On the dynamics of Dominated Splitting

Enrique R. Pujals* and Martín Sambarino†

To Jose Luis Massera, in memorian

Abstract

Let $f : M \to M$ be a C^2 diffeomorphism of a compact surface. We prove a Spectral Decomposition Theorem for the limit set $L(f)$ under the assumption of dominated splitting. As a fundamental step, we show that in any compact invariant set having dominated splitting, the periods of the non-hyperbolic periodic points are bounded.

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1 Introduction

In the theory of differentiable dynamics, i.e. the study of the asymptotic behaviour of orbits $\{f^n(x)\}$ when $f : M \to M$ is a diffeomorphism of a compact riemaniann manifold M, one may say that a fundamental problem is to understand how the dynamics of the tangent map Df controls or determines the underlying dynamics of f.

So far, this program has been solved for hyperbolic dynamics in the so called Spectral Decomposition Theorem, where it is given a satisfactory (complete) description of the dynamics of a system within the assumption that the tangent map has a hyperbolic structure. More precisely, under the assumption that the tangent bundle over $L(f)$ (the minimum closed invariant set that contains the ω and α limit set of any orbit) splits into two subbundles, $T_{L(f)}M = E^s \oplus E^u$, invariant under Df and vectors in E^s are contracted by positive iteration of the tangent map (the same holding for E^u but under negative iteration), Newhouse [N1] proved that $L(f)$ can be decomposed into the disjoint union of finitely compact invariant and transitive sets. Moreover, the periodic points are dense in $L(f)$ and the asymptotic behaviour of any point in the manifold is represented by an orbit in $L(f)$. These sets can be regarded as (the orbit of) a homoclinic class of a hyperbolic periodic point (that is, the closure of the transversal intersection between the stable and unstable manifold of a hyperbolic periodic point). The Smale's Ω-Spectral Decomposition Theorem for Axiom A [S] systems is obtained from the above since in this case the nonwandering set $\Omega(f)$ is equal to the limit set $L(f)$.

There were, basically, two ways to relax hyperbolicity. One, called partial hyperbolicity, allows the tangent bundle to split into Df-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, and the behaviour of vectors in E^s, E^u is similar to the hyperbolic case, but vector in E^c may be neutral for the action of the tangent map. In the other, non-uniform hyperbolicity (or Pesin theory), where the tangent bundle splits for points a.e. with respect to some invariant measure, and vectors are asymptotically contracted or expanded in a rate that may depend on the base point.

Since the latter case starts on a measure-theoretical setting, one cannot expect to obtain a description from the topological dynamic point of view. In the former, there is no general theory regarding its topological dynamic consequences (although there are many important results from the ergodic point of view, see for instance [BP], [PuSh], [ABV], [CY]).

There is also another category which includes the partial hyperbolic systems: *dominated splitting*. An f-invariant set Λ is said to have dominated splitting if we can decompose its tangent bundle in two invariant subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\|\|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

with $C > 0$ and $0 < \lambda < 1$.

Although partial hyperbolic systems arose in a natural way (time one maps of Anosovo flows, frame flows, group extensions), the concept of dominated splitting was introduced independently by Mañé, Liao and Pliss, as a first step in the attempt to prove that structurally stable systems satisfy a hyperbolic condition on the tangent map. However, during the last decades, there have been a large amount of research on this subject, mostly form the ergodic point of view.

A natural question arises: what is the feedback of a system having dominated splitting? In other words, is it possible to describe the dynamics of a system having dominated splitting? The aim of this paper is to give a positive answer (as satisfactory as in the hyperbolic case) to this question when M is a compact surface.

Before stating our main theorem, let us say that in solving the above problem, another kind of differentiable dynamic problem arose, namely: how is affected the dynamic of a system, under some given conditions, as the smoothness of it is improved? This is the case, for example, of the classical Denjoy theory about diffeomorphism in the circle having irrational rotation number.

We will give a complete description of the (topological) dynamics of a system having dominated splitting as far as the system is smooth enough (C^2) :

Main Theorem: *Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{L}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is a set of periodic points with bounded periods and contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*

2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*

3. *$\tilde{L}(f)$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets. The periodic points are dense in $\tilde{L}(f)$ and contains at most finitely many non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f/\tilde{L}(f)$ is expansive.*

Roughly speaking, the above theorem says that the dynamic of a C^2 diffeomorphism having dominated splitting can be decomposed into two parts: one where the dynamic consists on periodic and almost periodic motions ($\mathcal{I}$, $\mathcal{R}$) and the diffeomorphism acts equicontinuously, and another one where the dynamics is expansive and similar to the hyperbolic case. Moreover, the set $\tilde{L}(f)$ can be characterized as the set of point in $L(f)$ that can be approximated by periodic points with unbounded periods. We will make a more precise statement of the theorem above in Section 4.

One may ask if the above theorem holds if we replace the limit set $L(f)$ by the non-wandering set $\Omega(f)$. The Spectral Decomposition Theorem does not hold, in general, when $\Omega(f)$ is hyperbolic but not equal to the limit set $L(f)$. However, in [NP] it is proved that if $\Omega(f)$ is hyperbolic and f is a diffeomorphism of a compact surface then $\Omega(f) = L(f)$. This also holds in our case: if $\Omega(f)$ has dominated splitting and f is a C^2 diffeomorphism of a compact surface then $\Omega(f) = L(f)$ and hence the main theorem holds for $\Omega(f)$ (see section 6).

A consequence of our main theorem is that any C^2 diffeomorphism with dominated splitting over $L(f)$ with a sequence of periodic points with unbounded periods, must exhibits a nontrivial homoclinic class, and hence its topological entropy is nonzero.

Corollary 1: *The topological entropy of a C^2 diffeomorphism of a compact surface having dominated splitting over $L(f)$ and having a sequences of periodic points with unbounded periods is positive.*

Moreover, in [PS1] (see also [PS2]) we showed that a system that cannot be C^1 approximated by another that exhibits a homoclinic tangency (that is, a diffeomorphism such that for some hyperbolic periodic point the stable and unstable manifolds have a non-transverse intersection), the nonwandering set $\Omega(f)$ has dominated splitting. Hence, under the scope of the theorem above, the following result follows.

Corollary 2: *Let $f \in Diff^2(M^2)$ having infinitely many sinks or sources with unbounded period. Then, f can be C^1-approximated by a diffeomorphism exhibiting a homoclinic tangency.*

Let us comment briefly the proof of the main theorem. The starting point is Theorem B in [PS1] (see Theorem 2.1 in section 2): the breakdown of hyperbolicity of a system with dominated splitting is due either to the presence of irrational rotations or to the presence of non-hyperbolic periodic points (and they could be extremely degenerated). The following theorem provides a way to deal with the presence of non-hyperbolic periodic points.

Theorem A: *Let $f : M \to M$ be a C^2-diffeomorphism of a two dimensional compact riemannian manifold M and let Λ be a compact invariant set having dominated splitting. Then, there exists an integer $N_1 > 0$ such that any periodic point $p \in \Lambda$ whose period is greater than N_1, is a hyperbolic periodic point of saddle type.*

Continuing with the outline of the proof of the main theorem, lets denote by $Per_h(f)$ the set of hyperbolic periodic points of saddle type and by Per_h^N the set of hyperbolic periodic point with period greater than N.

Theorem B: *Let $f \in Diff^2(M^2)$ and assume that $\overline{Per_h(f)}$ has a dominated splitting. Then, there exists $N > 0$ such that $\overline{Per_h^N(f)}$ can be decomposed into the disjoint union of finitely many homoclinic classes. Moreover, $\overline{Per_h^N(f)}$ contains at most finitely many non-hyperbolic periodic points and $f_{/\overline{Per_h^N(f)}}$ is expansive.*

Thus, the final step in the proof of the main theorem is to show that $\tilde{L}(f) \subset \overline{Per_h(f)}$. As the reader might guess, these final steps seem to be similar to the hyperbolic case. In fact they are, but let us explain why. In the hyperbolic case, the description of the dynamics follows from a fundamental tool: at each point there are transverse invariant manifolds of uniform size and these manifolds have a dynamic meaning (points in the "stable" one are asymptotic to each other in the future, and points in the "unstable" one are asymptotic to each other in the past). Under the sole assumption of dominated splitting, even if locally invariant manifolds do exist, they do not have any dynamic meaning at all. However, in the two dimensional case, using the fact that these locally invariant manifolds are one-dimensional together with smoothness, we are able to prove that these manifolds already have a dynamic meaning, perhaps not of uniform size, but enough to proceed to a description of the dynamics.

There is another important property of hyperbolic sets which is called analytic continuation: if $\Lambda \subset \Omega(f)$ is a hyperbolic set then, for any nearby diffeomorphism g there is a set Λ_g homeomorphic to Λ and such that the dynamics of f/Λ and g/Λ_g are conjugated. We may wonder if sets having dominated splitting also exhibit an analytic continuation. In the full generality, the answer is no. For instance, an isolated saddle node fixed point is a set having dominated splitting, but this point might disappear after a small perturbation. However, it is also possible to perturb the system in such a way that the fixed point not only persists but also becomes hyperbolic. If the set having dominated splitting contains a non-trivial homoclinic class, it can not disappear, but may "explode" (see section 5). However, we may perturb the system in such a way that the set has an analytical continuation and becomes hyperbolic as well. More precisely:

Theorem C: *Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting and let Λ be a basic piece of the spectral decomposition of $\tilde{L}(f)$. Then there exists a connected open set $\mathcal{V} \subset Diff^2(M)$ such that*

1. *$f \in \overline{\mathcal{V}}$.*
2. *$\forall g \in \mathcal{V}$ there is a set $\Lambda_g \subset \tilde{L}(g)$ homeomorphic to Λ such that Λ_g is a basic hyperbolic set for g (i.e locally maximal transitive hyperbolic set) and f/Λ and g/Λ_g are conjugated.*

We want to point out that the continuation of hyperbolic sets can be done through analytic methods (implicit function theorem in Banach spaces, see [HPS]) but this is not the case for dominated splitting: *we have to understand first the topological behaviour of the initial system and then show that this structure is "rigid" in the C^2-topology.*

Let us remark that a fundamental step towards the proof of Theorem C is to show that the periods of non hyperbolic periodic points is bounded in a "suitable" neighbourhood of f, i.e., the number N_1 in Theorem A can be chosen uniformly for any appropriate perturbation of f. A remarkable consequence of this fact is that, in the absence of saddle-node periodic points (but in the presence of other non-hyperbolic periodic points), there are sequences of f-periodic points with one (normalized) eigenvalue converging to one and they can not be perturbed to obtain a non-hyperbolic periodic point, i.e., they are C^2-stably hyperbolic:

Theorem C': *There exist a C^2 diffeomorphism $f : M \to M$ and a neighourghood $\mathcal{U}$ of f in the C^2 topology such that f has a sequence of periodic points p_n with unbounded periods and one normalized eigenvalue of p_n converging to 1 and any periodic point of $g \in \mathcal{U}$ of period greater than 2 is hyperbolic.*

This theorem implies that the Franks' lemma in [F], which is extremely useful in the C^1-topology, *is no longer valid in the C^2-topology.* (see Section 5).

In view of Theorem C, we may also ask if $L(f)$ has a hyperbolic analytic continuation. This fail to be true if f exhibits a simple closed curve supporting an irrational rotation or if f has infinitely many periodic points with bounded periods: there is no way to "unfold" an irrational rotation without passing through other irrational rotations, and there is no way to "unfold" infinitely many periodic points with bounded periods (unless in some very degenerate situations) without passing through another bifurcation.

Does $\tilde{L}(f)$ have a hyperbolic analytic continuation? Even in the hyperbolic case this is not true without the no-cycle condition. Thus, if $L(f)$ satisfy a no-cycle condition (see Section 6) we get a hyperbolic continuation of $\tilde{L}(f)$

Theorem D: *Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting and assume that the no-cycle condition holds. Then there exists a connected open set $\mathcal{V} \subset Diff^2(M)$ such that*

1. *$f \in \overline{\mathcal{V}}$.*
2. *$\forall g \in \mathcal{V}$ the set $\tilde{L}(g)$ is hyperbolic and $f/\tilde{L}(f)$ and $g/\tilde{L}(g)$ are conjugated.*

Finally, we will use the theorems above to obtain results regarding the topological entropy. Although systems with dominated splitting are not stable, and in a one parameter family many bifurcations may take part, the topological entropy does not change.

Theorem E: *Let $\mathcal{U} \subset Diff^\infty(M^2)$ such that for any $f \in \mathcal{U}$, $\Omega(f)$ has dominated splitting. Then the topological entropy on $\mathcal{U}$ is constant.*

By the observation made just before corollary 2, as a consequence of Theorem E, we have as a corollary a result proved in [PS2].

Corollary 3: *Let $f : M \to M$ be a C^∞ diffeomorphism such that for any C^∞-neighbourhood $\mathcal{U}$ of f, the map $\mathcal{U} \ni g \to h_{top}(g)$ is not constant. Then f can be C^1-approximated by a diffeomorphism exhibiting a homoclinic tangency.*

The paper is organized as follows: in section 2 we state some results and tools that will be used to prove the theorems above; in section 3 we prove Theorem A; in Section 4 we prove Theorem B and Main Theorem; Section 5 is devoted to prove Theorem C and C'. Finally, in Section 6, Theorem D and Theorem E are proved.

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2 Preliminaries

Let $f : M \to M$ be a diffeomorphism of a compact riemannian manifold M. Recall that a compact f-invariant set Λ is said to have a *dominated splitting* if its tangent bundle splits into two Df-invariant subbundles $T_\Lambda M = E \oplus F$, and such that:

$$\|Df^n_{/E(x)}\|\|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0$$

with $C > 0$ and $0 < \lambda < 1$.

It follows that these subbundles are continuous as in the hyperbolic splitting. We observe that in the compact invariant subset of Λ where one of the subbundles is trivial, the other one must be hyperbolic (contracting or expanding) and hence this subset is finite and consist just on periodic attractors or repellors. In other words, the dominated splitting is interesting when none of the subbundles are trivial and we shall assume that this is the case through our paper.

We will assume also that $C = 1$. It is not a major assumption since we can replace f by a power of itself and notice that all the theorems that we will prove, if they are true for a power of f then they are also true for f. We shall refer λ (in the above definition) as a *constant of domination.* And through the paper, M will denote a two dimensional compact riemannian manifold, i.e., a compact surface (unless otherwise indicated).

2.1 Sufficient conditions for hyperbolicity

Here we state a slight modification of Theorem B of [PS1].

Theorem 2.1 *Let f be a C^2-diffeomorphism on a compact surface, and let $\Lambda \subset \Omega(f)$ be a compact f-invariant set having a dominated splitting $T_{/\Lambda}M = E \oplus F$ and such that all the periodic points in Λ are hyperbolic. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ...\mathcal{C}_n$, normally hyperbolic and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

The mentioned theorem in [PS1] requires that the periodic points in Λ are hyperbolic *of saddle type*. However, this is a superfluous assumptions as we will see. Denote by P_0 the set of periodic attractors and by F_0 the set of periodic repellors. Observe that P_0 and F_0 are isolated in Λ (since $\Lambda \subset \Omega(f)$) and hence $\Lambda_0 = \Lambda \backslash \{P_0 \cup F_0\}$ is compact, invariant, contained in $\Omega(f)$ and all the hyperbolic periodic points in Λ_0 are hyperbolic of saddle type. Therefore, we can apply Theorem B of [PS1] and obtain the desired decomposition for Λ_0 (that is, Λ_0 is the union of a hyperbolic set and a finitely many periodic curves supporting an irrational rotation). It follows that P_0 and F_0 must be finite (otherwise all their limit points belong to a hyperbolic set, which is impossible). Thus, our theorem follows from Theorem B of [PS1].

2.2 Central stable and unstable manifolds: I

A fundamental tool that we use through the paper is the existence of locally invariant manifolds. These locally invariant manifolds already exist under the assumption of dominated splitting. However, we will need these manifolds to be of class C^2. For this purpose, a sufficient condition is the 2-domination: let us say that a compact invariant set Λ having dominated splitting is 2-dominated if there exist $C > 0$ and $\sigma < 1$ such that

$$\|Df^n_{/E(x)}\| \|Df^{-n}_{/F(f^n(x))}\|^2 < C\sigma^n$$

and

$$\|Df^n_{/E(x)}\|^2 \|Df^{-n}_{F(f^n(x))}\| < C\sigma^n$$

hold for any point $x \in \Lambda$ and $n \geq 0$. The presence of a periodic attractor or repeller could be an obstruction for the 2-domination. Let explain this. Consider $\mu, 0 < \mu < \lambda$ and assume that a periodic attractor (sink) p has (normalized) eigenvalues μ and μ^2. This periodic attractor may exists in a set having dominated splitting with λ as a constant of domination. However, if m is the period of p then

$$\|Df^m_{/E(p)}\| \|Df^{-m}_{/F(f^m(p))}\|^2 = 1$$

and the 2-domination fails.

In order to bypass this possible obstruction, let us make the following definition: given $\mu,\ 0 < \mu < 1$, we say that a hyperbolic periodic point p of period m is a *μ-sink* (respec. *μ-source*) if the modulus of the eigenvalues of $Df^m_p : T_pM \to T_pM$ are less (respec. greater) than μ^m (respec.μ^{-m}). Notice that μ-sinks or sources are isolated in the nonwandering set (or in the limit set).

Theorem 2.2 *Let $f : M \to M$ be a C^1 diffeomorphism and Λ be a set with dominated splitting $T_{/\Lambda}M = E \oplus F$. Then, for any $0 < \mu < 1$ the set of μ-sinks in Λ is finite. The same holds for μ-sources.*

Proof: Let $0 < \mu < 1$ and fix $\gamma, \mu < \gamma < 1$. Let p be a μ-sink in Λ and denote its period by m. We claim that there exists $p_i = f^i(p)$ in the orbit of p such that

$$\|Df^n_{/F(p_i)}\| \leq \gamma^n,\ 1 \leq n \leq m. \tag{1}$$

Arguing by contradiction, assume that this is not true. Then for every $p_i = f^i(p)$ there exists $n(p_i), 1 \le n(p_i) \le m$ such that $\|Df^{n(p_i)}_{/F(p_i)}\| > \gamma^{n(p_i)}$. Let $C = \sup_{x\in M}\{\|Df^i_x\| : i = 1, ..., m\}$. Take k_0 such that for $k \ge k_0$ we have

$$\frac{C\mu^{km}}{\gamma^m} < \gamma^{km} \tag{2}$$

Take $k \ge k_0$ and consider $n_1 = n(p), n_2 = n(f^{n_1)}(p), ..., n_i = n(f^{n_1+...+n_{i-1}}(p))$. There is some i so that $km \le n_1 + ... + n_i \le km + m$. Set $N = n_1 + ... + n_i$. Then

$$\begin{aligned} \|Df^N_{/F(p)}\| &= \|Df^{km}_{/F(p)}\|\|Df^{N-km}_{/F(p)}\| \\ &\le c\mu^{km} < \gamma^{km+m} < \gamma^N < \|Df^N_{/F(p)}\|, \end{aligned}$$

a contradiction. This proves our claim. (Notice that, since γ is arbitrary, it follows that (1) holds for $\gamma = \mu$, but this is not necessary for our purpose). Thus, we have proved that if p is a μ-sink, then there is p_i in the orbit of p satisfying (1). We may assume that $p = p_i$. Since m is the period of p it follows that $\|Df^n_{/F(p)}\| \ge \gamma^n,\ n \ge 0$. By the domination and since $angle(E, F)$ is bounded away from zero in Λ, we conclude that there is a constant K such that

$$\|Df^n_p\| \le K\gamma^n,\ n \ge 0.$$

Let $c > 0$ such that $\gamma(1 + c) < 1$ and let n_0 be such that $K\gamma^{n_0}(1 + c)^{n_0} < 1$.

