Rational curves of minimal degree and characterizations of $\mathbb{P}^n$

Carolina Araujo
IMPA
Rio de Janeiro, Brazil
caraujo@impa.br

Abstract

In this paper we investigate complex uniruled varieties X whose rational curves of minimal degree satisfy a special property. Namely, we assume that the tangent directions to such curves at a general point $x \in X$ form a linear subspace of T_xX. As an application of our main result, we give a unified geometric proof of Mori's, Wahl's, Campana-Peternell's and Andreatta-Wiśniewski's characterizations of $\mathbb{P}^n$.

1 Introduction

Let X be a smooth complex projective variety, and assume that X is uniruled, i.e., there exists a rational curve through every point of X. Let H be a covering family of rational curves on X having minimal degree with respect to some fixed ample line bundle. For each $x \in X$ denote by $\mathcal{C}_x$ the subvariety of the projectivized tangent space at x consisting of tangent directions to rational curves from H passing through x. We are interested in varieties X for which $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX)$ for general $x \in X$. We prove the following result.

Theorem 1.1. *Suppose $\mathcal{C}_x$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX)$ for a general point $x \in X$. Then there is a dense open subset X^0 of X and a $\mathbb{P}^{d+1}$-bundle $\varphi^0 : X^0 \to T^0$ such that any curve from H meeting X^0 is a line on a fiber of φ^0.*

In fact we prove a stronger result, as we allow $\mathcal{C}_x$ to be a union of linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$ (see Theorem 3.1). We remark that the variety $\mathcal{C}_x$ has been studied in a series of papers by Hwang and Mok (see [Hwa01]).

As an application, we provide a unified geometric proof of the following characterization of $\mathbb{P}^n$.

Theorem 1.2. *Let X be a smooth complex projective n-dimensional variety. Assume that the tangent bundle T_X contains an ample locally free subsheaf E of rank r. Then $X \cong \mathbb{P}^n$ and either $E \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ or $r = n$ and $E = T_{\mathbb{P}^n}$.*

The first instance of this theorem, namely the case $E \cong T_X$, was proved by Mori in [Mor79]. In his proof, Mori recovers the projective space by studying rational curves of minimal degree passing through a general point of X. Then, in [Wah83], Wahl settled the case that E is a line bundle. Wahl's proof is very different from Mori's. It relies on the theory of algebraic derivations in characteristic zero. It does not make any use of the geometry of rational curves on X. Recently Druel gave a geometric proof of Wahl's theorem in [Dru04]. His proof is based on studying the foliation by curves defined by the inclusion $E \hookrightarrow T_X$, and applying a criterion for algebraicity of the leaves. In [CP98], Campana and Peternell proved the theorem in the cases $r = n$, $n-1$ and $n-2$. The proof was finally completed by Andreatta and Wiśniewski in [AW01]. Their proof uses the geometry of minimal covering family of rational curves on X. It relies, on one side, on Mori's theorem, and on the other side, on Wahl's theorem.

Our proof follows the lines of Mori's proof of the Hartshorne conjecture in [Mor79]. Here is the outline. An n-dimensional variety X whose tangent bundle contains an ample locally free subsheaf is uniruled. So we fix a covering family of rational curves of minimal degree on X, and consider the variety $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ of tangent directions to curves passing through $x \in X$. We translate the existence of an ample locally free subsheaf of T_X into projective properties of the embedding $\mathcal{C}_x \hookrightarrow \mathbb{P}(T_xX)$, and show that $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX)$ for general $x \in X$. By Theorem 1.1, there is a $\mathbb{P}^{d+1}$-fibration on a dense open subset of X, which can be extended in codimension 1 following an argument in [AW01]. If $d + 1 < n$, then the relative tangent bundle of such fibration inherits the ampleness properties of T_X. We reach a contradiction by applying a result by Campana and Peternell ([CP98]).

Throghout the paper we work over $\mathbb{C}$. In section 2 we gather some properties of minimal families of rational curves and the embedding $\mathcal{C}_x \hookrightarrow \mathbb{P}(T_xX)$. In section 3 we investigate varieties X for which $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$. In section 4 we give a unified proof of Theorem 1.2.

Notation. In our discussion on rational curves we follow the notation of [Kol96]. By a general point of a variety X, we mean a point in some dense open subset of X. If E is a vector bundle on a variety X, we denote by $\mathbf{P}(E)$ the Grothendieck projectivization $\mathrm{Proj}_X(\mathrm{Sym}(E))$. If V is a complex vector space, we denote by $\mathbb{P}(V)$ the natural projectivization of V. (So $\mathbf{P}(V) = \mathbb{P}(V^\vee)$.)

2 Tangent directions to rational curves of minimal degree

Let X be a smooth complex projective variety, and assume that X is uniruled. Let H be an irreducible component of $\mathrm{RatCurves}^n(X)$. We say that H is a *covering family* if the corresponding universal family dominates X. A covering family H of rational curves on X is called *minimal* if, for a general point $x \in X$, the subfamily of H parametrizing curves through x is proper. It is called *unsplit* if H itself is proper.

Fix a minimal covering family H of rational curves on X (for instance, one can take H to be a covering family having minimal degree with respect to some fixed ample line bundle on X).

Let $x \in X$ be a general point and denote by H_x the normalization of the subscheme of H parametrizing rational curves passing through x. Let $\pi_x : U_x \to H_x$ and $\eta_x : U_x \to X$ be the universal family morphisms,

$$\begin{array}{ccc} U_x & \xrightarrow{\eta_x} & X, \\ {\scriptstyle \pi_x}\downarrow & & \\ H_x & & \end{array}$$

so that U_x is normal and π_x is a $\mathbb{P}^1$-bundle (see [Kol96, II.2.12]). Denote by $\mathrm{locus}(H_x)$ the closure of the image of η_x (with the reduced scheme structure). We remark that a rational curve smooth at x is parametrized by at most one element of H_x.

