) \, ,$$

and the following stronger relation

$$u \prec v \, (v \succ u) \text{ for } u, v \in \mathbb{R}^m \text{ iff } v - u \in \mathrm{int}(K) \text{ (see Fig.1}(c)) \, .$$

Figure 1: In (a) we have the cone K. In (b) the vectors in the above cone are $\succeq u$, while those in the other cone are $\preceq u$. In (c) we have vectors which are $\succ u$ and $\prec u$ in the above and the below cone, respectively.

Among the advantages of the notation "$0 \preceq w$" over "$w \in K$", we mention that K-inequalities can be handled as regular ones, e.g., two of such inequalities can be added up, or multiplied by nonnegative numbers, etc.

Given a subset Ω of $\mathbb{R}^n$ and a mapping $F : \Omega \to \mathbb{R}^m$, the vector optimization problem, understood in the weak Pareto sense ([7, 8]),

$$(P) \qquad\qquad \min_{x \in \Omega} F(x) \, ,$$

consists of finding a *feasible* point x^* ($x^* \in \Omega$) such that $F(x^*)$ is *weakly efficient* (or a *weak Pareto minimal element*) for $F(\Omega)$, i.e. such that

$$F(x) \prec F(x^*)$$

does not hold for any feasible x. We recall that $x^* \in \Omega$ is *efficient* (or *Pareto minimal element*) for $F(\Omega)$ if there does not exist $x \in \Omega$ such that $F(x) \preceq F(x^*)$, with $F(x) \neq F(x^*)$ ([7, 8]). Trivially, efficient points are also weakly efficient.

Scalarization techniques [4, 6, 8, 9] for solving problem (P) substitute the original vector problem by a suitable scalar one, in such a way that the optimal solutions of the new problem are also optimal for the original one. The main advantage of this approach, from a practical point of view, is that we can use a large number of fast and reliable methods developed for single-valued optimization in order to solve vector problems.

One of the most widely used scalarization techniques in multicriteria (i.e., in the Paretian cone case or, in other words, the point-wise partial order) is the weighting method, which consists of minimizing a weighted sum of the different objectives. The weights, which are critical for the method, in general are not known in advance, so computational implementations of this technique are not always straightforward.

In this paper we study an extension of the weighting method for vector optimization, and we also present its geometrical analysis, which, for the best of our knowledge, has not been done yet. This approach sheds new light on this strategy, and provides conditions for the existence of suitable vectors of weights.

2 The Weighting Method Scalarization

In this work, we will say that a scalar minimization problem is a *scalarization* of (P) if its optimal solutions are weakly efficient for (P). For some authors (see, e.g., [7] and [8]) a scalarization is a family of scalar problems whose optima are exactly all optimal solutions for the vector-valued problem.

We will focus our attention on problems for which the convex hull of $F(\Omega)$, after a suitable rigid movement, does not touch $-K$, i.e.,

$$[\mathrm{conv}(F(\Omega)) + u_0] \cap (-K) = \emptyset\,, \quad \text{for some } u_0 \in \mathbb{R}^m\,. \tag{1}$$

Our next result shows that we do not loose generality on assuming $u_0 = 0$, that is to say,

$$\mathrm{conv}(F(\Omega)) \cap (-K) = \emptyset\,. \tag{2}$$

Lemma 2.1 *Suppose that* (P) *satisfies condition* (1). *For* $u \in \mathbb{R}^m$ *define* $F_u(x) := F(x) + u$ *and consider* (P_u) *the corresponding vector problem in* Ω*,*

$$(P_u) \qquad \min_{x\in\Omega} F_u(x).$$

Then, (P_u) *satisfies condition* (2) *for all* $u \succeq u_0$ *(with* F_u *replacing* F*), where* u_0 *is given by* (1).

Moreover, these problems ((P) and (P_u)) have the same optima.

