Contracting Lorenz attractors through resonant double homoclinic loops

C.A.Morales, M.J.Pacifico & B. San Martin *

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Abstract

A contracting Lorenz attractor is obtained from the geometric Lorenz attractor replacing the usual expanding condition $\lambda_3 + \lambda_1 > 0$ in the eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ at the origin by a *contracting* condition $\lambda_3 + \lambda_1 < 0$ [R]. In this paper we analyse the appearence of contracting Lorenz attractors in the unfolding of certain resonant double homoclinic loops in dimension three. We prove that the corresponding unfolding yields contracting Lorenz attractors in a positive Lebesgue measure set of parameters, answering a question posed in [Rob1].

1 Introduction

A contracting Lorenz attractor is obtained from the geometric Lorenz attractor replacing the usual expanding condition $\lambda_3 + \lambda_1 > 0$ in the eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ at the origin by a *contracting* condition $\lambda_3 + \lambda_1 < 0$ [R]. In this paper we analyse the appearence of contracting Lorenz attractors in the unfolding of certain resonant double homoclinic loops in dimension three. We prove that the corresponding unfolding yields contracting Lorenz attractors in a positive Lebesgue measure set of parameters, answering a question posed in [Rob1, Remark 5.1, p. 138]. Indeed, in the serie of papers [Rob1, Rob2, Rob3] Robinson studied the existence of transitive attractors of Lorenz type in generic unfoldings of resonant double homoclinic loops in dimension three. For instance, Theorem 3.1, p. 130, in [Rob1] says that under certain conditions such unfoldings produce transitive weak attractors containing the singularity. This result was generalized in [MPS], where we obtained attractors insteady of weak attractors and

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enlarged the region where there are expanding Lorenz attractors in the unfolding. Here we shall consider parametrized families of vector fields unfolding a resonant double homoclinic loop at $\eta = \eta_0$ as in [MPS]. We assume the same hypotheses (A1)-(A7) of [MPS], except for (A4) and (A5) that we replace, respectively, by

$$\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + 2\lambda_u(\eta_0) < 0, \quad \text{and} \quad \lambda_{ss}(\eta_0) < 2\lambda_s(\eta_0), \tag{1}$$

and

$$B = \frac{C^+_{\eta_0} + C^-_{\eta_0}}{C^+_{\eta_0} C^-_{\eta_0}} < 1,$$

where $C^{\pm}_{\eta_0}$ are defined as in that paper: the constants $C^{\pm}_{\eta_0}$ measure the change in area within a certain bundle over Γ, the resonant double homoclinic loop. Condition (A4) here implies the existence of C^2 strong stable foliations in a neighborhood of the loop. The proof is based first on rescaling techniques [PT] to obtain convergence to non-continuous maps, and second on a BC-type argument [BC1] to show that the parameters corresponding to contracting Lorenz attractors have positive Lebesgue measure.

We point out that besides the papers cited above, there are many other concerning Lorenz attractors and its bifurcations. We refer to the interested reader [MPS] and the references therein for a survey on this.

In order to describe our results in a precise way, let us introduce some notations and definitions. By *attractor* we mean a transitive set which is maximal invariant in a positively invariant open set. A set is *transitive* if it is the omega-limit set of one of its orbits.

1.1 Our hypotheses

Let us state our hypothesis in a precise way. In what follows X_η is a family of C^r, $r \geq 1$, vector fields on $\mathbb{R}^3$ unfolding a resonant double homoclinic loop at $\eta = \eta_0$, see Figure **??**, satisfying the following conditions:

(A1) For every η, X_η has a hyperbolic singularity Q_η such that the eigenvalues of $DX_\eta(Q_\eta)$ are real with $\lambda_{ss}(\eta) < \lambda_s(\eta) < 0 < \lambda_u(\eta)$, and with eigenvectors v^{ss}, v^s, and v^u, respectively.

With this assumption, there are several invariant manifolds for the singularity Q_η. We denote the one-dimensional unstable manifold tangent to v^u by $W^u(Q_\eta, \eta)$, and the two-dimensional stable manifold tangent to v^{ss} and v^s by $W^s(Q_\eta, \eta)$. Next, there is a one-dimensional strong stable manifold $W^{ss}(Q_\eta, \eta)$. This latter manifold is made of points which converge to Q_η at an asymptotic rate determined by the eigenvalue λ_{ss}. All these manifolds are C^r if the vector field is C^r. Finally, there is a two-dimensional central stable manifold tangent to v^s and v^u, which we denote by $W^{cu}(Q_\eta, \eta)$. The later manifold is at least C^1. With this notation we can make the second assumption about the existence of a double homoclinic connection.

(A2) For the bifurcation value η_0, there is a double homoclinic connection with the unstable manifold of Q_{η_0} contained in the stable manifold but outside the strong stable manifold,

$$\Gamma = W^u(Q_{\eta_0}, \eta_0) \subset W^s(Q_{\eta_0}, \eta_0) \setminus W^{ss}(Q_{\eta_0}, \eta_0).$$

In fact, we assume that the two branches $\Gamma^{\pm}$ of $\Gamma \setminus \{Q_{\eta_0}\}$ are contained in the same component of $W^s(Q_{\eta_0}, \eta_0) \setminus W^{ss}(Q_{\eta_0}, \eta_0)$. Note that $\Gamma = \{Q_{\eta_0}\} \cup \Gamma^+ \cup \Gamma^-$.

(A3) For η_0, the central manifold $W^{cu}(Q_{\eta_0}, \eta_0)$ is transverse to the stable manifold $W^s(Q_{\eta_0}, \eta_0)$ along Γ.

Let

$$P(q) = T_q W^{cu}(Q_{\eta_0}, \eta_0) \quad \text{for} \quad q \in \Gamma.$$

The transversality condition in (A3) with condition

$$W^u(Q_{\eta_0}, \eta_0) \cap W^{ss}(Q_{\eta_0}, \eta_0) = Q_{\eta_0}$$

in Assumption (A2) implies that $P(q)$ converges to $P(Q_{\eta_0})$ as q converges to Q_{η_0} along Γ by the Inclination Lemma [dMP]. Therefore, $\{P(q) : q \in \Gamma\}$ is a continuous bundle over Γ. Considering one half of the homoclinic connection $\Gamma^+ \cup Q_{\eta_0}$, let $\nu^+ = 1$ if the bundle $\{P(q) : q \in \Gamma^+ \cup Q_{\eta_0}\}$ is orientable and $\nu^+ = -1$ if the bundle is nonorientable. In the same way considering the other half of the homoclinic connection $\Gamma^- \cup Q_{\eta_0}$, let $\nu^- = \pm 1$ whenever the bundle $\{P(q) : q \in \Gamma^- \cup Q_{\eta_0}\}$ is orientable or nonorientable respectively.

(A4) We assume that

$$\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + 2\lambda_u(\eta_0) < 0, \quad \text{and} \quad \lambda_{ss}(\eta_0) < 2\lambda_s(\eta_0).$$

We shall use the notation $\alpha(\eta) = -\frac{\lambda_s(\eta)}{\lambda_u(\eta)}$ and $\beta(\eta) = -\frac{\lambda_{ss}(\eta)}{\lambda_u(\eta)}$.

These are open conditions and so do not add a codimension to the bifurcation. The first inequality in (A4) assures the existence of C^2 strong stable foliations in a neighborhood of the loop while the second one assures that $W^{cu}(Q_{\eta_0}, \eta_0)$ is C^2.

Let $q^{\pm}(t)$ be a parametrization of the solution along $\Gamma^{\pm}$ and $\text{div}_2(q^{\pm}(t))$ the Jacobian of X_{η_0} at t restricted to $T_{\Gamma^{\pm}} W^{cu}$. Define $C^{\pm}_{\eta_0}$ by

$$C^{\pm}_{\eta_0} = \exp\left(\int_{-\infty}^{\infty} \text{div}_2(q^{\pm}(t))dt\right).$$

The quantity $C^{\pm}_{\eta_0}$ is the change in area within the -planes $P(q)$ along the whole length of $\Gamma^{\pm}$.

(A5)

$$B = \frac{C^+_{\eta_0} + C^-_{\eta_0}}{C^+_{\eta_0} C^-_{\eta_0}} < 1.$$

(A6) There is a one-to-one resonance between the unstable and weak stable eigenvalue for η_0:

$$\lambda_u(\eta_0) + \lambda_s(\eta_0) = 0.$$

Observe that condition (A6) means $\alpha(\eta) = 1$. This condition is needed to have (A5) satisfied see [Rob1]. This resonance condition is a co-dimension one condition; in total, the conditions of η_0 are co-dimension three, (two conditions are from the double homoclinic connection and resonance gives the third and final co-dimension). The final assumption is related to the unfolding of the bifurcation. We assume that the parameter space is is large enough in order to break the double homoclinic loop in a correct way.

