

Matrix Bispectrality and Huygens' Principle for Dirac Operators

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ABSTRACT. We explore relations among Huygens' principle for Dirac operators, rational solutions of the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy, and matrix bispectrality. We show how such properties are connected by relating the Hadamard expansion coefficients to the expressions of the nonlinear AKNS flows. The matrix properties above have natural scalar counterparts obtained by reduction to certain manifolds, in which case we get Huygens' principle for wave operators, rational solutions of the Korteweg-de Vries equation and scalar bispectrality. As a by-product we give an alternative proof to a classical result of Lagnese and Stellmacher.

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1. Introduction

J. Hadamard in his celebrated *Lectures on Cauchy's Problem* [**Had53**] focused on a strong version of Huygens' principle that is valid for solutions of the Cauchy

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problem for the wave equation

$$(1.1) \quad \begin{cases} \square\psi \stackrel{\text{def}}{=} \partial_0^2\psi - \sum_{i=1}^n \partial_i^2\psi = 0 . \\ (\psi, \partial_0\psi)\Big|_{x^0=0} = (f, g) \end{cases}$$

provided the number of spatial dimensions $n = 3, 5, 7, \dots$. To wit, the domain of dependence of $\psi(x^0, x^1, \dots, x^n)$ is precisely the sphere in \mathbb{R}^n centered at (x^1, \dots, x^n) of radius $|x^0|$, and does not include its interior. See Figure 1. This is equivalent to saying that the distributional solution $\Phi(\cdot, y)$ to $\square\Phi = \delta_y$ is supported on the light cone

$$(1.2) \quad \mathcal{C}(y) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^{n+1} \mid (x^0 - y^0)^2 = \sum_{i=1}^n (x^i - y^i)^2 \right\} .$$

The localized aspect of the fundamental solution to the unperturbed wave operator \square that results from the strict Huygens' principle is crucial for the meaningful transmission of information. Indeed, if n is even the effect of a localized disturbance in space-time would be felt indefinitely after such event has ceased and its wave front reached the observer. Courant and Hilbert [CH89] expressed the importance of this property in the following:

Thus our physical world, in which acoustic and electromagnetic signals are the basis of communication seems to be singled out among other mathematically conceivable models by intrinsic simplicity and harmony.

The main goal of this article is to analyze some unexpected connection of the strict Huygens property (henceforth Huygens property) mentioned above with the theory of completely integrable systems and solitons at the matrix level. Namely, by looking at Huygens property for Dirac operators and rational solutions of the AKNS hierarchy. This will bring in another seemingly unrelated question, namely the *bispectral property*.

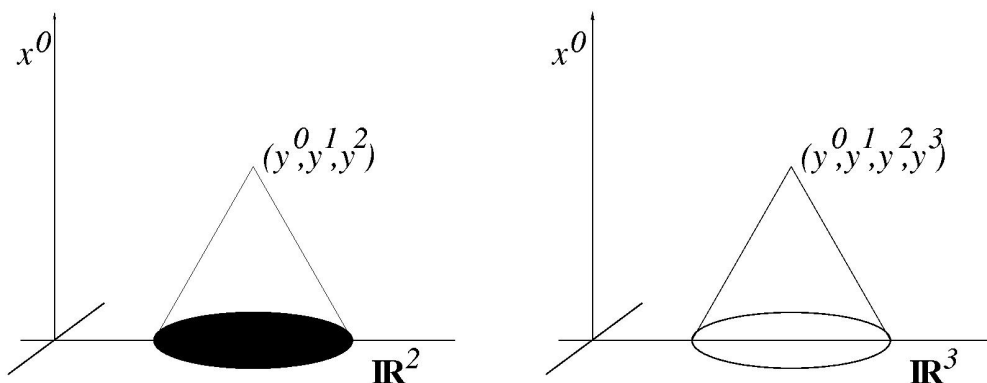


FIGURE 1. Comparison of the domain of dependence for the wave equation solutions in two (left) and three (right) spatial dimensions.

We say that a (linear ordinary) differential operators $L(x, \partial_x)$ is *bispectral* if it possesses a family of eigenfunctions $\Psi(x, k)$ also satisfying a differential equation in the spectral parameter k of the form

$$B(k, \partial_k)\Psi = \Theta(x)\Psi ,$$

where B is a (positive order linear ordinary) differential operator in k independent of x and Θ is independent of k . The motivation for the concept of bispectrality stems from problems in signal analysis such as the time-and-band limiting problem [Sle83, SP61] and the limited-angle tomography problem [Grü84]. For a historical overview, the reader is referred to [Grü94, Grü98, Zub98].

Despite progress on several directions and efforts by a number of authors on the issue of bispectrality the general problem of classifying all the bispectral operators of arbitrary order has only been settled in the scalar case up to order 2 [DG86]. This is not so even for the order 1 matrix case.

In the scalar case, the link between Huygens property and bispectrality is a consequence of an earlier discovery due to Stellmacher and Lagnese [LS67]. They found a class of potentials $u = u(x^0)$ such that $\square + u$ is Huygens. This class can be described in terms of yet another remarkable and seemingly unrelated problem, namely the rational solutions of the Korteweg-de Vries equation [AMM77, AM78, CC77]. It turns out that such rational solutions are also bispectral potentials for the Schrödinger operator $-d^2/dx^2 + u(x)$.

Motivated by the above chain of connections, one is naturally led to consider the relation between matrix analogues of the above objects. One possibility being the following:

- M1 Matrix bispectral operators of AKNS type.
- M2 Rational solutions of the AKNS hierarchy.
- M3 Huygens' property for Dirac operators.

The relation between topics M1 and M2 above has been the subject of [Zub92b]. See also [SZ01] for a related problem and generalization. The relation between topics M2 and M3 was the subject of [CZ03a] which we discuss herein from another viewpoint, namely the termination of the Hadamard series.

One of our objective in the present work is to remark how the three objects above lead to counterparts at the scalar level in a systematic way. In particular, we establish the results of Lagnese and Stellmacher in an alternative way. We also provide proofs to some remarks that are not easily found in the literature, such as the relation between the Adler-Moser polynomials, which yield the rational solutions of the KdV and mKdV hierarchies, and the polynomial τ -functions of AKNS, which are directly constructed in terms of Schur functions.

The general plan is the following:

In Section 2 we describe the AKNS hierarchy of nonlinear evolution equations. This hierarchy is directly connected to Dirac operators and plays a fundamental role in the theory of solitons. We also construct the modified-Korteweg-de Vries (mKdV)

hierarchy as a reduction of the AKNS-hierarchy. Section 3 starts by reviewing some key definitions and results concerning Huygens' principle for wave and Dirac operators. We then present some recent results by the authors on Huygens' principle for Dirac operators. In Section 4 we construct a class of rational solutions to the AKNS hierarchy. This class includes, upon particularization, the rational solutions of the mKdV hierarchy, which in turn, by means of the Miura transformation yields the rational solutions of the KdV hierarchy. Section 5 presents a proof of the above mentioned Lagnese-Stellmacher result that uses only the tools and constructions of the earlier sections. We conclude with a section on the bispectral property, which is the unifying theme of the present work.

2. Construction of the AKNS and mKdV hierarchies

In the present section, we describe the AKNS hierarchy of nonlinear evolution equations. This hierarchy is a generalization of the Korteweg-de Vries equation and played an important role in the development of inverse scattering method. It was proposed by Ablowitz, Kaup, Newell, and Segur [AKNS74] as a way of tackling a number of important equations in Mathematical Physics, such as the cubic Schrödinger equation, the mKdV equation, and the sine-Gordon equation as compatibility conditions of linear systems of equations.

Our approach will be pedestrian and operational. We refer the reader to the many works that deal with soliton theory for a more complete survey [New85, AS81, CD82] as well as motivations.