Proof. Note that

$$\operatorname{conv}(F(\Omega)) + u = \operatorname{conv}(F(\Omega) + u) = \operatorname{conv}(F_u(\Omega)).$$

So $\operatorname{conv}(F(\Omega)) + u_0 = \operatorname{conv}(F_{u_0})$ and (P_{u_0}) satisfies (2). Now take $u \succeq u_0$. Then, $u - u_0 \in K$,

$$\operatorname{conv}(F_u(\Omega)) = [\operatorname{conv}(F(\Omega)) + u_0] + (u - u_0)\,,$$

and therefore (P_u) satisfies (2).

The last part of the claim holds trivially. ∎

At this point, a natural question is the following: which vector optimization problems satisfy condition (1)?

Lemma 2.2 *Assume that* $F(\Omega) + K$ *is a convex subset of* $\mathbb{R}^m$.

If problem (P) *has an optimal solution, then it satisfies condition* (1).

Proof. Consider x^* a weakly efficient solution for (P). Just take $u_0 \succ -F(x^*)$.

∎

As we shall see, functions which are "convex with the respect to K" satisfy the assumption on Lemma 2.2. Later on, in Section 3, we will come back on the matter of when we can perform a separating rigid movement as in (1).

Let us now consider the problem of how to find weakly efficient solutions for problem (P). Recall that we are assuming condition (1). Observe that the optimality condition $F(x) \not\prec F(x^*)\ \forall\, x \in \Omega$ means $F(x^*) - F(x) \notin \operatorname{int}(K)\ \forall\, x \in \Omega$, which in turn can be written as

$$[F(x^*) - \operatorname{int}(K)] \cap F(\Omega) = \emptyset\,. \tag{3}$$

Note that $F(x^*) - \operatorname{int}(K)$ is the interior of the cone obtained by translating $-K$ till the moment $F(x^*)$ becomes its vertex (see Fig.2). Note also that other points on the intersection of the boundaries of $F(\Omega)$ and shifted cones of the form $F(x^*) - \operatorname{int}(K)$ are also weakly efficient (Fig.2(a)). Whenever

Figure 2: The vertex of the shifted cone K in (a) is weak Pareto minimal for $F(\Omega)$, as well as all points in the line supported by the cone. In (b) the shifted cone touches $F(\Omega)$ just at its vertex, which is (strong) Pareto optimal.

the vertex is the only point in that intersection, it is Pareto optimal for $F(\Omega)$ (Fig.2(b)).

We will devote our efforts to locate those weakly efficient points x^*, which are "F's pre-images of shifted cones' vertices satisfying (3). But how can we effectively locate such vertices?

Let us now give an answer to the above question. Since we are assuming that $\text{conv}(F(\Omega)) \cap (-K) = \emptyset$, using the Convex Separation Theorem (see [1], Theorem 4.7 or [2], Proposition B13), there exists a hyperplane, say $L_w := \{y \in \mathbb{R}^m \mid \langle w, y \rangle = \alpha\}$, that separates $-K$ and $\text{conv}(F(\Omega))$. We therefore have that $\text{conv}(F(\Omega))$ and $-K$ lie on the different half-spaces determined by L_w in the following way:

$$\langle w, y \rangle \leq \alpha \leq \langle w, z \rangle \quad \forall\, y \in -K \text{ and } \forall\, z \in \text{conv}(F(\Omega))\,. \tag{4}$$

Since K is closed, we have that $0 \in -K$, so $0 \leq \alpha$. If there exists $\bar{y} \in -K$ such that $\langle w, \bar{y} \rangle > 0$, then, for large enough $\lambda > 0$ we will have $\langle w, \lambda\bar{y} \rangle > \alpha$, with $\lambda\bar{y} \in -K$, in contradiction with (4). Therefore, $\langle w,\ y \rangle \leq 0\ \ \forall\, y \in -K$; consequently, we can take $\alpha = 0$. Hence the hyperplane $H_w := \{y \in \mathbb{R}^m \mid \langle w, y \rangle = 0\}$ separates $-K$ and $\text{conv}(F(\Omega))$. In particular,

$$\langle w, y \rangle \leq 0 \text{ for all } y \in -K \tag{5}$$

and

$$\langle w, F(x) \rangle \geq 0 \text{ for all } x \in \Omega\,. \tag{6}$$

By normalizing w, if necessary, we may assume that $||w|| = 1$, where $|| \cdot ||$ stands for the euclidean norm.