Since f is C^1 there is some $\eta > 0$ such that

$$\text{if } d(f^j(x), f^j(y)) < \eta,\ j = 0, 1, ..., n-1 \text{ then } \|Df^n_y\| < (1 + c)^n\|Df^n_x\| \tag{3}$$

Take $\epsilon > 0$ such that if $d(x, y) < \epsilon$ then $d(f^j(x), f^j(y)) < \eta, j = 0, 1, ..., n_0 - 1$. Consider any $y \in B_\epsilon(p)$. It follows that

$$\|Df^{n_0}_y\| < (1 + c)^n\|Df^{n_0}_p\| < K\gamma^{n_0}(1 + c)^n_0 < 1.$$

Therefore $d(f^{n_0}(y), f^{n_0}(p)) < \eta$. By (3), using induction we get that $\|Df^n_y\| \le K\gamma^n(1 + c)^n$ and $d(f^n(y), f^n(p) < k\gamma^n(1 + c)^n d(y, p),\ n \ge n_0$. In other words, $B_\epsilon(p)$ is in the basin of attraction of the μ-sink p. Since ϵ does not depend on p and different sinks have disjoint basin of attraction, it follows that there can be only finitely many μ-sinks in Λ.

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Let us point out that this theorem is no longer valid if M is an n-dimensional manifold with $n \ge 3$ without supplementary hypothesis. Further details exceed the purpose of this paper.

The next lemma is an adaptation of Lemma 3.0.3 of [PS1] and in its proof we shall indicate the main lines, details can be found in the lemma cited above. For $\mu, 0 < \mu < 1$ and Λ a compact invariant set, denote by $P_\mu(\Lambda)$ ($F_\mu(\Lambda)$) the set of μ-sinks (resp. μ-sources) in Λ.

Lemma 2.2.1 *Let $f : M \to M$ be a C^1 diffeomorphism and let Λ be a compact invariant set having a dominated splitting $T_{/\Lambda}M = E \oplus F$. There exists μ, $0 < \mu < 1$ such that any compact invariant subset $\Lambda_0, \Lambda_0 \subset \Lambda \backslash \{P_\mu(\Lambda) \cup F_\mu(\Lambda)\}$ is 2-dominated.*

Proof:

Since $angle(E, F) > \gamma > 0$ for every point in Λ there exist a positive constant K such that

$$\|Df^n(z)\| \leq K \sup\{\|Df^n_{/E(z)}\|, \|Df^n_{/F(z)}\|\} \leq K\|Df^n_{/F(z)}\|$$

for all $z \in \Lambda$ and for all positive integer n, where the last inequality follows by the dominated splitting again.

Take σ_0, $\lambda < \sigma_0 < 1$ and q such that $2K\lambda^q < \sigma_0$. Let μ be such that $\sigma_0 < \mu^q$.

We now prove, for this choose of μ, and Λ_0 as in the statement of the lemma that the conclusion is true. We shall prove only the first item of the 2-domination, the second one being analogous. It is enough to show the existence of a positive integer m such that for every $x \in \Lambda$ we have

$$\|Df^m_{/E(x)}\|\|Df^{-m}_{/F(f^m(x))}\|^2 < \frac{1}{2}.$$

Now, arguing by contradiction, for each positive integer n there exist $x_n \in \Lambda_0$ such that

$$\|Df^j_{/E(x_n)}\|\|Df^{-j}_{/F(f^j(x_n))}\|^2 \geq \frac{1}{2}$$

for all $0 \leq j \leq n$. We may assume that $x_n \to x$ for some $x \in \Lambda_0$. For this x, by the domination and the way we'd chosen q we get

$$\prod_{j=0}^{n} \|Df^q(f^{qj}(x))\| \leq K^n 2\lambda^{qn} = 2(K\lambda^q)^n < \sigma_0^n$$

for all $n \geq 0$. Let $g = f^q$. Thus

$$\prod_{j=0}^{n} \|Dg(g^j(x))\| \leq \sigma_0^n \; \forall n \geq 0$$

Consider $0 < \lambda < \sigma_0 < \sigma_1 < \sigma_2 < \mu^q < 1$. Then, there exist a sequence of integers $n_k \to \infty$ such that, for any k and for every positive n,

$$\|Dg^n(g^{n_k}(x))\| \leq \prod_{j=0}^{n} \|Dg(g^j(g^{n_k}(x)))\| < {\sigma_1}^n.$$

Thus, it can be proved that there exist $\eta > 0$, independent of k, such that for any $y \in B_\eta(g^{n_k}(x))$,

$$\|Dg^n(g^{n_k}(x))\| \leq {\sigma_2}^n$$

holds. Let j_0 be such that for every $j \geq j_0$ we get ${\sigma_2}^j < \frac{\eta}{4}$. Now, take n_i and n_l such that $n_l - n_i > j_0$ and $dist(g^{n_l}(x), g^{n_i}(x)) < \frac{\eta}{4}$. Setting $r = n_l - n_i$, it follows that $g^r(B_\eta(g^{n_i}(x))) \subset B_\eta(g^{n_i}(x))$ and $g_{/B_\eta(g^{n_i}(x))}$ is a contraction. Then there is a point $p \in B_\eta(g^{n_i}(x))$ which is fixed under g^r. Since $g = f^q$ we conclude that p is an attracting fixed point under f^{qr}. Therefore p is a sink, attracting the point $z = g^{n_i}(x)$ and so $p \in \Lambda_0$. Moreover

$$\|Df_p^{qr}\| = \|Dg_p^r\| \leq {\sigma_2}^r < \mu^{rq}.$$

Hence p is a μ-sink in Λ_0, a contradiction.

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Let $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^2(I_1, M)$ the set of C^2-embeddings of I_1 on M. The next result can be found in [HPS].

Lemma 2.2.2 *Let $f : M \to M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having dominated splitting which is also 2-dominated. Then, there exist two continuous functions $\phi^{cs} : \Lambda \to Emb^2(I_1, M)$ and $\phi^{cu} : \Lambda \to Emb^2(I_1, M)$ such that if define $W^{cs}_\epsilon(x) = \phi^{cs}(x)I_\epsilon$ and $W^{cu}_\epsilon(x) = \phi^{cu}(x)I_\epsilon$ the following properties hold:*

a) $T_xW^{cs}_\epsilon(x) = E(x)$ and $T_xW^{cu}_\epsilon(x) = F(x)$

b) for all $0 < \epsilon_1 < 1$ there exist ϵ_2 such that

$$f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$$

and

$$f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x))$$

From now on, whenever we have a set Λ as in the Lemma 2.2.2, we shall assume the functions ϕ^{cs} and ϕ^{cu} to be chosen and fixed. We call the manifold W^{cs} the (local) center stable manifold and W^{cu} the (local) center unstable manifold. Observe that property *b)* means that $f(W^{cs}_\epsilon(x))$ contains a neighborhood of $f(x)$ in $W^{cs}_\epsilon(f(x))$ and $f^{-1}(W^{cu}_\epsilon(x))$ contains an neighborhood of $f^{-1}(x)$ in $W^{cu}_\epsilon(f^{-1}(x))$. In particular, we have:

Corollary 2.2.1 *Given ϵ, there exist a number $\delta = \delta(\epsilon)$ with the following property:*

1. *if $y \in W^{cs}_\epsilon(x)$ and $dist(f^j(x), f^j(y)) \leq \delta$ for $0 \leq j \leq n$ then $f^j(y) \in W^{cs}_\epsilon(f^j(x))$ for $0 \leq j \leq n$.*

2. *if* $y \in W^{cu}_\epsilon(x)$ *and* $dist(f^{-j}(x), f^{-j}(y)) \leq \delta$ *for* $0 \leq j \leq n$ *then* $f^{-j}(y) \in W^{cu}_\epsilon(f^{-j}(x))$ *for* $0 \leq j \leq n$.

Another consequence is corollary 3.2 of [PS1] that we will use later and we state here for the seek of completeness.

Corollary 2.2.2 *Let* Λ *be as in Lemma 2.2.2 and let* $\gamma, 0 < \gamma < 1$. *Then, there exists* $\epsilon = \epsilon(\gamma)$ *such that for any* $x \in \Lambda$ *satisfying*

$$\|Df^{-n}_{/F(x)}\| \leq \gamma^n, \forall n \geq 0$$

then, the length

$$\ell(f^{-n}(W^{cu}_\epsilon(x))) \to_n 0$$

i.e., the central unstable manifold of size ϵ *is in fact an unstable manifold.*

A fundamental fact that we will use, regarding the C^2 smoothness of the central manifolds is the following lemma, which is classical in one dimensional dynamics (see for example [dMS]). We will state it for the central unstable manifolds, being clear a similar result for the central stable ones. Let us remark that the center manifolds vary continuously in the C^2 topology and hence there is a uniform Lipschitz constant K_0 of $\log(Df)$ along these manifolds. If J is an arc, i.e., an embedding of the unit interval, we will denote by $\ell(J)$ its *length*.

Lemma 2.2.3 *There exists* K_0 *such that for all* $x \in \Lambda$ *and any arc* $J \subset W^{cu}_\epsilon(x)$ *such that, if* $f^{-j}(J) \subset W^{cu}_\epsilon(f^{-j}(x))$ *for* $0 \leq j \leq n$ *then*

1. $\frac{\|Df^{-n}_{/\tilde{F}(y)}\|}{\|Df^{-n}_{/\tilde{F}(z)}\|} \leq \exp(K_0 \sum_{j=0}^{n-1} \ell(f^{-j}(J)));\;\; y, z \in J,\;\; \tilde{F}(y) = T_yJ,\;\; \tilde{F}(z) = T_zJ.$

2. $\|Df^{-n}_{/\tilde{F}(y)}\| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{j=0}^{n-1} \ell(f^{-j}(J))).$

We end this section making some comments about sets having dominated splitting (for a more precisely discussion see the beginning of section 3.2 of [PS1]). Assume that Λ is a set having dominated splitting. Similar to the hyperbolic case, we can find a family of cone fields (central unstable and central stable cone fields) which the property that the diffeomorphism leaves these cone fields invariant (indeed, the central unstable cones are mapped strictly into them in the future, the same for the central stable ones in the past), but without the property of expansion and contraction of vectors. Nevertheless, these cone fields can be extended to a neighbourhood of Λ having the same property where it makes sense. We will say that an arc in this neighbourhood is transversal to the E-direction (respec. F-transversal) if the tangent space at any point lies in the central unstable (respec stable) cone field. Finally, since for $x \in \Lambda$, the tangent space at x of the central unstable (respec. stable) manifold is F (repec. E), there is some ϵ_0 (fixed from now on) such that $W^{cu}_{\epsilon_0}(x)$ (respec. $W^{cs}_{\epsilon_0}(x)$) is an arc transversal to the E- direction (respec. F-direction).

2.3 Boxes and Distortion

Through this section Λ denotes a compact invariant set having dominated splitting and it is 2-dominated.

Definition 2.3.1 . Boxes *A box B is a (small) open rectangle such that $B\cap\Lambda \neq \emptyset$ having the boundary transversal to the E and F-direction. More precisely*

$$B = int(h([-1,1]^2))$$

where $h : [-1,1]^2 \to M$ is a diffeomorphism onto its image such that if we define the central stable boundary as

$$\partial^{cs}(B) = h(\{-1,1\} \times [-1,1]),$$

we required that the two components (arcs) of it are transversal to the E-direction. For the central unstable boundary

$$\partial^{cu}(B) = h([-1,1] \times \{-1,1\})$$

we required transversality to the F-direction. The axis of the box are the arcs $h([-1,1] \times \{0\})$ and $h(\{0\} \times [-1,1])$.

A vertical strip is a subrectangle $S \subset B$ such that $\partial^{cu}(S) \subset \partial^{cu}(B)$ (we do not require that $S \cap \Lambda \neq \emptyset$).

Definition 2.3.2 Subboxes

We say that $B' \subset B$ is a cu-subbox if it is a box and $\partial^{cu}(B') \subset \partial^{cu}(B)$. In a similar way we define cs-subboxes.

In the sequel we assume that the diameter of a box B is much smaller than ϵ_0 so that if $y \in B \cap \Lambda$ then any component of $W^{cu}_{\epsilon_0}(y) - \{y\}$ intersects the boundary of B. Now, we will introduce some particular boxes exhibiting a kind of Markov property.

Definition 2.3.3 . Adapted boxes

Let B be a box. For $y \in B \cap \Lambda$ lets denote by $J^{cu}_B(y)$ the connected component of $W^{cu}_{\epsilon_0}(y) \cap B$ that contains y. We say that a box B is ϵ-cu-adapted (or adapted only) if for every $y \in B \cap \Lambda$ the following conditions are satisfied:

1. $\overline{J^{cu}_B(y)} \cap \partial^{cs}(B) = \emptyset$
2. $\ell(f^{-n}(J^{cu}_B(y))) \leq \epsilon$ *for all* $n \geq 0$
3. $f^{-n}(J^{cu}_B(y)) \cap B = \emptyset$ *or* $f^{-n}(J^{cu}_B(y)) \subset B$ *for all* $n \geq 0$

Definition 2.3.4 . Returns

Let B be an adapted box. A map $\psi : S \to B$, where $S \subset B$, is called a cu-return of B associated to Λ if:

- $S \cap \Lambda \neq \emptyset$
- *there exist* $k > 0$ *such that* $\psi = f^{-k}_{/S}$
- $\psi(S) = f^{-k}(S)$ *is a connected component of* $f^{-k}(B) \cap B$
- $f^{-i}(y) \notin B,\ 1 \leq i < k$ *for any* $y \in S \cap \Lambda$.

Let $\psi : S \to B$ be a cu-return, $\psi = f^{-k}_{/S}$ and let $y \in S \cap \Lambda$. Since B is adapted, it follows that $f^{-i}(J^{cu}_B(y)) \cap B = \emptyset, 1 \leq i < k$ and $f^{-k}(J^{cu}_B(y)) \subset B$. Thus, $J^{cu}_B(y) \subset S$.

We will denote the set of cu-returns of B associated to Λ by $\mathcal{R}^{cu}(B,\Lambda)$. Moreover, we say that a return $\psi \in \mathcal{R}^{cu}(B,\Lambda), \psi : S \to B$ is hyperbolic if we have $|\psi'| < \xi < 1$, that is, if $\|Df^{-k}_{/\tilde{F}(z)}\| < \xi$ for all $z \in J^{cu}_B(y), y \in S \cap \Lambda$, where $\psi = f^{-k}_{/S}$ and $\tilde{F}(z) = T_z J^{cu}_B(y) = T_z W^{cu}_{\epsilon_0}(y)$.

For our purposes, we need a refinement of the definition of an adapted box. Before doing so, let us make an observation that may help the reader to understand the following definition. If B is an adapted box such that $\Lambda \cap \partial^{cs}(B) = \emptyset$ then we may found a cu-subbox B' so that $B - B' = S_1 \cup S_2$ where S_1, S_2 are vertical strips with $S_i \cap \Lambda = \emptyset, i = 1, 2$. However the condition $\Lambda \cap \partial^{cs}(B) = \emptyset$ is not always possible (for instance, if $\Lambda = M$).

Definition 2.3.5 . Well adapted boxes

Let B be an adapted box. We say that B is well adapted if there exist a cu-subbox B' and two disjoint vertical strips S_1, S_2, such that

$$B - B' = S_1 \cup S_2$$

where S_1, S_2 satisfy either

a) $S_i \cap \Lambda = \emptyset$.

or

b) S_i is a domain of a cu-return $\psi_i \in \mathcal{R}(B,\Lambda)$ and $\psi_i(S_i)$ is a cs-subbox. Moreover, if k_i is such that $\psi_i = f^{-k_i}/S_i$ then we require that $f^{-j}(S_i) \cap B = \emptyset$ for $1 \leq j < k$.

At some point we will have to compare the volume of a box, and the length of the axis. And this is well performed if we have some kind of "distortion" properties.

Definition 2.3.6 . Distortion

We say that a box B has distortion (or cu-distortion) C if for any two arcs $J_1,\ J_2$ in B transversal to the E-direction and whose endpoints are in $\partial^{cu}(B)$ the following holds:

$$\frac{1}{C} \leq \frac{\ell(J_1)}{\ell(J_2)} \leq C.$$

Remark 2.3.1 *If an adapted box has distortion C, then, for any $y, z \in B \cap \Lambda$,*

$$\frac{1}{C} \leq \frac{\ell(J_B^{cu}(z))}{\ell(J_B^{cu}(y))} \leq C.$$

Notice that, in order to guarantee distortion C on a box B it is sufficient to find a C^1 foliation close to the E-direction in the box, such that, for any two arcs $J_1,\ J_2$ (taken as in the definition of distortion),

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

holds, where $\Pi = \Pi(J_1, J_2)$ is the projection along the foliation between these arcs.

2.4 Denjoy property

Let Λ be a set with dominated splitting, and let V be an admissible neighbourhood of Λ, that is, any compact invariant set in V has dominated splitting. Take U another neighborhood of Λ, $\Lambda \subset U \subset \overline{U} \subset V$. Denote by $\Lambda_1 = \cap_{n\in\mathbb{Z}} f^n(\overline{U})$ the maximal invariant set in $\overline{U}$ (the closure of U) and by $\Lambda^+ = \cap_{n\geq 0} f^{-n}(\overline{U})$ the set of points which remains in $\overline{U}$ in the future and by $\Lambda^- = \cap_{n\geq 0} f^n(\overline{U})$ the set of points which remains in $\overline{U}$ in the past. Notice that Λ_1 has a dominated splitting $T_{\Lambda_1} = E \oplus F$ since V is admissible. Moreover, for every point $x \in \Lambda^+$ we have a uniquely determined E-direction. Recall that an open arc $I \subset M$ mean an embedding of the real line (or the open unit interval) in M and $\ell(I)$ denotes its *length.* A simple closed curve $S \subset M$ will mean an embedded of the circle S^1 in M. The ω-limit set of an arc I, $\omega(I)$, is the collection of the ω-limit set of any point in I.

Definition 2.4.1 *We say that an open C^2arc I in M is a δ-E-arc if the next two conditions hold:*

1. *$I \subset \Lambda_1^+$ and $\ell(f^n(I)) \leq \delta$ for all $n \geq 0$.*
2. *$f^n(I)$ is always transversal to the E-direction.*

In an analogous way we define δ-F-arc.

In order to prove dynamics properties on the central manifolds, we recall Proposition 3.1 from [PS1]. Although items $2a$ and $2b$ are not included in the original statement, they are consequence of the proof of the cited proposition. Before state it, let us say that a compact arc $\mathcal{J} \subset \Lambda_1^+$ is E-normally hyperbolic if it is transversal to the E-direction and moreover, for all $z \in \mathcal{J}$ we have that $\|Df^n_{E(z)}\| \leq C\gamma^n$ for some $0 < \gamma < 1$. In this case, for all $z \in \mathcal{J}$ there is stable manifold $W^s(z)$ which is tangent to $E(z)$. We define the basin of attraction of $\mathcal{J}$ as $\cup_{z\in\mathcal{J}} W^s(z)$. Notice that if $\mathcal{J}$ is periodic, i.e. $f^m(\mathcal{J}) \subset \mathcal{J}$ for some positive integer m, then the only non-wandering points in the interior of its basin of attraction are just the periodic points in $\mathcal{J}$.