Let us recursively define three sequences of differential polynomials $\{e_l\}_{l=0}^\infty$, $\{f_l\}_{l=0}^\infty$, and $\{h_l\}_{l=0}^\infty$ on the functions $q = q(x)$ and $r = r(x)$ as follows:

$$(2.1) \quad e_0 \stackrel{\text{def}}{=} 0 \quad f_0 \stackrel{\text{def}}{=} 0 \quad h_0 \stackrel{\text{def}}{=} 1 ,$$

and for $l = 0, 1, \dots$

$$(2.2) \quad e_{l+1} = qh_l + \frac{1}{2}\partial_x e_l ,$$

$$(2.3) \quad f_{l+1} = rh_l - \frac{1}{2}\partial_x f_l ,$$

$$(2.4) \quad h_{l+1} = -\frac{1}{2} \sum_{\substack{m+n=l+1 \\ m,n \geq 1}} (e_n f_m + h_n h_m)$$

DEFINITION 2.1. The system of (nonlinear partial differential) equations

$$(2.5) \quad \begin{cases} q_{t_\ell} = 2e_{\ell+1}(q, \dots, \partial_x^\ell q; r, \dots, \partial_x^{\ell-1} r) \\ r_{t_\ell} = -2f_{\ell+1}(q, \dots, \partial_x^{\ell-1} q; r, \dots, \partial_x^\ell r) . \end{cases}$$

will be called the ℓ -th equation (or the ℓ -th flow) of the AKNS hierarchy.

Thus, $e_1 = q$, $f_1 = r$, $h_1 = 0$,

$$e_2 = \frac{1}{2}q_x, \quad f_2 = -\frac{1}{2}r_x, \quad h_2 = -\frac{1}{2}qr,$$

$$\begin{aligned}
 e_3 &= \frac{1}{4}(-2q^2r + q_{xx}), & f_3 &= \frac{1}{4}(-2qr^2 + r_{xx}), & h_3 &= \frac{1}{4}(-rq_x + qr_x), \\
 e_4 &= \frac{1}{8}(-6qrq_x + q_{xxx}), \\
 f_4 &= \frac{1}{8}(6qrr_x - r_{xxx}), \\
 h_4 &= \frac{1}{8}(3q^2r^2 + q_xr_x - rq_{xx} - qr_{xx}).
 \end{aligned}$$

The first two flows of the AKNS hierarchy take the form:

$$\begin{cases} q_{t_0} &= 2q, \\ r_{t_0} &= -2r, \end{cases}$$

which corresponds to a scaling of the dependent variables q and r , and

$$\begin{cases} q_{t_1} &= q_x, \\ r_{t_1} &= r_x, \end{cases}$$

which corresponds to a translation in the x variable. The first nontrivial flows are given by

$$\begin{cases} q_{t_2} &= -q^2r + \frac{1}{2}q_{xx}, \\ r_{t_2} &= -qr^2 + \frac{1}{2}r_{xx}. \end{cases}$$

For $\ell = 3$, we get

$$\begin{cases} q_{t_3} &= \frac{1}{4}(q_{xxx} - 6qrq_x), \\ r_{t_3} &= \frac{1}{4}(r_{xxx} - 6qrr_x). \end{cases}$$

It is immediate to check that in the $\ell = 3$ case above, $q = \pm r$ gives the modified KdV equation. Furthermore, such reduction is compatible with all the odd flows of the AKNS hierarchy.

The construction above can be motivated as follows:

Let

$$(2.6) \quad R^{(\ell)} \stackrel{\text{def}}{=} (R_0k^\ell + R_1k^{\ell-1} + \cdots + R_\ell),$$

where

$$(2.7) \quad R_\ell \stackrel{\text{def}}{=} e_\ell E + f_\ell F + h_\ell H$$

with

$$H = \text{diag}[1, -1] \quad \text{and} \quad E = F^\top = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One can show [New85, Zub90] that equation (2.5) implies the compatibility condition of

$$(2.8) \quad \partial_x \Psi = (kH + Q)\Psi \stackrel{\text{def}}{=} (kH + qE + rF)\Psi,$$

and

$$(2.9) \quad \partial_{t_\ell} \Psi = R^{(\ell)} \Psi.$$

It is well-known that the flows generated by (2.5) commute with one another and thus it makes sense to consider q and r as functions of arbitrarily many time variables t_1, t_2, \dots . In fact, they are Hamiltonian vector fields whose corresponding Hamiltonian are in involution with one another [New85, AKNS74]. Furthermore, the AKNS hierarchy comes as the compatibility condition of equations (2.8) and (2.9) for arbitrary ℓ . Such compatibility condition can be expressed for $l = 0, 1, \dots$

$$(2.10) \quad e_{l+1} = qh_l + \frac{1}{2}\partial_x e_l ,$$

$$(2.11) \quad f_{l+1} = rh_l - \frac{1}{2}\partial_x f_l ,$$

$$(2.12) \quad \partial_x h_{l+1} = f_{l+1}q - e_{l+1}r .$$

REMARK 2.2. The recursion (2.2-2.4) is one possible solution of such recursive system of equations. In fact, one can characterize the recursion (2.2-2.4) as the only such solution whose term h_k is an isobaric differential polynomial on q and r with weight k if we give $\partial_x^l q$ and $\partial_x^l r$ weight $l + 1$. In this sense, we shall refer to Equation (2.5) with (e_ℓ, f_ℓ) as in (2.2-2.4) as the *canonical* AKNS hierarchy. Other choices of the integration constant in Equation (2.12) lead to linear combinations of the k -th canonical flow with lower order flows.

An important reduction of the AKNS hierarchy is the modified-Korteweg-de Vries (mKdV) one. It is obtained by restricting the flows to $q = r$ and considering only the the odd time-variables. The following lemma confirms that such a reduction makes sense:

LEMMA 2.3. *If we impose $q = r$ in the recursion (2.2), (2.3), (2.4), we have that*

$$(2.13) \quad \begin{cases} e_{2j+1} = f_{2j+1} \\ h_{2j+1} = 0 \\ e_{2j+2} = -f_{2j+2} \end{cases} .$$

PROOF. A straightforward induction in j . For $j = 0$ this is obvious. Suppose valid for $k \leq j - 1$. Then

$$e_{2j+1} - f_{2j+1} = \frac{1}{2}\partial_x(e_{2j} + f_{2j}) = 0 .$$

For the second one, we see in Equation (2.4) that $h_n h_m = 0$ (as $n + m$ is odd, n or m is odd) and then we use the symmetry and anti-symmetry relations between e_j and f_j to conclude that $h_{2j+1} = 0$. Finally,

$$e_{2j+2} + f_{2j+2} = 2qh_{2j+1} + \frac{1}{2}\partial_x(e_{2j+1} - f_{2j+1}) = 0 .$$

□

So, we can write the AKNS recursion, in this particular case

$$(2.14) \quad e_{2k+1} = qh_{2k} + \frac{1}{2}\partial_x e_{2k} ,$$

$$(2.15) \quad e_{2k} = \frac{1}{2}\partial_x e_{2k-1} ,$$

$$(2.16) \quad h_{2k} = -\frac{1}{2} \sum_{n=1}^{2k-1} [(-1)^{n+1} e_n e_{2k-n} + h_n h_{2k-n} ,]$$

with $e_0 = 0$, $h_0 = 1$.

The reader familiar with the mKdV hierarchy as the one obtained from the KdV hierarchy via the Miura transformation $u = \pm q_x + q^2$ can easily check that the two definitions coincide by means of the following argument:

Assuming $q = r$, and using the Lemma 2.3 we can write

$$(2.17) \quad e_{2k} = \frac{1}{2}\partial_x (qh_{2k-2} + \frac{1}{2}\partial_x e_{2k-2})$$

Using (2.12) we have that $\partial_x h_{2k} = -2qe_{2k}$. Denoting by ∂_x^{-1} any right inverse¹ of ∂_x we get $h_{2k} = -2\partial_x^{-1}(qe_{2k})$. This, upon substitution in (2.17), yields the recursion formula

$$(2.18) \quad e_{2k} = \left(\frac{1}{4}\partial_x^2 - q^2 - q_x\partial_x^{-1} \circ q \right) e_{2k-2} , k \geq 1$$

where \circ means composition of operators, $e_0 = 0$ and $e_2 = q_x/2$. Similarly, we have

$$(2.19) \quad e_{2k+1} = \left(\frac{1}{4}\partial_x^2 - q^2 + q\partial_x^{-1} \circ q_x \right) e_{2k-1} , k \geq 1$$

with $e_1 = q$. Equation (2.18) is precisely the recursion operator formula for the mKdV hierarchy. See, for instance, Section 2.3 of [ZM91].