Figure 3: The closest points in $F(\Omega)$ to H_w are weakly efficient.

According to Fig.3, good candidates for vertices $F(x^*)$ of shifted cones of the above mentioned form, which touch $F(\Omega)$ without overlapping image's points other than the boundary ones, are the closest points from H_w at $F(\Omega)$.

We will now prove that such points are indeed weakly efficient for $F(\Omega)$; in other words, that x^* is a weak Pareto minimal element of problem (P) if $F(x^*)$ realizes the minimal distance between $F(\Omega)$ and H_w. We begin by establishing some notation which we will use on the sequel. We will call P_w the orthogonal projector onto the separating hyperplane H_w, i.e.,

$$P_w(y) := y - \langle w, y \rangle w \,. \tag{7}$$

Clearly, the distance between a point $y \in \mathbb{R}^m$ and H_w is given by $||y - P_w(y)||$. In our next theorem we will see that, as we claimed, every point in Ω that minimizes (over Ω) $||F(x) - P_w(F(x))||$ is an optimal solution for problem (P).

Before stating and proving the announced result, let us make some comments. First, (5) is telling us that w is an element of K^*, the *positive dual* (or *polar*) of K, the (convex) cone of vectors that form an acute angle with every element of K, i.e.,

$$K^* := \{y \in \mathbb{R}^m \mid \langle v, y \rangle \geq 0 \ \forall \, v \in K\} \,. \tag{8}$$

Second, it is well known that it is possible to describe the topological interior of K as:

$$\operatorname{int}(K) = \{y \in \mathbb{R}^m \mid \langle y, v \rangle > 0, \ \forall \, v \in K^* \setminus \{0\}\} \,. \tag{9}$$

And finally, if $y \in \operatorname{int}(K)$ then $P_w(y) \neq y$. Indeed, if $P_w(y) = y$, then, from the definition of P_w given in (7), it follows that $\langle w, y \rangle = 0$, in contradiction with (9), since $w \in K^* \setminus \{0\}$.

Theorem 2.3 *If* (2) *holds and the unitary m-vector w satisfies* (5)-(6)*, then every optimal solution of the scalar problem*

$$(S) \qquad\qquad \min_{x \in \Omega} ||F(x) - P_w(F(x))||$$

is weakly efficient for the vector problem

$$(P) \qquad\qquad \min_{x \in \Omega} F(x) \,.$$

Proof. Let $x^* \in \Omega$ be an optimal solution for (S), that is to say

$$||F(x^*) - P_w(F(x^*))|| \leq ||F(x) - P_w(F(x))|| \text{ for all } x \in \Omega \,. \tag{10}$$

Suppose that x^* is not a weakly efficient point for problem (P). Then, the optimality condition (3) does not hold, i.e.

$$[F(x^*) - \mathrm{int}(K)] \cap F(\Omega) \neq \emptyset\,.$$

So, there exist $\bar{y} \in \mathrm{int}(K)$ and $\bar{x} \in \Omega$ such that

$$F(x^*) - \bar{y} = F(\bar{x})\,.$$

Therefore

$$\begin{aligned} ||F(x^*) - P_w(F(x^*))||^2 &= ||F(\bar{x}) + \bar{y} - P_w(F(\bar{x}) + \bar{y})||^2 \\ &= ||F(\bar{x}) - P_w(F(\bar{x}))||^2 + ||\bar{y} - P_w(\bar{y})||^2 \\ &\quad + 2\langle F(\bar{x}) - P_w(F(\bar{x})), \bar{y} - P_w(\bar{y})\rangle \\ &> ||F(\bar{x}) - P_w(F(\bar{x}))||^2 \\ &\quad +2\langle F(\bar{x}) - P_w(F(\bar{x})), \bar{y} - P_w(\bar{y})\rangle \\ &= ||F(\bar{x}) - P_w(F(\bar{x}))||^2 + 2\langle w, F(\bar{x})\rangle\langle w, \bar{y}\rangle, \end{aligned} \tag{11}$$