Theorem 2.3 *(**Denjoy's property**) There exists $\delta_0 (\leq \epsilon_0)$ such that if I is a δ-E-arc with $\delta \leq \delta_0$, then one of the following properties holds:*

1. *$\omega(I)$ is a periodic simple closed curve $\mathcal{C}$ normally hyperbolic and $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (where m is the period of $\mathcal{C}$) is conjugated to an irrational rotation,*
2. *$\omega(I) \subset Per(f_{/V})$ where $Per(f_{/V})$ is the set of the periodic points of f in V. More precisely: either*
   - *(a) There is a periodic closed arc $\mathcal{J}$ E-normally hyperbolic and I is in its basin of attraction*
     or
   - *(b) $\omega(I)$ is a sink or a saddle-node periodic point.*

Remark 2.4.1 *for future purpose, let us remark that V can be chosen so small such that for any g in a suitable neighbourhood $\mathcal{U}$ of f the set $\cap_{n\in\mathbb{Z}} g^n(\overline{U})$ has dominated splitting and the same family of cone fields is appropiate for $g \in \mathcal{U}$. Then, the constant δ_0 in Theorem 2.3 can be chosen uniformly on $\mathcal{U}$.*

Note: In the sequel we say that a set Λ does not have a closed curve supporting an irrational rotation if there is no (periodic) simple closed curve in Λ normally hyperbolic $\mathcal{C}$ such that $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (where m is the period of $\mathcal{C}$) is conjugated to an irrational rotation.

3 Proof of Theorem A

Let us start to prove Theorem A. Arguing by contradiction, assume that the conclusion of Theorem A is not true. Then, there exists a sequence p_n of periodic points whose periods are unbounded and they are not hyperbolic periodic points of saddle type. Let Λ_0 be the set of limits points of the orbits of the points p_n, i.e.:

$$\Lambda_0 = \cap_{m\geq 0}\overline{\cup_{n\geq m}\mathcal{O}(p_n)}.$$

This set is compact invariant and, since it is a subset of Λ, has a dominated splitting.

Assume first that either all the periodic points in Λ_0 are hyperbolic or Λ_0 does not contain any periodic point at all. Then, by Theorem 2.1, we conclude that Λ_0 is a union of a hyperbolic set and a finite union of periodic simple closed curves normally hyperbolic. Since given a neighbourhood of Λ_0 there exists n_0 such that, for any $n \geq n_0$, the orbit of p_n it is contained in this neighborhood, we get a contradiction. In fact, the orbits of p_n can not accumulate on the periodic simple closed curves since they are normally hyperbolic (attracting or repelling curves). Thus, Λ_0 is a hyperbolic set and so the maximal invariant set in an admissible compact neighbourhood of Λ_0 is hyperbolic as well. In particular, for sufficient large n, p_n lies on this maximal invariant set and so it must be a hyperbolic periodic point of saddle type, a contradiction and so our assumption is false.

Therefore, Λ_0 must contain a non-hyperbolic periodic point p, and we have that the orbits of a subsequence of $\{p_n\}$ accumulates on p with unbounded periods. This contradicts the following theorem:

Theorem 3.1 *Let f be a C^2 diffeomorphism of a compact surface M and $\Lambda \subset \Omega(f)$ be a compact set having a dominated splitting. Let $p \in \Lambda$ be a non-hyperbolic periodic point and denote by N_p its period. Then, there exists a neighbourhood U_p of p such that any periodic point $q \in \Lambda$ with period greater than $2N_p$ and whose orbit intersects U_p is a hyperbolic periodic point of saddle type.*

Summarizing, the proof of Theorem A is reduced to the proof of above theorem, which is postponed to section 3.4. Nevertheless, in the next section we will give a rough outline of it.

3.1 Idea of the proof of Theorem 3.1

Let p be a non-hyperbolic periodic point and q a periodic point whose orbit goes through a very small neighborhood of p. We would like to show that q is a hyperbolic of saddle type, that is $||Df^{n_q}_{|F_q}|| > 1$ and $||Df^{n_q}_{|E_q}|| < 1$ where n_q is the period of q.

The idea is to split the orbit of q into pieces where either is outside the neighborhood of p or inside of it.

On one hand, we show that outside any neighborhood of p, the derivative along the F-direction for any trajectory is uniform bounded away from zero, i.e.: $||Df^n_{/F_x}|| > c > 0$ for $f^i(x) \notin U_p$, $i = 1, .., n$ (notice that this does not contradict that q might be a periodic attractor).

On the other hand, when a trajectory is going through a tiny neighborhood of p, not only does not loose expansion (although the derivative of p along the $F-$direction might be one) but it has a good expansion along the $F-$direction from the first time that the point goes into U_p until the last time that remains in it (even if the exponential rate is close to one), i.e.: for x such that $f(x) \notin U_p$, $x, ..., f^{-n}(x) \in U_p$ and $f^{-(n+1)}(x) \notin U_p$ then $||Df^n_{|F_x}|| > 2/c$.

Let us explain the latter. First we consider a small central unstable segment J containing x. Observe that since a long trajectory of x is inside a U_p, then J is close to the central unstable manifold of p. Let us consider a segment I in a fundamental domain of the central unstable manifold of p, obtained as the "projection of J over the central unstable manifold of p along the central stable foliation". We show that the lengths of $f^{-k}(I)$ and $f^{-k}(J)$ are uniformly comparable for any $1 \leq k \leq n$ and we conclude that

$$||Df^{-n}_{/F(x)}|| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{j=0}^{n-1} \ell(f^{-j}(J))) \approx$$

$$\frac{\ell(f^{-n}(I))}{\ell(I)} \exp(K_0 \sum_{j=0}^{n-1} \ell(f^{-j}(I))) \leq \frac{\ell(f^{-n}(I))}{\ell(I)} \exp(K_0 \ell(W^{cu}_\epsilon(p))).$$

For the first part, we will also use an argument of summability, more precisely, showing that there is a uniform constant K such for any x verifying that $f^{-j}(x) \notin U_p, j = 0, 1, ..., n$, then there is a central unstable segment J containing x with the property that $\sum_{j=0}^n \ell(f^{-j}(I)) < K$. This is enough to find a uniform lower bound for the derivative along the F-direction.

To develop these ideas we need to understand first the structure of the non-hyperbolic periodic points in a set having dominated splitting and the dynamic behaviour of the central manifolds of points nearby. These will be done in the next two sections, before giving the complete proof of Theorem 3.1.

3.2 Non-hyperbolic periodic points

Recall that in the case of sets having dominated splitting, we may have to deal with non-hyperbolic periodic points. However, due to the dominated splitting, at least one eigenvalue of the periodic point has modulus far from one. Hence, a non-hyperbolic periodic point in a set with dominated splitting has only one eigenvalue with modulus one (and so it is 1 or -1). We will say that a non-hyperbolic periodic point is E-non-hyperbolic (respec. F-non-hyperbolic) if the eigenspace associated to the eigenvalue with modulus 1 is the E-space (respec. F-space).

Notice that if p is an F-non-hyperbolic periodic point then, for some ϵ, $W^{cs}_\epsilon = W^{ss}_\epsilon$, i.e., the local central stable manifold coincides with the local strong stable manifold. Analogously, if p is an E-non-hyperbolic periodic point then, for some ϵ, $W^{cu}_\epsilon = W^{uu}_\epsilon$, i.e., the local central unstable manifold coincides with the local strong unstable manifold.

Let p be a F-non-hyperbolic fixed (periodic of period m) point and consider the following statement: there exists some ϵ_1 such that for any $\epsilon < \epsilon_1$ there exists γ such that $f^{-n}(W^{cu}_\gamma(p)) \subset W^{cu}_\epsilon(p)\ \forall n \geq 0$.

In case this statement holds then, either for some γ any point in $W^{cu}_\gamma(p)$ converges to p by backwards iterates or for any γ this not happens. In the former, $W^{cu}_\gamma(p)$ is in fact an unstable manifold. In the latter, if for some γ there is a component of $W^{cu}_\gamma(p) - \{p\}$ such that any point in this component converges to p by backwards iterates, then we conclude that points in one component of W^{cu}_γ converges to p by backwards iterates and on the other component there is a sequence of fixed (periodic of period m) points converging to p. If for any γ and any component of W^{cu}_γ there are points that do not converges to p by backwards iterates, then in both components there is a sequence of fixed or 2-periodic (periodic with period m or $2m$) points converging to p.

On the other hand, if the statement above does not hold, we may ask if it is true replacing W^{cu}_γ by a component of $W^{cu}_\gamma - \{p\}$. If it is true for one component, we may conclude on this component that either any point converges by backwards iterates to p or there is a sequence of fixed (periodic of period m) point converging to p. Notice that on the other component, point must converge to p in the future. Now, if the statement is not true for any component, then any point in a neighbourhood converges to p in the future. Thus, we have proved the following lemma (see also sections 3.3 and 4.1).

Lemma 3.2.1 *Let p be an $F-$non-hyperbolic periodic point. Then one and only one of the following situation holds:*

1. *For some $\epsilon > 0$ and any $x \in W^{cu}_\epsilon(p)$, $f^{-n}(x) \to_{n\to\infty} p$ holds.(i.e. W^{cu}_ϵ is an unstable manifold). This point will be called periodic point of saddle-type.*
2. *For some $\epsilon > 0$ and any $x \in W^{cu}_\epsilon(p)$, $f^{n}(x) \to_{n\to\infty} p$ holds.(i.e. W^{cu}_ϵ is an stable manifold). This point will be called periodic point of sink-type.*
3. *For some $\epsilon > 0$, on one component of $W^{cu}_\epsilon(p) - \{p\}$ we have $f^{-n}(x) \to_{n\to\infty} p$ and on the other one either there is sequence of fixed points converging to p*

or point in this component converges to p by forward iterates. This will be called F-saddle-node-type periodic point.

4. *Either on one component there is a sequence of periodic points converging to p and points on the other one converges to p in the future, or in both components there is sequence of periodic points converging to p (the periods are, in this case, equal to or double of the period of p). This will be called sink-node type periodic point.*

The same result we get for E-non-hyperbolic periodic point.

Lemma 3.2.2 *Let p be an $E-$non-hyperbolic periodic point. Then one and only one of the following situation holds:*

1. *For some $\epsilon > 0$ and any $x \in W_\epsilon^{cs}(p)$, $f^n(x) \to_{n\to\infty} p$ holds.(i.e. W_ϵ^{cs} is an unstable manifold). This point will be called periodic point of saddle-type.*

2. *For some $\epsilon > 0$ and any $x \in W_\epsilon^{cs}(p)$, $f^{-n}(x) \to_{n\to\infty} p$ holds.(i.e. W_ϵ^{cs} is an unstable manifold). This point will be called periodic point of source-type.*

3. *For some $\epsilon > 0$, on one component of $W_\epsilon^{cs}(p) - \{p\}$ we have $f^n(x) \to_{n\to\infty} p$ and on the other one either there is sequence of periodic points converging to p or point in this component converges to p by forward iterates. This will be called E-saddle-node-type periodic point.*

4. *Either on one component there is a sequence of periodic points converging to p and points on the other one converges to p in the past, or in both components there is sequence of periodic points converging to p. This will be called source-node type periodic point.*

Corollary 3.2.1 *Let p be a non-hyperbolic periodic point. Then,*

1. *if p is F-saddle-node (respec E-saddle node) then, for any small ball $B(p)$, any point in one component of $B(p)\backslash W_{loc}^{ss}(p)$ (respec.$B(p)\backslash W_{loc}^{uu}(p)$)converges in the future (respect. in the past) to p or to a periodic point with the same period of p. Moreover, given m, if the ball $B(p)$ is small enough, any periodic point in the other component has period greater than m and the orbit must leave the ball $B(p)$.*

2. *if p is sink or sink-node type (respec. source or source-node type) then, any point in a small neighbourhood $B(p)$ converges in the future (respec. in the past) to p or to a periodic point with either the same period of p or double of it.*

3. *if p is saddle type then, given m there is a small neighbourhood of p such that any periodic point in this neighbourhood has period greater than m.*

3.3 Central stable and unstable manifolds II

In this section we study the dynamic behaviour of the central manifolds of points in a set Λ (having dominated splitting) near periodic points.

Definition 3.3.1 . Boxes around periodic points *Let $p \in \Lambda$ be periodic point and let $\delta^s < \delta_0$, $\delta^u < \delta_0$ be given. A box around p is a box (see definition 2.3.1) with p in its interior and axis $W^{cu}_{\delta^u}(p)$ and $W^{cs}_{\delta^s}(p)$ (and small enough so that Corollary 3.2.1 applies with m twice the period of p).*

Lets call *branch* a component of $W^{cj}_{\delta^j(p)}(p) - \{p\}$, $j = u, s$. This branches divide the box $B_{(\delta^s,\delta^u)}(p)$ into four quadrants $B^i_{(\delta^s,\delta^u)}$ $1 \leq i \leq 4$. A quadrant is said to be *non-isolated* (with respect to Λ) if there are points of Λ in this quadrant that are either non-periodic or periodic points with period greater than $2N_p$ where N_p is the period of p. In particular the central stable branch and the central unstable branch bounding this quadrants are contained in the stable manifold and in the unstable manifold respectively of p (even if p is a non-hyperbolic periodic point).

For any $y \in B^i_{(\delta^s,\delta^u)}(p) \cap \Lambda$, set $J^{cu,i}_{\delta^u} = J^{cu}_{B^i_{(\delta^s,\delta^u)}}$ and $J^{cs,i}_{\delta^s} = J^{cs}_{B^i_{(\delta^s,\delta^u)}}$. Now, using Theorem 2.3, we will prove that these central unstable (stable) arcs defined previously, do not increase the size by negative (positive) iteration.

Lemma 3.3.1 *Let $\Lambda \subset \Omega(f)$ be a compact set having dominated splitting and 2-dominated without closed curves supporting irrational rotations and let $p \in \Lambda$ be a periodic point. Let $B_{(\delta^s,\delta^u)}(p)$ be as above, and let $B^i_{(\delta^s,\delta^u)}(p)$ be a non-isolated quadrant.*

Then, for any ϵ, $0 < \epsilon < \delta_0$, there is $\delta^u_* = \delta^u_*(\epsilon) < \delta^u$ such that for any $x \in B^i_{(\delta^s,\delta^u_*)}(p) \cap \Lambda$ different from p

$$f^{-n}(J^{cu,i}_{\delta^u_*}(x)) \subset W^{cu}_\epsilon(f^{-n}(x))$$

holds for any $n \geq 0$. A similar statement we get for the central stable manifolds, more precisely: for any $\epsilon > 0$, there is $\delta^s_* = \delta^s_*(\epsilon) < \delta^s$ such that for any $x \in B^i_{(\delta^s_*,\delta^u)}(p) \cap \Lambda$ different from p

$$f^n(J^{cs,i}_{\delta^s_*}(x)) \subset W^{cs}_\epsilon(f^n(x))$$

holds for any $n \geq 0$.

Proof: We shall prove the lemma only for the central unstable manifolds. The case for central stable is completely similar. For convenience, we will forget the index i in $J^{cu,i}$.

Assume that the lemma is not true. Then, setting $\delta = \delta(\epsilon)$ from corollary 2.2.1 there exist sequences $\gamma_n \to 0$, $x_n \in B^i_{(\delta^s, \gamma_n)} \cap \Lambda$, and $m_n \to \infty$ such that, for $0 \leq j \leq m_n$,

$$\ell(f^{-j}(J^{cu}_{\gamma_n}(x_n))) \leq \delta$$

and

$$\ell(f^{-m_n}(J^{cu}_{\gamma_n}(x_n))) = \delta.$$

Letting $I_n = f^{-m_n}(J^{cu}_{\gamma_n}(x_n))$ we can assume (taking a subsequence if necessary) that $I_n \to I$ and $f^{-m_n}(x_n) \to z$, $z \in \Lambda$, $z \in \overline{I}$ (the closure of I).

Now, we have that $\ell(f^n(I)) \leq \delta$ for all positive n, and since $I \subset W^{cu}_{\epsilon_0}(z)$, we conclude that I is a δ-E-arc. From Theorem 2.3 $\omega(\overline{I})$ is a periodic orbit q (a sink or a saddle-node) or $\overline{I}$ is in the basin of an invariant segment $\mathcal{J}$.

We have two possibilities, either z is an interior point of I or it is not the case. In the former we get that, for large n, x_n is a nonwandering point if and only if x_n is the periodic point q or is a periodic point in $\mathcal{J}$. Since $\gamma_n \to 0$ we get that $dist(x_n, W^{cs}_{\delta^s}(p)) \to 0$. If $x_n = q$ then we conclude that $q \in W^{cs}_{\delta^s}(p)$ and hence $x_n = q = p$, a contradiction. On the other hand, if x_n is a periodic point in $\mathcal{J}$, then we also conclude that $p \in \mathcal{J}$. This is a contradiction, because $x_n \in \mathcal{J}$ and on the other hand x_n belongs to a non-isolated quadrant.

If z is in the boundary of I, we get that either $z \in W^s(q_1)$ or $z \in W^s(q_2)$ where q_1 and q_2 are the periodic points in the boundaries of $\mathcal{J}$. Assume that $z \in W^s(q_1)$. In case

$$f^{-m_n}(J^{cu}_{\gamma_n}(x_n)) \cap W^s(q_1) \neq \emptyset$$

we get, since $x_n \in J^{cu}_{\gamma_n}(x_n) = f^{m_n}(f^{-m_n}(J^{cu}_{\gamma_n}(x_n)))$, that $x_n \to q_1$ and so $q_1 = p$. But then $J^{cu}_{\gamma_n}(x_n) \cap W^{cs}(p) \neq \emptyset$ which contradicts the definition of J^{cu}.

Finally, if

$$f^{-m_n}(J^{cu}_{\gamma_n}(x_n)) \cap W^s(q_1) = \emptyset$$

then, $\omega(f^{-m_n}(x_n))$ is a point in $\mathcal{J}$. Thus, x_n is a nonwandering point if and only if it is a periodic point of $\mathcal{J}$. As we showed above, this is contradiction.

∎

Remark 3.3.1 *As a consequence of the previous, by [HPS] we get coherence inside each non-isolated quadrants of $B_{(\delta^s,\delta^u)}(p)$, that is, the central unstable manifolds either are disjoint or coincide.*

3.4 Proof of the Theorem 3.1

Let $p \in \Lambda$ be a periodic point. We will prove Theorem 3.1 by contradiction. Assume then that the conclusion of the theorem is false, that is, there exist a sequence of periodic points $\{q_n\} \subset \Lambda$ accumulating at p whose periods increase to infinity and such they are not hyperbolic of saddle type, i.e., either

$$\|Df^{-m_n}_{/F(q_n)}\| \geq 1 \text{ or } \|Df^{m_n}_{/E(q_n)}\| \geq 1$$

where m_n is the period of q_n. Let us assume that

$$\|Df^{-m_n}_{/F(q_n)}\| \geq 1$$

holds for any n. We will show that is not the case for sufficient large n, leading to a contradiction.