3. Huygens' Principle

Huygens' Principle is one of the unifying threads of the present work. We start by reviewing it in the case of perturbations of the D'Alembertian operator, i.e., operators of the form $\square + u$. Section 3.2 concerns the case of Dirac operators.

3.1. Riesz Kernels and Wave Operators. Our main tool for the construction of fundamental solutions to perturbations of wave and Dirac equations is the notion of Riesz kernels. To define them in our context, we let λ denote the "time-like geodesic distance"

$$\lambda(x) = \begin{cases} \sqrt{(x^0)^2 - \sum_{i=1}^n (x^i)^2}, & \text{if } (x^0)^2 - \sum_{i=1}^n (x^i)^2 \geq 0 , \\ 0 & \text{otherwise.} \end{cases}$$

¹The choice of the constant of integration can be done, for example, by introducing boundary conditions at infinity. It can also be done by requiring that the currents satisfy an isobaric condition with respect to the weights introduced in Remark 2.2. Different choices will give lead to additions by lower order flows in the hierarchy.

For values of $\alpha \in \mathbb{C}$ with positive real part we define the Riesz kernel² as the distribution

$$(3.1) \quad \Lambda^\alpha = N(\alpha)\lambda^\alpha ,$$

where the numerical normalization factor is given by

$$(3.2) \quad N(\alpha) = \frac{1}{2} [2^{\alpha+n} \pi^{(n-1)/2} \Gamma(\frac{\alpha+n+1}{2}) \Gamma(\frac{1}{2}\alpha+1)]^{-1} .$$

The numerical factor $N(\alpha)$ obeys the important recursion rule

$$(3.3) \quad (\alpha+2)(\alpha+n+1)N(\alpha+2) = N(\alpha) .$$

If the real part of α is sufficiently large, with the help of preceding formula, one proves [Fol197]:

$$\square \Lambda^{\alpha+2} = \Lambda^\alpha .$$

So, we can extend by analytic continuation the definition of Riesz kernels for all complex values of α where Λ^α belongs to the space of distributions $\mathcal{D}(\mathbb{R}^{n+1})$.

From equation (3.2) we see that $N(\alpha) = 0$ when $\alpha = -2, -4, -6, \dots$ or $\alpha = -n-1, -n-3, -n-5, \dots$. In these cases Λ^α is supported on the light cone with vertex in the origin. Furthermore, one can show that

$$\Lambda^{-n-1} = \delta_0 .$$

In other words, Λ^{-n+1} is a fundamental solution of \square at $y = 0$. Since \square is invariant by translations it follows that $\Lambda^{-n+1}(x-y)$ is a fundamental solution of \square , i.e.,

$$\square \Lambda^{-n+1}(x-y) = \delta_y(x) .$$

Thus, we conclude that \square satisfies the Huygens property if, and only if, n is odd and greater than 1.

We now consider the case of perturbations of the D'Alembertian: $\square + u$. Here, as before, $\square = \partial_0^2 - \sum_{i=1}^n \partial_i^2$ and $u = u(x^0, \dots, x^n)$ is a given potential in n spatial dimensions (x^1, \dots, x^n) and time x^0 .

In order to construct a fundamental solution Φ of the operator $\square + u$, we consider a series expansion in the above defined Riesz kernels,

$$(3.4) \quad \Phi(x, y) = \sum_{k=0}^{\infty} \Lambda^{-n+1+2k}(x-y) w_k(x, y) .$$

Applying the operator $\square + u$ to Φ and equating to $\Lambda^{-n-1}(x-y) = \delta_y(x) = \delta(x-y)$ we find the recursion

$$(3.5) \quad \begin{aligned} w_0 &= 1 , \\ w_k + \frac{1}{k} \sum_{\mu=0}^n (x^\mu - y^\mu) \partial_\mu w_k &= -(\square + u) w_{k-1} , \end{aligned}$$

²Note that α is an exponent in λ and an index in Λ .

where the derivatives ∂_μ and \square are taken with respect to the x variable³. The coefficients w_k , $k = 1, 2, \dots$, are uniquely determined by the condition that $w_k(x, y)$ be bounded when $x \rightarrow y$ [**Had53**]. For a proof see, for example, Sec. 2 of [**Ber97**].

The recursion (3.5) is known as Hadamard's recursion. Its *termination* at k , i.e., $w_{k'} = 0$, for every $k' > k$, implies the validity of Huygens' principle for $n = 2k + 3$, as can readily be seen from the Riesz kernels properties stated before. The termination requirement amounts to a highly nonlinear condition over u , which makes it very difficult to find examples of Huygens' potentials.

The relations between Huygens' principle for wave operator and the KdV Hierarchy can be established by means of the following result: ⁴

THEOREM 3.1. *Assume that u depends only on the time variable x^0 and let $W_k(x^0)$ denote the Hadamard coefficient $w_k(x, y)$ of Equation (3.5) along the diagonal $y = x$. Then, for $k = 1, 2, \dots$, $W_k(x^0) = W_k[u](x^0)$ is a differential polynomial in u such that*

$$(3.6) \quad \mathbf{X}_k[u](x^0) \stackrel{\text{def}}{=} \partial_{x^0} W_k[u](x^0) ,$$

is the k -th vector field of the KdV hierarchy.

PROOF. First notice that because of Equation (3.5), the assumption that u depends only on x^0 , and the fact that the Hadamard coefficients must be bounded when $x \rightarrow y$ one can show by induction that the quantity $w_k(x, y)$ only depends on (x^0, y^0) . Now we set

$$W_k^p \stackrel{\text{def}}{=} \lim_{y^0 \rightarrow x^0} \partial_0^p w_k(x^0, y^0) .$$

and differentiate p times the second equation in (3.5) with respect to x^0 . Upon taking the limit $y^0 \rightarrow x^0$ we get

$$(3.7) \quad \frac{k+p}{p} W_k^p = -W_{k-1}^{p+2} - \sum_{i=0}^p \binom{p}{i} u^{(p-i)} W_{k-1}^i .$$

Equation (3.7) defines a recursion with initial condition given by $W_0^p = \delta_0^p$. Such recursion coincides, modulo numerical constants, with that for the KdV-hierarchy differential polynomials as described in the proof of Theorem 5.3 of [**Sch95**]. \square

REMARK 3.1. As a corollary to the above result it follows that if $u(x^0)$ is a stationary solution of all the flows of the KdV hierarchy of order (strictly) higher than k

³Many authors consider the ansatz (3.4) with the forward or backward Riesz potentials Λ_\pm^α , where $\Lambda_\pm^\alpha(x) \stackrel{\text{def}}{=} \Lambda^\alpha(x) \chi_{\pm x^0 > 0}(x)$, χ_I is the characteristic function of I , and $\Re\alpha$ is sufficiently large [**Fol95**]. The analytic extension proceeds in the same way described above. It turns out that the recursion relation (3.5) is precisely the same and all the statements concerning the validity of Huygens principle are independent of such restriction. The choice of the $-$ or $+$ sign is related to solutions of the wave equation that propagate forward (respec. backwards). Our choice of ansatz is motivated by the Dirac operator case which we will deal with in the sequel and differs from that in [**Fol95**] by a constant factor of $1/2$.