where the inequality holds by virtue of the last observation we made just before this theorem, since $\bar{y} \in \mathrm{int}(K)$, and the last equality follows from (7) and the fact that $||w|| = 1$. Now we claim that

$$\langle w, \bar{y}\rangle > 0\,. \tag{12}$$

As we have mentioned just before this theorem, the vector w is an element of K^*. Since $||w|| = 1$ and $\bar{y} \in \mathrm{int}(K)$, from (9) it follows that (12) holds.

Combining now (6), (11) and (12), we get

$$||F(x^*) - P_w(F(x^*))||^2 > ||F(\bar{x}) - P_w(F(\bar{x}))||^2\,,$$

in contradiction with (10). Whence, (3) holds and x^* is weakly efficient for problem (P). □

Observe now that from the definition of the orthogonal projector P_w given in (7), from (6) and the fact that $||w|| = 1$, it follows that

$$||F(x) - P_w(F(x))|| = \langle w, F(x)\rangle \text{ for all } x \in \Omega\,. \tag{13}$$

Hence, we have the following corollary.

Corollary 2.4 *Under the assumptions of Theorem 2.3, all optimal solutions, if any, of*

$$\min_{x\in\Omega} \langle w, F(x)\rangle \,, \tag{14}$$

are weakly efficient for (P).

Proof. The result follows combining Theorem 2.3 and (13). □

Problem (14) is a scalarization of (P). The procedure which consists of choosing some $w \in K^*$ and then solving the scalar problem (14) is known as the *weighting method* [5, 11].

Note that (14) may be an unbounded problem and, even if it is bounded, it may lack minimizers. Here we will not study which of such scalarizations (i.e., which w's) are "adequate". Of course, there are some simple cases for which this is not an issue; for instance, if $F(\Omega)$ is compact, all scalarizations as (14) will be useful, since this kind of problems always has optimal solutions.

Up to now, we know that by performing a rigid movement, if necessary, we can try to compute (at least theoretically) a weakly efficient solution of problem (P), by minimizing the distance between $F(\Omega)$ (or $F(\Omega)+u$, for some $u \in \mathbb{R}^m$) and a hyperplane that separates this set from $-K$. Next we will prove a sort of converse, namely that, when the objective's image is convex, every weakly efficient point can be obtained in such a way. Afterward, we will study a more general condition, related to convexity, which guarantees the same result.

Theorem 2.5 *Assume that $F(\Omega)$ is a convex set and let $x^* \in \Omega$ be a weakly efficient point for (P). Then, there exists a vector $u \in \mathbb{R}^m$ such that x^* minimizes the distance between $F(\Omega) + u$ and a hyperplane that separates this set and $-K$. Furthermore, this hyperplane is of the form $H_w = \{y \in \mathbb{R}^m \mid \langle w, y\rangle = 0\}$, where $w \in K^*$ and $||w|| = 1$. In particular,*

$$x^* \in \mathrm{argmin}_{x\in\Omega}\langle w, F(x)\rangle \,.$$

Proof. If $x^* \in \mathrm{argmin}_{x\in\Omega}||P_w(F(x)) - F(x)||$ for some norm one vector $w \in K^*$, where P_w is the orthogonal projector onto H_w, then the result is obviously true for $u = 0$.