There is no loss of generality if we assume that p is a fixed point and that the eigenvalues of Df_p are positive. Consider

$$\Lambda_0 = \overline{\{\mathcal{O}(q_n) : n \geq 0\}}$$

the subset of Λ which is the closure of the orbits of q_n. There is also no loss of generality if we assume that $\Lambda = \Lambda_0$ and so Λ does not contains closed curves supporting an irrational rotation. Moreover, using Theorem 2.2 and Lemma 2.2.1, we may assume that $\Lambda = \Lambda_0$ is 2-dominated.

Let $B_{(\delta^s,\delta^u)}(p)$ be a box and let B^i be a non-isolated quadrant, that is $\mathcal{O}(q_n) \cap B^i \neq \emptyset$ for each n (take a subsequence if necessary). This quadrant B^i is determined by branches of $W^{cs}_{\delta^s}(p)$ and $W^{cu}_{\delta^u}(p)$. Lets denote this branches by $W^{cs,+}_{\delta^s}(p)$ and $W^{cu,+}_{\delta^u}(p)$, and order them in some way (since they are arcs). Let $\delta^s_* < \delta^s$ be such that

$$\ell(f^n(J^{cs}_{\delta^s_*})) \leq \delta^s/2$$

for $n \geq 0$ (see lemma 3.3.1). Let $x \in W^{cs,+}_{\delta^s_*}(p) - \{p\}$ be an accumulation point of the orbits of q_n.

Lemma 3.4.1 *Let $x \in W^{cs,+}_{\delta^s_*}$ be as above and let $\epsilon > 0$ be given. Then, there exist a well ϵ-adapted box $B = B(x)$ such that*

1. *x belongs to a component of $\partial^{cu}(B(x))$ which is also contained in a fundamental domain of $W^{cs}_{\delta^s_*}(p)$.*
2. *For any large n, the orbits of q_n have nonempty intersection with $B(x)$.*

Proof:

Take $\delta^u_* = \delta^u_*(\epsilon)$ from lemma 3.3.1, i. e.

$$\ell(f^{-n}(J^{cu}_{\delta^u_*})) \leq \epsilon$$

holds for $n \geq 0$. Take q_{n_1} and q_{n_2} such that $z_1 < x < z_2$ where z_i, $i = 1, 2$ is the endpoint of $J^{cu}_{\delta^u_*}(q_{n_i})$ that belongs to $W^{cs,+}_{\delta^s_*}(p)$. These periodic points can be taken such that the points between z_1 and z_2 (denote this arc by $J_{(z_1,z_2)}$) is contained in a fundamental domain of $W^{cs}_{\delta^s_*}(p)$. Let m_i be the period of q_{n_i}. Take $d>0$ such that

$$\operatorname{dist}(f^{-n}(J^{cu}_{\delta^u_*}(q_{n_i})), J_{(z_1,z_2)}) > d \text{ for } 0 < n \neq km_i,\ i=1,2\ k=1,2,...$$

Take some periodic point $q = q_{n_k} \in B_{(\delta^*_s,\delta^u_*)}(p)$ such that, if we consider the box B bounded by $J_{(z_1,z_2)}$, $J^{cu}_{\delta^u_*}(q_{n_1})$, $J^{cu}_{\delta^u_*}(q_{n_2})$ and $J^{cs}_{\delta^s_*}(q)$, then

- $\operatorname{dist}(z, J_{(z_1,z_2)}) < d$ for any point z in this box B.
- There are no points of the orbit of q in B.
- $W^{cs,+}_{\delta^s_*}(p) \cap B = \emptyset$.

Notice that we may take the point q belonging to the boundary of B. Lets denote by $\partial^{cu}_q B$ the component of the cu-boundary of B that contains q. By the definition of B, it follows that $\partial^{cu}_q B \subset J^{cs}_{\delta^s_*}(q)$. Lets prove that this box is ϵ-adapted. Condition 1) of the definition of ϵ-adapted boxes is already satisfied by Remark 3.3.1 and condition 2) holds by the election of δ^u_*. Thus, we only have to check condition 3) of this definition. Let $y \in \Lambda \cap B$ and assume that for some $n>0$ we have that $f^{-n}(J^{cu}_B(y)) \cap B \neq \emptyset$. We have to prove that $f^{-n}(J^{cu}_B(y)) \subset B$. Arguing by contradiction assume that this is not the case. Then, by the coherence of the local central manifolds within $B_{(\delta^s_*,\delta^u_*)}(p)$ we conclude that

$$f^{-n}(J^{cu}_B(y)) \cap J_{(z_1,z_2)} \neq \emptyset \quad \text{or} \quad f^{-n}(J^{cu}_B(y)) \cap \partial^{cu}_q B \neq \emptyset$$

In the former, set $z = f^{-n}(J^{cu}_B(y)) \cap J_{(z_1,z_2)}$. Now, $f^n(z) \in J^{cu}_B(y) \subset B$ and, on the other hand, since $z \in J_{(z_1,z_2)} \subset W^{cs,+}_{\delta^s_*}(p)$, we conclude that also $f^n(z) \in W^{cs,+}_{\delta^s_*}(p)$, a contradiction with the definition of B. In the latter, that is, if $f^{-n}(J^{cu}_B(y)) \cap \partial^{cu}_q B \neq \emptyset$ then, $f^n(\partial^{cu}_q B) \cap B \neq \emptyset$ and we claim that $f^n(\partial^{cu}_q B) \subset B$, leading to a contradiction because, if this is the case, $f^n(q) \in B$, that is, there are points of the orbit of q in B. So, lets prove the claim. Arguing by contradiction, assume that $f^n(\partial^{cu}_q B)$ is not a subset of B. This implies that $f^n(\partial^{cu}_q B) \cap J^{cu}_{\delta^u_*}(q_{n_i}) \neq \emptyset$ for $i=1$ or 2, (say $i=1$.) Therefore, $f^{-n}(J^{cu}_{\delta^u_*}(q_{n_1})) \cap B \neq \emptyset$, and in particular we have that $n \neq km_1\ k=1,2,...$ (m_1 is the period of q_{n_1}). This contradicts the election of d and we have proved the claim. This finished the proof that B is ϵ-adapted. Moreover, since the periodic points q_n accumulate at x we also have that for any large n, this periodic points intersects the box B.

However, this box may fail to be well-adapted. In this case, we will find a cu-subbox satisfying the thesis of the lemma.

Order the arc $J_{(z_1,z_2)}$ in some way. Consider the map $\Pi : B \cap \Lambda \to J_{(z_1,z_2)}$ where $\Pi(z)$ is the endpoint of $J^{cu}_B(z)$ that belongs to $J_{(z_1,z_2)}$. Let $H = \Pi(B \cap \Lambda)$.

We claim that x is not a boundary point of a component of the interior of H. Arguing by contradiction, assume that there is a component of the interior of H, say l, so that x is a boundary point of it. Let q_n be a periodic point so that $\Pi(q_n) \in l$ and denote by l^+ the arc whose endpoints are $\Pi(q_n)$ and x. Let $W = W^{cs}_\epsilon(q_n) \cap \{\cup_{z\in\Pi^{-1}(l^+)} J^{cu}_B(z)\}$. Let k be the double of the period of q_n and so $f^k(W) \cap W \neq \emptyset$. We have three possibilities: $W \subset f^k(W)$, $f^k(W) \subset W$ or $f^k(W) = W$. In the first two it is not difficult to see that we get a contradiction, indeed it follows that x is an interior point of H. In case $f^k(W) = W$ we get a contradiction as follows: consider the continuous monotone map $P : l^+ \to l^+$ by $P(z) = \Pi(f^{-k}(w_z))$ where w_z is some point in $\Pi^{-1}(z)$. Since B is adapted, $P(z)$ does not depend on the election of w_z and hence P is well defined. P can not be the identity: otherwise the periods of the periodic points in $\Pi^{-1}(l^+)$ are uniformly bounded (recall that $\Lambda = \overline{\{\mathcal{O}(q_n) : n \geq 0\}}$). On the other hand, if P is not the identity, there is a fixed point of P, say y, attracting some subarc of l^+ containing y in its boundary. However there is a periodic point in $B \cap \Lambda$, say $\bar{q}$ such that $\Pi(\bar{q})$ belongs to the interior of this subarc; let $n_{\bar{q}}$ be the period of $\bar{q}$. We get a contradiction since $\Pi(\bar{q})$ is fixed by $P^{n_{\bar{q}}}$ and on the other hand must converge to y under iteration of P. The proof of our claim is complete.

Now, if x is not an interior point of H it is not difficult to find two strips S_1, S_2 to the left and to the right of x such that $S_i \cap \Lambda = \emptyset$, $i = 1, 2$ and we find in this way a well adapted cu-subbox of B. In case x is an interior point of H (taking a subbox if necessary) we may assume that without loss of generality that $H = J_{(z_1,z_2)}$. It is not difficult to find two periodic points $\hat{q}, \tilde{q} \in B \cap \Lambda$ (they might be in the same orbit) in such a way that $\Pi(\hat{q}) < x < \Pi(\tilde{q})$ and for any other point $y \in B \cap (\mathcal{O}(\hat{q}) \cup \mathcal{O}(\tilde{q}))$ we have either $\Pi(y) < \Pi(\hat{q})$ or $\Pi(\tilde{q}) < \Pi(y)$. The subbox $\hat{B}$ of B whose central stable boundary are the arcs $J^{cu}_B(\hat{q})$ and $J^{cu}_B(\tilde{q})$ is well adapted. Let us prove it. Denote by $J^{cs}_1 = W^{cs}_\epsilon(\hat{q}) \cap \hat{B}$ and let $k_0 = \min\{j \geq 1 : f^j(J^{cs}_1) \cap \hat{B} \neq \emptyset\}$. Denote by S_1 the connected component of $f^{k_0}(\hat{B}) \cap \hat{B}$ that contains $f^{k_0}(J^{cs}_1)$. We will show that S_1 is the domain of a return in the condition of a well adapted box. By the way we choose the points $\hat{q}, \tilde{q}$ and that $B, \hat{B}$ are cu-adapted, we conclude that one endpoint of $f^{k_0}(J^{cs}_1)$ belongs to the central stable boundary of $\hat{B}$. On the other hand, it is not difficult to see that

$$S_1 = \bigcup \{J^{cu}_{\hat{B}}(z) : z \in \hat{B} \cap \Lambda, J^{cu}_{\hat{B}}(z) \cap f^{k_0}(J^{cs}_1) \neq \emptyset\}$$

and hence S_1 is the domain of a return. Arguing similarly with $\tilde{q}$ we find the other vertical strip S_2.

■

On the other hand, let $y \in W^{cu,+}_{\delta^u_*}(p) - \{p\}$ be an accumulation point of the orbits of q_n, say by $\hat{q}_n = f^{k_n}(q_n)$. Notice that we may assume that $f^j(q_n) \in$

$B^i_{(\delta^s,\delta^u)}(p)$, $0 \leq j \leq k_n$. Similar to the previous lemma, we can construct an adapted box for y.

Lemma 3.4.2 *Let $y \in W^{cu,+}_{\delta^u_*}$ be as above. Then, there exist a cu-adapted box $B = B(y)$ such that*

1. *y belongs to a component of $\partial^{cs}(B(y))$ which is also contained in a fundamental domain of $W^{cu}(p)$.*
2. *For any large n, the orbits of q_n have nonempty intersection with $B(y)$.*

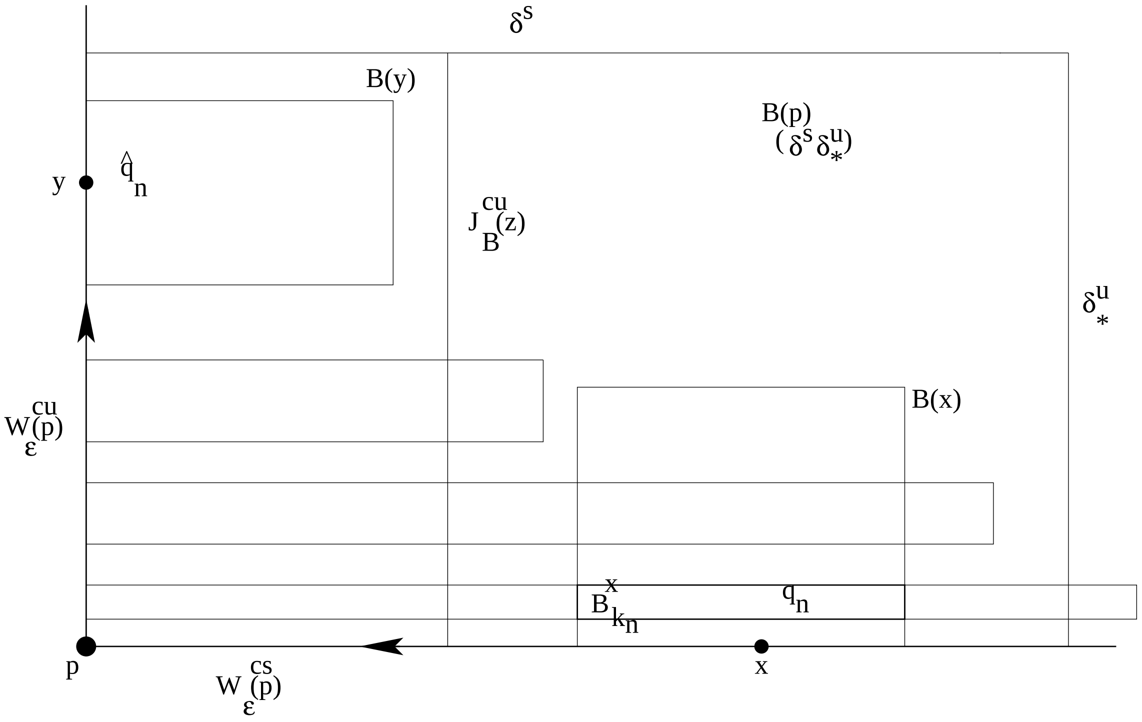
From now on, fix a C^1- foliation $\mathcal{F}^{cs}$ on $B(y)$ close to the E-direction, that is, take a C^1-vector field X in $B(y)$, C^0-close to the E-direction ($X(z)$ lies in a central stable cone), and such that, for $z \in \partial^{cu}(B(y))$, $X(z) \in T_z\partial^{cu}$. Consider the foliation $\mathcal{F}^{cs}$ (or the flow) generated by this vector field. For any $z \in B(y)$ let $\mathcal{F}^{cs}(z)$ the leave passing through z. Notice that there exists C such that

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

where $\Pi = \Pi(J_1, J_2)$ is the projection along this foliation between two arcs transversal to the E-direction; this means that the box $B(y)$ has distortion C.

Definition 3.4.1 Boxes II *Recalling that $y \in W^{cu}_{\delta^s_*}(p)$ and that $J^{cu}_{B(y)}(y)$ lies in a fundamental domain of $W^{cu}_{\delta^u_*}(p)$, let B_1 be the connected component of $f^{-1}(B(y)) \cap B_{(\delta^s,\delta^u)}(p)$ that contains $f^{-1}(y)$. For $k \geq 2$, we define B_k as the connected component of $f^{-1}(B_{k-1}) \cap B_{(\delta^s,\delta^u)}(p)$ that contains $f^{-k}(y)$.*

Moreover, given some periodic point $\hat{q}_n$ in $B(y)$, let $k_n = \min\{k \geq 0 : f^{-k}(\hat{q}_n) \in B(x)\}$. We define $B^x_{k_n}$ as the component of $B_{k_n} \cap B(x)$ that contains $f^{-k_n}(\hat{q}_n)$ (the boxes B_k defined as above). See figure below.



Proposition 3.1 *Given $r > 0$, there exists $s > 0$ such that if $dist(\hat{q}_n, y) < s$ the following hold*

1. *$B^x_{k_n}$ is a r-cu-adapted cs-subbox in $B(x)$.*

2. *If r is small enough, then any return to $B^x_{k_n}$ is a hyperbolic return. Indeed, $|\psi'| < \frac{1}{2}$ for $\psi \in \mathcal{R}^{cu}(B^x_{k_n}, \Lambda)$.*

Let us show how the last proposition implies Theorem 3.1. In fact, we get a contradiction to the assumption that the periodic point were not F-expanding. To show this, take the periodic point $q_n = f^{k_n}(\hat{q}_n) \in B^x_{k_n}$ and assume that the period is m. Let $0 < m_1 < m_2 < ... < m_l = m$ be the successive returns of the point q_n to $B^x_{k_n}$ until return to itself. Then,

$$\|Df^{-m}_{/F(q_n)}\| \leq \left(\frac{1}{2}\right)^l < 1,$$

a contradiction to our assumption.

In order to prove Proposition 3.1 we have to deal with arguments involving distortion and summability. We need two lemmas.

Lemma 3.4.3 *Let $B(y)$ be the box in Lemma 3.4.2 having distortion C. Then,*

1. *there exists $K > 0$ such that for any cu-subbox $B' \subset B(y)$ and $n > 0$ such that $f^{-k}(B') \subset B_k$ for $0 \le k \le n$ then*

$$\sum_{i=0}^{n} \ell(f^{-k}(J)) \le K$$

holds for any arc $J \subset B'$ transversal to the E-direction with endpoints in $\partial^{cu}(B')$. A similar result holds for a cs-subbox in $B(x)$.

2. *there exists $C_2 = C_2(C)$ such that $B^x_{k_n}$ is a cu-adapted cs-subbox in $B(x)$ having distortion C_2.*

Lemma 3.4.4 *Let B' be a cu-adapted cs-subbox of $B(x)$ having distortion C_2. Then, there exist $K_1 > 0$ such that for $z \in B' \cap \Lambda$ we have that*

$$\sum_{j=0}^{n} \ell(f^{-j}(J^{cu}_{B'}(z))) \le K_1$$

whenever $f^{-j}(z) \notin B'$, $1 \le j \le n$.

Before proving Lemma 3.4.3 and 3.4.4, let us show that they imply Proposition 3.1. To prove item 1), notice that if $\hat{q}_n$ is close enough to y, then $B^x_{k_n}$ is a *cs*-subbox. Moreover, it is *cu*-adapted since $B(y)$ is adapted. It remains to prove that $B^x_{k_n}$ is r-adapted. Take $\delta^u_1 = \delta^u_*(\epsilon)$ with $\epsilon = r$ from lemma 3.3.1. If $\hat{q}_n$ is close enough to y, then $B^x_{k_n} \subset B_{k_n} \subset B_{(\delta^s_*, \delta^u_1)}(p)$ and hence $B^x_{k_n}$ is r-adapted. This completes the proof of item 1).

Let us prove item 2). Let C the distortion of $B(y)$, and C_2 from lemma 3.4.3. Also, consider K_0 from lemma 2.2.3, $K_1 = K_1(C_2)$ from lemma 3.4.4, K from lemma 3.4.3 and let $L = \min\{\ell(J_B(z)) : z \in B(y) \cap \Lambda\}$.