⁴F.A.C.C.C. thanks Prof. Schimming for mentioning such result.

vanishing at infinity with all its derivatives, then the operator $\square + u(x^0)$ is Huygens' in $n = 2k + 3$ spatial dimensions. Thus establishing a direct link between Huygens' principle and the rational solutions of the KdV hierarchy decaying at infinity which will be pursued further in Section 6. Such rational solutions of the KdV hierarchy in turn provide also a large class of bispectral potential for the Schrödinger operator.

3.2. Huygens' Principle for the Dirac Operator. Dirac operators play a fundamental role in the relativistic quantum mechanics [Tha92]. We start this section by describing such operators in detail.

Let $g^{\mu\nu} = \text{diag}[1, -1, \dots, -1]$ denote the Minkowski tensor. Associated to $g^{\mu\nu}$ we can construct a Clifford Algebra. It is an associative algebra (with identity I) over the reals generated by all linear combinations of the form

$$(\gamma^0)^{m_0} (\gamma^1)^{m_1} \dots (\gamma^n)^{m_n}, \quad m_\mu \in \{0, 1\},$$

where the matrices $\{\gamma^\mu\}$, obey the relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I$. The matrix

$$\bar{\gamma} \stackrel{\text{def}}{=} (-1)^{(n-1)/4} \gamma^0 \gamma^1 \dots \gamma^n$$

will play an important role in the sequel. The Dirac matrices $\{\gamma^\mu\}$, are linearly independent [GM91, Mar98]. They satisfy

$$(3.8) \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \quad (\bar{\gamma})^\dagger = \bar{\gamma}.$$

Dirac operators are defined by [Tha92]:

$$\mathcal{D} = \gamma^\mu \partial_\mu + v,$$

where the summation for repeated indices is implied. We shall adopt the notation $\not{\partial} = \gamma^\mu \partial_\mu$ and restrict ourselves to the case where v is a linear combination of I and $\bar{\gamma}$. It is easy to see that

$$(3.9) \quad \not{\partial}^2 = \square I.$$

For odd n (and only in this case), $\bar{\gamma}$ is uniquely defined, modulo a sign, from the relations $\bar{\gamma} \gamma^\mu + \gamma^\mu \bar{\gamma} = 0$, $\mu = 0, 1, \dots, n$ and $\bar{\gamma}^2 = I$.

By a fundamental solution of the Dirac operator we mean a solution of

$$(\not{\partial} + v)\Psi(\cdot, y) = \delta_y,$$

where δ_y denotes the Dirac-delta distribution supported in an arbitrary point y in space-time. In agreement with Equation (1.2), we denote the light-cone with vertex in y by

$$\mathcal{C}(y) = \{x | (x^\mu - y^\mu)(x_\mu - y_\mu) = 0\}$$

where, as usual we write $x^\mu = g^{\mu\nu} x_\nu$.

We say that a Dirac operator $\not{\partial} + v$ (and by extension, its fundamental solution) obeys *Huygens' principle* (or is of *Huygens' type*) if Ψ satisfies

$$(3.10) \quad \text{supp } \Psi(\cdot, y) \subset \mathcal{C}(y),$$

for every y .

More generally, if we take \mathcal{L} a (strictly) hyperbolic operator defined on a causal domain Ω , we say that \mathcal{L} obeys *Huygens' principle* (or is of *Huygens' type*) if it admits a fundamental solution supported on the characteristic conoid emanating from y , $\forall y \in \Omega$.

From equation (3.9) we can see that $\not\partial\Lambda^\alpha$ is a fundamental solution of the free Dirac operator $\not\partial$ whenever Λ^α is a fundamental solution of wave operator. This motivates the definition of a family of distributions

$$(3.11) \quad \Theta^\alpha \stackrel{\text{def}}{=} \not\partial\Lambda^\alpha ,$$

defined in the same way the Riesz kernels Λ^α were in Section 3.1. We shall refer to (3.11) as Dirac kernels. It is easy to see that

$$(3.12) \quad \Theta^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha + n - 1} \Gamma ,$$

for $\alpha \neq -n + 1$, with $\Gamma \stackrel{\text{def}}{=} \gamma^\mu(x_\mu - y_\mu)$.

REMARK 3.2. The free Dirac operator in n spatial dimensions obeys Huygens' principle if, and only if, n is odd.

It is natural to seek transformations that trivially preserve Huygens' property. In a similar way to what is done for the wave equation, one is led to consider transformations that trivially preserve Huygens' principle.

DEFINITION 3.3. We call the following operations *trivial transformations* for Dirac operators (in the sense of Huygens principle):

1. Change of independent variables by a smooth (C^∞) diffeomorphism: $\tilde{x}^\mu = f^\mu(x^0, \dots, x^n)$, $\mu = 0, \dots, n$, with $\det(\partial_\mu f^\nu)_{\mu, \nu=0, \dots, n} \neq 0$.

2. Left multiplication:

Take $\mathcal{D} \mapsto \bar{\mathcal{D}} = \Xi(x)\mathcal{D}$ and $\Psi \mapsto \bar{\Psi} = \Psi\Xi(y)^{-1}$, where $\Xi \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{N \times N})$ and $\Xi(x)$ is a non-singular matrix for all x .

3. Factor transformations:

Let ρ be a non-singular smooth matrix-valued function, i.e.,

$$\rho = \rho_\phi I + \rho_\mu \gamma^\mu + \bar{\rho}_\mu \gamma^\mu \bar{\gamma} + \bar{\rho} \bar{\gamma} ,$$

where ρ_ϕ , ρ_μ , $\bar{\rho}_\mu$ and $\bar{\rho}$ are smooth functions with $\det(\rho(x)) \neq 0$, for all x in the domain under consideration. The factor transformation consists in the mapping $\mathcal{D} \mapsto \bar{\mathcal{D}} = \rho(x)\mathcal{D}\rho(x)^{-1}$ and $\Psi \mapsto \bar{\Psi} = \rho(x)\Psi\rho(y)^{-1}$.

We can thus state the following:

LEMMA 3.1. *The transformations in Definition 3.3 preserve the validity of Huygens' principle.*

Two operators that are related by means of compositions of the above transformations above will be called *trivially equivalent*.

We look for a solution Ψ to (3.10) that takes the form of a series expansion such as

$$(3.13) \quad \Psi = \sum_{k=0}^{\infty} \left\{ \Theta^{\alpha_0+2k} s_{2k} + \Lambda^{\alpha_0+2k} s_{2k+1} \right\},$$

where $s_k = s_k(x, y)$ is a smooth matrix coefficient and $\alpha_0 = -n + 1$.

We consider $v = aI + \bar{a}\bar{\gamma}$ and $v^* = aI - \bar{a}\bar{\gamma}$. Applying $\not\partial + v$ to Ψ we find

$$\begin{aligned} (\not\partial + v)\Psi &= \Lambda^{\alpha_0-2} \left[s_0 + \frac{2}{\alpha_0 + n - 1} (x^\mu - y^\mu) \partial_\mu s_0 \right] + \\ &\quad + \sum_{k=0}^{\infty} \Theta^{\alpha_0+2k} \left[-\not\partial s_{2k} + s_{2k+1} + v^* s_{2k} \right] + \\ &\quad + \sum_{k=1}^{\infty} \Lambda^{\alpha_0+2k-2} \left[s_{2k} + \frac{2}{\alpha_0 + 2k + n - 1} (x^\mu - y^\mu) \partial_\mu s_{2k} + \not\partial s_{2k-1} + v s_{2k-1} \right]. \end{aligned}$$

Equating $(\not\partial + v)\Psi = \Lambda^{-n-1}$ we find *Hadamard's recursion*:

$$(3.14) \quad \begin{aligned} s_0 &= 1, \\ s_{2k+1} &= (\not\partial - v^*) s_{2k}, \\ s_{2k} + \frac{1}{k} (x^\mu - y^\mu) \partial_\mu s_{2k} &= -(\not\partial + v) s_{2k-1}. \end{aligned}$$