Otherwise, by taking $u = -F(x^*)$, we have, from the definition of weak efficiency, that

$$(F(\Omega) + u) \cap (-\mathrm{int}(K)) = \emptyset \,.$$

Therefore, by the non-strict Convex Separation Theorem, there exists $0 \neq w \in \mathbb{R}^m$ such that

$$\langle w, F(x) - F(x^*)\rangle \geq 0 \;\; \forall\, x \in \Omega\,, \tag{15}$$

$$\langle w, y\rangle \leq 0 \;\; \forall\, y \in \; -\text{int}(K)\,. \tag{16}$$

Clearly, there is no loss of generality if we assume that $||w|| = 1$. Moreover, using (16) and the continuity of the inner product, we conclude that $\langle w, y\rangle \leq 0$ for all $y \in -K$; that is to say, $w \in K^*$. Finally, by virtue of (15) we get

$$x^* \in \text{argmin}_{x\in\Omega}\langle w, F(x)\rangle\,.$$

The whole result follows now from (13). ◻

Observe that $F(\Omega)$ may be convex but, nevertheless, problem (P) may lack weak Pareto optimal solution. Consider, for instance, $n = 1, m = 2$, $K = \mathbb{R}^2_+$, the nonnegative orthant, $\Omega = (0, 1) \subset \mathbb{R}$ and $F(t) := te$, where $e^t = (1, 1) \in \mathbb{R}^2$. Clearly, Ω and $F(\Omega)$ are convex sets, but the last set has no weak Pareto minimal elements.

We will now see that if $F(\Omega) + K := \{F(x) + y \mid x \in \Omega \,\text{and}\, y \in K\}$ is a convex subset of $\mathbb{R}^m$, then similar results as those of Theorem 2.5 can be proved. First, we establish a condition under which the set $F(\Omega) + K$ is convex. For that, we need to introduce the notion of convexity of a function in relation to a cone. We say that $F : \Omega \to \mathbb{R}^m$ is *K-convex* if

$$F(\lambda x + (1-\lambda)y) \;\preceq\; \lambda F(x) + (1-\lambda)F(y) \;\; \forall\, x, y \in \Omega \text{ and } \forall\, \lambda \in [0, 1].$$

Note that this definition extends the classical concept of convexity of a scalar valued function; in fact, convex real valued functions are K-convex, with $K = \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the nonnegative half-line in $\mathbb{R}$.

In general, K-convexity of F does not imply convexity of it's image, but (together with convexity of Ω) it does imply convexity of the set $F(\Omega) + K$, as we will see now.

Lemma 2.6 *Let Ω be a convex subset of $\mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^m$ a K-convex mapping. Then $F(\Omega) + K$ is a convex set.*

Proof. Let $\text{epi}(F)$ be the (K-)epigraph of F,

$$\text{epi}(F) := \{(x, y) \in \Omega \times \mathbb{R}^m \mid F(x) \preceq y\}\,,$$

and let $\Pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projector onto $\mathbb{R}^m$, i.e.,

$$\Pi(x, y) := y\,, \quad \forall\,(x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

We claim that

$$\Pi(\mathrm{epi}(F)) = F(\Omega) + K\,. \tag{17}$$

Let us see that (17) is true. If $(x, y) \in \mathrm{epi}(F)$, we have that $y - F(x) \in K$, so $\Pi(x, y) = y = F(x) + (y - F(x)) \in F(\Omega) + K$. Conversely, if $u \in F(\Omega) + K$, then there exists $x \in \Omega$ and $y \in K$ such that $u = F(x) + y$. Whence, $u - F(x) = y \in K$, that is to say, $F(x) \preceq u$, and $(x, u) \in \mathrm{epi}(F)$; so $u = \Pi(x, u) \in \Pi(\mathrm{epi}(F)$.

Now observe that K-convexity of F trivially implies convexity of $\mathrm{epi}(F)$. Therefore, since Π is linear and Ω convex, $\Pi(\mathrm{epi}(F))$ is convex and the proof is complete by virtue of (17).