(A7) Let $\mathcal{N} \subset \mathcal{X}^1(\mathbb{R}^3)$ be the three-submanifold defined by conditions (A1)-(A6). We assume that the family $\{X_\eta\}$ is transverse to $\mathcal{N}$ at η_0.

1.2 The main result

It is now possible to announce our main result. Given a set A, $\mathrm{Cl}(A)$ denotes the closure of A.

Theorem A. *Let $\{X_\eta\}$ be a C^k-parametrized family of C^r-vector fields $(r, k \geq 3)$ satisfying (A1) to (A7). Then, there is a positive Lebesgue measure set $\mathcal{L}$ in the parameter space with $\eta_0 \in \mathrm{Cl}(\mathcal{L})$ such that X_η has a contracting Lorenz attractor for all $\eta \in \mathcal{L}$.*

The tools used in the proof are reduction of the dynamics to a one-dimensional Poincaré map and the existence of a suitable rescaling for such maps with a well defined limit dynamics. The limit map obtained is peacewise differentiable, with two critical values corresponding to only one critical point. We use these maps to control both the forward orbit as well the derivative along it of the critical values. Once we have that, we use a BC-type argument [BC1, BC2] to choose a positive Lebesgue measure set of parameters such that the corresponding maps present a contracting Lorenz attractor. Convergence to non-continuous maps via rescaling was also considered in [MPu, MSV, MPS], while a BC-type argument for discontinuous maps was also considered in [R, MSV].

1.3 Sketch of the proof

Let us present the idea of the proofs. As in [Rob1], we observe that (A1)-(A3) imply the existence of a strong stable invariant foliation close to the loop. By (A4) the C^r Section Theorem [S] implies that such a foliation is C^2 and varies C^2 with the parameters. As usual we consider the Poincaré map along the homoclinic loop. Using the strong stable foliation we reduce the dynamics of the return map to a one-dimensional map $f_\eta(\tau)$. We denote α the order of $f_\eta(\tau)$ at the discontinuity point. Clearly α depends on η. In Lemma 2.2 we fix α and prove the existence of *good parameters values*, i.e., parameters for which the critical values $f_\eta(0^\pm)$ of f_η are either fixed or pre-fixed or periodic (with period 2) expanding points. Such parameters are solution of certain equations that can be solved only for $\alpha > 1$ because of (A5). Using the critical values we construct, for those good parameters, a f_η-invariant closed interval $[p, q]$ containing $\tau = 0$. Afterward we use rescaling techniques [PT]: we take a suitable parameter-depending change of coordinates in a neighborhood of $[p, q]$ and, at the same time, we rescale the parameter space in a small neighborhood of those good parameters. This yields a new family $g_\alpha(\mu, \nu, x)$ and new good parameters $(\mu(\alpha), \nu(\alpha))$. In Lemma 3.5 we prove the following bounds for the derivative of $g_\alpha(\mu, \nu, x)$:

$$K_1 \mid x \mid^{\alpha-1} \leq \mid \frac{\partial}{\partial x} g_\alpha(\mu, \nu; x) \mid \leq K_2 \mid x \mid^{\alpha-1} \quad \text{and} \tag{2}$$

$$K_1 \mid x \mid^{\alpha-2} \leq \mid \frac{\partial^2}{\partial x} g_\alpha(\mu, \nu; x) \mid \leq K_2 \mid x \mid^{\alpha-2} . \tag{3}$$

These bounds are used to prove that $g_\alpha(\mu, \nu, \cdot)$ converges (in a C^2-sense to be defined below) to a map $g(\mu, \nu, \cdot)$ as $\alpha \to 1^+$. The limit map $g(\mu, \nu, \cdot)$ is piecewise linear expanding that looks like the one in Figure **??**. In the same lemma we show that the limit $\lim_{\alpha \to 1^\pm}(\mu(\alpha), \nu(\alpha)) = (\mu(1), \nu(1))$ exists.

The maps $g(\mu, \nu, \cdot)$ above do not have trapping regions, but they can be approximated in our family by ones having them. This is proved in Theorem 4.1 where we verify that $g_\alpha(\mu, \nu, \cdot)$ has trapping regions for $\alpha > 1$ close to 1 and for (μ, ν) close to $(\mu(\alpha), \nu(\alpha))$. The existence of trapping regions is an open property, and so, trapping regions of the flow do exist in an open set of parameters accumulating η_0. To finish the proof of Theorem A we use Theorem 5.1 that imply transitiveness of the maximal invariant set in the trapping region for a positive Lebesgue measure set.

2 One-dimensional reduction and good parameters

Consider a cross-section Σ of X_{η_0} close to Q_{η_0} transversal $W^s(Q_{\eta_0})$ intersecting both branches of $W^u(Q_{\eta_0})$. There is a neighborhood V of $\Sigma \cap W^s(Q_{\eta_0})$ in Σ such that the positive orbit of every point at $V \setminus W^s(Q_{\eta_0})$ intersects Σ for every parameter η near enough η_0, defining in this way a Poincaré map

$$F_\eta : V \setminus W^s(Q_{\eta_0}) \subset \Sigma \to \Sigma\,.$$

As X_{η_0} satisfies conditions (A1)-(A4), the standard stable manifold theory applies to show the existence of a C^2 stable foliation in a small neighborhood (that for convenience we assume equals to V) of $W^s(Q_\eta)$ varying C^2 with the parameter. As in [Rob2] the existence of a C^r stable foliation ($r \geq 1$) depends on the relation

$$C_3 e^{T(\lambda_{ss}(\eta_0)-\lambda_s(\eta_0))}(e^{T\lambda_u(\eta_0)})^r < 1\,.$$

By the first eigenvalue inequality in (A4) we have the above relation for $r = 2$. Then, using [S, Theorem 5.18] in the same way as in [Rob2], via projection along the leaves of the strong stable foliation, the problem is reduced to a one-dimensional Poincaré map

$$f_\eta : V' \setminus \{c_\eta\} \subset [-1,1] \to [-1,1]\,.$$

Here c_η is the projection of $W^s(Q_{\eta_0}) \cap V$ onto V'. We assume $c_\eta = 0$ for every η. Denote $a_\eta^\pm = \lim_{\tau\to\pm 0} f_\eta(\tau)$, $\tau \in [-1,1]$. As in [Rob2] we have the following

Lemma 2.1. *There is an interval J, $0 \in J$ such that for every η sufficiently near to η_0, the map $f_\eta : J \subset [-1,1] \to [-1,1]$ has the following form:*

$$f_\eta(\tau) = \begin{cases} a_\eta^+ + \nu^+ C_\eta^+ |\tau|^{\alpha_\eta} + O_{\eta,1}(|\tau|^{\alpha_\eta}) & \text{if} \quad \tau > 0 \\ a_\eta^- - \nu^- C_\eta^- |\tau|^{\alpha_\eta} + O_{\eta,2}(|\tau|^{\alpha_\eta}) & \text{if} \quad \tau < 0, \end{cases}$$

where $O_{\eta,i}$ are C^2, varying C^2 with respect to η, and $\lim_{x\to 0} \frac{O_{\eta,i}(x)}{x} = 0$ uniformly on η. Moreover, $C_\eta^\pm$ depends C^2 on η.

From now on we assume that X_η is a three-parameter family, for which there is an open set $U \subset \mathbb{R}^3$ such that for $\eta \in U$, X_η satisfies conditions (A1)-(A7).

From the transversality hypothesis (A7) we have that the map $\eta \mapsto (\alpha_\eta, a_\eta^+, a_\eta^-)$ is a diffeomorphism from a neighborhood of η_0 onto a neighborhood of $(1,0,0)$. So, we can reparametrize X_η by $(\alpha, a^+, a^-) \mapsto \eta(\alpha, a^+, a^-)$ in such away that $\alpha_{\eta(\alpha,a^+,a^-)} = \alpha$, $a^+_{\eta(\alpha,a^+,a^-)} = a^+$ and $a^-_{\eta(\alpha,a^+,a^-)} = a^-$.

The next lemma is analogous to the corresponding one [Lemma 3.2] in [MPS], and so we shall not give its proof. We point out that here the good parameters depend on $\alpha > 1$ while there the good parameters depend on $\alpha < 1$. The reason for this difference comes from the fact that here $B < 1$ while there $B > 1$, where B is given in condition (A5).