We now recall that the Riesz (and therefore also the Dirac) kernels of order α in n spatial dimensions are supported on the light-cone when $\alpha = -2, -4, -6, \dots$ or $\alpha = -(n+1), -(n+3), \dots$. By making use of the Hadamard series expansion we get the following useful result:

THEOREM 3.2. *Let $n \geq d \in \mathbb{N}$ be an odd integer. If the Hadamard coefficients s_k for the operator $\not\partial + v$ vanish for all $k \geq d$, then $\not\partial + v$ is a Huygens' type operator.*

As in the wave operator case, the difficulty in using the above result lies in the fact that the vanishing of s_k for $k \geq 2$ is a highly nonlinear system of partial differential equations on v . Nevertheless, we were able to use Theorem 3.2 as a way of explicitly constructing examples of Huygens potentials for Dirac operators. Indeed, in [CZ03a] we constructed Dirac operators of Huygens' type by making use of Schlesinger transformations from soliton theory [New85] and to prove the following:

THEOREM 3.3. *Let $q(x^0)$ and $r(x^0)$ be rational solutions of the AKNS hierarchy given by (4.4) and*

$$(3.15) \quad \mathcal{D} = \not\partial - \frac{q+r}{2} I + \frac{q-r}{2} \bar{\gamma}.$$

Then, the Hadamard series for \mathcal{D} is terminating and therefore \mathcal{D} has the Huygens' property for any sufficiently high number of odd space dimensions.

PROOF. See [CZ03a]. □

It turns out that the potentials q and r mentioned in Theorem 3.3 are also bispectral potentials for the corresponding AKNS operators, as we shall see in Section 6.

As in the wave operator case, we can generate differential polynomials associated to the Hadamard coefficients along the diagonal $x = y$. In the present case, the polynomials will be related, as one would expect, to the AKNS hierarchy. Hadamard's recursion (3.14) can be solved by matrix coefficients depending only on (x^0, y^0) and thus can be re-written as

$$\begin{aligned} s_0(x^0, y^0) &= I, \\ s_{2k+1}(x^0, y^0) &= \gamma^0 \partial_0 s_{2k}(x^0, y^0) - v^*(x^0) s_{2k}(x^0, y^0), \\ s_{2k}(x^0, y^0) + \frac{1}{k}(x^0 - y^0) \partial_0 s_{2k}(x^0, y^0) &= -\gamma^0 \partial_0 s_{2k-1}(x^0, y^0) - v(x^0) s_{2k-1}(x^0, y^0). \end{aligned}$$

We solve the first terms to get

$$(3.16) \quad s_0(x^0, y^0) = I,$$

$$(3.17) \quad s_1(x^0, y^0) = -a(x^0)I + \bar{a}(x^0)\bar{\gamma},$$

$$(3.18) \quad s_2(x^0, y^0) = \frac{1}{x^0 - y^0} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz I + \frac{a(x^0) - a(y^0)}{x^0 - y^0} \gamma^0 - \frac{\bar{a}(x^0) - \bar{a}(y^0)}{x^0 - y^0} \gamma^0 \bar{\gamma},$$

$$(3.19) \quad s_3(x^0, y^0) = \left(\frac{a'(x^0)}{x^0 - y^0} - \frac{a(x^0) - a(y^0)}{(x^0 - y^0)^2} - \frac{a(x^0)}{x^0 - y^0} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz \right) I + \left(-\frac{1}{(x^0 - y^0)^2} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz + \frac{a(x^0)a(y^0)}{x^0 - y^0} - \frac{\bar{a}(x^0)\bar{a}(y^0)}{x^0 - y^0} \right) \gamma^0 + \left(-\frac{\bar{a}'(x^0)}{x^0 - y^0} + \frac{\bar{a}(x^0) - \bar{a}(y^0)}{(x^0 - y^0)^2} + \frac{\bar{a}(x^0)}{x^0 - y^0} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz \right) \bar{\gamma} + \frac{a(x^0)\bar{a}(y^0) - a(y^0)\bar{a}(x^0)}{x^0 - y^0} \gamma^0 \bar{\gamma}.$$

We define $p_k(x^0) = s_k(x^0, x^0)$ and find that

$$\begin{aligned} p_0 &= I, \\ p_1 &= -aI + \bar{a}\bar{\gamma}, \\ p_2 &= (a^2 - \bar{a}^2)I + a'\gamma^0 - \bar{a}'\gamma^0\bar{\gamma}, \\ p_3 &= \left(\frac{a''}{2} - a^3 + a\bar{a}^2 \right) I - \left(\frac{\bar{a}''}{2} + \bar{a}^3 - a^2\bar{a} \right) \bar{\gamma} - (a\bar{a}' - a'\bar{a}) \gamma^0 \bar{\gamma}. \end{aligned}$$

The relations between the components of p_k in the directions given by the elements of the Clifford algebra and the initial AKNS hierarchy flows given in Section 2 can

be explicitly verified, modulo some constants, by substituting

$$\begin{aligned} q &= -(a - \bar{a}) , \\ r &= -(a + \bar{a}) . \end{aligned}$$

In particular from these examples, it becomes plain that termination of the Hadamard expansion is equivalent to having (q, r) stationary solutions of all sufficiently high canonical AKNS flows. Such examples turn out to be particular cases that lead to the following:

THEOREM 3.4 ([CZ03b]). *If a Dirac operator of the form*

$$(3.20) \quad \not{D} - \frac{q+r}{2}I + \frac{q-r}{2}\bar{\gamma} ,$$

has a terminating Hadamard expansion, then (q, r) is a stationary solution of the canonical AKNS hierarchy flows. Furthermore, if (q, r) is a stationary solution of the AKNS hierarchy (2.10), (2.11), (2.12), i.e., $e_k = f_k = h_k = 0$, $k \geq k_0$, with

$$(3.21) \quad \lim_{|x| \rightarrow \infty} q^{(i)}r^{(j)} = 0 , \quad i, j \geq 0$$

then the operator given by (3.20) is of Huygens type.

For a proof see [CZ03b].

A consequence of Theorem 3.4 is the fact that any rational solution of the AKNS hierarchy with the decaying property (3.21) and associated with τ -functions that only depend on finitely many variables leads to Dirac operators of Huygens' type. We shall discuss examples of this situation in Section 4

4. Special Solutions of the AKNS and mKdV Hierarchies

In this section we present explicit expressions for classes of rational solutions to the KdV hierarchy and the AKNS hierarchy that yield operators of Huygens type.

We start by introducing an infinite sequence of polynomials $\{\mathcal{Q}_j(y_1, y_2, \dots, y_j)\}_{j=0}^{\infty}$ defined by the requirement that

$$(4.1) \quad \sum_{j \geq 0} \lambda^j \mathcal{Q}_j = \exp \left(\sum_{j \geq 1} \lambda^j y_j \right) .$$

We extend the definition of Schur polynomials as zero for negative indices. Using this convention it is easy to check that

$$(4.2) \quad \partial_{y_k} \mathcal{Q}_j = \mathcal{Q}_{j-k} .$$

Now, we introduce the so-called Schur functions that are given by the Wronskians

$$(4.3) \quad \tau_j^d = W[\mathcal{Q}_d, \mathcal{Q}_{d-1}, \dots, \mathcal{Q}_{d-j}] ,$$

where the derivatives should be interpreted with respect to the first variable y_1 . We set $y_1 = x$ and take

$$(4.4) \quad q = \frac{\tau_{j-1}^d}{\tau_j^d}, \quad r = -\frac{\tau_{j+1}^d}{\tau_j^d}.$$

For convenience, we extend the definition (4.3) of τ_j^d as follows:

$$\tau_j^d \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j = -1 \text{ and } d = 0, 1, \dots, \\ 0 & \text{if } j < -1 \text{ or } d < 0. \end{cases}$$

In order to relate the variables y_k in the Schur polynomials with the time variables t_k of the AKNS hierarchy, we set

$$(4.5) \quad y_k = \frac{1}{(-2)^{k-1}} t_k, \quad k = 1, 2, \dots.$$

We have the following result:

THEOREM 4.1 (R.Sachs [Sac88]). *For every pair of integers d and j , such that $d \geq j > -1$, the pair of fields (q, r) defined by equations (4.3), (4.4), and (4.5) is a rational solution of the AKNS hierarchy.*

For an alternative proof of Theorem 4.1 see Section 5 of [Zub92b].