Let $r > 0$ be such that

$$r\frac{C_2}{L}\exp(K_0K_1 + K_0K) < \frac{1}{2}.$$

Take $\hat{q}_n$ close to y such that $B^x = B^x_{k_n}$ is r-*cs*-subbox. Let show that any return to this box is hyperbolic. For this purpose, take any $z \in B^x \cap \Lambda$ and let m be the first return of z to the box, i.e., $f^{-m}(z) \in B^x$ and $f^{-i}(z) \notin B^x$ for $0 < i < m$. Notice that $m > k_n$ and $f^{k_n - m}(z) \in B(y)$. Set $h = m - k_n$. We get that

$$\|Df^{-m}_{/F(z)}\| \le \|Df^{-k_n}_{/F(f^{-h}(z))}\| \|Df^{-h}_{/F(z)}\|$$

Using lemma 3.4.4 we get

$$\|Df^{-h}_{/F(z)}\| < \frac{\ell(f^{-h}(J^{cu}_{B^x}(z)))}{\ell(J^{cu}_{B^x}(z))}\exp(K_0K_1)$$

On the other hand, by lemma 3.4.3

$$\|Df^{-k_n}_{/F(f^{-h}(z))}\| \leq \frac{\ell(f^{-k_n}(J^{cu}_{B(y)}(f^{-h}(z))))}{\ell(J^{cu}_{B(y)}(f^{-h}(z)))}\exp(K_0K)$$

and

$$\ell(f^{-k_n}(J^{cu}_{B(y)}(f^{-h}(z)))) = \ell(J^{cu}_{B^x}(f^{-m}(z))).$$

Thus,

$$\begin{aligned}
\|Df^{-m}_{/F(z)}\| &\leq \|Df^{-k_n}_{/F(f^{-h}(z))}\|\|Df^{-h}_{/F(z)}\| \\
&\leq \frac{\ell(f^{-k_n}(J^{cu}_{B(y)}(f^{-h}(z))))}{\ell(J^{cu}_{B(y)}(f^{-h}(z)))}\exp(K_0K_1)\frac{\ell(f^{-h}(J^{cu}_{B^x}(z)))}{\ell(J^{cu}_{B^x}(z))}\exp K_0K \\
&= \ell(f^{-h}(J^{cu}_{B^x}(z)))\frac{\ell(J^{cu}_{B^x}(f^{-m}(z)))}{\ell(J^{cu}_{B^x}(z))\ell(J_{B(y)}(f^{-h}(z)))}\exp(K_0K_1+K_0K) \\
&\leq r\frac{C_2}{L}\exp(K_0K_1+K_0K) < \frac{1}{2}.
\end{aligned}$$

This proves that any cu-return is hyperbolic and finish the proof of Proposition 3.1. It only remains to prove Lemma 3.4.3 and Lemma 3.4.4. We now proceed to do it:

Proof of Lemma 3.4.3:

Claim: *Assume that $\|Df_{/E(p)}\| < \lambda_1 = \lambda^{\frac{1}{2}} < 1$ and let B_k be as in Definition 3.4.1. Then, there exits $C_1 = C_1(C)$ such that, for any k, B_k has distortion C_1.*

Proof of the claim: Let $\mathcal{F}^{cs}_k$ be the foliation in B_k which is the pull-back foliation $\mathcal{F}^{cs}$ in $B(y)$. Let J^k_1 and J^k_2 be two arcs in B_k transversal to the E-direction whose endpoints are in $\partial^{cu}(B_k)$. We have to show that there exists C_1 such that

$$\frac{1}{C_1} \leq \|\Pi'_k\| \leq C_1$$

where Π_k is the projection along $\mathcal{F}^{cs}_k$ between J^k_1 and J^k_2. Notice that $J_1 = f^k(J^k_1)$ and $J_2 = f^k(J^k_2)$ are also two arcs in $B(y)$ transversal to the E-direction with endpoints in $\partial^{cu}(B(y))$. For a point $x \in f^j(J^k_i),\ i = 1, 2$ set $\tilde{F}(x) = T_xf^j(J^k_i),\ 0 \leq j \leq k$.

By the equality

$$\Pi_k \circ f^{-k}_{/J_1} = f^{-k} \circ \Pi$$

we conclude, for $z \in J_1$, that

$$\|\Pi_k'(f^{-k}(z))\|.\|Df^{-k}_{/\tilde{F}(z)}\| = \|Df^{-k}_{/\tilde{F}(\Pi(z))}\|.\|\Pi'(z)\|$$

Hence

$$\|\Pi_k'(f^{-k}(z))\| = \frac{\|Df^{-k}_{/\tilde{F}(\Pi(z))}\|}{\|Df^{-k}_{/\tilde{F}(z)}\|}.\|\Pi'(z)\|$$

Thus, to finish the proof of the lemma it suffices to find M such that

$$\frac{1}{M} \leq \frac{\|Df^{-k}_{/\tilde{F}(\Pi(z))}\|}{\|Df^{-k}_{/\tilde{F}(z)}\|} \leq M$$

which is the same, setting $x = f^{-k}(z)$, as

$$\frac{1}{M} \leq \frac{\|Df^{k}_{/\tilde{F}(x)}\|}{\|Df^{k}_{/\tilde{F}(\Pi_k(x))}\|} \leq M.$$

Observe that for any pair of point z_1, z_2 belonging to the same central leaf of $\mathcal{F}_k^{cs}$, we get that there is a constant $\lambda_2 < 1$ such that

$$dist(f^j(z_1), f^j(z_2)) \leq \lambda_2^j dist(z_1, z_2)$$

for $j \leq k$ and so, given some constant α, there is a constant A such that

$$\Sigma_{i=0}^k \ell(f^j(\mathcal{F}_k^{cs}(x)))^\alpha < A.$$

With the same arguments as in [Sh] pags 45-46, it is possible to prove that there exist $\tau > 0$ and $\alpha > 0$ such that

$$\Big| \|Df_{/\tilde{F}(f^j(w_1))}\| - \|Df_{/\tilde{F}(f^j(w_2))}\| \Big| \leq \eta^j D + dist(f^j(w_1), f^j(w_2))^\alpha$$

for some constant $0 < \eta < 1$ and D whenever $\tilde{F}$ lies in the central unstable cone and $dist(f^j(w_1), f^j(w_2)) \leq \tau,\ 0 \leq j \leq k$. (This is, roughly speaking, a consequence of the fact that the distribution F is α-holder and any other direction converges exponentially fast to F.)

Therefore, if the diameter of $B_{(\delta^s,\delta^u)}(p)$ is less than τ, it follows that

$$\frac{\|Df^{n}_{/\tilde{F}(x)}\|}{\|Df^{n}_{/\tilde{F}(\Pi_k(x))}\|} \leq \exp\left(\frac{D}{1-\eta} + \sum_{j=0}^{j=k} dist(f^j(x), f^j(\Pi_k(x)))^\alpha\right)$$

Since x and $\Pi_k(x)$ belongs to $\mathcal{F}_k^{cs}(x)$, we conclude that

$$\sum_{j=0}^{k} dist(f^j(x), f^j(\Pi_k(x)))^\alpha \leq \sum_{j=0}^{n} \ell(f^j(\mathcal{F}_k^{cs}(x)))^\alpha \leq A.$$

Thus

$$\frac{\|Df^k_{/\tilde{F}(x)}\|}{\|Df^k_{/\tilde{F}(\Pi_k(x))}\|} \leq \exp(\frac{D}{1-\eta} + A).$$

Finally, taking $M = \exp(\frac{D}{1-\eta} + A)$, we have that $C_1 = C.M$ satisfies the claim.

Let us prove item 1) of Lemma 3.4.3. Assume that $\|Df_{/E(p)}\| < \lambda_1$. By the claim, there is some C_1 such that B_k has distortion C_1 and hence

$$\ell(f^{-k}(J)) \leq C_1 \ell(f^{-k}(J^{cu}_{B(y)}(y)))$$

and so

$$\sum_{i=0}^{n} \ell(f^{-i}(J)) \leq C_1 \sum_{i=0}^{n} \ell(f^{-i}(J^{cu}_{B(y)}(y))) \leq C_1 \ell(W^{cu}_{\delta^s_*}(p)) = D_1.$$

On the other hand, assume that $\|Df_{/E(p)}\| \geq \lambda_1 = \lambda^{\frac{1}{2}}$. It follows by the domination that $\|Df^{-1}_{/F(p)}\| < \lambda_1 < 1$. Hence, if the box $B_{(\delta^s,\delta^u)}(p)$ is small enough, we have that $\|Df^{-1}_{/\tilde{F}(z)}\| < \lambda_2 < 1$ for some $\lambda_2 > \lambda_1$ and any $z \in B_{(\delta^s,\delta^u)}(p)$ and any $\tilde{F}$ close the F-direction. Therefore,

$$\sum_{i=0}^{n} \ell(f^{-i}(J)) \leq \sum_{i=0}^{n} \lambda_2^i \ell(J) \leq \frac{C}{1-\lambda_2} \ell(J^{cu}_{B(y)}(y)) = D_2.$$

Setting $K = \max\{D_1, D_2\}$ we conclude the proof of item 1).

Now we proceed to prove item 2) of the Lemma. In case $\|Df_{/E(p)}\| < \lambda_1$ then, by the claim, we conclude setting $C_2 = C_1$. On the other hand, if $\|Df_{/E(p)}\| \geq \lambda_1$ we shall argue as in the proof of the previous claim, together with the fact that F is expansive in a neighbourhood of p. Let $\mathcal{F}^{cs}_{k_n}$ be the foliation in $B^x_{k_n}$ which is the pull-back foliation $\mathcal{F}^{cs}$ in $B(y)$. Let $J_1^{k_n}$ and $J_2^{k_n}$ be two arcs in $B^x_{k_n}$ transversal to the E-direction whose endpoints are in $\partial^{cu}(B^x_{k_n})$. We have to show that there exists C_2 such that

$$\frac{1}{C_2} \leq \|\Pi'_{k_n}\| \leq C_2$$

where Π_{k_n} is the projection along $\mathcal{F}^{cs}_{k_n}$ between $J_1^{k_n}$ and $J_2^{k_n}$. Notice that $J_1 = f^{k_n}(J_1^{k_n})$ and $J_2 = f^{k_n}(J_2^{k_n})$ are also two arcs in $B(y)$ transversal to the E-direction with endpoints in $\partial^{cu}(B(y))$. As in the proof of the claim, it is enough to show that there exists M such that

$$\frac{1}{M} \leq \frac{\|Df^k_{/\tilde{F}(x)}\|}{\|Df^k_{/\tilde{F}(\Pi_k(x))}\|} \leq M.$$

Again, with the same arguments as in [Sh] pages 45-46, and the fact that F is expansive in a neighbourhood of p, it is possible to prove that there exists $\tau > 0$ (and hence assume that the diameter of $B_{(\delta^s,\delta^u)}(p)$ is less than τ) such that

$$\left| \|Df_{/\tilde{F}(f^j(w_1))}\| - \|Df_{/\tilde{F}(f^j(w_2))}\| \right| \leq \eta^j D + dist(f^j(w_1), f^j(w_2))$$

and so

$$\frac{\|Df^{k_n}_{/\tilde{F}(x)}\|}{\|Df^{k_n}_{/\tilde{F}(\Pi_{k_n}(x))}\|} \leq \exp\left(\frac{D}{1-\eta} + \sum_{j=0}^{j=k_n} dist(f^j(x), f^j(\Pi_{k_n}(x))) \right).$$

In order to conclude the proof, we only need to bound the previous sum. From item 1), applied to $B^x_{k_n} \subset B(x)$, we know that there exists K such that

$$\sum_{j=0}^{j=k_n} dist(f^j(x), f^j(\Pi_{k_n}(x))) \leq \sum_{j=0}^{j=k_n} \ell(\mathcal{F}^{cs}_{k_n}(x)) \leq K.$$

Therefore, setting $M = exp(\frac{D}{1-\eta} + K)$ and $C_2 = CM$ we finished the proof that $B^x_{k_n}$ has distortion C_2. This finish the proof of lemma 3.4.3.

Proof of Lemma 3.4.4:

Since $B(x)$ is a well adapted box, there exist a subbox $\hat{B}$ and two disjoint vertical strip S_1, S_2 such that $B(x) - \hat{B} = S_1 \cup S_2$ and S_i is either a domain of a return or $S_i \cap \Lambda = \emptyset$.

Let $z \in B' \cap \Lambda$ as in the hypothesis of the lemma, i.e.: $f^{-j}(z) \notin B'$, $1 \leq j \leq n$ and let $0 < n_1 < n_2 < ... < n_k \leq n$ be the set $\{0 < j \leq n : f^{-j}(z) \in B(x)\}$.

Consider (if exists) the sequence $0 = m_0 < m_1 < m_2 < ... < m_l \leq n$ such that

$$\|Df^j_{/E(f^{-m_i}(z))}\| < \lambda_2^j,\ 0 \leq j \leq m_i,\ \forall i = 1, ..., l$$

We claim that there exists D such that

$$\sum_{i=0}^{l} \ell(f^{-m_i}(J^{cu}_{B'}(z))) \leq D.$$

To prove the claim, assume first that $z \notin S_1 \cup S_2$.

Notice that there exists ϵ_1 such that if $y \in \hat{B} \cap \Lambda$ (i.e., $y \notin S_1 \cup S_2$), then $W^{cs}_{\epsilon_1}(y) \subset B(x)$.

Set $\epsilon_2 = \frac{\epsilon_1}{2}$, $\epsilon_3 = \frac{\epsilon_2}{2}$. For any point $w \in \Lambda$ consider a box (not necessary adapted) $B(w)$ with axis $W^{cu}_\gamma(w)$ and $W^{cs}_{\epsilon_3}(w)$. Since Λ is compact we can cover Λ by a finite number of such boxes. We will denote these ones by $B_1, ..., B_r$. Set $C_3 = \sum_{k=1}^r \ell(W^{cu}_{2\gamma}(w_k))$.

For $1 \leq i \leq l$ let B^{m_i} be a box of axis $W^{cs}_{\epsilon_2}(f^{-m_i}(z))$ and $f^{-m_i}(J^{cu}_{B'}(z))$ contained in $f^{-m_i}(B')$. Notice that (using similar arguments as the proof of the claim in Lemma 3.4.3) there exists C_4 such that all these boxes have distortion C_4.

It follows for $i \neq j$ that

$$B^{m_i} \cap B^{m_j} = \emptyset.$$

Otherwise, we would have (if $i < j$) $f^{-(m_j - m_i)}(z) \in B'$ which is a contradiction since $m_j - m_i \leq n$, or we would contradict the fact that B' is an adapted box as well. Since $B_1, ..., B_r$ covers Λ, we have that $f^{-m_i}(z)$ belong to some of this box, say B_k (if belongs to more than one we choose one in an arbitrary way). Let $J_{m_i} = B^{m_i} \cap W^{cu}_{2\gamma}(w_k)$. It follows that for every i,

$$\frac{1}{C_4} \leq \frac{\ell(f^{-m_i}(J^{cu}_{B'}(z)))}{\ell(J_{m_i})} \leq C_4.$$

Moreover, since $B^{m_i} \cap B^{m_j} = \emptyset$ we conclude that

$$J_{m_i} \cap J_{m_j} = \emptyset.$$

Hence

$$\sum_{i=0}^{l} \ell(f^{-m_i}(J^{cu}_{B'}(z))) \leq \sum_{i=0}^{l} C_4 \ell(J_{m_i}) \leq C_3 C_4.$$

If $S_1 \cap \Lambda = \emptyset$ and $S_2 \cap \Lambda = \emptyset$, we are done. If not, we consider the case such that $z \in S_1 \cup S_2$. Notice that, in this situation S_1 and/or S_2 are domain of return $\psi_r = f^{-k_r}_{/S_r}$ $r = 1, 2$ as in the definition of well adapted box.

Let $i_0 = \min\{i : f^{-n_i}(z) \notin S_1 \cup S_2\}$, and let $j_0 = \min\{j : m_j \geq n_{i_0}\}$. As we did before we can conclude that

$$\sum_{j=j_0}^{l} \ell(f^{-m_j}(J^{cu}_{B'}(z))) \leq C_3 C_4.$$

Fix some $z_1 \in S_1 \cap \Lambda$ and $z_2 \in S_2 \cap \Lambda$.

Take $i < i_0$. Then $f^{-n_i}(z) \in S_1 \cup S_2$ and let $B(n_i)$ be the connected component of $f^{-n_i}(B') \cap (S_1 \cup S_2)$ which contains $f^{-n_i}(z)$. Let us assume, for instance, that $f^{-n_i}(z)$ is in S_1. Then, for every m_j such that $n_i \leq m_j < n_{i+1}$, consider the box $B^{m_j} = f^{-(m_j - n_i)}(B(n_i))$, and $J_{m_j} = B^{m_j} \cap f^{-(m_j - n_i)}(J^{cu}_{B}(z_1))$.

As before, we have that B^{m_j} has distortion C_4 and $B^{m_j} \cap B^{m_k} = \emptyset$ for every $0 \leq m_j, m_k < n_{i_0}$. Thus $J_{m_j} \cap J_{m_k} = \emptyset$. Therefore

$$\sum_{j=0}^{j_0 - 1} \ell(f^{-m_j}(J^{cu}_{B'}(z))) \leq \sum_{j=0}^{j_0 - 1} C_4 \ell(J_{m_j}) \leq 2C_4 M$$

where M is such that $\sum_{j=0}^{k_i} \ell(f^{-j}(J_B^{cu}(z_i))) \leq M$. Set $D = C_3C_4 + 2C_4M$. Then we have

$$\sum_{j=0}^{l} \ell(f^{-m_j}(J_{B'}^{cu}(z))) \leq D,$$

and the claim is proved.

Now, to complete the proof of the lemma, we must control the sum between consecutive $m_i's$ (or when the $m_i's$ do not exist). To do that we use a reformulation of a lemma due to Pliss [Pl] that we include the statement here for the sake of completeness:

Pliss' Lemma: *There exist $N = N(\lambda_1, \lambda_2, f)$ with the following property: given $x \in \Lambda$ such that for some $n \geq N$ we have*

$$\|Df^n_{/E(x)}\| \leq \lambda_1^n$$

then there exist $0 \leq n_1 < n_2 < .... < n_l \leq n$ such that

$$\|Df^j_{/E(f^{n_i}(x))}\| \leq \lambda_2^{j-n_r};\ r = 1, ..., l;\ n_r \leq j \leq n.$$

Similarly, for f^{-1} and the F-direction.