Lemma 2.2. *There are $\Lambda > 0$, an open full Lebesgue measure set $O \subset (1, 1+\Lambda)$, and C^1 maps $a^+, a^-, p, q : O \to \mathbb{R}$ with $p(\alpha) < 0 < q(\alpha)$ such that for $\eta = \eta(\alpha, a^+(\alpha), a^-(\alpha))$ the following hold:*

(a) if $\nu^+ = \nu^- = 1$ then $f_\eta(p(\alpha)) = p(\alpha)$, $f_\eta(q(\alpha)) = q(\alpha)$, $f_\eta(0^+) = p(\alpha)$, and $f_\eta(0^-) = q(\alpha)$,

(b) if $\nu^+ = \nu^- = -1$ then $f_\eta(p(\alpha)) = q(\alpha)$, $f_\eta(q(\alpha)) = p(\alpha)$, $f_\eta(0^+) = q(\alpha)$, and $f_\eta(0^-) = p(\alpha)$,

(c) if $\nu^+ = -\nu^- = 1$ then $f_\eta(p(\alpha)) = q(\alpha)$, $f_\eta(q(\alpha)) = q(\alpha)$, and $f_\eta(0^+) = f_\eta(0^-) = p(\alpha)$,

(d) if $\nu^+ = -\nu^- = -1$ then $f_\eta(p(\alpha)) = p(\alpha)$, $f_\eta(q(\alpha)) = p(\alpha)$, and $f_\eta(0^+) = f_\eta(0^-) = q(\alpha)$

In any case, $\lim_{\alpha\to1} |p(\alpha)|/q(\alpha) = C^+_{\eta_0}/C^-_{\eta_0}$, $\lim_{\alpha\to1} q(\alpha) = 0 = \lim_{\alpha\to1} p(\alpha)$, *and* $\lim_{\alpha\to1} q(\alpha)^{\alpha-1} = \lim_{\alpha\to1} |p(\alpha)|^{\alpha-1} = B$.

Notation 2.3. *Let O be as in Lemma 3.2. We shall call $\eta = \eta(\alpha, a^+(\alpha), a^-(\alpha))$ for $\alpha \in O$ the* good parameters *of X_η.*

3 Rescaling

In this section we perform rescaling techniques [PT]. Keeping the notation p, q in Lemma 3.2 we take suitable parameter-depending change of coordinates in a neighborhood of $[p, q]$ and, at the same time, we rescale the parameter space in a small neighboorhod of the good parameters in Notation 2.3. This yields a new family $g_\alpha(\mu, \nu, x)$ and new good parameters $(\mu(\alpha), \nu(\alpha))$. The goal of this section is to prove that $g_\alpha(\mu, \nu, \cdot)$ converges to a map $g(\mu, \nu, \cdot)$ in some sense to be described below.

To start, consider a parametrized family $\{X_\eta\}$ satisfying the hypotheses described in Section 1.1.

Given $\alpha \in O$ let $a^-(\alpha)$, $a^+(\alpha)$, $p(\alpha)$ and $q(\alpha)$ be as in the previous lemma. Define $(\mu(\alpha), \nu(\alpha)) = (\frac{a^+(\alpha)}{q(\alpha)}, \frac{a^-(\alpha)}{q(\alpha)})$ and $(\mu, \nu) = (\frac{a^+}{q(\alpha)}, \frac{a^-}{q(\alpha)})$ in a neighborhood of $(a^+(\alpha), a^-(\alpha))$ onto a neighborhood of $(\mu(\alpha), \nu(\alpha))$, and the family of maps

$$g_\alpha(\mu, \nu, x) = \frac{1}{q(\alpha)} f_\eta(q(\alpha)x), \tag{4}$$

where $\eta = \eta(\alpha, q(\alpha)\mu, q(\alpha)\nu)$. We set $\mathrm{Dom}(g_\alpha)$ for the domain of g_α.

Remark 3.1. *Observe that for each fixed* α, *this change of variables renormalizes the parameters* $a^{\pm}$ *with* $a^{+} = q(\alpha)\mu$ *and* $a^{-} = q(\alpha)\nu$. *Moreover, by Lemma 2.2,* $\lim_{\alpha\to 1}\mu(\alpha)$ *and* $\lim_{\alpha\to 1}\nu(\alpha)$ *exist and we denote them by* $\mu(1)$ *and* $\nu(1)$ *respectively.*

Definition 3.2. *Let* $g:\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})\to\mathbb{R}$. *We say that* $g_\alpha\to g$ *in the* C^0 *topology in compact sets of* $\mathbb{R}^3$ *as* $\alpha\to 1$ *if*

(a) $\mathrm{Dom}(g_\alpha)\to\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *as* $\alpha\to 1$, *that is, for all* $R>0$ *there is* $1<\alpha_0$ *such that if* $1<\alpha<\alpha_0$ *then* $B_R(0)\cap(\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})\subset\mathrm{Dom}(g_\alpha)$, *where* $B_R(0)$ *is the ball of radius* R *centered at* $(0,0,0)$.

(b) for every compact set $K\subset\mathbb{R}^3$ *and every* $\epsilon>0$ *there is* $\delta>0$ *such that if* $|\alpha-1|<\delta$ *then*

$$\sup_{y\in K\cap(\mathbb{R}^2\times(\mathbb{R}\setminus\{0\}))}|g_\alpha(y)-g(y)|<\epsilon.$$

Definition 3.3. *Let* $g:\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})\to\mathbb{R}$. *We say that* $g_\alpha\to g$ *in the* C^1 *topology in compact sets of* $\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *if*

(a) $\mathrm{Dom}(g_\alpha)\to\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *as* $\alpha\to 1$,

(b) for every compact set $K\subset\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *and every* $\epsilon>0$ *there is* $\delta>0$ *such that if* $|\alpha-1|<\delta$ *then*

$$\sup_{i\in\{0,1\},y\in K}|D^i g_\alpha(y)-D^i g(y)|<\epsilon.$$

Note that with these notions, C^1 convergence does not imply C^0 convergence.

Definition 3.4. *We say that* $g_\alpha\to g$ *in the* C^2 *topology in x-compact sets of* $\mathbb{R}\setminus\{0\}$ *uniformly in compact set of* $\mathbb{R}^2$ *if*

(a) $\mathrm{Dom}(g_\alpha)\to\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *as* $\alpha\to 1$,

(b) for every compact $K\subset\mathbb{R}^2\times(\mathbb{R}\setminus\{0\})$ *and every* $\epsilon>0$ *there is* $\delta>0$ *such that if* $|\alpha-1|<\delta$ *then*

$$\sup_{i\in\{0,1,2\},(\rho,x)\in K}|\partial_x^i g_\alpha(\rho,x)-\partial_x^i g(\rho,x)|<\epsilon.$$

We have the following result.

Lemma 3.5. *Let* g_α *be as in (4) and define*

$$g(\mu,\nu,x)=\begin{cases}\mu+\nu^{+}C_{\eta_0}^{+}Bx & \text{if} \quad x>0\\ \nu+\nu^{-}C_{\eta_0}^{-}Bx & \text{if} \quad x<0\end{cases}$$

where $\eta_0=\eta(1,0,0)$. *Then*

(i) $g_\alpha \to g$ *in the* C^0 *topology in compact sets of* $\mathbb{R}^3$ *as* $\alpha \to 1$, $\alpha \in O$,

(ii) $g_\alpha \to g$ *in the* C^1 *topology in compact sets of* $\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})$ *as* $\alpha \to 1$, $\alpha \in O$.

(iii) $g_\alpha \to g$ *in the* C^2 *topology in compact sets of* $\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})$ *as* $\alpha \to 1$, $\alpha \in O$.

Moreover, for any $c > \max\{1, C^+_{\eta 0}/C^-_{\eta 0}\}$*, there are constants* $\Delta_0 > 0$, $0 < K_1 < K_2$ *such that for* $\alpha \in O \cap [1, 1+\Delta_0]$ *we have*

(a) $[-c,c]^2 \times ([-c,c] \setminus \{0\}) \subset \mathrm{Dom}(g_\alpha)$,

(b) $K_1|x|^{\alpha-1} \leq \left|\frac{\partial}{\partial x} g_\alpha(\mu,\nu;x)\right| \leq K_2|x|^{\alpha-1}, \quad \forall (\mu,\nu,x) \in [-c,c]^2 \times ([-c,c]\setminus\{0\})$.

(c) $K_1|x|^{\alpha-2} \leq \left|\frac{\partial^2}{\partial x} g_\alpha(\mu,\nu;x)\right| \leq K_2|x|^{\alpha-2}, \quad \forall (\mu,\nu,x) \in [-c,c]^2 \times ([-c,c]\setminus\{0\})$.

Proof: The proof of (i), (ii), (a) and (b) are similar to the proofs in [MPS, Lemma 3.3] and we shall not do them here.