As we mentioned in Section 2, the mKdV case is a consistent reduction of the general AKNS hierarchy. On the other hand, if we let q be a rational solution of the mKdV equation vanishing at infinity, then $u = \pm q_x + q^2$ is a rational solution of the KdV vanishing at infinity. The full classification of the rational solutions of the KdV decaying at infinity is well known. See [AMM77, AM78, CC77]. Thus, it follows from [AMM77, AM78] that the rational solutions of the KdV can be expressed in the form ⁵

$$(4.6) \quad u = 2\partial_x^2 \log \vartheta_k(x),$$

where ϑ_k is the k -th Adler-Moser polynomials. It is not hard to check that the corresponding rational solution of the mKdV takes the form

$$\pm \partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x),$$

for some k and a suitable choice of parameters for the Adler-Moser polynomials ϑ_{k+1} and ϑ_k .

4.1. Reduction of the AKNS to the mKdV.

We start by proving that

LEMMA 4.1. *If $y_{2k} = 0$ for all $k \in \mathbb{N}$ then*

$$\tau_{j+1}^{2j+1} + \tau_{j-1}^{2j+1} = 0.$$

⁵Our choice for the Schrödinger operator $L = \partial_x^2 + u(x)$ leads to the + sign in Equation (4.6).

PROOF. Define the “infinite vector” $\vec{y} = (y_1, y_2, \dots)$ and its scaled version as

$$\vec{y}_a = (ay_1, a^2y_2, \dots, a^ky_k, \dots) ,$$

for any real valued a . We easily find that

$$\mathcal{Q}_j(\vec{y}_a) = a^j \mathcal{Q}_j(\vec{y}) .$$

If we choose $a = -1$ and fix all even terms $y_{2k} = 0$, we have $\vec{y}_a = -\vec{y}$ and, consequently,

$$\mathcal{Q}_j(-\vec{y}) = (-1)^j \mathcal{Q}_j(\vec{y}) .$$

From the definition (4.1), we have

$$\left[\sum_{j \geq 0} \lambda^j \mathcal{Q}_j(-\vec{y}) \right] \left[\sum_{j \geq 0} \lambda^j \mathcal{Q}_j(\vec{y}) \right] = 1 .$$

We equate term-by-term the above Taylor expansion in λ to find, with the help of previous equations, that (we omit the parameter \vec{y})

$$(4.7) \quad \mathcal{Q}_0^2 = 1 ,$$

$$(4.8) \quad \sum_{p=0}^j (-1)^p \mathcal{Q}_p \mathcal{Q}_{j-p} = 0 \quad j = 1, 2, \dots .$$

Let us define the $(j+2) \times (j+2)$ matrix

$$A^{(0)} = (a_{n,m}^{(0)}) ,$$

where $n, m = 0, \dots, j+1$ and

$$a_{n,m}^{(0)} = \mathcal{Q}_{2j+1-n-m} .$$

In this case

$$\tau_{j+1}^{2j+1} = \det A^{(0)} .$$

We define

$$A^{(1)} = (a_{n,m}^{(1)}) ,$$

with

$$a_{n,m}^{(1)} = \sum_{p=0}^{j+1-m} (-1)^p a_{n,m+p} \mathcal{Q}_p ,$$

for $n, m = 0, \dots, j+1$. Column m of matrix $A^{(0)}$ is substituted by a linear combination of all the columns, where the weight of the column m is \mathcal{Q}_0 . So

$$\det A^{(1)} = \mathcal{Q}_0^{j+1} \det A^{(0)} .$$

We redefine the left upper block

$$A^{(2)} = (a_{n,m}^{(2)}) = (a_{n,m}^{(1)}) ,$$

with $n, m = 0, \dots, j-1$. From properties (4.7) and (4.8) we find

$$A^{(1)} = \begin{pmatrix} & & & \vdots & * & * \\ & & & \vdots & * & * \\ & & & \vdots & * & * \\ \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 0 \end{pmatrix} .$$

Consequently

$$\det A^{(1)} = -\det A^{(2)} .$$

Again, from property (4.8), we find

$$a_{n,m}^{(2)} = (-1)^{n+m} \sum_{p=0}^{j-1-n} (-1)^p \mathcal{Q}_{2j+1-n-m-p} \mathcal{Q}_p .$$

We construct the matrix $A^{(3)}$ by multiplying the row n by $(-1)^n$ and the column m by $(-1)^m$,

$$a_{n,m}^{(3)} = \sum_{p=0}^{j-1-n} (-1)^p \mathcal{Q}_{2j+1-n-m-p} \mathcal{Q}_p ,$$

with $n, m = 0 \dots, j-1$. So $\det A^{(3)} = \det A^{(2)}$.

Now, define a new matrix

$$\begin{aligned} \bar{A}^{(0)} &= (\bar{a}_{n,m}^{(0)}) , \\ \bar{a}_{n,m}^{(0)} &= \mathcal{Q}_{2j+1-n-m} , \end{aligned}$$

with $n, m = 0 \dots, j-1$. We construct the matrix $\bar{A}^{(1)}$ taking linear combinations of the rows in the matrix $\bar{A}^{(0)}$,

$$\bar{a}_{n,m}^{(1)} = \sum_{p=0}^{j-1-n} (-1)^p \bar{a}_{n+p,m}^{(0)} \mathcal{Q}_p .$$

We easily find that

$$\begin{aligned} \bar{A}^{(1)} &= A^{(3)} , \\ \det \bar{A}^{(0)} &= \tau_{j-1}^{2j+1} \end{aligned}$$

and

$$\det \bar{A}^{(1)} = \mathcal{Q}_0^{j-1} \det \bar{A}^{(0)} .$$

We conclude

$$\begin{aligned} \tau_{j-1}^{2j+1} &= \det \bar{A}^{(0)} = \frac{1}{\mathcal{Q}_0^{j-1}} \det \bar{A}^{(1)} = \frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(3)} = \\ &= \frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(2)} = -\frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(1)} = -\mathcal{Q}_0^2 \det A^{(0)} = -\tau_{j+1}^{2j+1} . \end{aligned}$$

This finishes the proof. \square

LEMMA 4.2. *If $y_k = 0$ for $k \geq 2$ then*

$$\tau_{j+1}^{2j+1} = (-1)^{j+1}(j+1)\tau_j^{2j+1}\frac{1}{y_1}.$$

PROOF. From equation (4.1) we have

$$\mathcal{Q}_j = \frac{y_1^j}{j!}.$$

From now on we use the convention that $1/j! = 0$ for $j < 0$. The Hirota variables τ_j^d , given by equation (4.3), are monomials in y_1 of degree $(d-j)(j+1)$. Thus, we can suppose $y_1 = 1$ without loss of generality. We define the $(j+2) \times (j+2)$ matrix

$$B^{(0)} = (b_{n,m}^{(0)})$$

where $n, m = 0, \dots, j+1$ and

$$b_{m,n}^{(0)} = \frac{1}{(2j+1-m-n)!}.$$

We write

$$b_{n,m}^{(1)} = \begin{cases} -(j-n)!b_{n,m}^{(0)} + (j-n-1)!b_{m,n+1}^{(0)} & n = 0, \dots, j+1, \quad m = 0, \dots, j-1; \\ -b_{n,m}^{(0)} & n = 0, \dots, j+1, \quad m = j; \\ b_{n,m}^{(0)} & n = 0, \dots, j+1, \quad m = j+1. \end{cases}$$

Then

$$\det B^{(1)} = (-1)^{j+1} \left[\prod_{n=0}^{j-1} (j-n)! \right] \det B^{(0)},$$

where $B^{(1)} = (b_{n,m}^{(1)})$, with $n, m = 0, \dots, j+1$.