□

Observe that the converse of our lemma does not hold in general; in other words, convexity of $F(\Omega) + K$ (and of Ω) does not necessarily imply K-convexity of F. To see this assertion, just consider $n = m = 1$, $\Omega = \mathbb{R}$, $K = \mathbb{R}_+$ and $F(x) = -|x|$. In this case, $F(\Omega) + K = -\mathbb{R}_+ + \mathbb{R}_+ = \mathbb{R}$, however F is not $(K-)$convex.

The vector optimization problem (P) is said to be *convex* if its feasible set Ω is convex and its objective F is K-convex. We can now state and prove the result we were looking for.

Theorem 2.7 *Let $x^* \in \Omega$ be a weakly efficient point for problem (P). If $F(\Omega) + K$ is convex , then there exists a norm one vector $w \in K^*$ such that*

$$x^* \in \mathrm{argmin}_{x \in \Omega} \langle w, F(x) \rangle\,.$$

In particular, if problem (P) is convex (i.e., if Ω is convex and F is K-convex), such $w \in K^*$ exists.

Proof. We claim that

$$\Big(F(\Omega) + K\Big) \cap \Big(F(x^*) - \mathrm{int}(K)\Big) = \emptyset\,. \tag{18}$$

Indeed, if there exist $x \in \Omega$, $y \in K$ and $y' \in \mathrm{int}(K)$ such that $F(x) + y = F(x^*) - y'$, then $F(x^*) - F(x) = y + y' \succ 0$, in contradiction with the optimality of x^*. Therefore (18) holds.

So, in view of (18) and the Convex Separation Theorem, there exist an m-vector $w \neq 0$ and a real number α such that

$$\langle w, F(x^*) - y \rangle \leq \alpha \ \text{ for all } y \in \text{int}(K) \tag{19}$$

and

$$\alpha \leq \langle w, F(x) + z \rangle \ \text{ for all } x \in \Omega \text{ and for all } z \in K. \tag{20}$$

Using the continuity of the inner product, we conclude that (19) holds for $y \in K$. So, taking $y = 0$ in (19) and $z = 0$, $x = x^*$ in (20), we get $\alpha = \langle w, F(x^*) \rangle$. Therefore, taking again $z = 0$ in (20), we verify that

$$x^* \in \text{argmin}_{x \in \Omega} \langle w, F(x) \rangle \, .$$

On the other hand, letting z be an arbitrary element of K and taking $x = x^*$ in (20), we conclude that $w \in K^*$. As $w \neq 0$, we can assume that $\|w\| = 1$.

If Ω is convex and F is K-convex, in view of Lemma 2.6, $F(\Omega) + K$ is convex and therefore the conclusion holds again.

□

We finish this section mentioning that some wellknown results for the weighting method in the Paretian case can be easily extended to the general one (i.e., when the ordering cone is closed, convex, pointed and has nonempty interior). For instance,

- If $w \in \text{int}(K^*)$, then all optima of $\min_{x \in \Omega} \langle w, F(x) \rangle$ are (strong) Pareto optima for the vector problem (P).
- If $\min_{x \in \Omega} \langle w, F(x) \rangle$ has a unique optimal solution, then it is a (strong) Pareto optimal point for problem (P).

3 On the Existence of Suitable Rigid Movements

Finally, we go back to the very beginning of our discussion and study when we can perform a rigid movement which prevents the convex hull of F's image from touching $-K$. That is to say, we will see in which cases there exists u_0 such that (1) holds.

The following fact will be needed.

$$K \cap \Big(K^* \setminus \{0\}\Big) \neq \emptyset\,. \tag{21}$$

In order to verify it, note that if (21) does not hold, then, by the nonstrict Convex Separation Theorem, there exists $0 \neq w \in \mathbb{R}^m$ such that

$$\langle w, v\rangle \geq 0 \ \ \forall\ v \in K, \tag{22}$$

$$\langle w, v\rangle \leq 0 \ \ \forall\ v \in \ K^* \setminus \{0\}. \tag{23}$$

By (22), $w \in K^*$. Since $w \neq 0$, by virtue of (23), $||w||^2 = \langle w, w\rangle \leq 0$, in contradiction with $w \neq 0$.