Notation 2.1. Let $f : \mathbb{P}^1 \to X$ be a morphism birational onto its image such that $f(o) = x$. We denote by $[f]$ the element of V (or V_x) parametrizing f. Sometimes we also denote by $[f]$ the point $\varphi_x([f]) \in H_x$ parametrizing the image of f. It should be clear from the context whether we view $[f]$ as a member of $\mathrm{Hom}(\mathbb{P}^1, X)$ or $\mathrm{RatCurves}^n(X)$.

Next we gather some important properties of minimal covering families of rational curves.

Proposition 2.2. *Let the notation be as above and $x \in X$ a general point .*

1. *For every $[f] \in H_x$, $f^*T_X \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$, with all $a_i \geq 0$.*
2. *H_x is a smooth projective variety of dimension $d := \deg(f^*T_X) - 2$.*
3. *If $[f]$ is a general member of any irreducible component of H_x, then $f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{n-d-1}$.*
4. *Every curve parametrized by H_x is immersed at x (i.e., df_o is nonzero for every $o \in \mathbb{P}^1$ such that $f(o) = x$).*
5. *The dimension of the subscheme of H_x parametrizing curves singular at x is at most the dimension of the subscheme of H_x parametrizing curves with cuspidal singularities (not necessarily at x).*
6. *If all the curves parametrized by H_x are smooth at x, then the restriction of η_x to each irreducible component of U_x is birational onto its image.*

Proof. Property (1) follows from [Kol96, II.3.11] and the assumption that x is a general point. Property (2) follows from [Kol96, II.1.7, II.2.16] and the assumption that H is a minimal covering family. Property (3) follows from [Kol96, IV.2.9].

Properties (4) and (5) can be found in [Keb02]. Property (5) is not explicitly stated in [Keb02], but follows from the proof of [Keb02, Theorem 3.3].

Property (6) is due to Miyaoka (see [Kol96, V.3.7.5]). □

Definition 2.3. Define the tangent map $\tau_x : H_x \dashrightarrow \mathbb{P}(T_xX)$ by sending a curve that is smooth at x to its tangent direction at x.

Define $\mathcal{C}_x$ to be the closure of the image of τ_x in $\mathbb{P}(T_xX)$.

Theorem 2.4. *Let the notation be as above. Then*

1. **([Keb02])** $\tau_x : H_x \to \mathcal{C}_x$ *is a finite morphism,*
2. **([HM04])** $\tau_x : H_x \to \mathcal{C}_x$ *is birational, and thus*
3. $\tau_x : H_x \to \mathcal{C}_x$ *is the normalization.*

Notice that $\mathcal{C}_x$ comes with a natural projective embedding into $\mathbb{P}(T_xX)$. It turns out that, for a general member $[f] \in H_x$, the "positive" directions of f^*T_X at x determine the tangent space of $\mathcal{C}_x$ at $\tau_x([f])$. This is made precise in the next proposition.

Definition 2.5. Let $f : \mathbb{P}^1 \to X$ be a morphism birational onto its image such that $x = f(o)$. Define the *positive tangent space at $x \in X$ with respect to f* to be the following linear subspace of T_xX:

$$T_xX_f^+ \ := \ \mathrm{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \to (f^*T_X(-1))_o \cong T_xX].$$

Proposition 2.6. *Let $[f] \in H_x$ be a general element. Then $\mathbb{P}(T_xX_f^+) \subset \mathbb{P}(T_xX)$ is the projective tangent space of $\mathcal{C}_x$ at $\tau_x([f])$.*

Proof. See [Hwa01, Proposition 2.3] or [AW01, Lemma 2.1]. □

From the splitting type of f^*T_X one can check whether τ_x is an immersion at $[f]$.

Proposition 2.7. *The morphism τ_x is an immersion at $[f] \in H_x$ if and only if $f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-d-1}$, where $d = \deg(f^*T_X) - 2$.*

Proof. Let H'_x be an irreducible component of H_x. Let V_x be the corresponding irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X, o \mapsto x)$, i.e., V_x parametrizes morphisms (birational onto their images) whose images are parametrized by H'_x. By Proposition 2.2(4), every morphism parametrized by V_x is an immersion at o. So we can define the morphism $\mathcal{T}_x : V_x \to \mathbb{P}(T_xX)$ by setting $\mathcal{T}_x([f]) = \mathbb{P}(df_o(T_o\mathbb{P}^1))$.

We have the following commutative diagram:

$$\begin{tikzcd} V_x \arrow[r, "\mathcal{T}_x"] \arrow[d, "\varphi_x"'] & \mathbb{P}(T_xX), \\ H'_x \arrow[ur, "\tau_x"'] & \end{tikzcd}$$

where φ_x is a smooth morphism with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1, o)$.

Fix $[f] \in V_x$ and write $f^*T_X \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$, with $a_1 \geq \cdots \geq a_n \geq 0$, $a_1 \geq 2$, and $\sum_{i=1}^{n} a_i = 2 + d$. Let us describe the tangent map $d\mathcal{T}_x([f]) : T_{[f]}V_x \to T_{\mathcal{T}_x([f])}\mathbb{P}(T_xX)$ explicitly.