We easily prove that if $n = j+1$ and $m = 0, \dots, j-1$, then $b_{j+1,m}^{(1)} = 0$. Furthermore $b_{j+1,j}^{(1)} = -1$ and $b_{j+1,j+1}^{(1)} = 0$.

We construct a new matrix,

$$B^{(2)} = (b_{n,m}^{(2)})$$

with $n, m = 0, \dots, j$ and

$$b_{n,m}^{(2)} = \begin{cases} b_{n,m}^{(1)} & n = 0, \dots, j, \quad m = 0, \dots, j-1 \\ b_{n,m+1}^{(1)} & n = 0, \dots, j, \quad m = j. \end{cases}$$

Then

$$\det B^{(2)} = \det B^{(1)}.$$

We rewrite, after a simple computation,

$$b_{n,m}^{(2)} = \begin{cases} \frac{(j-m-1)!(j+1-n)}{(2j+1-n-m)!} & n = 0, \dots, j, \quad m = 0, \dots, j-1; \\ \frac{1}{(j-m)!} & n = 0, \dots, j, \quad m = j. \end{cases}$$

We define a new matrix $B^{(3)} = (b_{n,m}^{(3)})$ with $n, m = 0, \dots, j$ such that

$$b_{n,m}^{(3)} = \begin{cases} \frac{1}{(j+1-n)} \frac{1}{(j-1-m)!} b_{n,m}^{(2)} & n = 0, \dots, j, \quad m = 0, \dots, j-1; \\ \frac{1}{(j+1-n)} b_{n,m}^{(2)} & n = 0, \dots, j, \quad m = j. \end{cases}$$

It is immediate to see that

$$\det B^{(3)} = \left[\prod_{n=0}^j (j+1-m) \prod_{m=0}^{j-1} (j-1-m)! \right]^{-1} \det B^{(2)}.$$

From the definition we have that

$$b_{n,m}^{(3)} = \frac{1}{(2j+1-n-m)!},$$

with $n, m = 0, \dots, j$. Then

$$\tau_j^{2j+1} = \det B^{(3)}.$$

We conclude noting that

$$\begin{aligned} \tau_{j+1}^{2j+1} = \det B^{(0)} &= (-1)^{j+1} \left[\prod_{m=0}^{j-1} (j-n)! \right]^{-1} \det B^{(1)} = (-1)^{j+1} \left[\prod_{m=0}^{j-1} (j-n)! \right]^{-1} \det B^{(2)} = \\ &= (-1)^{j+1} \frac{(j+1)!}{j!} \det B^{(3)} = (-1)^{j+1} (j+1) \tau_j^{2j+1}. \end{aligned}$$

Relaxing the hypothesis that $y_1 = 1$ we finish the proof. \square

With the help of the preceding lemmas, we can prove

THEOREM 4.2. *If $y_{2k} = 0$ for all $k \in \mathbb{N}$, then*

$$\frac{\tau_{j+1}^{2j+1}}{\tau_j^{2j+1}}(x, 0, y_3, 0, y_5, \dots) = (-1)^{j+1} \partial_x \log \frac{\vartheta_{j+1}}{\vartheta_j}(x, \chi_2, \chi_3, \dots),$$

where ϑ_j is the j -th Adler-Moser polynomial and for a suitable choice of the free variables y_3, y_5, \dots and χ_2, χ_3, \dots .

PROOF. From the Lemma 4.1 and Equation (4.4) we get $q = r$, so we are working with solutions to the mKdV hierarchy. All the solutions have been classified and are of the form

$$\pm \partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x),$$

for $k \in \mathbb{N}$. See [AM78]. So, for each $j \in \mathbb{N}$, there is a k such that

$$\frac{\tau_{j+1}^{2j+1}}{\tau_j^{2j+1}}(x) = \pm \partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x),$$

for a particular choice of the variables χ_2, χ_3, \dots . Let us consider $|y_1| \gg |y_i|$ for $2 \leq i \leq j$. In this case,

$$Q_j \sim \frac{y_1^j}{j!}.$$

Then, from Lemma 4.2 the LHS is asymptotically equal to $(-1)^{j+1}(j+1)/y_1$. To the RHS, consider the asymptotic behavior in $\chi_1 \rightarrow \infty$. In this case, $\vartheta_k \sim \chi_1^{k(k+1)/2}$, and then

$$\partial_{\chi_1} \log \frac{\vartheta_{k+1}}{\vartheta_k}(\chi) \sim \frac{k+1}{\chi_1} .$$

Remembering that both k and j are positive, and upon choosing the correct sign we conclude that $k = j$. Furthermore $\chi_1 = x = y_1$ and as the above approximation should be exact, there exists suitable χ_k for $k \geq 2$ such that the identity holds. \square

5. Lagnese and Stellmacher Revisited

In their seminal paper [LS67] Lagnese and Stellmacher developed (independently of [AM78]) a construction of what is known in the soliton literature as Darboux transformations. The purpose of the present section is to show that the results of Lagnese and Stellmacher can be reproduced *without the use of Darboux transformations*. Using the results from the previous sections this can be done in fact in different instructive ways. One way is by using that the rational solutions of KdV hierarchy are stationary of sufficiently high flows and then using Theorem 3.1. The other way is by using the general results for rational solutions of AKNS and Huygens' property for Dirac operators, then reducing to the mKdV, and projecting onto the KdV. Thus, following one of the main themes of the present work, namely obtaining the scalar level properties from those at the matrix level.

LEMMA 5.1. *Let $v(x^0)$ be a scalar potential, i.e., $\bar{a} = 0$. If $\not\partial + v$ is terminating (and consequently, has the Huygens' property) and u^\pm are the Miura transforms of v , i.e., $u^\pm = \pm v' + v^2$, then $\square - u^\pm$ has the Huygens' property.*

PROOF. First we note that if $\not\partial + v$ is terminating, so is $\not\partial - v$.

This is consequence of the identity

$$-\bar{\gamma}(\not\partial + v)\bar{\gamma} = \not\partial - v .$$

Consider Ψ^\pm and Φ^\pm , the fundamental solutions of $\not\partial \pm v$ and $\square - u^\pm$ respectively. In the Pauli-Dirac representation

$$(\not\partial - v)(\not\partial + v) = \square + \gamma^0 v' - v^2 = \begin{pmatrix} \square - u^- & 0 \\ 0 & \square - u^+ \end{pmatrix} = (\square - u^-) \oplus (\square - u^+) .$$

In a similar way

$$(\not\partial + v)(\not\partial - v) = (\square - u^+) \oplus (\square - u^-) .$$

Consider the Hadamard series given by (3.13) and (3.4) where the coefficients s_k^\pm and w_k^\pm obey recursions (3.14) and (3.5) respectively, with s_k^\pm and w_k^\pm depending only on x^0 and y^0 . The even coefficients obey the recursion

$$s_{2k}^\pm + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k}^\pm = -(\not\partial \pm v)(\not\partial \mp v)s_{2k-2}^\pm = -(\square - u^\pm) \oplus (\square - u^\mp)s_{2k-2}^\pm .$$

We conclude that

$$s_{2k}^\pm = w_k^\pm \oplus w_k^\mp .$$

Then s_k^\pm are terminating if, and only if, w_k^\pm are terminating. Noticing that if s_k^+ is terminating, so is s_k^- , we conclude the proof. \square

THEOREM 5.1. *If $u_j = -2\partial_0^2 \log \vartheta_j(x^0)$, where ϑ_j is the j -th Adler-Moser polynomial, then $\square - u_j(x^0)$ is a Huygens type operator in dimension $n = 2j + 3$.*