Continuing with the proof of our lemma, consider $M_1 = \sup\{\|Df^j\| : 1 \leq j \leq N\}$. There are two possibilities: $m_{i+1} - m_i < N$ or $m_{i+1} - m_i \geq N$. If $m_{i+1} - m_i < N$, then

$$\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_{B'}^{cu}(z))) \leq NM_1 \ell(f^{-m_i}(J_{B'}^{cu}(z))).$$

On the other hand, if $m_{i+1} - m_i \geq N$, then

$$\|Df^j_{/E(f^{-m_i-j}(z))}\| \geq \lambda_1^j \text{ for } N \leq j \leq m_{i+1} - m_i.$$

Thus, by the dominated splitting,

$$\|Df^{-j}_{/F(f^{-m_i}(z))}\| \leq \lambda_1^j \text{ for } N \leq j \leq m_{i+1} - m_i.$$

Applying Pliss'lemma, there exists $\tilde{n}_i, \tilde{n}_i - m_i < N$ such that

$$\|Df^{-j}_{/F(f^{-\tilde{n}_i}(z))}\| \leq \lambda_2^j \text{ for } 0 \leq j \leq m_{i+1} - \tilde{n}_i$$

and so, for any $y \in f^{-\tilde{n}_i}(J_{B'}^{cu}(z))$ we have for some λ_3, $\lambda_2 < \lambda_3 < 1$, setting $\tilde{F}(y) = T_y f^{-\tilde{n}_i}(J(z))$, that

$$\|Df^{-j}_{/\tilde{F}(y)}\| \leq \lambda_3^j \text{ for } 0 \leq j \leq m_{i+1} - \tilde{n}_i.$$

Hence

$$\begin{aligned}
\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J^{cu}_{B'}(z))) &\leq \sum_{j=m_i}^{\tilde{n}_i-1} \ell(f^{-j}(J^{cu}_{B'}(z))) + \sum_{j=\tilde{n}_i}^{m_{i+1}-1} \ell(f^{-j}(J^{cu}_{B'}(z))) \\
&\leq NM_1 \ell(f^{-m_i}(J^{cu}_{B'}(z))) \\
&\quad + \sum_{j=0}^{m_{i+1}-\tilde{n}_i-1} M_1 \ell(f^{-m_i}(J^{cu}_{B'}(z)))\lambda_3^j \\
&\leq \left(NM_1 + M_1 \frac{1}{1-\lambda_3}\right) \ell(f^{-m_i}(J^{cu}_{B'}(z))).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j=0}^{n} \ell(f^{-j}(J^{cu}_{B'}(z))) &= \sum_i \sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J^{cu}_{B'}(z))) \\
&\leq \left(NM_1 + M_1 \frac{1}{1-\lambda_3}\right) \sum_i \ell(f^{-m_i}(J^{cu}_{B'}(z))) \\
&\leq \left(NM_1 + M_1 \frac{1}{1-\lambda_3}\right) D = M_2.
\end{aligned}$$

Finally, if the sequence $m_i's$ does not exist, the same argument shows that

$$\begin{aligned}
\sum_{j=0}^{n} \ell(f^{-j}(J^{cu}_{B'}(z))) &\leq \left(NM_1 + M_1 \frac{1}{1-\lambda_3}\right) \ell(J^{cu}_{B'}(z)) \\
&\leq \left(NM_1 + M_1 \frac{1}{1-\lambda_3}\right) L = M_3
\end{aligned}$$

where $L = \sup\{\ell(J^{cu}_{B'}(z)) : z \in B' \cap \Lambda\}$. Taking $K = \max\{M_2, M_3\}$ we conclude the proof of Lemma 3.4.4.

The proof of Theorem 3.1 is complete.

4 Proof of Theorem B and Main Theorem

Regarding Theorem A, we know that the period of the non-hyperbolic periodic points are bounded in a set of dominated splitting. This turns out to be very helpful to give a better description of the central stable and unstable manifolds from the dynamic point of view.

4.1 Central stable and unstable manifolds: III

Let p be a non-hyperbolic periodic point. Assume also that it is a F-nonhyperbolic periodic point of saddle-node type. In this case (recall lemma 3.2.1) we have that, for some $\gamma_0 = \gamma_0(p)$, one component of $W^{cu}_{\gamma_0}(p) - \{p\}$, say $W^{cu,+}_{\gamma_0}(p)$, is in fact an unstable manifold, and on the other one, points converges to p in the future or there is a sequence in it of periodic points (in this case, $W^{cu,-}_{\gamma_0}(p)$ is an invariant arc normally attractive). For $\delta^u < \gamma_0$ and δ^s consider the box $B_{(\delta^s,\delta^u)}(p)$ as in Definition 3.3.1. Lets call $B^+_{(\delta^s,\delta^u)}(p)$ the connected component of $B_{(\delta^s,\delta^u)}(p) - W^{cs}_{\delta^s}(p)$ that contains $W^{cu,+}_{\gamma_0}(p)$. Notice that point on the other component converge in the future to a point in $W^{cu,-}_{\gamma}(p)$. Similar properties and notations hold for an E-nonhyperbolic periodic point of saddle-node type.

Lets recall the definition of local stable and unstable *sets*:

$$W^s_\epsilon(x) = \{y \in M : \lim_{n\to+\infty} \mathrm{d}(f^n(x), f^n(y)) = 0, \text{ and } \mathrm{d}(f^n(x), f^n(y)) \le \epsilon \ n \ge 0\}$$

$$W^u_\epsilon(x) = \{y \in M : \lim_{n\to-\infty} \mathrm{d}(f^n(x), f^n(y)) = 0, \text{ and } \mathrm{d}(f^n(x), f^n(y)) \le \epsilon \ n \le 0\}$$

Theorem 4.1 *Let $f : M \to M$ be a C^2 diffeomorphism and Λ be a set having dominated splitting which is also 2-dominated and without closed curves supporting an irrational rotation and having no sink or sink-type periodic points. Assume that Λ has only finitely many non-hyperbolic periodic points and let $p_1, ...p_r$ be the F-saddle-node type non-hyperbolic periodic points, and let $q_1, ..., q_t$ be the E-saddle-node ones. Let $N_1 = N_1(\Lambda)$ from Theorem A and set $N = 2N_1$. Then, given $\epsilon < \delta_0$ there exist $\delta^u = \delta^u(\epsilon), \delta^s = \delta^s(\epsilon)$ and $\gamma = \gamma(\delta^u, \delta^s, \epsilon)$ such that for any $x \in \Lambda$ satisfying that neither $\omega(x)$ nor $\alpha(x)$ is a periodic orbit with period $\le N$, the following hold:*

1. *If $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$ then $W^{cu}_\gamma(x) \subset W^u_\epsilon(x)$.*
2. *If $x \in B^+_{(\delta^s,\delta^u)}(p_i)$ then $J^{cu,+}_{B^+}(x) \subset W^u_\epsilon(x)$*
3. *If $x \notin \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$ then $W^{cs}_\gamma(x) \subset W^s_\epsilon(x)$.*

4. If $x \in B^+_{(\delta^s,\delta^u)}(q_i)$ then $J^{cs,+}_{B^+}(x) \subset W^s_\epsilon(x)$.

Proof: We will prove the theorem only for the central unstable manifolds (i.e. items 1 and 2). We claim first that there exists and admissible compact neighbourhood V of Λ such that if p is a non-hyperbolic periodic point in $\Lambda(V) = \cap_{n\in\mathbb{Z}} f^n(V)$ then its period is $\leq N_1$. Otherwise, there exist a sequence V_n, $\cap_n V_n = \Lambda$ and a sequence of non-hyperbolic periodic points $p_n \in \Lambda(V_n)$ with periods greater than N_1 (but uniformly bounded by Theorem A applied to $\Lambda(V_1)$). Let p be an accumulation point of p_n. Then $p \in \Lambda$ and it is a non-hyperbolic period point (and hence $per(p) \leq N_1$). On the other hand, since p is accumulated by periodic points with bounded periods, it follows that p is either of sink-node or saddle-node type. The former is not possible since Λ does not contain sink-node periodic points. Thus p is of saddle-node type. Then, for large n we have $per(p_n) = per(p) \leq N_1$, a contradiction. This proves our claim.

Assume that $\epsilon \leq \delta_0$ and also that $\{x : dist(x,\Lambda) < \epsilon\} \subset V$. Take $\delta^s = \delta^s_*(\epsilon)$ and $\delta^u = \delta^u_*(\epsilon)$ from Lemma 3.3.1. Assume that they are small enough such that $B_{(\delta^s,\delta^u)}(i) \cap B_{(\delta^s,\delta^u)}(j) = \emptyset$ for $i,j = p_1,\dots,p_r,q_1,\dots,q_t$.

Let $x \in \Lambda$, and assume that $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$. Set $\delta = \delta(\epsilon)$ from Corollary 2.2.1. Let show first that there is some γ such that

$$\ell(f^{-n}(W^{cu}_\gamma(x))) \leq \delta, \quad n \geq 0.$$

If such γ does not exist, then there are sequences $x_n \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$, $\gamma_n \to 0$ and $m_n \to \infty$ such that, for $0 \leq j \leq m_n$,

$$\ell(f^{-j}(W^{cu}_{\gamma_n}(x_n))) \leq \delta$$

and

$$\ell(f^{-m_n}(W^{cu}_{\gamma_n}(x_n))) = \delta$$

Letting $I_n = f^{-m_n}(W^{cu}_{\gamma_n}(x_n))$ we can assume (taking a subsequence if necessary) that $I_n \to I$ and $f^{-m_n}(x_n) \to z$, $z \in \Lambda$, $z \in \overline{I}$ (the closure of I).

Now, we have that $\ell(f^n(I)) \leq \delta \leq \delta_0$ for all positive n, and since $I \subset W^{cu}_\epsilon(z)$, we conclude that I is a δ-E-interval. Thus, $\omega(z)$ is a periodic orbit p because $z \in \overline{I}$. Since $z \in \Lambda$ we conclude that $p \in \Lambda$. We claim that E_p is contractive. Otherwise, we conclude that one of the component of $W^u(p) - \{p\}$ has length less than δ_0, which contradicts (if δ_0 is assumed small enough) Corollary 2.2.2, proving our claim. Hence $z \in W^s(p)$.

If the point p is hyperbolic, we conclude that, at least, one of the components of $W^u(p) - \{p\}$ has length less than ϵ. Thus, in case

$$f^{-m_n}(W^{cu}_{\gamma_n}(x_n)) \cap W^s(p) \neq \emptyset$$

we get a contradiction with the inclination lemma (or λ-lemma, see [P]) because this intersection is transversal and

$$\ell(f^{m_n}(f^{-m_n}(W^{cu}_{\gamma_n}(x_n)))) = \ell(W^{cu}_{\gamma_n}(x_n)) \to 0.$$

On the other hand, if

$$f^{-m_n}(W^{cu}_{\gamma_n}(x)) \cap W^s(p) = \emptyset$$

it follows, for sufficiently large n, that $\omega(f^{-m_n}(x_n))$ is the other endpoint (say q) of the component of $W^u(p) - \{p\}$ having length less than δ. By the lemma 3.3.1 of [PS1], it is a sink or a non-hyperbolic periodic point. This implies that $\omega(f^{-m_n}(x_n)) = \omega(x_n) = q$ for n large enough, and by theorem A the period of q is $\leq N_1$ contradicting the assumption in the theorem.

In case p is a non-hyperbolic period point of saddle type, the same argument as before also applies and we also get a contradiction. Assume now that $p = p_i$ is a saddle node. Then, it follows that $f^{m_n}(f^{-m_n}(x_n))$ is arbitrarily near p and so $x_n \in B_{(\delta^s,\delta^u)}(p_i)$ contradicting our assumption that $x_n \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$.

To finish the proof of item 1) it remains to prove that

$$\ell(f^{-n}(W^{cu}_{\gamma}(x))) \to 0$$

for $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$. Arguing by contradiction, assume that this is not the case. Then, there exist $\eta > 0$ and a sequence $n_k \to \infty$ such that

$$\ell(f^{-n_k}(W^{cu}_{\gamma}(x))) > \eta$$

for some $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$

Letting $I_{n_k} = f^{-n_k}(W^{cu}_{\gamma}(x))$ we can assume that $I_{n_k} \to I$ and $f^{-n_k}(x) \to z \in \overline{I}, z \in \Lambda$. As we did above, we get that I is a δ_0-E-interval, and so $\omega(z)$ is a periodic point $p \in \Lambda$. Again, $E(p)$ must be contractive.

Assume that the point p is hyperbolic. If $z \in int(I)$, then, since I is transversal to $W^s(p)$, it follows, by the inclination lemma, that $\ell(W^u(p)) \leq \delta$ and hence the endpoints q_1, q_2 of $W^u(p)$ are not hyperbolic periodic points of saddle type. Therefore, they have periods less than N_1 and so the period of p is at most $2N_1 = N$. On the other hand, for large n, $\omega(f^{-m_n}(x)) = \omega(x) \subset \{p, q_1, q_2\}$, a contradiction.

On the other hand, if $z \notin int(I)$, again, the inclination lemma implies that one of the components of $W^u(p) - \{p\}$ has length less than δ. As we did above, the case

$$f^{-n_k}(W^{cu}_{\gamma}(x)) \cap W^s(p) = \emptyset$$

leads to a contradiction. So

$$f^{-n_k}(W^{cu}_{\gamma}(x)) \cap W^s(p) \neq \emptyset.$$

Using the inclination lemma, the fact that $\ell(f^j(f^{-n_k}(W^{cu}_\gamma(x)))) \leq \delta,\ 0 \leq j \leq n_k$ together with $f^{-n_k}(x) \to z$ imply that $x \in W^u(p)$. Therefore $\alpha(x) = p$, a periodic point with period $\leq N$, which is, as before, a contradiction.

Now assume that p is not hyperbolic. It follows that p can not be either sink or sink-node type, otherwise (as before) $\omega(x)$ is a periodic point of period less than N. Thus, p is saddle-type or saddle-node type. The case saddle type is similar to the hyperbolic case that we've discussed above. So, assume that p is of saddle-node type. It follows that $z \in W^{ss}(p)$. In case $z \in int(I)$ it is not difficult to see that again $\omega(x)$ is a periodic point with period less than N_1, a contradiction. Also, arguing as before, in case $z \notin int(I)$ we get that $\omega(x)$ is periodic point with period less than N_1, or $\alpha(x) = p$. Both cases lead to a contradiction.

To prove item 2), notice that, by lemma 3.3.1, we know that

$$\ell(f^{-n}(J^{cu,+}_{B^+}(x))) \leq \delta(\epsilon)$$

To finish the proof of the theorem remains to prove that

$$\ell(f^{-n}(J^{cu,+}_{B^+}(x))) \to 0.$$

The arguments are very similar to those we've already done and so we leave to the reader the proof of it.

■

Remark 4.1.1 *It is not assumed in the previous theorem that $\Lambda \subset \Omega(f)$.*

Corollary 4.1.1 *Let Λ be as in Theorem 4.1. Then, there exists $\eta > 0$ such that if z_1 and z_2 are two hyperbolic periodic point inΛ with period greater than N and $d(z_1, z_2) < \eta$ then there is a transverse intersection between $W^s(z_1)$ and $W^u(z_2)$.*

Proof: Let ϵ be small and take δ^s, δ^u and $\gamma(\delta^u/2, \delta^s/2, \epsilon)$ from the previous theorem. Let $\eta > 0$ be such that for any $x, y \in \Lambda$ and $d(x, y) < \eta$ then $W^{cu}_\gamma(x)$ and $W^{cs}_\gamma(y)$ have a (unique) non-empty transverse intersection. Moreover, assume that η is small enough so that if $x \in B_{(\delta^s/2,\delta^u/2)}(p_i)$ for some i then $y \in B_{(\delta^s,\delta^u)}(p_i)$, the same for the points q_i.

Let z_1 and z_2 be as in the statement of the lemma. In case

$$z_1, z_2 \notin \cup_{i=1}^r B_{(\delta^s/2,\delta^u/2)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s/2,\delta^u/2)}(q_i)$$

then $W^{cs}_\gamma(z_1)$ and $W^{cu}_\gamma(z_2)$ have a nonempty intersection and since $W^{cs}_\gamma(z_1) \subset W^s(z_1)$ and $W^{cu}_\gamma(z_2) \subset W^u(z_2)$ the result follows. On the other hand, if $z_1 \in B_{(\delta^s/2,\delta^u/2)}(q_i)$ (or $z_2 \in B_{(\delta^s/2,\delta^u/2)}(p_i)$) then $z_1, z_2 \in B_{(\delta^s,\delta^u)}(q_i)$ (respec. $z_1, z_2 \in B_{(\delta^s,\delta^u)}(p_i)$). Since the periods of z_1 and z_2 are greater than N, it follows that $z_1, z_2 \in B^+_{(\delta^s,\delta^u)}(q_i)$ (respec. $z_1, z_2 \in B^+_{(\delta^s,\delta^u)}(p_i)$) and the result follows from the theorem.

■

An important consequence of Theorem 4.1 is that the central stable and unstable manifolds are locally unique (or coherent):

Lemma 4.1.1 *Let Λ be a set having dominated splitting and let $\epsilon > 0$. Assume that for some $x \in \Lambda$ there is some γ such that $W^{cs}_\gamma(x) \subset W^s_\epsilon(x), n \geq 0$. Then, if W is any 1-submanifold containing x, $T_xW = E(x)$ and such that $W \subset W^s_\epsilon(x)$ we get that $W \cap W^{cs}_\gamma(x)$ is relatively open for both W and $W^{cs}_\gamma(x)$.*

Proof: We'll just sketch the proof, details are left to the reader. We may assume (taking a forward iterate of x if necessary) that ϵ is arbitrarily small. If $W \cap W^{cs}_\gamma(x)$ is not relatively open, then there are points $z \in W, y \in W^{cs}_\gamma(x)$, $z \neq y$ such that they can be joined by an arc $A = a(z, y)$ transversal to the E-direction. Moreover, $dist(z, y) \approx \ell(a(z, y))$. Since the forward iterate of an arc transversal to the E-direction is also tranversal to the E-direction (as long as remains of small length), we conclude that $A_n = f^n(a(z, y))$ is an arc transversal to the E-direction joining $f^n(z)$ and $f^n(y)$ and such that $\ell(f^n(a(z, y)) \approx dist(f^n(z), f^n(y))$.

Pick $c > 0$ so that $(1 + c)^2\lambda < 1$. Then, if ϵ is small enough, for any large n, there is $w \in A_n = f^n(a(z, y))$ such that

$$\begin{aligned}
\ell(A) &= \|Df^{-n}_{/T_wA_n}\|\ell(A_n) \leq (1 + c)^n\|Df^{-n}_{/F(f^n(x))}\|\ell(A_n) \\
&\leq (1 + c)^n\|Df^{-n}_{/F(f^n(x))}\|(K\ell(f^n(W)) + K\ell(f^n(W^{cs}_\gamma(x)))) \\
&\leq (1 + c)^n\|Df^{-n}_{/F(f^n(x))}\|(1 + c)^n\|Df^n_{/E(x)}\|K(\ell(W) + \ell(W^{cs}_\gamma(x)))
\end{aligned}$$

Therefore, by the domination, we get:

$$0 < \frac{\ell(A)}{2K \max\{\ell(W), \ell(W^{cs}_\gamma(x))\}} \leq (1 + c)^{2n}\lambda^n \to_{n\to\infty} 0,$$

a contradiction.

■

Let $\beta > 0$ and Λ be a set having dominated splitting. We denote by Λ_β the maximal invariant set in a β-neighbourhood of Λ, i.e. $\Lambda_\beta = \cap_{n\in\mathbb{Z}} f^n(\{x : d(x, \Lambda) \leq \beta\})$. Also, denote by Λ^+_β (Λ^-_β) the set of points that remains in $\{x : d(x, \Lambda) \leq \beta\}$ under positive (respec. negative) iteration.