There are isomorphisms $T_{[f]}V_x \cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_o)$, where $\mathcal{I}_o$ denotes the ideal sheaf of o in $\mathbb{P}^1$ (see [Kol96, II.1.7]), and $T_{\mathcal{T}_x([f])}\mathbb{P}(T_xX) \cong T_xX/\hat{\mathcal{T}}_x([f])$, where $\hat{\mathcal{T}}_x([f])$ denotes the 1-dimensional subspace of T_xX corresponding to the point $\mathcal{T}_x([f]) \in \mathbb{P}(T_xX)$.

Fix a local parameter t of the local ring of o on $\mathbb{P}^1$. If v is a global section of f^*T_X vanishing at o, then $d\mathcal{T}_x([f])(v)$ is given by

$$d\mathcal{T}_x([f])(v) = \left[\frac{d}{dt} v(t) \bigg|_{t=o} \right] \in (f^*T_X)_o / T_o\mathbb{P}^1 \cong T_xX/\hat{\mathcal{T}}_x([f]).$$

So we see that $\operatorname{im} d\mathcal{T}_x([f]) \cong T_xX_f^+/\hat{\mathcal{T}}_x([f])$.

Set $l = \dim \operatorname{im} d\mathcal{T}_x([f]) = \sharp\{a_i | a_i > 0\} - 1$. Then $l \leq d = \sum_{i=1}^{n} a_i - 2$, and equality holds if and only if τ_x is an immersion at $\varphi_x([f])$. Since $a_1 \geq 2$ and $a_i \geq 1$ for $1 \leq i \leq l+1$, this is equivalent to $a_1 = 2$ and $a_i = 1$ for $1 < i \leq d+1$, i.e., $f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus n-d-1}$. $\square$

Corollary 2.8. *If every irreducible component of $\mathcal{C}_x$ is smooth, then*

1. *all curves parametrized by H_x are smooth at x, and*
2. *the restriction of the universal family morphism $\eta_x : U_x \to X$ to each irreducible component of U_x is birational onto its image.*

Proof. Since every irreducible component of $\mathcal{C}_x$ is smooth, τ_x is an immersion by Theorem 2.4 (in fact, the restriction of τ_x to each irreducible component of H_x is an isomorphism). Thus, by Proposition 2.7, $f^*T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-d-1}$ for every member $[f] \in H_x$. From the splitting type of f^*T_X we see that no curve parametrized by H_x has a cuspidal singularity. The corollary then follows from Proposition 2.2(5)–(6). $\square$

3 The distribution defined by linear $\mathcal{C}_x$

Let X be a smooth uniruled complex projective variety. Let H be a minimal covering family of rational curves on X, and let $\mathcal{C}_x$ be the subvariety of $\mathbb{P}(T_xX)$ defined in section 2. In this section we study varieties X for which $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$.

Denote by $\pi : U \to H$ and $\eta : U \to X$ the universal family morphisms. Consider the Stein factorization of η:

$$\begin{tikzcd} & X' \arrow[d, "\rho"] \\ U \arrow[ur, bend left, "\eta'"] \arrow[r, "\eta"] \arrow[d, "\pi"'] & X. \\ H & \end{tikzcd}$$

We may view H as a minimal covering family of rational curves on X'.

The main result in this section is the following.

Theorem 3.1. *Suppose that $\mathcal{C}_x$ is a union of d-dimensional linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$. Let X' be as defined above. Then there is a dense open subset U^0 of X' and a $\mathbb{P}^{d+1}$-bundle $\varphi^0 : U^0 \to T^0$ such that any curve on X' parametrized by H and meeting U^0 is a line on a fiber of φ^0.*

For a general point $x \in X$, denote by H_x^i, $1 \le i \le k$, the irreducible components of H_x, and by $\mathcal{C}_x^i$ the image of H_x^i under τ_x. Suppose that each $\mathcal{C}_x^i$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX)$.

Viewing H as a minimal covering family of rational curves on X', $H_{x'}$ is irreducible and $\mathcal{C}_{x'}$ is a linear subspace of $\mathbb{P}(T_{x'}X')$ for a general point $x' \in X'$. Moreover X' is smooth along $\mathrm{locus}(H_{x'})$ (for η is smooth along $\pi^{-1}(H_{x'})$ by [Kol96, II.3.5.3, II.2.15]).

We obtain a rank $d+1$ distribution D on a dense open subset of X' as follows. For a general point $x' \in X'$, set $D_{x'} = \hat{\mathcal{C}}_{x'}$, the linear subspace of $T_{x'}X'$ corresponding to $\mathcal{C}_{x'} \subset \mathbb{P}(T_{x'}X')$.

Lemma 3.2. *Suppose that $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$. Let X' and D be as defined above. Then the distribution D is tangent to* $\mathrm{locus}(H_{x'})$ *for a general point $x' \in X'$. In particular,* $\mathrm{locus}(H_{x'})$ *is smooth at x'.*

Proof. Let $x' \in X'$ be a general point and set $Y := \mathrm{locus}(H_{x'})$. By Frobenius' Theorem, we are done if we can show that $T_yY = D_y = \hat{\mathcal{C}}_y$ for a general point $y \in Y$.