To prove (iii), put $k = q(\alpha)$, where $q(\alpha)$ is given by Lemma 2.2. Recall $k \to 0$ as $\alpha \to 1^+$. Now note that

$$\partial_x^2 g_\alpha(\mu,\nu,x) = \begin{cases} \nu^+ C^+_\eta \alpha(\alpha-1)k^{\alpha-1}|x|^{\alpha-2} + O''_{\eta,1}(|kx|^\alpha)\alpha^2 k^{2\alpha}|x|^{2(\alpha-1)} + \\ O'_{\eta,1}(|kx|^\alpha)\alpha(\alpha-1)|kx|^{\alpha-2}k \quad \text{for} \quad x > 0 \\ -\nu^- C^-_\alpha \alpha(\alpha-1)k^{\alpha-1}|x|^{\alpha-2} - O''_{\eta,2}(|kx|^\alpha)\alpha^2 k|kx|^{2(\alpha-1)} - \\ O'_{\eta,2}(|kx|^\alpha)\alpha(\alpha-1)|kx|^{\alpha-2}k \quad \text{for} \quad x < 0\,. \end{cases}$$

By Lemma 2.1 $\mathcal{O}_{\eta,i}$ is C^2 and so $\mathcal{O}''_{\eta,i}(|kx|^\alpha)$ is uniformly bounded in K. By Lemma 2.2 we have that $k^{2\alpha} \to 0$, $k^{\alpha-1} \to B$, and $(\alpha - 1) \to 0$ as $\alpha \to 1$. Since $|x|^{2\alpha-1}$ is uniformly bounded in K we finally obtain that $\partial_x^2 g_\alpha(\mu,\nu,x) \to 0$ as $\alpha \to 1$ in compact sets of $\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})$. Now note that the expression above for $\partial_x^2 g_\alpha(\mu,\nu,x)$ together with the bounds for k, $k^{2\alpha}$, $k^{\alpha-1}$ imply (c). All together conclude the proof of Lemma 3.5.

□

Remark 3.6. *Observe that* $|g'| = |C^\pm_{\eta 0} B|$*. Since* $|\nu^\pm C^\pm_{\eta 0} B| > 1$ *we obtain that* g *is an expanding map.*

4 Trapping region

In this section we prove the existence of trapping regions for $g_\alpha(\mu,\nu,\cdot)$, for $\alpha \in O$ close to 1 and (μ,ν) close to $(\mu(\alpha),\nu(\alpha))$ chosen in an appropriated way (recall the notation in Section 4). A trapping region for $g_\alpha(\mu,\nu,.)$ is a closed interval J such that $g_\alpha(\mu,\nu,J) \subset Int(J)$, where $Int(J)$ stands for the interior of J. We shall prove the following theorem.

Theorem 4.1. *There is an open set $\mathcal{O} \subset I\!R^3$ such that the properties below hold:*

1. $(\alpha, \mu(\alpha), \nu(\alpha)) \in \mathrm{Cl}(\mathcal{O})$ *for all* $\alpha \in O$*;*
2. *If* $\theta = (\alpha, \mu, \nu) \in \mathcal{O}$ *then there is a closed interval* $I_\theta \subset I\!R$ *with* $0 \in \mathrm{int}(I_\theta)$ *such that* $g_\alpha(\mu, \nu, x) \subset \mathrm{int}(I_\theta)$ *for all* $x \in I_\theta$*.*

To prove Theorem 4.1 we proceed as in [MPS]: we define an auxiliary function $F_\alpha : I\!R^2 \to I\!R^2$, which is used to find the required trapping regions. This function involves the critical and the fixed points of $g_\alpha(\mu, \nu, \cdot)$. As in [MPS] it is possible to prove that F_α is locally one-to-one. Once we have that, the parameters V_α corresponding to maps with trapping regions are obtained as $F_\alpha(V_\alpha) = W_\alpha \cap \{(x, y), x > 0, y < 0\}$ where W_α is a neighborhood of $(0, 0)$. For the detailed proof see [MPS, Theorem 4.1].

Remark 4.2. *Observe that for any* $\alpha \in O$ *the set* $\mathcal{O}_\alpha = \{(\mu, \nu) : (\alpha, \mu, \nu) \in \mathcal{O}\}$ *contains a cone with vertices at* $(\mu(\alpha), \nu(\alpha))$*. This property will be a key point in the next section.*

5 One-dimensional analysis

In this section we will prove a one-dimensional theorem from which Theorem A follows. To announce in a precise way this theorem recall that by hypothesis (A7) the map $\eta \mapsto (\alpha_\eta, a_\eta^+, a_\eta^-)$ is a submersion from a neighborhood of η_0 onto a neighborhood of $(1, 0, 0)$. So, we can assume that X_η is a three-parameter family and we can reparametrize X_η by $(\alpha, a^+, a^-) \mapsto \eta(\alpha, a^+, a^-)$ in such away that $\alpha_{\eta(\alpha,a^+,a^-)} = \alpha$, $a^+_{\eta(\alpha,a^+,a^-)} = a^+$ and $a^-_{\eta(\alpha,a^+,a^-)} = a^-$. Moreover it was obtained an invariant strong stable foliation that induces a one-dimensional family f_η for which a^+ and a^- are the critical values, recall Lemma 2.1. In Lemma 2.2 it was obtained a set O such that for each $\alpha \in O$, it was defined the α-dependent points $p(\alpha)$ and $q(\alpha)$, and α-dependent parameters $a^+(\alpha)$ and $a^-(\alpha)$. Using these we rescaled the family f_η and obtained a new family $g_\alpha(\mu, \nu, \cdot)$ with new parameter space. In Sections 3 and 4 we defined the α-dependent points $x(\alpha)$ and $y(\alpha)$, and α-dependent parameters $\mu(\alpha)$ and $\nu(\alpha)$ in a such way that $x(\alpha)$ and $y(\alpha)$ are fixed or pre-fixed points of $g_\alpha(\mu(\alpha), \nu(\alpha), \cdot)$, and the dynamics of $g_\alpha(\mu(\alpha), \nu(\alpha), \cdot)$ is one of the displayed at Figure **??**. Furthermore, for the limit function $g(\mu, \nu, \cdot)$ there are a unique limit parameters values $\mu(1), \nu(1)$ and unique limit points $x(1), y(1)$, which are fixed or pre-fixed by the expansive piecewise linear map $g(\mu(1), \nu(1), \cdot)$, see Figure **??**.

Before we announce the main theorem in this section let us recall that f_η has positive Lyapunov exponent at a if there exists $\lambda > 1$ such that $\left|(f_\eta^k)'(a)\right| > \lambda^k$ for every $k > 0$. Given $A \subset \mathbb{R}$, *convexh*(A) denotes the convex hull of the set A and $\mathrm{Cl}(A)$ denotes the closure of A.

For the family f_η described above we have the following:

Theorem 5.1. *For each $\alpha \in O$ close enough to 1, there exists a Lebesgue positive measure set E_α in the (a^+, a^-)-parameter space such that:*

1. *For $(a^+, a^-) \in E_\alpha$, the critical values a^+ and a^- of the function f_η, where $\eta = (\alpha, a^+, a^-)$, have positive Lyapunov exponents.*
2. *For every $(a^+, a^-) \in E_\alpha$, we have*

$$\mathrm{Cl}(\{f_\eta)^k(a^-)\}_{k\in\mathbb{N}}) = \mathrm{Cl}(\{f_\eta^k(a^-)\}_{k\in\mathbb{N}}) = convexh\{f_\eta(a^+), f_\eta(a^-), a^+, a^-\}\,.$$

Before we proof the above theorem, let us finish the proof of Theorem A.

Proof of Theorem A.
For simplicity, denote $\Theta_\eta = convexh\{f_\eta(a^+), f_\eta(a^-), a^+, a^-\}$. For each η in the positive Lebesgue parameter subset given by $\{\eta = (\alpha, a^+, a^-) : \alpha \in O, (a^+, a^-) \in E_\alpha\}$, let us consider the projection π along the invariant stable foliation for f_η. Then the set

$$Q_\eta = \bigcap_{T>0} \bigcup_{t>T} X_\eta^t(\pi^{-1}(\Theta_\eta))$$

is the required attractor. Indeed, existence of a trapping region for Q_η follows from the fact that Θ_η is an attractor for f_η and the fact that the invariant foliation defining f_η is contracting. That Q_η is transitive follows from the same reasons as before.