PROOF. Let d be equal $2j+1$ and set all even times to zero in the AKNS hierarchy, i.e., $t_{2k} = 0 = y_{2k}$. From Lemma 4.1 and from Theorem 4.2 (changing signs, if necessary) we find that the Dirac operator in equation (3.15) is

$$\partial - \partial_0 \log \frac{\vartheta_{j+1}}{\vartheta_j}(x^0)I ,$$

which is Huygens for dimension $n = 2j + 3$ by Theorem 3.3, i.e., $s_{2j+4} = 0$. From the Lemma 5.1, we have that $\square - u^\pm$ has the Huygens' property, where u^\pm are the Miura transforms of v . Then

$$\begin{aligned} u^+ &= -2\partial_0^2 \log \vartheta_j , \\ u^- &= -2\partial_0^2 \log \vartheta_{j+1} \end{aligned}$$

are both Huygens in $n = 2j+5$, since $w_{j+2}^\pm = 0$. We conclude that $\square - u_j$ is of Huygens type in dimension $2j + 3$, exactly the classical Lagnese and Stellmacher result. \square

6. The Bispectral Property and Conclusions

The simple fact that a certain evolution equation admits families of rational solutions is remarkable *per se*. The fact that such families turn out to be solutions to apparently unrelated problems deserves due investigation. One of such problems turns out to be the bispectral problem, which was originally posed by A. Grünbaum in connection with certain specific problems in the spectral theory of integral operators and tomography. We now explore such connections “vis-à-vis” Huygens property for Dirac operators.

In Theorem 5.1 we have seen that the j -th Adler-Moser polynomial yields potentials of the form

$$u_j(x^0) = -2\partial_x^2 \log \vartheta_j(x^0)$$

whose corresponding perturbations of the D'Alembertian are operators of Huygens' type for suitable spatial dimensions. In [DG86], it was shown that Schrödinger operators $L(x, \partial_x) = -\partial_x^2 + u(x)$ with the potentials of the form $u(x) = u_j(x)$

are bispectral. In fact, Duistermaat and Grünbaum [DG86] characterized all the bispectral potentials for Schrödinger operators. The ones associated to the Adler-Moser polynomials are only “half” of such bispectral potentials. They can be characterized by the condition that (generically) L is associated to a commutative algebra of differential operators of rank 1. Another way of saying this is that the space of

functions $\Psi(x, k)$ that satisfy simultaneously the equation $L(x, \partial_x)\Psi = k\Psi$ and the equation in the spectral parameter $B(k, \partial_k)\Psi = \Theta(x)\Psi$ is at most one dimensional⁶.

A very natural question, once one is familiar with the results in [DG86], is whether rational solutions of other hierarchies of integrable PDEs would lead to bispectral operators. This was answered affirmatively in different directions, both at the higher order operator case as well as in the matrix operator case. Indeed, in [Zub92a] it was shown that this is indeed the case for a large class of rational solutions of the Kadomtsev-Petviashvili hierarchy. See also [Wil93b]. On the matrix level, in [Zub92b], it was shown that polynomial τ -functions for the AKNS hierarchy constructed in Section 2 lead to bispectral operators of the form

$$(6.1) \quad \mathbf{L} = \begin{bmatrix} \partial_x & -q \\ r & -\partial_x \end{bmatrix}$$

The operator \mathbf{L} in Equation (6.1) is associated to the construction of the AKNS hierarchy of integrable equations we developed in Section 2. Indeed, the compatibility condition approach in the AKNS hierarchy motivation given in that section is equivalent to a Lax-pair construction, in which case the key operator is the one in (6.1).

Furthermore, the operator \mathbf{L} is directly connected to the Dirac operators operators we have been studying. Indeed, let ⁷

$$(6.2) \quad \mathbf{H} \stackrel{\text{def}}{=} \bar{\gamma}\partial_0 - \frac{1}{2}(q+r)\bar{\gamma}\gamma^0 - \frac{1}{2}(q-r)\gamma^0.$$

A simple computation shows that the operators $\mathbf{H} + \sum_{j=1}^n \bar{\gamma}\gamma^0\gamma^j\partial_j$ and

$$(6.3) \quad \mathcal{D} = \partial - \frac{q+r}{2}I + \frac{q-r}{2}\bar{\gamma}$$

are trivially equivalent. The Weyl representation of \mathbf{H} is

$$(6.4) \quad \mathbf{H} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \partial_0 - q \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} + r \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix},$$

where I is the identity matrix of order $2^{(n-1)/2} \times 2^{(n-1)/2}$.

It is clear that \mathbf{H} is bispectral if, and only if, \mathbf{L} is bispectral. Furthermore, the AKNS hierarchy associated to \mathbf{L} in (6.1) admits a family of stationary solutions of the form

$$(6.5) \quad q = \sigma/\tau$$

and

$$(6.6) \quad r = \rho/\tau,$$

where the τ -function triple (σ, τ, ρ) is composed of Schur functions in the variables (x, t_2, t_3, \dots) .

⁶At least generically in the parameters that enter the Adler-Moser polynomials

⁷The operator in \mathbf{H} in (6.2) plays an important role in the proof of Theorem 3.3 given in [CZ03a].

The bispectrality associated to rational solutions of the AKNS hierarchy alluded above was established in [Zub92b] by means of the following:

THEOREM 6.1. *Let (q, r) be a rational solution of the AKNS hierarchy such that the associated Hirota variables σ , τ and ρ are polynomials in $t_1 = x, t_2, \dots, t_m$. Suppose t_2, \dots, t_m is such that σ , ρ , and τ have disjoint roots. Then, for any polynomial $\Theta(x)$, satisfying $\deg(\Theta) \geq \max\{\deg(\sigma), \deg(\tau)\}$, there exists non-degenerate operators $B^\pm(k, \partial_k)$, independent of x , such that*

$$B^\pm(k, \partial_k)\Psi^\pm = \Theta(x)\Psi^\pm ,$$

where Ψ^\pm are independent eigenfunctions of \mathbf{L} .

We saw in Section 3 that the above rational solutions of the AKNS hierarchy yield operators of the form (6.3) of Huygens type.

REMARK 6.1. The previous result also holds when we restrict to the mKdV hierarchy. This in turn, allows us to connect the present work with the bispectral property for the Schrödinger operator as studied in [DG86].

REMARK 6.2. The bispectral eigenfunctions Ψ^\pm can be written down explicitly in terms of the so-called vertex operators [Zub92b].

6.1. Conclusions. We have illustrated how the bispectral property for differential operators, Huygens principle, and rational solutions of integrable PDEs are related both at the scalar and matrix cases.

At the scalar level the example centered around known results for the Schrödinger operator, the perturbed D'Alembertian, and the family of rational solutions of the KdV hierarchy (vanishing at infinity). We presented, however, a new proof of the classical result of Lagnese-Stellmacher. Such result was demonstrated as a consequence of a property of termination of the Hadamard series. We believe this property lies in the heart of Huygens' principle, at least for the cases we have focused so far. The connection with the bispectral property followed then from the seminal work of Duistermaat-Grunbaum [DG86].

At the matrix level, we dealt with the AKNS-type operator \mathbf{L} of Equation (6.1). The corresponding hierarchy of nonlinear evolution of equations is the AKNS hierarchy. This, as we have seen, connects with certain perturbations of Dirac operators. Huygens property for such perturbations was obtained from rational solutions of the AKNS hierarchy in our recent work [CZ03a] and was the subject of Section 3. The reduction and connection between the rational solutions at the matrix level and scalar level were studied in Section 4.

The connection between bispectrality for AKNS and Schrödinger operators was explored in [Zub92b]. In particular, it was described therein the role of bispectrality for the mKdV reduction of AKNS and its connection to the bispectral property at scalar level as in [DG86].

A natural follow-up of the present work would be to consider more general perturbations of $\bar{\gamma}\partial_0$ than that in Equation (6.4). In fact, one could try to substitute

the AKNS hierarchy by a more general hierarchy using the construction of integrable hierarchies of AKNS-Zakharov-Shabat type as in [Wil93a]. Its rational solutions [Kri79] would lead to natural candidates to perturbations of Dirac operators for which we conjecture Huygens property might hold.

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