Theorem 4.2 *Let $\epsilon > 0$ and let $\Lambda, p_1, ..p_r, q_1, ..., q_t$, N, $\delta^u = \delta^u(\epsilon), \delta^s = \delta^s(\epsilon)$ be as in Theorem 4.1. Then, there exists β_0 such that for any $0 < \beta < \beta_0$ the following hold:*

1. *If p is a non-hyperbolic periodic point in $\Lambda_\beta \backslash \Lambda$ then $p \in B_{(\delta^s,\delta^u)}(w)$ for some $w = p_1, ..p_r, q_1, ..., q_t$ and $per(p) \leq N$.*

2. *There exists $\gamma = \gamma(\delta^s, \delta^u, \epsilon)$ such that for any y which neither $\omega(y)$ nor $\alpha(y)$ is a periodic orbit with period $\leq N$ the following is true:*

   (a) *if $y \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i)$ and $y \in \Lambda_\beta^-$ then $W_\gamma^{cu}(y) \subset W_\epsilon^u(y)$.*

   (b) *if $y \notin \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$ and $y \in \Lambda_\beta^+$ then $W_\gamma^{cs}(y) \subset W_\epsilon^s(y)$.*

3. *For $x \in \Lambda$ such that $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$ and y satisfies that $d(f^n(x), f^n(y)) < \beta$ for $n \geq 0$ ($n \leq 0$) the following is true:*

   (a) *if $\omega(x)$ (respec. $\alpha(x)$) in none of the periodic points $p_1, ..., p_r, q_1, ..., q_t$ then $y \in W_\gamma^{cs}(x)$ (respec. $W_\gamma^{cu}(x)$).*

   (b) *if $\omega(x)$ is one of the points p_i, q_i and x, y are non-wandering then $y \in W_\gamma^{cs}(x)$ (respec. $W_\gamma^{cu}(x)$).*

Proof: (1) follows as the beginning of the proof of Theorem 4.1. Indeed, arguing by contradiction, assume that there for $\beta_n \to 0$ there exist non-hyperbolic periodic points $p_n \in \Lambda_{\beta_n} \backslash \Lambda$ and $p_n \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$. Let p be an accumulation point of the sequence p_n. It follows that $p \in \Lambda$ is a non-hyperbolic periodic point of saddle-node type but $p \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$, a contradiction.

The proof of item (2) goes along the same lines as the proof of Theorem 4.1 and we leave it to the reader.

Lets prove item (3a): let $\gamma = \gamma(\delta^s, \delta^u, \epsilon)$ be as item (2). Notice, from corollary 2.2.1, that there exits η such that, for any $w \in \Lambda$, if $z \in W_\gamma^{cu}(w)$ but $f^{-1}(z) \notin W_\gamma^{cu}(f^{-1}(w))$ then $d(z, w) > \eta$. Let $\gamma_1 < \gamma(\delta^s, \delta^u, \eta/2)$, i.e., for any $z \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$ and $z \in \Lambda_\beta^+$ then $\ell(f^n(W_{\gamma_1}^{cs}(w))) \leq \eta/2$, $n \geq 0$. We may assume that β_0 is small enough so that $\beta < \eta/2$ and such that if $d(z, w) < \beta_0, z \in \Lambda, w \in \Lambda_\beta^+$ then $W_{\gamma_1}^{cu}(z)$ and $W_{\gamma_1}^{cs}(w)$ have a non-empty intersection. Let x and y be as in the statement item (3a). Since $\omega(x)$ is not none of the points p_i or q_i, it follow that there is a sequence $n_k \to \infty$ such that $f^{n_k}(x) \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$. Now, assume that the conclusion is not true. Then, the point $z = W_{\gamma_1}^{cs}(y) \cap W_{\gamma_1}^{cu}(x)$ is different from x (otherwise, $x \in W_{\gamma_1}^{cs}(y)$ and follows that $T_x W_{\gamma_1}^{cs}(y) = E(x)$ and by Lemma 4.1.1 this would imply that $y \in W_{\gamma_1}^{cs}(x)$ as we wish). Hence, for n_k large enough, we get $z \notin f^{-n_k}(W_\gamma^{cu}(f^{n_k}(x)))$. Let $m > 0$ be the least positive integer such that

$$z \notin f^{-m}(W_\gamma^{cu}(f^m(x))).$$

Therefore,

$$\begin{aligned} \beta &> d(f^{m-1}(x), f^{m-1}(y)) \geq \\ &\geq d(f^{m-1}(x), f^{m-1}(z)) - d(f^{m-1}(z), f^{m-1}(y)) \geq \\ &\geq \eta - \eta/2 > \beta, \end{aligned}$$

a contradiction.

The proof of item (3b) is very similar. Assume that $\omega(x) = p_i$. Since x is non-wandering then $x \in W^{ss}(p_i)$. Using the same notation as above, if $z \in f^{-m}(W^{cu}_\gamma(f^m(x)))$ for any positive integer m it follows, for m large enough, that $f^m(z)$ (and so $f^m(y)$) belongs to $B^-_{(\delta^s,\delta^u)}(p_i)$. But all the points in B^- are wandering and so will be y, a contradiction.

$\blacksquare$

Corollary 4.1.2 *Let Λ be as in Theorem 4.1 and let $x \in \Lambda$ be such that neither $\omega(x)$ nor $\alpha(x)$ is a periodic orbit with period less than N. Then there are $\beta(x), \gamma(x), \epsilon$ such that $W^s_\beta(x) \subset W^{cs}_\gamma(x) \subset W^s(x)$ and so $W^s(x)$ is a smooth manifold tangent to $E(x)$ at x. If $\omega(x)$ is a periodic orbit p of period $\leq N$ and p is of saddle type (hyperbolic or not) then $W^s(x)$ is also a smooth manifold tangent to $E(x)$ at x.*

4.2 Proof of Theorem B

We say that a compact invariant set $\Lambda \subset L(f)$ admits a *spectral decomposition* if $\Lambda = \Lambda_1 \cup ... \cup \Lambda_r$ such that Λ_i, $i = 1, ..., r$ are compact invariant and transitive sets and $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$ and furthermore $\Lambda_i = \Lambda_{i_1} \cup ... \cup \Lambda_{i_{n_i}}$ such that $f(\Lambda_{i_j}) = \Lambda_{i_{j+1}}(\mathrm{mod}(n_i))$ and $f^{n_i}_{/\Lambda_{i_j}}, j = 1, ..., n_i$ is topologically mixing. The sets Λ_i are called basic pieces and lets call the sets Λ_{i_j} the subbasic pieces.

On the other hand, recall that for a hyperbolic periodic point p of saddle type, the *Homoclinic class* of p is defined as

$$H_p = \overline{\{W^s(p) \pitchfork W^u(p)\}}.$$

In order to prove Theorem B we have to show that for some positive integer N the following holds:

1. $\overline{Per^N_h(f)}$ has at most finitely many non-hyperbolic periodic points.

2. $\overline{Per^N_h(f)}$ admits a spectral decomposition such that the subbasic pieces are homoclinic classes.

3. $f_{/\overline{Per^N_h(f)}}$ is expansive.

Let us prove item (1): since $\overline{Per_h(f)}$ has dominated splitting, by theorem A we conclude that there exists some N_1 such that any non-hyperbolic point in $\overline{Per_h(f)}$ has period at most N_1. Set $N > 2N_1$ and lets show that the number of non-hyperbolic periodic points in $\overline{Per^N_h(f)}$ is finite. Arguing by contradiction, assume that this is not the case, and let p_n be a sequence of non-hyperbolic periodic points. Take q an accumulation of this sequence. Since the periods of p_n are bounded by N_1, it follows that q is a periodic point (and a non-hyperbolic one). Since the points p_n are accumulated by periodic points with periods greater than $N = 2N_1$, using corollary 3.2.1, we conclude that q must be of saddle-node type (say F-saddle node). In this case the sequence p_n, for large n, belongs $W^{cu,-}_{\gamma}(p)$, and also any point in $B^-(p)$ is asymptotic to a periodic point in $W^{cu,-}$ with period at most $2N_1$. This is a contradiction, because $p_n \in \overline{Per^N_h(f)}$. This completes the proof of item (1).

To prove item (2) notice that, taking $N > 2N_1$ large enough, we may (and will) assume, by Theorem 2.2 and Lemma 2.2.1, that the dominated splitting over $\overline{Per^N_h(f)}$ is also 2-dominated. In particular, $\Lambda = \overline{Per^N_h(f)}$ satisfies the hypothesis of Theorem 4.1.

Now, we shall proceed as in the hyperbolic case. For $p, q \in Per^N_h(f)$ define the equivalence relation

$$p \sim q \text{ iff } W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Denote by $H(p)$ the closure of the equivalent class of p. We claim that, for $p, q \in Per^N_h(f)$ we have $H(p) \cap H(q) = \emptyset$ or $H(p) = H(p)$. Assume that $H(p) \cap H(q) \neq \emptyset$ and let x be a point in this intersection. Let $p_n \in H(p)$ and $q_n \in H(q)$ be two sequences of periodic points converging to x. From corollary 4.1.1, since p_n and q_n have periods $\geq N$, we get that the stable and unstable manifolds of p_n and q_n have, for sufficiently large n, non-empty intersection. Thus, $p \in H(q)$ and $q \in H(p)$ and so $H(p) = H(q)$. The same argument also shows that there are only finitely many equivalent classes. The rest of the proof of the spectral decomposition is similar to the hyperbolic case (see, for example [Sh]). Indeed, the subbasic pieces are exactly the sets $H(p)$ (the closure of the equivalent class defined above) and it is well known that $H(p)$ coincides with the homoclinic class H_p.

It remains to prove item (3), that is, $f_{/\overline{Per^N_h(f)}}$ is expansive. We have to show that there exists $\alpha >$ such that if $x, y \in \overline{Per^N_h(f)}$ and $d(f^j(x), f^j(y)) \leq \alpha \ \forall j \in \mathbb{Z}$ then $x = y$. We shall argue by contradiction. Assume that f is not expansive, i.e., there exist $\alpha_n \to 0$ and $x_n, y_n \in \overline{Per^N_h(f)}$, $x_n \neq y_n$ such that $d(f^j(x_n), f^j(y_n)) \leq \alpha_n \ \forall j \in \mathbb{Z}$.

Recall that there are only finitely many nonhyperbolic periodic points in $\overline{Per^N_h(f)}$, and hence finitely many periodic point with periods less than N. Moreover there are a small neighbourhoods of these periodic points such that any other orbit in $\overline{Per^N_h(f)}$, must leave them in the future or in the past. Therefore, we may assume that x_n is not one of these periodic points, and there is no loss of generality if we assume that x_n is far from the nonhyperbolic periodic points, that is,

$$x_n \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i)$$

where p_i and q_i are saddle-node periodic points in $\overline{Per_h^N(f)}$. Let β be as in Theorem 4.2. Then, from item (3a) and (3b) of that theorem and $\alpha_n < \beta$ we conclude that

$$y_n \in W_\gamma^{cs}(x_n) \cap W_\gamma^{cu}(x_n) = x_n,$$

a contradiction. This completes the proof of Theorem B.

4.3 Isolated periodic points

Through this subsection, we shall assume that $L(f)$ has dominated splitting, and let λ the constant of domination and choose $1 > \lambda_2 > \lambda_1 = \lambda^{\frac{1}{2}}$. Since the periodic points are in $L(f)$ we know, from Theorem B, that for some N, $\overline{Per_h^N(f)}$ admits a spectral decomposition.

Definition 4.3.1 *We say that a periodic point p is $\Omega\backslash P$-isolated if it is an interior point of $Per(f)\backslash\{\Omega(f) - Per(f)\}$, that is, if there is a neighbourhood U_p such that $U_p \cap \Omega(f) \subset Per(f)$.*

Lemma 4.3.1 *The periods of the $\Omega\backslash P$ isolated points are bounded.*

Proof: If the period of a $\Omega\backslash P$ isolated point p is greater that N, then $p \in \overline{Per_h^N(f)}$. Moreover, p is an isolated basic piece of $\overline{Per_h^N(f)}$. Since there are finitely many basic pieces, we conclude the proof.

■

Let $\Lambda = L(f)$ and consider V and admissible neighbourhood and let U another neighbourhood such that $U \subset \overline{U} \subset V$. Let Λ_1, Λ^+, and Λ^- as in the beginning of section 2.4.

Definition 4.3.2 *Let I be a closed arc or a simple closed curve. We say that $I \subset \Lambda^+$ is an (E, P_ϵ)-arc (or (E, P_ϵ) simple closed curve) if for some m we have $f^m(I) \subset I, f^j(I) \cap I = \emptyset$ $1 \le j < m$, is transversal to the E direction and the periodic points in I are ϵ-dense, that is, any subinterval of I of length ϵ contains a periodic point. We call $m \ge 1$ the period of such an arc. In a similar way, we define (F, P_ϵ)-arcs and (F, P_ϵ) simple closed curve.*

Lemma 4.3.2 *There exists $\epsilon, C > 0$ and $\lambda_3 < 1$ such that if I is an (E, P_ϵ)-arc or simple closed curve then it is (C, λ_3)-normally hyperbolic (attractive), i.e., for each $x \in I$, $T_xM = E \oplus T_xI$ and $\|Df^j_{/E_x}\| < C\lambda_3^j$, $j \ge 0$.*

Proof: First notice that there exists N_1 such that if $I \subset \Lambda^+$ is an arc (or simple closed curve not supporting an irrational rotation) transversal to the E-direction and $f^m(I) \subset I, f^j(I) \cap I = \emptyset, 1 \le j < m$ then $m \le N$.

Indeed, let N_1 be from Theorem A applied to $\Lambda = L(f)$. Since $f^m(I) \subset I$ then I contains a periodic point. Moreover, for any periodic point $p \in I$, $T_pI = F(p)$ holds since I is transversal to the E-direction. Also, not all the periodic points in I can be repelling (within I), and hence there is a periodic point $p \in I$ that is either a sink or not hyperbolic of saddle type, and therefore the period of p is bounded by N_1. Since $m \leq per(p)$ (in fact $m = per(p)$ if I is an arc or $per(p)$ is multiple of m if I is a simple closed curve not supporting an irrational rotation), we conclude that $m \leq N_1$. Furthermore, it follows that the period of any periodic point in I is bounded by $N = 2N_1$ (indeed they are $\leq N_1$ unless $f^m : I \to I$ reverse orientation in I).

Let ϵ be such that if p is a periodic point of period less than N and $\|Df^{-m}_{/F(p)}\| \leq \lambda_1^m$ (m is the period of p) then $\ell(W^{uu}(p)) > 2\epsilon$. Assume also that ϵ is such if $x, y \in \Lambda^+$, $d(x,y) < \epsilon$ and $\|Df^m_{/E(x)}\| \leq \lambda_1^m$ then $\|Df^m_{/E(y)}\| \leq \lambda_2^m$.

Let I be an (E, P_ϵ) arc or simple closed curve. Let p be a periodic point in I. Since the periodic points in I are ϵ-dense, it follows that $\|Df^{-m}_{/F(p)}\| \geq \lambda_1^m$. Otherwise, $W^{uu}(p) \subset I$ and $\ell(W^{uu}(p)) \geq 2\epsilon$, and hence some component of $W^{uu}(p) - \{p\}$ has length greater than ϵ and does not contains a periodic point, contradicting that the periodic points in I are ϵ-dense.

By the domination, we conclude that

$$\|Df^m_{/E(p)}\| \leq \lambda_1^m.$$

Since for any point $x \in I$ there is a periodic point p such that $d(x,p) \leq \epsilon$ we get that

$$\|Df^m_{/E(x)}\| \leq \lambda_2^m$$

This, together with the periodicity of I, imply that there exist $C > 0$ and $\lambda_3 < 1$ such that

$$\|Df^j_{/E(x)}\| \leq C\lambda_3^j, \;\; j \geq 0.$$

$\blacksquare$

Remark 4.3.1 *A similar result holds for (F, P_ϵ) arc or simple closed curve.*

Corollary 4.3.1 *If I is a (E, P_ϵ) arc or simple closed curve (ϵ from the previous lemma) then for any $x \in I$ there is a local strong stable manifold $W^{ss}_{loc}(x)$ which is tangent to E at x of uniform size. Moreover any point in $W^{ss}_{loc}(I) = \cup_{x \in I} W^{ss}_{loc}(x)$ is asymptotic to a periodic point in I. In particular any periodic point p in the interior of I is $\Omega \backslash P$ isolated.*

Lemma 4.3.3 *There exist $\eta > 0$ such that if p and q are $\Omega \backslash P$ isolated and $d(p,q) < \eta$ then p and q belong to either a (E, P_ϵ) or (F, P_ϵ) arc or simple closed curve.*

Proof: Assume that the conclusion of the lemma is false. Then, there exist sequences $\eta_n \to 0$, p_n, q_n $\Omega \backslash P$-isolated periodic point with $d(p_n, q_n) < \eta_n$ and p_n, q_n does not belong to any such arc or simple closed curves.

By lemma 4.3.1, the period of p_n and q_n are bounded. We may assume (taking a subsequence if necessary) that p_n and q_n converges. They must converges to a periodic point p, and since the periods of p_n and q_n are bounded, p is a non-hyperbolic periodic point. Assume that p is F-nonhyperbolic (the other case is similar). From corollary 3.2.1 we conclude that p is of saddle-node type or sink-node type. In any case we conclude that, given $\gamma < \epsilon$, for large n, p_n and q_n belongs to $W^{cu}_{\gamma}(p)$. If p_n, q_n accumulate to p at one branch of $W^{cu}_{\gamma}(p)$ then, take some p_{n_0} in this branch and it follows that for large n, p_n, q_n belongs to the arc in this branch determined by p_{n_0} and p. This arc is an (E, P_ϵ)-arc, which contradicts our assumption. If p_n and q_n accumulate to p from different branches of $W^{cu}_{\gamma}(p)$ then take the arc determined by p_{n_0} and q_{n_0} and again we get a contradiction. This completes the proof of the lemma.

∎

Definition 4.3.3 *We say that a (E, P_ϵ) arc is maximal if it is not a proper sub-arc of a (E, P_ϵ)-arc or simple closed curve. Similarly, define (F, P_ϵ) maximal arc.*

Notice that any (E, P_ϵ) arc which is not contained in a (E, P_ϵ) simple closed curve, it is contained in a (unique) maximal (E, P_ϵ)-arc. Moreover any two maximal (E, P_ϵ) arc are disjoint or coincide.

Lemma 4.3.4 *There are finitely many maximal (E, P_ϵ) $((F, P_\epsilon))$ arcs and (E, P_ϵ) $((F, P_\epsilon))$ simple closed curves.*

Proof: Assume that the assertion of the lemma is not true, that is, there are infinitely many distinct maximal (E, P_ϵ) arc I_n. For each n, choose a periodic point $p_n \in I_n$. Then there exist n_1 and n_2 such that $d(p_{n_1}, p_{n_2}) < \eta$ (η from lemma 4.3.3). Therefore $I_{n_1} \cap I_{n_2} \neq \emptyset$, a contradiction.

∎

We are able to state the main consequence of the results in this section. Lets denote by $\mathcal{I}$ the set of $\Omega \backslash P$ isolated periodic points.

Theorem 4.3 *Assume that $L(f)$ has dominated splitting and let $\mathcal{I}$ be the set of $\Omega \backslash P$ isolated periodic points. Then the periods of the periodic points in $\mathcal{I}$ is bounded and $\mathcal{I}$ is a subset of disjoint union of periodic points and normally hyperbolic arcs and closed curves, i.e.*

$$\mathcal{I} \subset \Gamma_1 \cup \ldots \cup \Gamma_r$$

where $\Gamma_i, i = 1, ..., r$ *is compact invariant and* $\Gamma_i \cap \Gamma_j = \emptyset, i \neq j$. *Moreover* Γ_i *is a periodic point or a normally hyperbolic (attractive or repelling) 1-dimensional manifold (closed arc or simple closed curve).*

Proof: Let η be from lemma 4.3.3. Lets say that a periodic point p is η-isolated if $B_\eta(p) \cap \Omega(f) = \{p\}$. It is not difficult to see that the are finitely many η-isolated points. From lemma 4.3.3 it follows that if a periodic point is not η-isolated then it is in a normally hyperbolic ((E, P_ϵ)or (F, P_ϵ)) arc or simple closed curve. By lemma 4.3.4 there are finitely many of such arcs or closed curves, and they are disjoint or coincide.