5.1 Proof of Theorem 5.1.

First we outline the proof of Theorem 5.1. For each $\theta > 0$ consider the straight line $L_{\alpha,\theta}$ in the (μ, ν)-plane given by

$$L_{\alpha,\theta} = \{(\mu, \nu)\,/\,\nu = \theta(\mu - \mu(\alpha)) + \nu(\alpha)\}.$$

By Remark 4.2 there is a positive measure set of angles θ such that some subset of $L_{\alpha,\theta}$ is contained in $\mathcal{O}_\alpha$. Let m and m_θ be the Lebesgue measure in the (μ, ν)-plane and in $L_{\alpha,\theta}$ respectively. Using Benedicks-Carleson techniques, for $\alpha \in O$ close to 1 (see [BC1, BC2, MV, R]) we will be able to show the existence of a m_θ-positive measure subset $\tilde{E}_{\alpha,\theta}$ of $L_{\alpha,\theta}$, having $(\mu(\alpha), \nu(\alpha))$ as a density point and satisfying Theorem 5.1 in the (μ, ν)-parameter setting. Once we have this, we consider $\tilde{E}_\alpha = \bigcup_\theta \tilde{E}_{\alpha,\theta}$. Note that Fubini's theorem implies $m(\tilde{E}_\alpha) > 0$. Now, since the reparametrization $(a^+, a^-) \longrightarrow (\mu, \nu)$ is an affine map, we get a positive measure set E_α in the (a^+, a^-)-parameter space satisfying Theorem 5.1 because f_η and $g_\alpha(\mu, \nu, \cdot)$ for $\eta = (\alpha, a^+, a^-)$ are conjugated by an affine map.

To set up $\tilde{E}_{\alpha,\theta}$ described above, we follow the steps in [R] proving first that the maximal orbits outside a neighbourhood of the critical point have exponential growth. This fact is established in Lemma 5.5. Here we point out that this fact is a key step in order to apply Benedicks-Carleson techniques. In Lemma 5.6 we will show that under the basic assumption (BA) the increasing in the derivative after a binding period (see definitions before Lemma 5.6) fully compensates the small factor introduced in the derivative along the segment of orbit of the critical value passing close to the critical point. As a consequence, we can establish, via an inductive argument, condition (FA), given before Lemma 5.7, to guarantee the exponential growth of the derivative at the critical values. This is stablished in Lemma 5.7.

To prove that the maximal orbits outside a neighbourhood of the critical values have exponential growth we follow [R]. We observe that the result is obtained easier here due to the expanssiviness of the limit dynamics.

Now we start with the proof of Theorem 5.1. From now on (α, μ, ν) will be always a parameter in the set $\mathcal{O}$ given by Theorem 4.1.

The next lemma assures the growth of the derivative of g_α with respect to the space variable for points outside a small interval containing the critical value for parameters close to $(\mu(\alpha), \nu(\alpha))$. Its proof is an easy consequence from the fact that g_α converges to a piecewise expanding linear map.

Lemma 5.2. *There is $\lambda_1 > 1$ such that for every small δ_0 there are constants $\alpha_1 = \alpha_1(\delta_0) > 1$ and $a_1 = a_1(\delta_0)$ such that for every $\alpha < \alpha_1$ in O, for every (μ, ν) with $|(\mu, \nu) - (\mu(\alpha), \nu(\alpha))| < a_1,$ and x such that $|x| \geq \delta_0$, then $|(g_\alpha(\mu, \nu, x))'| > \lambda_1$.*

Proof: By Lemma 3.5, $g_\alpha(\cdot, \cdot, \cdot) \to g(\cdot, \cdot, \cdot)$ with $g(\cdot, \cdot, \cdot)$ piecewise expanding linear map with slope $|C^{\pm}_{\eta_0} B| > 1$ because $B = \frac{C^+_{\eta_0} + C^-_{\eta_0}}{C^+_{\eta_0} C^-_{\eta_0}}$, see Figure **??**. Let $1 < \lambda < min\{|C^+_{\eta_0} B|, |C^-_{\eta_0} B|\}$. So, given $\delta_0 > 0$ take $\alpha_1 = \alpha_1(\delta_0)$ such that for $\alpha < \alpha_0, the map g_\alpha(\cdot, \cdot, \cdot)$ is sufficiently near $g(\cdot, \cdot, \cdot)$. Taking a_1 small enough we conclude the proof of Lemma 5.2 $\square$.

Remark 5.3. *Lemma 1.2 in [R] is the corresponding to Lemma 5.2 and its proof is much more involved because the limit map there is not expanding.*

The next result gives that uniformly along finite number of iterates outside a small interval containing the critical point the derivative grows exponentially, for parameter values sufficiently near $\mu(\alpha), \nu(\alpha)$.

Lemma 5.4. *Given δ_0 small enough there exist $\alpha_2 = \alpha_2(\delta_0) > 1$ and $\lambda_2 > 1$ independing on δ_0, with the following property: for $\delta < \delta_0$ there is $a_2 = a_2(\delta) > 0$ such that for every $\alpha < \alpha_2$ in O and for any (μ, ν) with $|(\mu, \nu) - (\mu(\alpha), \nu(\alpha))| < a_2$, and x, $\delta \leq |x| < \delta_0$, there is $l = l(a_2, x, \delta_0)$*

such that $\left|g_\alpha^j(\mu,\nu,x)\right| > \delta_0$ *for* $j = 1,...,l$ *and* $\left|(g_\alpha^l(\mu,\nu,x))'\right| > \lambda_2^l$ *. Furthermore, there are constants* $L = L(\delta_0)$ *and* $M = M(\delta,\,\delta_0)$ *such that* $L < l < \ M$ *.*

Proof: Let us suppose that $\nu^+ = -1$ and $\nu^- = +$. The other cases are similar. Define

$$M_\varepsilon = M_\varepsilon(\alpha,\mu,\nu) = \max\{\left|(g_\alpha(\mu,\nu,x))'\right| : x(\alpha) - \varepsilon \le x \le \ x(\alpha) + \varepsilon\}$$

and

$$m_\varepsilon = m_\varepsilon(\alpha,\mu,\nu) = \min\{\left|(g_\alpha,(\mu,\nu,x)'\right| : x(\alpha) - \varepsilon \le x \le x(\alpha) + \varepsilon\}\,.$$

When $\alpha \to 1$, we have that $g_\alpha(\mu,\nu,\cdot)$ converges uniformly on compact sets to $g_1(\mu,\nu,\cdot)$. Hence, given $\zeta > 0$, there are constants $\tau > 0$, $\overline{\varepsilon} > 0$ and $\alpha_2 > 1$ such that $|(\mu,\nu) - (\mu(1),\nu(1))| < \tau$, $\varepsilon < \overline{\varepsilon}$, $\alpha < \alpha_2$, which implies that $1 - \zeta < \frac{m_\varepsilon}{M_\varepsilon} < 1$.

From this, and since $\alpha > 1$, we can choose constants $a_2 > 0$, $\varepsilon > 0$, and $\lambda > 1$, such that if $\alpha \ge \alpha_2$ and $|(\mu,\nu) - (\mu(\alpha),\nu(\alpha))| < a_2$ then $m_\varepsilon M_\varepsilon^{\frac{1-\alpha}{\alpha}} > \lambda > 1$. Note that λ does not dependent on δ_0. Next, let δ_0 be a small positive constant and fix δ, $0 < \delta < \delta_0$.

To simplify notation, write $g = g_\alpha(\mu,\nu,\cdot)$. Given x and $\delta \le y \le \delta_0$, let l be the first positive integer such that $g^l(g^2(x)) > x(\alpha) + \varepsilon$. Notice that from the uniform convergence of $g_\alpha(\mu(\alpha),\nu(\alpha),\cdot)$ to $g_1(\mu(1),\nu(1),\cdot)$, recall Lemma 3.5, there are constants $\alpha_2 > 1$, $L = L(\delta_0)$ and $M = M(\delta,\,\delta_0)$ such that for $\alpha \le \alpha_2$ we have $L < l < M$. Now observe that for α_2 close to 1 and a_2 small enough get $|g(0^\pm) - y(\alpha)| + \left|g^l(g(y(\alpha))) - x(\alpha)\right|$ arbitrarily small because $g_\alpha(\mu(\alpha),\nu(\alpha),x(\alpha)) = x(\alpha)$ and $g_\alpha(\mu(\alpha),\nu(\alpha),y(\alpha)) = x(\alpha)$. Recall that here $g(0^+) = \lim_{x\to 0^+} g(x) = \mu$ and $g(0^-) = \lim_{x\to 0^-} g(x) = \nu$ which are near $y(\alpha)$.