Let $[f] \in H_{x'}$ be a general member and let y be a general point in the image of f. Let $o, p \in \mathbb{P}^1$ be such that $f(o) = x'$ and $f(p) = y$. Let $V_{x'}$ be the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X, o \mapsto x')$ corresponding to $H_{x'}$. We have the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{P}^1 \times V_{x'} \arrow[r] \arrow[d] \arrow[rr, bend left, "F"] & U_{x'} \arrow[r, "\eta_{x'}"] \arrow[d, "\pi_{x'}"] & X'. \\
V_{x'} \arrow[r, "\varphi_{x'}"] & H_{x'} &
\end{tikzcd}
$$

By generic smoothness, the tangent space T_yY is the image in T_yX' of the differential $dF_{(p,[f])}$. From the description of $dF_{(p,[f])}$ given in [Kol96, II.3.4], together with Proposition 2.6, we see that this is precisely $T_yX'^+_f = \hat{\mathcal{C}}_y$. $\square$

Next we describe $\mathrm{locus}(H_{x'})$ for general $x' \in X'$.

Lemma 3.3. *Suppose that $\mathcal{C}_x$ is a union of d-dimensional linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$. Let X' be as defined above. Then, for a general point $x' \in X'$, the normalization of $\mathrm{locus}(H_{x'})$ is isomorphic to $\mathbb{P}^{d+1}$. Under this isomorphism, the rational curves on $\mathrm{locus}(H_{x'})$ parametrized by $H_{x'}$ come from lines on $\mathbb{P}^{d+1}$ passing through a fixed point.*

Proof. Set $Y = \mathrm{locus}(H_{x'})$ and let $n : \tilde{Y} \to Y$ be the normalization.

The subfamily $H_Y = \{[f] \in H | f(\mathbb{P}^1) \subset Y\}$ is a minimal covering family of rational curves on Y. Moreover, x' is a general point of Y (indeed, $Y = \mathrm{locus}(H_y)$ for a general point $y \in Y$ by Lemma 3.2). The subfamily $H_{Y,x'}$ of H_Y parametrizing curves through x' is just $H_{x'} \cong \mathbb{P}^d$.

Denote by $\pi_{x'} : U_{x'} \to H_{Y,x'}$ and $\eta_{x'} : U_{x'} \to Y$ the universal family morphisms. Since $\pi_{x'}$ is a $\mathbb{P}^1$-bundle, $U_{x'}$ is smooth. Moreover, $\eta_{x'}$ is birational by Proposition 2.2(6). We have the commutative diagram

$$
\begin{tikzcd}
 & \tilde{Y} \arrow[d, "n"] \\
U_{x'} \arrow[ur, bend left, "\tilde{\eta}_{x'}"] \arrow[r, "\eta_{x'}"] \arrow[d, "\pi_{x'}"'] & Y. \\
H_{Y,x'} &
\end{tikzcd}
$$

Since Y is smooth at x', there is a unique point $\tilde{x} \in \tilde{Y}$ such that $n(\tilde{x}) = x'$, and $\tilde{Y}$ is smooth at $\tilde{x}$.

Let $\sigma \subset U_{x'}$ be the section of $\pi_{x'}$ that is contracted to $\tilde{x} \in \tilde{Y}$ by $\tilde{\eta}_{x'}$. Then $\tilde{\eta}_{x'} : U_{x'} \to \tilde{Y}$ is a surjective birational morphism and restricts to an isomorphism on $U_{x'} \setminus \sigma$. In particular $\tilde{Y}$ is smooth. In this setting, a standard argument by Mori (see [Kol96, V.3.7.8]) yields the result. $\square$

Proof of Theorem 3.1. Let X' be as defined above. By Lemmas 3.2 and 3.3, together with Frobenius' Theorem, there exists a dense open subset $U^0 \subset X'$ and a morphism $\varphi^0 : U^0 \to T^0$ such that the normalization of the closure of the general fiber of φ^0 is isomorphic to $\mathbb{P}^{d+1}$. By enlarging U^0 if necessary, we may assume that $X' \setminus U^0$ is the indeterminacy locus of φ^0. If $d = n-1$, there is nothing to prove. So we assume that $\dim T^0 \geq 1$.

Suppose that the general fiber of φ^0 is not proper (so that its closure intersects $X' \setminus U^0$). Let $t \in T^0$ be a general point. Then there exists a point $y \in X' \setminus U^0$ and a positive dimensional irreducible subvariety $T' \subset T^0$ containing t such that y lies in the closure of every fiber of φ^0 over T'. Let H_y be the subscheme of H parametrizing curves passing through y.

Let $t' \in T'$ be a general point, and x' a general point in the fiber over t'. By Lemma 3.3, the normalization of $\mathrm{locus}(H_{x'})$ is isomorphic to $\mathbb{P}^{d+1}$, and the curves parametrized by $H_{x'}$ come from lines in $\mathbb{P}^{d+1}$. This has two consequences. First, there is an element $[f] \in H_{x'}$ parametrizing a curve passing through y. Since x' is general, $f^*T_{X'} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-d-1}$, and thus $\dim_{[f]} H_y = d$ by [Kol96, II.1.7, II.2.16]. Second, $\mathrm{locus}(H_{x'}) \subset \mathrm{locus}(H_y)$. Since this holds for a point x' in a general fiber over T', we have that $\dim_{[f]} H_y \geq d+1$, contradicting the equality obtained above.

We conclude that the general fiber of φ^0 is proper. By shrinking U^0 and T^0 if necessary we get that $\varphi^0 : U^0 \to T^0$ is a $\mathbb{P}^{d+1}$-bundle. □

When H_x is irreducible and $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX)$, Theorem 3.1 yields a dense open subset $X^0 \subset X$ and a $\mathbb{P}^{d+1}$-bundle $\varphi^0 : X^0 \to T^0$. If we further assume that H is an unsplit family, then φ^0 can be extended in codimension 1, as we show below.