Recall now that the recession cone of a convex set $C \subset \mathbb{R}^m$ (see, e.g., [10]) is the set of all half-lines contained on the set, i.e.,

$$0^+C := \{v \in \mathbb{R}^m \ \mid C + tv \ \subset C \ \forall\ t \geq 0\}.$$

We begin the analysis with a sufficient condition for the existence of a separating translation. We will first study the general case of a convex set $C \subset \mathbb{R}^m$. As usual, $\mathrm{cl}(C)$ stands for the closure of C.

Lemma 3.1 *Let $C \subset \mathbb{R}^m$ be convex. If $0^+\mathrm{cl}(C) \cap (-K) = \{0\}$, then there exists $u \in \mathbb{R}^m$ such that*

$$[C + u] \cap (-K) = \emptyset.$$

Proof. Suppose that

$$[C + u] \cap (-K) \neq \emptyset, \quad \forall\, u \in \mathbb{R}^m. \tag{24}$$

It is enough to show that, under this assumption, $0^+\mathrm{cl}(C) \cap (-K) \neq \{0\}$.

By (21), we can take $e \in K \cap (K^* \setminus \{0\})$ and for each $k \in \mathbb{N}$, we use (24) to obtain

$$x^k \in [C + ke] \cap (-K).$$

Therefore,

$$x^k = c^k + ke, \qquad c^k \in C, \quad x^k \in -K.$$

Since $e \in K^*$ and $x^k \in -K$, we have $\langle e, x^k\rangle \leq 0$, i.e., $\langle e, c^k + ke\rangle \leq 0$, and so, using the Cauchy-Schwartz inequality,

$$k\|e\|^2 \leq \langle -e, c^k\rangle \leq \|e\|\,\|c^k\|\,,$$

which implies $\|c^k\| \geq k\|e\|$. So

$$\lim_{k\to\infty} \|c^k\| = \infty\,.$$

We claim that the accumulation points of the normalized sequence $\{c^k/\|c^k\|\}$ belong to $0^+\mathrm{cl}(C)$. Indeed, let $c = \lim_{j\to\infty} c^{k_j}/\|c^{k_j}\|$, $x \in \mathrm{cl}(C)$ and $t \geq 0$. Since $\{c^k\}$ diverges, we have that

$$x + tc = \lim_{j\to\infty}(1 - t/\|c^{k_j}\|)x + (t/\|c^{k_j}\|)c^{k_j}$$

Therefore, since x and c^k belong to the convex set $\mathrm{cl}(C)$, the above sequence of their convex combinations is in $\mathrm{cl}(C)$, and so it's limit belongs to $\mathrm{cl}(C)$, i.e., $x+tc \in \mathrm{cl}(C)$. Hence, $c \in 0^+\mathrm{cl}(C)$, as we claimed. Observe that the sequence $\{c^k/\|c^k\|\}$ has at least one accumulation point because it is bounded.

Note that

$$c^k = x^k - ke.$$

Since $-K$ is a closed convex cone, $c^k \in -K$ and the accumulation points of $\{c^k/\|c^k\|\}$ also belong to $-K$. Therefore, $0^+\mathrm{cl}(C) \cap (-K) \neq \{0\}$. □

Now we discuss a necessary condition for the existence of a separating translation.

Lemma 3.2 *Let $C \subset \mathbb{R}^m$ be convex. If there exists $u \in \mathbb{R}^m$ such that $[C+u] \cap (-K) = \emptyset$, then*

$$0^+C \cap \mathrm{int}(-K) = \{0\}.$$

Proof. We will show that, if

$$0^+C \cap \mathrm{int}(-K) \neq \{0\}\,, \tag{25}$$

then for any $u \in \mathbb{R}^m$,

$$[C+u] \cap (-K) \neq \emptyset\,. \tag{26}$$

Assume that (25) holds and take $v \in 0^+C \cap \mathrm{int}(-K)$. For any $u \in \mathbb{R}^m$ and $c \in C$, there exists some (large enough) $t > 0$ such that

$$(1/t)(c+u) + v \in -K\,.$$

Therefore $c + u + tv \in -K$. On the other hand, since $0^+C = 0^+[C + u]$, $v \in 0^+[C + u]$ and so we have $c + u + tv \in [C + u]$. Altogether, gives $c + u + tv \in [C + u] \cap (-K)$ and (26) holds for any u. $\square$