Thus, by the chain rule and the Mean Value Theorem we get

$$\begin{aligned}
\varepsilon \ &< g^l(g^2(x)) - x(\alpha) \\
&\le \left|g^l(g^2(x)) - g^l(g^2(0^\pm))\right| + \left|g^l(g^2(0^\pm)) - g^l(g(y(\alpha)))\right| + \left|g^l(g(y(\alpha))) - x(\alpha)\right| \\
&\le \ \ M_\varepsilon^l\, C_2\, K_2\, |x|^\alpha + M_\varepsilon^l\, C_2\, |g(0^\pm) - y(\alpha)| + \left|g^l(g(y(\alpha))) - x(\alpha)\right|,
\end{aligned}$$

where C_2 is a positive constant close to $|(g_\alpha(\mu(1),\nu(1),y(1))'|$, and K_1 and K_2 are given by Lemma 3.5.

Solving the inequality above for $|x|$ we obtain

$$|x| > \left\{\frac{1}{M_\varepsilon^l\, C_2\, K_2}[\varepsilon - M_\varepsilon^l\, C_2\left|g(0^\pm) - y(\alpha)\right| - \left|g^l(g(y(\alpha))) - y(\alpha)\right|]\right\}^{\frac{1}{\alpha}}.$$

Furthermore, for a positive constant C_1 close to $|(g_1(\mu(1),\nu(1),y(1))'|$ we also have $\left|(g^{l+2})'(x)\right| \ge m_\varepsilon^l C_1 K_1\, |x|^{\alpha-1}$ and hence,

$$\left|(g^{l+2})'(x)\right| \ge$$

$$m_\varepsilon^l\, C_1\, K_1 K_2^{\frac{1-\alpha}{\alpha}} \left\{ \frac{\varepsilon}{M_\varepsilon^l\, C_2} - \left|g(0^\pm) - y(\alpha)\right| - \frac{\left|g^l(g(y(\alpha))) - x(\alpha)\right|}{M_\varepsilon^l\, C_2\, K_2} \right\}^{\frac{\alpha-1}{\alpha}} .$$

Now, if we consider this last inequality for $\mu = \mu(\alpha)$, $\nu = \nu(\alpha)$ and L large (this condition occurs with δ_0 small), then for some $\lambda_1 > 1$ we obtain

$$\begin{aligned} \left|(g^{l+2})'(x)\right| &\geq \left(m_\varepsilon\ M_\varepsilon^{\frac{1-\alpha}{\alpha}}\right)^l\ \varepsilon^{\left(\frac{\alpha-1}{\alpha}\right)} K_2^{\frac{1-\alpha}{\alpha}}\ C_1 C_2^{\frac{1-\alpha}{\alpha}} K_1 \\ &\geq \overline{C}\lambda^l, \text{ with } \overline{C} = \varepsilon^{\frac{\alpha-1}{\alpha}} K_2^{\frac{1-\alpha}{\alpha}} C_1\, C_2^{\frac{1-\alpha}{\alpha}} K_1 \\ &\geq \lambda_1^{l+2}. \end{aligned}$$

Since $L < l < M$, taking a_2 small enough, we can find λ_2, $1 < \lambda_2 < \lambda_1$, independent of δ_0, such that for (μ, ν) with $|(\mu, \nu) - (\mu(\alpha), \nu(\alpha))| < a_2$ we have $\left|(g^{l+2})'(x)\right| \geq \lambda_2^{l+2}$. This ends the proof of the lemma. □

Lemmas 5.2 and 5.4 imply the next result that guarantees that the derivative of g_α with respect to the phase space is bigger than 1 at each return of the orbit to a small interval containing the critical point.

Lemma 5.5. *There exist $\lambda_0 > 1$ satisfying the following property: for every $\delta > 0$ small enough, there exist $a_0 = a_0(\delta)$ and $\alpha_0 = \alpha_0(\delta)$ such that for every $\alpha < \alpha_0$, and (μ, ν) with $|(\mu, \nu) - (\mu(\alpha), \nu(\alpha))| < a_0$, and x such that $\left|g_\alpha^i(\mu, \nu, x)\right| \geq \delta$ for $i = 0, ..., k-1$ but $\left|g_\alpha^k(\mu, \nu, x)\right| < \delta$, then*

$$\left|(g_\alpha^k(\mu, \nu, x)'\right| \geq \lambda_0^k.$$

Proof: The proof follows directly from Lemmas 5.2 and 5.4. Indeed, let $0 \leq k_1 < k_2 < ... < k_n < k_{n+1} = k$ the iterates of x such that $|g_\alpha^{k_i}(\mu, \nu, x)| < \delta_0$. By Lemma 5.4 there is l_i such that $\left|(g_\alpha^{l_i}(\mu, \nu, x))'\right| > \lambda_2^{l_i}$. On the other hand, $\left|(g_\alpha^{k_{i+1}-k_i-l_i}(\mu, \nu, g_\alpha^{k_i+l_i}(\mu, \nu, x))'\right| > |\lambda_1^{k_{i+1}-k_i-l_i}$ and $\left|(g_\alpha^{k_1}(\mu, \nu, x)'\right| > |\lambda_1^{k_1}$. Then choosing λ_0 smaller than λ_1 and λ_2 we obtain the result. □

Now we start the construction of the set $\tilde{E}_\alpha$ described in the sketch of the proof. To simplify notation, given $(\mu, \nu) \in L_{\alpha,\theta}$ set $g_\mu = g_\alpha(\mu, \nu, \cdot)$.

For a small $\gamma > 0$, let us define $\tilde{E}_{\alpha,\theta}(\gamma) \subset L_{\alpha,\theta} \cup \mathcal{O}_\alpha$ as the set of parameters (μ, ν) satisfying the following basic assumption:

$$\left|g_\mu^j\ (\mu)\right| \geq e^{-\gamma j} \quad \text{and} \quad \left|g_\mu^j\ (\nu)\right|\ \geq e^{-\gamma j} \qquad \forall\ j \geq 1. \qquad \text{(BA)}$$

Let $\beta > 0$, $(\mu, \nu) \in \tilde{E}_{\alpha,\theta}(\gamma)$, and $\delta > 0$ small. For a positive integer k such that $\left|g_\mu^k\ (\mu)\right| \leq \delta$, define the *binding period* associated to the parameter

(μ,ν) for the return $g^k_\mu(\mu)$ of the critical value μ as the maximal interval $[k+1,\, k+s]$ such that for $1 \le j \le s$

$$\left| g_\mu^{k+j}(\mu) - g_\mu^{j-1}(\nu) \right| < e^{-\beta j}\ , \quad \text{if} \quad g^k_\mu(\mu) < 0$$

or

$$\left| g_\mu^{k+j}(\mu) - g_\mu^{j-1}(\mu) \right| < e^{-\beta j}\ , \quad \text{if} \quad g^k_\mu(\mu) > 0$$

holds.

Thus, during the binding period, the orbit of $g_\mu^{k+1}(\mu)$ is close to that of μ or ν depending on above conditions. In the same way, we define the *binding period* associated to the parameter (μ,ν) for the return $g^k_\mu(\nu)$ of the critical value ν.

Lemma 5.6. *For a suitable choice of γ, β and δ, there are positive constants $\widetilde{K}$, $D > 1$, A, k_1, and τ, depending only on γ and β, such that for both critical values $\eta = \mu$ or $\eta = \nu$ of $g_\mu = g_\alpha(\mu,\nu,\cdot)$, with $(\mu,\nu) \in \tilde{E}_{\alpha,\theta}(\gamma)$, we have: If $g^k_\mu(\eta) \in (e^{-\gamma k}, \delta)$ for some $k > k_1$ and $\left|(g^j_\mu)'(\eta)\right| > \lambda_1^j$, $j = 1,\ldots,k-1$ for some $\lambda_1, 1 < \lambda_1 < \lambda_0$ (λ_0 as in Lemma 5.5), then*

1. $\dfrac{1}{A} < \dfrac{\left|(g^j_\mu)'(x)\right|}{\left|(g^j_\mu)'(y)\right|} < A$ *for all $x, y \in \overline{convexh\{g_\mu^{k+1}(\eta),\, \nu\}}$ and $1 \le j \le s$, where s is the binding period associated to the parameter (μ,ν) for the return $g^k_\mu(\eta)$ of the critical value η of g_μ.*
2. $\dfrac{r}{\beta + \log D} - 1 \le s \le \dfrac{r\,\alpha - \log(\widetilde{K}A^{-1})}{\beta + \log \lambda_1}$, *where $e^{-r} = \left|g^k_\mu(\eta)\right|$.*
3. $\left|(g_\mu^{s+1})'(g^k_\mu(\eta))\right| \ge \tau \exp\left[\left(\dfrac{\log \lambda_1}{\alpha} - \dfrac{\alpha-1}{\alpha}\beta\right)(s+1)\right] > 1.$

   Similar results can be obtained if $g^k_\mu(\eta) \in (-\delta, -e^{-k\gamma})$, but in this case we have to change $\overline{convexh\{g_\mu^{k+1}(\eta), \nu\}}$ by $\overline{convexh\{g_\mu^{k+1}(\eta),\, \mu\}}$ in conclusion 1 above.