Theorem 3.4. *Suppose H is an unsplit family and $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX)$ for general $x \in X$. Then there is an open subset $X^0 \subset X$ whose complement has codimension at least 2 in X, and a $\mathbb{P}^{d+1}$-bundle $\varphi^0 : X^0 \to T^0$ over a smooth base satisfying the following property. Every rational curve parametrized by H and meeting X^0 is a line on a fiber of φ^0.*

Proof. We follow an argument in [AW01].

Let $\varphi^0 : X^0 \to T^0$ be the $\mathbb{P}^{d+1}$-bundle from Theorem 3.1. Let $T \to \mathrm{Chow}(X)$ be the normalization of the closure of the image of T^0 in $\mathrm{Chow}(X)$, and let $\mathcal{U}$ be the normalization of the universal family over T. Denote by $p : \mathcal{U} \to T$ and $q : \mathcal{U} \to X$ the universal family morphisms.

Let $0 \in T$ be any point. Set $\mathcal{U}_0 = p^{-1}(0)$ and let $x, y \in \mathcal{U}_0$ be arbitrary points. We can find a 1-parameter family of fibers $\mathcal{U}_t = p^{-1}(t)$, together with points $x_t, y_t \in \mathcal{U}_t$, such that $\mathcal{U}_t \cong \mathbb{P}^{d+1}$ for $t \neq 0$, and $\lim_{t \to 0}(\mathcal{U}_t, x_t, y_t) = (\mathcal{U}_0, x, y)$.

Let $l_t \subset \mathcal{U}_t$ be the curve parametrized by H joining x_t and y_t. Since H is unsplit, the limit $\lim_{t\to 0}[l_t]$ lies in H. It parametrizes an irreducible (and reduced) rational curve $l \subset \mathcal{U}_0$ joining x and y. This shows that $\mathcal{U}_0$ is irreducible.

Notice that $q : \mathcal{U} \to X$ is birational and T has dimension $n-d-1$. Let $E \subset \mathcal{U}$ be an irreducible component of the exceptional locus E' of q. Since X is smooth, E has codimension 1 in $\mathcal{U}$. Set $p_E = p|_E$ and $\mathcal{E} = p(E) \subset T$.

Let $\mathcal{U}_t$ be an arbitrary fiber of p and assume that $\mathcal{U}_t \cap E \neq \varnothing$, i.e., $t \in \mathcal{E}$. Since E misses the general fiber of p, $\dim \mathcal{E} \leq \dim T - 1 = n-d-2$. Set

$E_t = p_E^{-1}(t)$. Then $d+1 = \dim \mathcal{U}_t \geq \dim E_t \geq \dim E - \dim \mathcal{E} \geq d+1$. Hence $\dim \mathcal{U}_t = \dim E_t$. Since $\mathcal{U}_t$ is irreducible, this implies that $\mathcal{U}_t = E_t$ and thus $\mathcal{U}_t \subset E$.

Set $S = q(E') \subset X$. This is a set of codimension at least 2 in X. The restriction $q|_{\mathcal{U} \setminus E'} : \mathcal{U} \setminus E' \to X \setminus S$ is an isomorphism. The proper morphism $p|_{\mathcal{U} \setminus E'} : \mathcal{U} \setminus E' \to T \setminus p(E')$ induces a proper morphism $X \setminus S \to T \setminus p(E')$ extending φ^0. We replace X^0 with $X \setminus S$ and T^0 with $T \setminus p(E')$, obtaining a proper morphism $\varphi^0 : X^0 \to T^0$ with $\operatorname{codim}(X \setminus X^0) \geq 2$, and whose general fiber is isomorphic to $\mathbb{P}^{d+1}$. By shrinking T^0 we may assume that it is smooth. (For this we need to remove from T^0 a subset of codimension at least 2, and so we still have $\operatorname{codim}(X \setminus X^0) \geq 2$.)

Let C be a curve in T obtained as the intersection of $n-d-2$ general very ample divisors. Set $C^0 = C \cap T^0$ and $X_{C^0} = (\varphi^0)^{-1}(C^0)$. By Bertini Theorem, both C^0 and X_{C^0} are smooth. Moreover, the general fiber of the induced fibration $\varphi_{C^0} : X_{C^0} \to C^0$ is isomorphic to $\mathbb{P}^{d+1}$. Since $\dim C^0 = 1$, there exists a φ_{C^0}-ample line bundle L on X_{C^0} such that the restriction of L to a general fiber of φ_{C^0} is isomorphic to $\mathcal{O}_{\mathbb{P}^{d+1}}(1)$. Thus we can apply [Fuj75, Corollary 5.4] and conclude that $\varphi_{C^0} : X_{C^0} \to C^0$ is in fact a $\mathbb{P}^{d+1}$-bundle. By Bertini, after removing from T^0 a subset of codimension at least 2, we may assume that $\varphi^0 : X^0 \to T^0$ is in fact a $\mathbb{P}^{d+1}$-bundle. $\square$

4 Proof of Theorem 1.2

Let X be a smooth complex projective n-dimensional variety. In this section we assume that the tangent bundle T_X contains a rank r ample locally free subsheaf E and prove that $X \cong \mathbb{P}^n$.

We begin by noticing that X is uniruled. This follows from a theorem by Miyaoka (see [Miy87] or Shepherd-Barron's article in [Kol92]). We fix a minimal covering family H of rational curves on X and set $d = \deg(f^*T_X) - 2$, where $[f]$ is any member of H. For a general point $x \in X$, consider the tangent map $\tau_x : H_x \to \mathcal{C}_x \subset \mathbb{P}(T_xX)$, defined in section 2. Denote by H_x^i, $1 \leq i \leq k$, the irreducible components of H_x, and by $\pi_x^i : U_x^i \to H_x^i$ and $\eta_x^i : U_x^i \to X$ the corresponding universal family morphisms. Denote by $\operatorname{locus}(H_x^i)$ the image of η_x^i, and by $\mathcal{C}_x^i$ the image of $\tau_x|_{H_x^i}$.