In particular, all these facts hold for $C = \mathrm{conv}(F(\Omega))$. Whence, concerning the existence of a $u_0 \in \mathbb{R}^m$ such that (1) holds, we have the following results.

Proposition 3.3 *If* $0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap (-K) = \{0\}$*, then there exists a vector* $u_0 \in \mathbb{R}^m$ *such that*

$$\mathrm{conv}(F_{u_0}(\Omega)) \cap (-K) = \emptyset\,.$$

Conversely, if such a u_0 *exists, then* $0^+\mathrm{conv}(F(\Omega)) \cap \mathrm{int}(-K) = \{0\}$.

Proof. Both facts follow immediately from our previous lemmata and the definition of F_{u_0} ($F_{u_0}(x) = F(x) + u_0$ for all $x \in \Omega$).

$\square$

We end this section by giving a straightforward interpretation of Proposition 3.3 from the weighting method's point of view.

Corollary 3.4 *If* $0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap (-K) = \{0\}$*, then there exists a vector* $w \in K^* \setminus \{0\}$ *such that the problem*

$$\min_{x\in\Omega}\langle w, F(x)\rangle$$

is bounded (from below). Conversely, if the above problem is bounded for some $w \in K^* \setminus \{0\}$*, then* $0^+\mathrm{conv}(F(\Omega)) \cap \mathrm{int}(-K) = \{0\}$.

Proof. First assume that $0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap (-K) = \{0\}$. By Proposition 3.3, there exists $u_0 \in \mathbb{R}^m$ such that $\mathrm{conv}(F_{u_0}(\Omega)) \cap (-K) = \emptyset$. Then, using once again the nonstrict Convex Separation Theorem and the fact that $-K$ is a closed cone (see (4)-(6)), we have that

$$\langle w, y\rangle \le 0 \le \langle w, F_{u_0}(x)\rangle \ \ \forall\, x \in \Omega, y \in -K\,,$$

for some $w \in K^* \setminus \{0\}$. Hence,

$$\langle w, -u_0\rangle \le \langle w, F(x)\rangle \ \ \forall\, x \in \Omega\,.$$

Let us now see the converse. Assume that, for some $w \in K^* \setminus \{0\}$,

$$\langle w, F(x) \rangle \geq \alpha \ \ \forall\, x \in \Omega, \tag{27}$$

Suppose that $0^+\mathrm{conv}(F(\Omega)) \cap \mathrm{int}(-K) \neq \{0\}$, and let v be a nonzero m-vector in that intersection. Then, for any $z \in \mathrm{conv}(F(\Omega))$ and $t \geq 0$, $z + tv \in \mathrm{conv}(F(\Omega))$. So, by Caratheodory's Theorem ([3], Proposition 1.3.1),

$$z + tv = \sum_{i=1}^{m+1} \lambda_i F(x^i)\,, \tag{28}$$

where $x^i \in \Omega$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots m+1$ and $\sum_{i=1}^{m+1} \lambda_i = 1$. From (28) and (27), we get

$$\langle z + tv, w \rangle \geq \alpha > -\infty \ \ \forall\, t \geq 0\,. \tag{29}$$

But, since $v \in \mathrm{int}(-K)$ and $w \in K^* \setminus \{0\}$, we have

$$\lim_{t \to +\infty} \langle z + tv, w \rangle = -\infty\,,$$

in contradiction with (29). So such v cannot exist and the conclusion follows.

□

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