Proof: Let us suppose that $\nu^+ = -1$ and $\nu^- = 1$, the other cases are similar. First, observe that taking $\gamma < \beta$ we get $e^{-(j+1)\beta} < e^{-j\gamma}$. Since $\left|g^j_\mu(\mu)\right| > e^{-j\gamma}$ and s is the binding period associated to (μ,ν) for the return $g^k_\mu(\eta)$ of the critical value η of g_μ, we conclude that $0 \notin convexh\{g_\mu^{k+1+j}(\eta), g^j_\mu(\mu)\}$ for $j < s$. Consequently, if $x \in (g^{k+1}_\mu(\eta), \mu)$ then $0 \notin convexh\{g^j_\mu(x), g^j_\mu(\mu)\} \subset convexh\{g_\mu^{k+1+j}(\eta), g^j_\mu(\mu)\}$. From (b) in Lemma 3.5, for all $j \le s-1$ we can find $\xi_j \in convexh\{g^j_\mu(x), g^j_\mu(\mu)\}$ such that

$$\begin{aligned} \left| g'_\mu(g^j_\mu(x)) - g'_\mu(g^j_\mu(\mu)) \right| &\le K_2\,|\xi|^{\alpha-2} \left| g^j_\mu(x) - g^j_\mu(\mu) \right| \\ &\le K_2\,|\xi|^{\alpha-2} \left| g_\mu^{k+1+j}(\eta) - g^j_\mu(\mu) \right|, \end{aligned}$$

and hence

$$\left|g'_\mu\,(g^j_\mu\,(x)) - g'_\mu\,(g^j_\mu\,(\mu))\right| \le K_2\, e^{-\gamma(\alpha-2)j}\,\left|h^{k+1+j}_\mu(\eta) - h^j_\mu(\mu)\right| \le K_2\, e^{-\gamma(\alpha-2)j-(j+1)\beta}.$$

Furthermore, $\left|g'_\mu\,(g^j_\mu\,(\eta))\right| \ge K_1 \left|g^j_\mu\,(\eta)\right|^{\alpha-1} \ge K_1 e^{-j\gamma(\alpha-1)}$.

Now,

$$\begin{aligned}
\frac{\left|(g^j_\mu)'(x)\right|}{\left|(g^j_\mu)'(\eta)\right|} &= \prod_{i=0}^{j-1}\left[1+\frac{\left|g'_\mu\,(g^i{}_\mu(x))\right| - \left|g'_\mu\,(g^i{}_\mu\,(\eta))\right|}{\left|g'_\mu\,(g^i{}_\mu\,(\eta))\right|}\right] \\
&\le \exp\left[\sum_{i=0}^{j-1}\frac{\left|g'_\mu\,(g^i{}_\mu\,(x)) - g'_\mu\,(g^i{}_\mu\,(\eta))\right|}{\left|g'_\mu\,(g^i{}_\mu\,(\eta))\right|}\right] \\
&\le \exp\left[\sum_{i=0}^{j-1}\frac{2K_2\, e^{-\gamma(\alpha-2)i-(i+1)\beta}}{K_1 e^{-i\gamma(\alpha-1)}}\right] \\
&< \exp\left[\frac{2K_2}{K_1}\, e^{-\beta}\sum_{i=0}^{\infty} e^{(\gamma-\beta)i}\right] \\
&< A.
\end{aligned}$$

In a similar way we obtain $\dfrac{\left|(g^j_\mu)'(x)\right|}{\left|(g^j_\mu)'(\eta)\right|} > \dfrac{1}{A}$, and assertion 1 of the lemma follows.

Integrating inequality (a) in Lemma 3.5, we obtain a constant $\widetilde{K}$, $0 < \widetilde{K} < 1$, such that $\left|g^{k+1}_\mu(\eta) - \mu\right| > \widetilde{K}\left|g^k_\mu(\eta)\right|^\alpha$. Now, applying the Mean Value Theorem to g^j_μ, assertion 1 obtained before, and the hypothesis we get

$$\left|g^{k+1+j}_\mu(\eta) - h^j_\mu(\mu)\right| > A^{-1}\lambda_1^j \widetilde{K} e^{-r\alpha}, \quad j < k.$$

From the binding period definition, for $j \le s$ and $j < k$ we have $e^{-\beta\gamma} \ge A^{-1}\lambda_1^j\widetilde{K}e^{-r\alpha}$, which implies that $j \le \dfrac{r\alpha - \log(\widetilde{K}A^{-1})}{\beta + \log(\lambda_1)}$.

Next, choosing γ and β such that $2\gamma\alpha < \beta$ and fixing $k_1 > \dfrac{-2\,\log(\widetilde{K}A^{-1})}{\log(\lambda_1)}$, then for j and k as in the hypotheses of the lemma we obtain $j \le \dfrac{r\alpha - \log(\widetilde{K}A^{-1})}{\beta + \log(\lambda_1)} < \dfrac{\gamma k\alpha - \log(\widetilde{K}A^{-1})}{\beta + \log(\lambda_1)} < \dfrac{k}{2}$.

If $s \ge k$ then the inequality above holds for $j = k-1$ but this implies $k - 1 < \dfrac{k}{2}$, which is a contradiction. Hence, $s < k$ and we can replace $j = s$ in the inequality above, obtaining the right inequality $s \le \dfrac{r\alpha - \log(\widetilde{K}A^{-1})}{\beta + \log(\lambda_1)}$.

To establish the left hand side of inequality in 2 of the lemma, let $= \max\{\left|g'_\mu(x)\right| : x(\alpha)-\varepsilon \le x \le y(\alpha)+\varepsilon\,,\ \mu\ \} > 1$. Applying the Mean Value Theorem and the binding period definition once again, we obtain a point $\xi \in (0, g^k_\mu(\eta))$ such that:

$$D^{s+1}\left|g^k_\mu(\eta)\right| \ge \left|(g^{s+1}_\mu)'(\xi)\right|\left|g^k_\mu(\eta)-0\right| = \left|g^{s+1}_\mu(g^k_\mu(\eta)) - g^s_\mu(\mu)\right| > e^{-(s+1)\beta}\,,$$

which implies $-(s+1)\beta < (s+1)\log(D) - r$, and the left hand side of inequality 2 follows.

In order to obtain assertion 3, put $t = \dfrac{\alpha}{\alpha-1} = 1+\dfrac{1}{\alpha-1}$ and take $\xi \in (h^{k+1}_\mu(\eta), \mu)$ such that $(g^s_\mu)'(\xi) = \dfrac{\left|g^{s+k+1}_\mu(\eta) - g^s_\mu(\mu)\right|}{\left|g^{k+1}_\mu(\eta)-\mu\right|}$.

Note that $\left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right| > A^{-1}\left|(g^s_\mu)'(\xi)\right|$ implies

$$\left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right| > A^{-1}\left|(g^s_\mu)'(\mu)\right| > A^{-1}\lambda_1^s$$

(since $s < k$). Thus,

$$\left|(g^{s+1}_\mu)'(g^k_\mu(\eta))\right|^t = \left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right|^t\left|g'_\mu(g^k_\mu(\eta))\right|^t$$

$$\ge \left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right|\left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right|^{\frac{1}{\alpha-1}} K_1^t\left|g^k_\mu(\eta)\right|^\alpha$$

$$\ge A^{-1}\left|(g^s_\mu)'(\xi)\right|\left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right|^{\frac{1}{\alpha-1}} K_1^t\widetilde{K}^{-1}\left|g^{k+1}_\mu(\eta)-\mu\right|$$

$$> A^{-1}K_1^t\widetilde{K}^{-1}\left|g^{s+k+1}_\mu(\eta) - g^s(\mu)\right|\left|(g^s_\mu)'(g^{k+1}_\mu(\eta))\right|^{\frac{1}{\alpha-1}}$$

$$> A^{-1}K_1^t\widetilde{K}^{-1}e^{-\beta(s+1)}A^{\frac{-1}{\alpha-1}}\lambda_1^{\frac{s}{\alpha-1}} = A^{-t}K_1^t\widetilde{K}^{-1}e^{-\frac{\log\lambda_1}{\alpha-1}}\exp\left[\left(\frac{\log\lambda_1}{\alpha-1}-\beta\right)(s+1)\right]$$

$$= \tau^t\exp\left[\left(\frac{\log\lambda_1}{\alpha-1}-\beta\right)(s+1)\right].$$

Finally, if β is small the coefficient of $(s+1)$ in the exponential above is positive, and the inequality in conclusion 3 of lemma follows for s large, which is obtained making δ small (compatible with the previous conditions imposed on α and β). □

A *free return index* is a return index that does not belong to a binding period associated to a previous return. Take γ, β and δ as in Lemma 5.6 and for any positive integer k let us consider $i_1 < i_2 < \ldots < i_l \le k$ all the free return indices, i.e., $e^{-\gamma i_t} < \left|g^{i_t}_\mu(\nu)\right| \le \delta$, and let $s_1, s_2.., s_{l-1}, (s_l)$ be the binding periods associated to returns $g^{i_1}_\mu(\nu), \ldots, g^{i_{l-1}}_\mu(\nu), g^{i_l}_\mu(\nu)$. Note that

s_l will be or not considered according to $i_l < i_l + s_l \leq k$ or $i_l \leq k < i_l + s_l$ respectively. We say that j, $j \leq k$ is a *free time* for the critical value ν associated to the parameter (μ, ν) if $\left|g^j_\mu(\nu)\right| > \delta$ and $j \notin [i_t, i_t + s_t]$, $t = 1, ..., l-1, (l)$. In the same way we define free time for the critical value μ associated to the parameter (μ, ν).