We use the description of $T_{\tau_x([f])}\mathcal{C}_x$ given in Proposition 2.6 to study the projective embedding $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ for general $x \in X$.

Proposition 4.1. *Let the notation and assumptions be as above. Then, for a general point $x \in X$, the following holds.*

1. *For every $i \in \{1, \dots, k\}$, $\mathcal{C}_x^i$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX)$. Moreover, $\mathbb{P}(E_x) \subset \bigcap_{i=1}^{k} \mathcal{C}_x^i$.*

2. *For every $i \in \{1, \dots, k\}$, the restriction $\tau_x|_{H_x^i} : H_x^i \to \mathcal{C}_x^i$ is an isomorphism. As a consequence, all curves parametrized by H_x are smooth at x and $\eta_x^i : U_x^i \to X$ is birational onto its image for every $i \in \{1, \dots, k\}$.*

Proof. Fix an irreducible component H_x^i of H_x, and let $[f] \in H_x^i$ be a general element. There is an injection of sheaves $f^*E \hookrightarrow f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-d-1}$. Since E is ample, f^*E is a subsheaf of the positive part of f^*T_X,

$$f^*T_X^+ = \mathrm{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes \mathcal{O} \to f^*T_X(-1)] \otimes \mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d},$$

and so $E_x \subset T_x X_f^+$. Hence for a general element $[f] \in H_x^i$ we have $\mathbb{P}(E_x) \subset \overline{T_{\tau_x([f])}\mathcal{C}_x^i} \subset \mathbb{P}(T_xX)$.

Now we apply Lemma 4.2 and conclude that each irreducible component $\mathcal{C}_x^i$ of $\mathcal{C}_x$ is a cone in $\mathbb{P}(T_xX)$ whose vertex contains $\mathbb{P}(E_x)$.

But we also know that H_x is smooth and that $\tau_x : H_x \to \mathcal{C}_x$ is the normalization. Therefore Lemma 4.3 implies that each $\mathcal{C}_x^i$ is a linear subspace of $\mathbb{P}(T_xX)$, and $\tau_x|_{H_x^i}$ is an isomorphism. The second part of (2) follows from Corollary 2.8. □

Lemma 4.2. *Let Z be an irreducible closed subvariety of $\mathbb{P}^m$. Assume there is a dense open subset U of the smooth locus of Z and a point $z_0 \in \mathbb{P}^m$ such that $z_0 \in \bigcap_{z \in U} T_zZ$. Then Z is a cone in $\mathbb{P}^m$ and z_0 lies in the vertex of this cone.*

Proof. We may assume that $\dim Z > 0$. Consider the projection from z_0, $\pi_{z_0} : Z \dashrightarrow \mathbb{P}^{m-1}$. Since $z_0 \in T_zZ$ for general $z \in Z$, the tangent map to π_{z_0} has rank $\dim Z - 1$ at a general point. So π_{z_0} has 1-dimensional fibers, and thus Z is a cone whose vertex contains z_0. (Notice that in this proof we use the characteristic 0 assumption). □

Lemma 4.3. *If Z is an irreducible cone in $\mathbb{P}^m$ and the normalization of Z is smooth, then Z is a linear subspace of $\mathbb{P}^m$.*

Proof. Let $x_0, \dots, x_m$ be the projective coordinates of $\mathbb{P}^m$. We may assume that Z is a cone with vertex $P = (0 : \dots : 0 : 1)$ over a closed irreducible subvariety V contained in the hyperplane section $(x_m = 0)$ of $\mathbb{P}^m$.

Let $I_V \subset \mathbb{C}[x_0, \dots, x_{m-1}]$ be the homogeneous ideal defining V in $(x_m = 0) \cong \mathbb{P}^{m-1}$. By changing m if necessary we may assume that V is nondegenerate in $\mathbb{P}^{m-1}$. Then $Z \setminus (x_m = 0)$ has affine coordinate ring $S(V) = \mathbb{C}[x_0, \dots, x_{m-1}]/I_V$, and the integral closure of $S(V)$ is $S' = \bigoplus_{l \geq 0} H^0(V, O_V(l))$. Moreover, S' can be written as $S' = \mathbb{C}[y_0, \dots, y_{M-1}]/I'$ for some $M > 1$ and some homogeneous ideal $I' \subset \mathbb{C}[y_0, \dots, y_{M-1}]$. By changing M we may assume that $V' = \mathrm{Proj}\, S'$ is nondegenerate in $\mathbb{P}^{M-1}$. Let $C(V') = \mathrm{Spec}\, S' \subset \mathbb{A}^M$ be the affine cone over V'. Then $C(V') \to Z \setminus (x_m = 0)$ is the normalization morphism.

Assume Z is nonlinear. Since V is nondegenerate in $\mathbb{P}^{m-1}$, this is the same as assuming that I_V is generated by elements of degree ≥ 2.