Now, for $\eta = \mu$ or $\eta = \nu$ denote by $F(\eta, k)$ the number given by:

$$F(\eta,k) = \#\{j \leq k \; : \; j \text{ is a free time for the critical value } \eta \text{ associated to the parameter } (\mu,\nu)\}.$$

Finally, fix $\varepsilon > 0$ and define $\tilde{E}_{\alpha,\theta} = \tilde{E}_{\alpha,\theta}(\gamma, \beta, \delta, \varepsilon)$ by

$$\tilde{E}_{\alpha,\theta} = \{(\mu,\nu) \in \tilde{E}_{\alpha,\theta}(\gamma) : \min\{F(\mu,k), F(\nu,k)\} \geq (1-\varepsilon)k, k \in \mathbb{N}\}.$$

As a matter-of-fact, $\tilde{E}_{\alpha,\theta}$ is the set of parameter values $(\mu, \nu) \in L_{\alpha,\theta}$ satisfying the basic assumption (BA) and, in addition, the number of free times smaller than k for the critical values ν, μ associated to the parameter (μ, ν) is "almost" k, in fact bigger than $(1-\varepsilon)k$, for any $k \in \mathbb{N}$. Note that this last condition, called *free time assumption* (FA) implies $\sum_{t=1}^{l} s_t < \varepsilon k$.

The set of free times is the union of the free intervals $V_1 = [1, ..., i_1 - 1]$, $V_2 = [i_1 + s_1 + 1, ..., i_2 - 1], ..., V_l = [i_{l-1} + s_{l-1} + 1, i_l - 1]$, $(V_{l+1} = [i_l + s_l + 1, k])$.

Now we can prove the following result.

Lemma 5.7. *For γ, β, δ and ε small enough there are λ, $1 < \lambda < \lambda_0$ (λ_0 as in Lemma 5.5), $\alpha_0 > 1$ and $a > 0$ such that:*
for $\alpha < \alpha_0$ in O and $|(\mu,\nu) - (\mu(\alpha), \nu(\alpha))| < a$, if $(\mu, \nu) \in \tilde{E}_{\alpha,\theta}$ then

$$\left|(g^k_\mu)'(\mu)\right| > \lambda^k \quad \textit{and} \quad \left|(g^k_\mu)'(\nu)\right| > \lambda^k, \quad k = 1, 2, ...$$

Proof: The proof is by induction on k.

Take $1 < \lambda < \lambda_0$ such that $|g'(\mu(1), \nu(1), x(1))| > \lambda$ (here $x(1)$ is the orientation preserving fixed point of $g(\mu(1), \nu(1), \cdot)$. Because $g_\mu = g_\alpha(\mu, \nu, \cdot)$ converges uniformly to $g(\mu, \nu, \cdot)$ we can find $\alpha_0 > 1$ and $a > 0$ such that for $\alpha < \alpha_0$ in O and (μ, ν) with $|(\mu,\nu) - (\mu(\alpha), \nu(\alpha))| < a$ then $\left|(g^k_\mu)'(\eta)\right| > \lambda^k$, $\eta = \mu, \nu$, $k \leq k_1$, where k_1 is taken as in Lemma 5.6. And then, the first step in the induction holds for $k \leq k_1$.

Next, take $k > k_1$ and assume that the conclusion holds for any iterate of critical value $\eta = \mu$ or $\eta = \nu$ of order less than k. Because $k > k_1$ and (μ, ν) satisfies the basic assumption (BA), Lemma 5.6 holds and it implies that during each binding period there is not lost of derivative, in fact, the total derivative during those periods is bigger than 1. By Lemma 5.5, it follows that during the free time intervals we have an increasing in the derivative of at least $\lambda_0^{\#V_i}$. We can write $k = \sum_{1\le i\le l or l+1} \#V_i + \sum_{1\le i\le l} s_i$ and from the choices above and (FA) we obtain

$$(g_\mu^k)'(\eta) \ge \prod_{1\le i\le l or l+1} \lambda_0^{\#V_i} \prod_{i=1}^{l} 1 > \lambda_0^{\sum_{1\le i\le l or l+1} \#V_i} > \lambda_0^{(1-\varepsilon)k} = \exp([(1-\varepsilon)\log\lambda_0]\, k).$$

Taking ε small we obtain $\log\lambda < (1-\varepsilon)\log\lambda_0$ (this choice works for any k) and then $\left|(g_\mu^k)'(\eta)\right| \ge \lambda^k$. This ends the proof of the Lemma 5.7.

□

To finish the proof of Theorem 5.1 we have to prove that it is possible to define a set $\tilde{E}_{\alpha,\theta}$ satisfying:

1. $m_\theta(\tilde{E}_{\alpha,\theta}) > 0$, where m_θ stands for the Lebesgue measure in $E_{\alpha,\theta}$, and
2. For $m_\theta-$almost every $(\mu,\nu) \in \tilde{E}_{\alpha,\theta}$, $\overline{\{g_\mu^k(\eta)\}}_{k\in\mathbb{N}}$ is equal to the convex hull of $\{g_\alpha(\mu,\nu,\mu), g_\alpha(\mu,\nu,\nu), \mu, \nu\}$, where $\eta \in \{\mu,\nu\}$.

The set $\tilde{E}_{\alpha,\theta}$ is obtained as in [BC2, Section 2]. We shall not do it here in details and we refer to [MV, Section 3] where this construction is nicely done. Each $\tilde{E}_{\alpha,\theta}$ is obtained as the intersection of some sets E_k contained in a small parameter interval having $\mu(\alpha)$ as the right extreme, $E_1 \supset E_2 \supset ... \supset E_k \supset ...$, satisfying the basic assumption (BA) and the free time assumption (FA). To define E_k, it is necessary to make some partition $\mathcal{P}_{k-1}$ of E_{k-1} to guaranty that the maps $\mu \longrightarrow g_\mu^k(\mu)$ and $\mu \longrightarrow g_\mu^k(\nu)$ (recall that $\nu = \theta(\mu - \mu(\alpha)) + \nu(\alpha)$) have bounded distorsion on each interval $\omega \in \mathcal{P}_{k-1}$ ([MV, Lemma 3.3] and [BC1, Lemma 5(7.2)]). This property and Lemma 5.6 enable us to estimate the Lebesgue measure of the set of points that will be excluded from E_{k-1} to define E_k, see also [R]. Thus, following the same steps as in [MV, Section 3] we obtain a set $\tilde{E}_{\alpha,\theta}$ with $m_\theta(\tilde{E}_{\alpha,\theta}) > 0$ satisfying the basic assumption (BA) and the free time assumption (FA).

Finally, in order to prove that the g_μ-orbit of μ and ν are dense in the convex hull of $\{g_\mu(\mu), g_\mu(\nu), \mu, \nu\}$ for almost every $(\mu,\nu) \in \tilde{E}_{\alpha,\theta}$, we observe that the set of parameters such that the g_μ-orbit of μ do not visit some fixed open interval is a zero Lebesgue measure set because $\mu \longrightarrow g_\mu^k(\mu)$ have bounded distortion for all k in any subinterval of $\mathcal{P}_{k-1}$. The same conclusion holds for the critical value ν. Finally, using the fact that the topology of $L_{\alpha,\theta}$ has a countable basis we conclude that the set of parameters such that the critical values are not dense is a zero Lebesgue measure set. Altogether proves Theorem 5.1.

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B. San Martín
Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta, Chile
e-mail: sanmarti@ucn.cl

C. A. Morales and M. J. Pacifico
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68.530, CEP 21.945-970
Rio de Janeiro, R. J. , Brazil
e-mail: morales@im.ufrj.br, pacifico@im.ufrj.br