Now consider the inclusion of graded rings $S(V) = \mathbb{C}[x_0, \dots, x_{m-1}]/I_V \hookrightarrow S' = \mathbb{C}[y_0, \dots, y_{M-1}]/I'$, and denote by φ_i the image of x_i in S'_1. Since $x_0, \dots, x_{m-1}$ are linearly independent in $S(V)_1$, $\varphi_0, \dots, \varphi_{m-1}$ are linearly independent in S'_1. But then $M \geq m > \dim Z = \dim C(V')$. Since we assume that V' is nondegenerate, this implies that $C(V')$ is a nonlinear cone, and hence not smooth. □

The next step is to prove that H_x is in fact irreducible for general $x \in X$. The idea is to produce a curve C through x such that, for every $i \in \{1, \ldots, k\}$, there exists an element $[f_i] \in H_x^i$ parametrizing C. Since C is smooth at x (by Proposition 4.1(2)) there exists a unique point in H_x parametrizing C, and H_x must be irreducible.

By Proposition 4.1(1), $\mathcal{C}_x$ is the union of linear subspaces of $\mathbb{P}(T_xX)$ for general $x \in X$. Fix $x \in X$ and let $i \in \{1, \ldots, k\}$. Set $Y_i = \mathrm{locus}(H_x^i)$. By Lemma 3.3, the normalization of Y_i is isomorphic to $\mathbb{P}^{d+1}$. Under this isomorphism, the rational curves on Y_i parametrized by H_x^i come from the lines on $\mathbb{P}^{d+1}$ passing through a fixed point $\tilde{x}_i \in \mathbb{P}^{d+1}$. Let $n_i : \mathbb{P}^{d+1} \to Y_i$ be the normalization morphism.

We claim that the vector bundle $E|_{Y_i}$ pulls back to a subsheaf of $T_{\mathbb{P}^{d+1}}$. Indeed, the injection $E \hookrightarrow T_X$ induces a map $\Omega_X \to E^\vee$ of maximal rank. By restricting to Y_i, we get a map $\Omega_X|_{Y_i} \to E^\vee|_{Y_i}$ of maximal rank. This map factors through $\Omega_{Y_i} \to E^\vee|_{Y_i}$. (This is because the composite map $I_{Y_i}/I_{Y_i}^2 \to \Omega_X|_{Y_i} \to E^\vee|_{Y_i}$ vanishes identically.) Lemma 4.4 below asserts that there is a map $\Omega_{\mathbb{P}^{d+1}} \to n_i^*E^\vee|_{Y_i}$ factoring $n_i^*\Omega_{Y_i} \to n_i^*E^\vee|_{Y_i}$. By dualizing we get a sheaf injection $n_i^*E|_{Y_i} \hookrightarrow T_{\mathbb{P}^{d+1}}$.

Thus, $n_i^*E|_{Y_i}$ is an ample vector bundle on $\mathbb{P}^{d+1}$ that is a subsheaf of $T_{\mathbb{P}^{d+1}}$. So either $n_i^*E|_{Y_i} \cong \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus r}$, or $r = d+1$ and $n_i^*E|_{Y_i} \cong T_{\mathbb{P}^{d+1}}$.

In any case, there exists an r-dimensional linear subspace M_i of $\mathbb{P}^{d+1}$, passing through $\tilde{x}_i$, for which $(n_i^*E|_{Y_i})|_{M_i} \hookrightarrow T_{\mathbb{P}^{d+1}}|_{M_i}$ factors through $T_{M_i} \hookrightarrow T_{\mathbb{P}^{d+1}}|_{M_i}$. Set $Z = n_i(M_i) \subset Y_i \subset X$. Then Z is an r-dimensional subvariety of X containing x and tangent to E along its smooth locus. By Frobenius' Theorem, Z is the unique subvariety of X with these properties. Hence $Z = n_i(M_i)$ for every $i \in \{1, \ldots, k\}$. Let C be the image in Z of a line through $\tilde{x}_i$ on M_i for some i. Then, for every i, C comes from a line through $\tilde{x}_i$ on M_i, and thus there exists an element $[f_i] \in H_x^i$ parametrizing C. This shows that H_x is irreducible as we noted above.

Lemma 4.4. *Let Z be a variety, $n : \tilde{Z} \to Z$ the normalization, and E a vector bundle on Z. Assume there is a map $\Omega_Z \to E^\vee$. Then the induced map $n^*\Omega_Z \to n^*E^\vee$ factors through $n^*\Omega_Z \to \Omega_{\tilde{Z}}$.*

Proof. This result follows from a theorem by Seidenberg, which asserts that a derivation of an integral domain over a ground field of characteristic 0 extends to its normalization.

Let $U = \operatorname{Spec} A \subset Z$ be an affine open subset over which E is trivial, and fix an isomorphism $E|_U \cong \mathcal{O}_U^{\oplus r}$, where $r = \operatorname{rank} E$.

Let $\tilde{A}$ be the integral closure of A.

The restricted map $\Omega_U \to E|_U \cong \mathcal{O}_U^{\oplus r}$ induces a homomorphism $\Omega_A \to A^{\oplus r}$. By composing with the r natural projections, $p_i : A^{\oplus r} \to A$, $1 \le i \le r$, we obtain r derivations $D_i : A \to A$, $1 \le i \le r$. By the main theorem in [Sei66], each of these derivations extends uniquely to a derivation $\tilde{D}_i : \tilde{A} \to \tilde{A}$. Such $\tilde{D}_i$'s determine a homomorphism $\Omega_{\tilde{A}} \to \tilde{A}^{\oplus r}$ extending $\Omega_A \to A^{\oplus r}$. $\square$

Theorem 3.1 yields a dense open subset X^0 of X and a $\mathbb{P}^{d+1}$-bundle φ^0 :

$X^0 \to T^0$. Since an ample vector bundle of rank r on a rational curve has degree at least r, either H is an unsplit family, or $d = 0$, $r = 1$, and $f^*E \cong \mathcal{O}_{\mathbb{P}^1}(2)$ for every $[f] \in H$. We analyse these two cases separately.

Case 1 (H is an unsplit family).

In this case we can apply Theorem 3.4 and assume that T^0 is smooth and $\operatorname{codim}(X \setminus X^0) \geq 2$.

Suppose that $\dim T^0 > 0$. Let $C' \subset X^0$ be a general smooth projective curve such that $C = \varphi^0(C')$ is also a smooth projective curve. (Such a curve exists because X and T^0 are smooth and $X \setminus X^0$ has codimension at least 2.) Then $X_C := (\varphi^0)^{-1}C \to C$ is a $\mathbb{P}^{d+1}$-bundle. Since C' is general, there is a sheaf inclusion $E|_{X_C} \hookrightarrow T_X|_{X_C}$. For general $x \in X_C$ we have $E_x \subset (T_{X_C/C})_x \subset T_xX$. The cokernel of the map $T_{X_C/C} \hookrightarrow T_X|_{X_C}$ is torsion free. Hence $E|_{X_C}$ is in fact a subsheaf of the relative tangent sheaf $T_{X_C/C}$. But this contradicts Lemma 4.5 below, due to Campana and Peternell. Therefore T^0 is a point, $X \cong \mathbb{P}^n$ and under this isomorphism either $E = T_{\mathbb{P}^n}$ or $E \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.

Lemma 4.5 ([CP98, Lemma 1.2]). *Let T be a smooth complex projective variety of positive dimension, E a vector bundle of rank $k+1$ on T, and $X = \mathbf{P}(E) \to T$ the corresponding $\mathbb{P}^k$-bundle. Then the relative tangent sheaf $T_{X/T}$ does not contain any ample locally free subsheaf.*

Case 2 (H is not proper, $d = 0$, $r = 1$ and $f^*E \cong \mathcal{O}_{\mathbb{P}^1}(2)$ for every $[f] \in H$).

We have $\dim H > 0$. Let $C' \subset H$ be a general curve. Let C be the normalization of the closure of C' in $\operatorname{Chow}(X)$. (Notice that, since H is not proper, some points of C may parametrize nonintegral curves.) Let S be the normalization of the universal family over C and denote by $p : S \to C$ and $n : S \to X$ the universal family morphisms.

Let $S' = n(S) \subset X$. Then $n : S \to S'$ is birational. Since $\dim H_x = 0$ for general $x \in X$, n does not contract any curve dominating C. Neither does it contract any curve contained in a fiber of p. Hence n is the normalization. By Lemma 4.4, the injection $E|_{S'} \hookrightarrow \Omega_{S'}^\vee$ lifts to an injection $n^*E \hookrightarrow \Omega_S^\vee$. For convenience set $L = n^*E$.

The idea is to reach a contradiction as follows. We look at the minimal resolution of S and contract the (-1)-curves that do not dominate C. In this way we obtain a $\mathbb{P}^1$-bundle over C. We show that L induces an ample line bundle on the resulting $\mathbb{P}^1$-bundle that is a subsheaf of the relative tangent sheaf. But this is impossible by Lemma 4.5.

So let $r : Y \to S$ be the minimal resolution of S and set $L_Y = r^*L$ (notice that L_Y is an ample line bundle on Y). By [BW74, Proposition 1.2], there is a natural isomorphism $r_*T_Y \xrightarrow{\cong} \Omega_S^\vee$. Therefore, from the natural isomorphism $\operatorname{Hom}_Y(L_Y, T_Y) \cong \operatorname{Hom}_S(L, r_*T_Y)$ (see [Har77, II.5]), we see that the map $L \to \Omega_S^\vee$ lifts to an injection $L_Y \hookrightarrow T_Y$.

The induced morphism $p_Y : Y \to C$ can be obtained from a suitable $\mathbb{P}^1$-bundle $p_Z : Z \to C$ by a composition of blowups, $q : Y \to Z$. Set $L_Z = q_*L_Y$.

By pushing forward to Z and applying the projection formula, we see that the inclusion $L_Y \hookrightarrow T_Y$ induces an inclusion $L_Z \hookrightarrow T_Z$.

To show that L_Z is an ample line bundle, it is enough to assume that $q : Y \to Z$ is the inverse of a single blowup (then use induction on the number of blowups). First note that L_Z is in fact a line bundle on Y (it is reflexive except possibly at finitely many points, and hence reflexive). So we can write $L_Y = q^*L_Z + aD$, where D is the exceptional curve, and $a = -aD^2 = -L_Y \cdot D$ is a negative integer. The ampleness of L_Z then follows from Nakai's criterion.

For any fiber F of p_Z we have $L_Z \cdot F > 0$. Hence, for a general fiber $F \cong \mathbb{P}^1$, the map $L_Z|_F \hookrightarrow T_Z|_F \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$ factors through $\mathcal{O}_{\mathbb{P}^1}(2) \cong T_F \hookrightarrow T_Z|_F$. Since the cokernel of the map $T_{Z/C} \hookrightarrow T_Z$ is torsion free, this implies that there is an inclusion $L_Z \hookrightarrow T_{Z/C}$ factoring $L_Z \hookrightarrow T_Z$.

We have shown that L_Z is an ample line bundle on Z that injects into $T_{Z/C}$. This is a contradiction as we noted above. So case 2 does not occur.

Acknowledgements. Most of the content of this article appears in my Ph.D. thesis. I am deeply grateful to my advisor, János Kollár, for his guidance, attention, many ideas and suggestions. Financial support during graduate school was provided by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico — Brazil) and Princeton University. This research was partially conducted during the period I was employed by the Clay Mathematics Institute as a Liftoff Fellow.

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