■

4.4 Proof of Main Theorem

Recall that $\mathcal{I}$ denotes the set of $\Omega \backslash P$-isolated periodic points. It follows from the definition that $\mathcal{I}$ is open in $L(f)$ (but may not be compact). On the other hand, lets denote by $\mathcal{R}$ the set of periodic simple closed curves normally hyperbolic supporting an irrational rotation. Notice that these curves are isolated, i.e., they are open in $L(f)$. We define $\tilde{L}(f)$ the complement in $L(f)$ of $\mathcal{I}$ and $\mathcal{R}$, that is

$$\tilde{L}(f) = L(f) \backslash (\mathcal{I} \cup \mathcal{R}).$$

It follows that $\tilde{L}(f)$ is compact an invariant. Notice that in this definition it is not assume that $L(f)$ has dominated splitting.

From now on, assume that $L(f)$ has dominated splitting and lets prove our Main Theorem. In section 3.1 of [PS1] it is showed that there are finitely many curves in $\mathcal{R}$. Also, in Theorem 4.3 it is proved the decomposition claimed on $\mathcal{I}$.

On the other hand, we have that $Per(f) \subset L(f)$ and so $\overline{Per_h(f)}$ has dominated splitting. From Theorem B we conclude that there is some N such that $\overline{Per_h^N(f)}$ has spectral decomposition where each subbasic piece is the homoclinic class of a hyperbolic periodic point, contains at most finitely many non-hyperbolic periodic points and moreover $f_{/\overline{Per_h^N(f)}}$ is expansive.

Therefore, to finish the proof of the Main Theorem is enough to show that $\tilde{L}(f) \subset \overline{Per_h^N(f)}$.

Lemma 4.4.1 *Let* $p \in \tilde{L}(f)$ *be a periodic point. Then there exists a sequence of periodic points* p_n *converging to* p *whose periods are unbounded, i.e.,* $p \in \overline{Per_h^M(f)}$ *for any positive integer* M.

Proof: Consider a box $B_{(\delta^s,\delta^u)}(p)$. Since $p \in \tilde{L}(f)$ it follows that it is not $\Omega \backslash P$-isolated and there is a sequence of points $y_n \in \tilde{L}(f)$ converging to p. This sequence converge to p through a non-isolated quadrant $B^i_{(\delta^s,\delta^u)}(p)$. Moreover, there is no loss of generality if we assume that $y_n \in \omega(x_n)$ for some point x_n Let $W^{cs,+}_{\delta^s}(p)$ and $W^{cu,+}_{\delta^u}(p)$ be the branches that bounds this non-isolated quadrant and recall that (no matter p is hyperbolic or not) they are included in the stable and unstable manifold of p respectively. Consider V a compact admissible neighbourhood of $L(f)$ and let L_1 be the maximal invariant set in V and L_1^+ the points that remains in V in the future. Set $x = x_n$ for n large enough. There is some positive integer m_0 so that $f^m(x) \in L_1^+$ for any $m \geq m_0$. Now, we may find $m_0 < m_1 < m_2$ such that $f^{m_j}(x) \in B^i_{(\delta^s,\delta^u)}(p)$, $j = 1, 2$. Let $w_1 = W^{cu,+}_{\delta^u}(p) \cap J^{cs}_{B_i(p)}(f^{m_1}(x))$. It follows that $f^{m_2-m_1}(w_1) \in B_i(p)$ therefore the point $w_2 = W^{cs,+}_{\delta^s}(p) \cap J^{cu}_{B_i(p)}(f^{m_2-m_1}(w_1))$ is a point of transverse intersection of the stable and unstable manifold of the point p. The result follows from standard arguments about the existence of periodic orbits associated to transversal homoclinic orbits.

■

Now, to prove that $\tilde{L}(f) \subset \overline{Per^N_h(f)}$, it is enough to prove that for any x such that $\omega(x)$ is not contained in $\mathcal{I} \cup \mathcal{R}$, $\omega(x) \subset \overline{Per^N_h(f)}$ holds.

Let $z \in \omega(x)$. We shall prove that $z \in \overline{Per^N_h(f)}$. In order to prove this, assume first that $\omega(z)$ contains a periodic point p. In particular p is not $\Omega\backslash P$-isolated, and hence, no matter p is hyperbolic or not, by the previous lemma, there are hyperbolic periodic points q_m arbitrarily near p. On the other hand, there is a sequence $n_k \to \infty$ such that $f^{n_k}(z) \to p$. Consider V a compact admissible neighbourhood of $L(f)$ and let L_1 and L_1^+ be as above. In particular, there is some m_0 such that $f^{m_0}(x) \in L_1^+$. Notice that if q_m is close enough to p and n_k is large enough then, there is a transversal intersection w_1 between the local stable manifold of q_m and the local unstable manifold of $f^{n_k}(z)$. On the other hand, since $f^m(x) \in L_1^+$ for $m \geq m_0$ then, there is a nonempty transversal intersection w_2 between the local unstable manifold of q_m and the local stable one of $f^{m_k}(x)$. Notice that $w_1, w_2 \in L_1$. So, $f^{-n_k}(w_1)$ is close to z and for some $l > 0$, $f^l(w_2)$ is close also to z. In particular $W^{cu}_\gamma(f^l(w_2))$ and $W^{cs}_\gamma(f^{-n_k}(w_1))$ have a transverse intersection (near z). Since $W^{cu}_\gamma(w_2) \subset W^u(q_m)$ and $W^{cs}_\gamma(w_1) \subset W^s(q_m)$ we conclude that the stable and unstable manifolds of some iterate of q_m have a nonempty transversal intersection near z, and therefore there are periodic points near z. Thus $z \in \overline{Per^N_h(f)}$.

In case that $\omega(z)$ does not contains periodic points then, by Theorem 2.1, it follows that $\omega(z)$ is a hyperbolic set. It follows $\omega(z) \subset \overline{Per^N_h(f)}$. We can apply the same argument as before, and we also conclude that $z \in \overline{Per^N_h(f)}$. This completes the proof of the Main Theorem.

Moreover, from the proof we get the following corollary (where is not assumed that $\tilde{L}(f)$ has dominated splitting).

Corollary 4.4.1 *If $\Lambda \subset \tilde{L}(f)$ and has dominated splitting then for any U neighbourhood of Λ we get that*

$$\Lambda \subset \overline{Per_h^N(f)} \bigcap \cap_{n\in\mathbb{Z}} f^n(\overline{U}).$$

4.5 More on the basic pieces of $\tilde{L}(f)$ and the Main Theorem restated.

Regarding the basic pieces of $\tilde{L}(f)$, there are some questions (and answers) that we want to address. Are these basic pieces locally maximal? Do they have local product structure? Do they have the shadowing property? Do they exhibit Markov partitions?.

Definition 4.5.1 *A set Λ having dominated splitting is said to have local product structure if there exist $\gamma > 0$ and $\eta > 0$ such that if $x, y \in \Lambda, d(x,y) < \eta$ then $W^{cs}_\gamma(x) \cap W^{cu}_\gamma(y) \in \Lambda$.*

Notice that if γ and η are small enough the above intersection is transversal and consists on one single point.

Lemma 4.5.1 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, Λ has local product structure.*

Proof: Since Λ is a basic piece of $\tilde{L}(f)$, the hyperbolic periodic points are dense. Thus, it is enough to show the local product structure among the hyperbolic periodic points.

If Λ does not contain any saddle-node point, then the local stable and unstable manifolds have uniform size (by theorem 4.1), and the local product structure follows by standard arguments.

On the other hand, assume Λ contains saddle-node type periodic points. Consider boxes around these points $B_{(\delta^s,\delta^u)}(p)$ small enough such that for the F-saddle node (respec. E saddle-node) one component of $B_{(\delta^s,\delta^u)}(p) - W^{cs}_{\delta^s}(p)$ (respec. $B_{(\delta^s,\delta^u)}(p) - W^{cu}_{\delta^u}(p)$) contains (at most) only $\Omega\backslash P$ periodic points. Consider γ (less than $\delta^u/2$ and $\delta^s/2$) from theorem 4.1. Let $\eta > 0$ such that for any two point x, y such that $d(x,y) < \eta$ the local center stable manifold $W^{cs}_\gamma(x)$ and the local center unstable manifold $W^{cu}_\gamma(y)$ have a (unique) nonempty (and transverse) intersection. Now, if x and y are hyperbolic periodic points in Λ and they are far from the saddle-node points, these local center stable and unstable manifolds are contained in the stable and unstable manifold, and the result follows by standard arguments. On the other hand if one (and hence both) of the points x, y are near a saddle-node point, since they are not isolated, the point of intersection of the central stable and unstable manifold of x and y respectively lies in the stable and unstable manifold of x and y (respec.) and hence, the result holds.

■

Remark 4.5.1 *Observe that in the above lemma the local stable and unstable manifolds may not have uniform size.*

Definition 4.5.2 *Let $\Lambda \subset \tilde{L}(f)$ be a compact and invariant set having dominated splitting. A markov partition of Λ is a collection of rectangles, i.e. diffeomorphic images of the square $Q = [-1, 1]^2$, say $R_1 = \psi_1(Q), ..., R_l = \psi_l(Q)$ such that:*

1. $\Lambda \subset \cup_i R_i$,
2. $int(R_i) \cap int(R_j) = \emptyset$ *if* $i \neq j$, *where* $int(R_i)$ *denotes the interior of* R_i,
3. $f(\partial_s R_i) \subset \cup_j \partial_s R_j$ *and* $f^{-1}(\partial_u R_i) \subset \cup_j \partial_u R_j$ *where* $\partial_s R_i = \psi_i(\{(x, y) : -1 \leq x \leq 1, |y| = 1\})$ *and* $\partial_u R_i = \psi_i(\{(x, y) : -1 \leq y \leq 1, |x| = 1\})$
4. *there is a positive integer* n *such that* $f^n(R_i) \cap R_j \neq \emptyset \; \forall \; 1 \leq i, j \leq l$

Moreover, we define the size of the markov partition as the maximum of the diameters of R_i.

Notice that item 3) means that the boxes are "adapted", i.e. for any $x \in R_i$ and $n \geq 0$ it follows that either $f^{-n}(J^{cu}_{R_i}) \cap R_j = \emptyset$ or $f^{-n}(J^{cu}_{R_i}) \subset R_j$ for any j. Similar to J^{cs} in the future. The proof of existence of Markov partitions in [PT] (appendix 2) without major modifications also proves the next lemma.

Lemma 4.5.2 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, there exists a Markov partition of Λ of arbitrarily small size.*

Fathi [Fa] has proved that an expansive homeomorphism on a compact space has an adapted metric (no necessarily coming from a riemannian structure) compatible with the topology. Indeed, there is some $\epsilon > 0$ and $0 < \lambda < 1$ such that if $dist(f^n(x), f^n) < \epsilon \; \forall n \geq 0$ then $dist(f^n(x), f^n(y)) < \lambda^n dist(x, y)$. Similarly for backward iterates. Therefore, the same proof of the shadowing lemma for hyperbolic sets with local product structure can be carried out in our case, since the basic pieces are expansive and have local product structure.

Lemma 4.5.3 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, it has the shadowing property, that is, given $\beta > 0$ there exists $\alpha > 0$ such that any α-pseudo orbit in Λ is β- shadowed by a true orbit in Λ.*

By the previous lemma and Theorem 4.2 (item 3a) we obtain the following:

Corollary 4.5.1 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, if $\omega(x) \subset \Lambda$ then there exits $y \in \Lambda$ such that $x \in W^s(y)$ i.e., $d(f^n(x), f^n(y)) \to_{n\to\infty} 0$*

In the hyperbolic theory local product structure is equivalent to be maximal invariant. For a basic piece in $\tilde{L}(f)$ this may not be the case.

Lemma 4.5.4 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, Λ is locally maximal if and only if Λ does not contains saddle-node points.*

Proof: If Λ does not contains saddle-node points, then the stable and unstable manifolds of any points have size bounded away from zero. From the local product structure and the shadowing lemma the result follows. On the other hand if Λ does contains saddle-node it is not difficult to see that is not locally maximal.

■

Definition 4.5.3 *We say that an invariant set Λ is an attractor if*

1. *there exists U a neighbourhood of Λ such that $f(U) \subset U$ and $\cap_{n\geq 0} f^n(U) = \Lambda$.*
2. *Λ is transitive.*

The next lemma follows with the same arguments as in the hyperbolic case.

Lemma 4.5.5 *A basic piece Λ of $\tilde{L}(f)$ is an attractor if and only if $W^u(x) \subset \Lambda$ for any $x \in \Lambda$.*

Remark 4.5.2 *When Λ is a transitive hyperbolic set the above definition of attractor is equivalent to say that there is a neighbourhood U of Λ such that $\omega(x) \subset \Lambda$ for any $x \in U$. This is not true in our case.*

Now, we summarize all the above results (see also figure below).

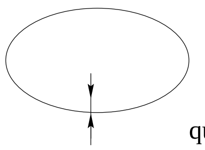
Theorem 4.4 Main Theorem restated *Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{L}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is the set of $\Omega\backslash P$-isolated periodic points. The periods of the points in $\mathcal{I}$ is bounded and $\mathcal{I} \subset \Gamma_1 \cup \ldots \cup \Gamma_r$ where $\Gamma_i, i = 1, ..., r$ is compact invariant and $\Gamma_i \cap \Gamma_j = \emptyset, i \neq j$. Moreover Γ_i is a periodic point or a normally hyperbolic (attractive or repelling) 1-dimensional manifold (closed arc or simple closed curve).*

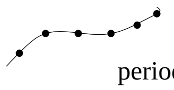
2. $\mathcal{R}$ *is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*

3. $f/\tilde{L}(f)$ *is expansive and admits a spectral decomposition* $\tilde{L}(f) = \Lambda_1 \cup \ldots \cup \Lambda_l$. *Each set* Λ_i *is compact, transitive and* $\Lambda_i \cap \Lambda_j = \emptyset$ *for* $i \neq j$. *These sets* Λ_i *are the (union of) homoclinc classes associated to hyperbolic periodic points. The hyperbolic periodic points are dense and* Λ_i *contains at most finitely many non-hyperbolic periodic points.* Λ_i *has local product structure, admits Markov partition of arbitrary small size and it is locally maximal if and only if does not contains saddle-node type periodic points.*

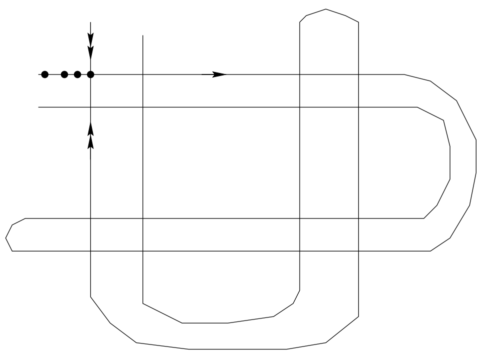
Furthermore, $M = \bigcup_{x \in L(f)} W^s(x) = \bigcup_{x \in L(f)} W^u(x)$ *and for any* $x \in L(f)$ *that is neither saddle-node nor sink-node,* $W^s(x)$ *is a smooth manifold.*



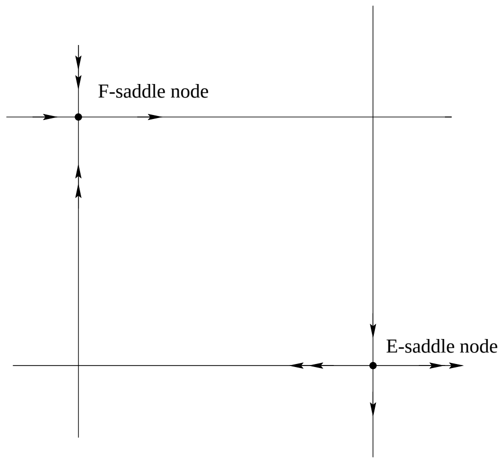
quaseperiodic motion



periodic arc



basic piece with periodic arc attached



basic piece

5 Proof of Theorem C

It is well known that a hyperbolic set Λ of a diffeomorphism f has an analytic continuation, meaning that for any diffeomorphism g close enough to f there is hyperbolic set Λ_g for g close to Λ (i.e. it is contained in a neighbouhood of it) and moreover they are homeomorphic and the respective dynamics on them are conjugated. This property turns out to be false for sets having dominated splitting, even when the set is a homoclinic class of a hyperbolic periodic point. Although the hyperbolic periodic point persist in a neighbourhood of the diffeomorphism, for certain diffeomorphism arbitrarily close, the homoclinic class of the continuation is not any more contained in a neighbourhood of the original one and furthermore what remains in a neighbourhood is not any more conjugated to the original one. For instance, take a "horseshoe" where instead of a hyperbolic fixed point p we have a saddle-node fixed one -but this horseshoe is the homoclinic class of another hyperbolic periodic point- (see figure a. below) and, after the disappearance of the saddle-node fixed point, the homoclinic class "moves" towards the point q in the referred figure. Furthermore, more pathological behaviour could appear in the case of figure b. (see [DRV], [C]).

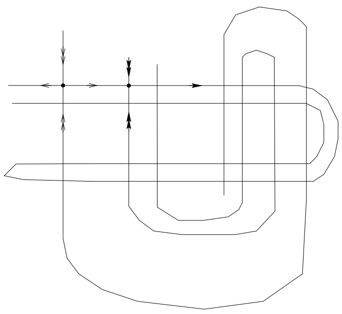


figure a.

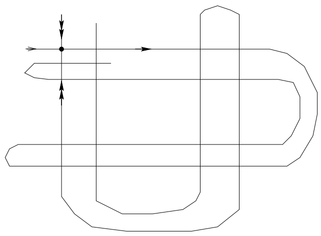


figure b.

However, in these examples if we perturb the saddle-node in such a way that has a hyperbolic continuation, the "horseshoe" becomes hyperbolic. This is the idea to prove Theorem C: *perturb the diffeomorphism f in such a way that the non-hyperbolic periodic points in a basic piece of $\tilde{L}(f)$ (which are finitely many) have a hyperbolic continuation.* To show that this is enough, we must guarantee that no other non hyperbolic periodic point appear after the perturbation. This is the content of the following section.

5.1 Theorem 3.1 revisited

Through this section Λ will denote a f-compact invariant set with dominated splitting and $\Lambda \subset \tilde{L}(f)$.

Definition 5.1.1 *Let p be a periodic point in Λ and let $\mathcal{V}$ be a connected subset of $Diff^2(M)$, $f \in \mathcal{V}$. We say that p has a continuation on $\mathcal{V}$ provided there exists a continuous map $\pi : \mathcal{V} \to M$ such that*

1. *$\pi(f) = p$.*
2. *For $g \in \mathcal{V}$, $\pi(g)$ is a g-periodic point with the same f-period of p.*
3. *$\pi(g)$ is not $\Omega \backslash P$-isolated.*

For $g \in \mathcal{V}$ the point $\pi(g)$ will be called an analytic continuation of p and will be denoted by $p(g)$.

Theorem 5.1 *Let f be a C^2-diffeomorphisms of a compact surface M^2, and let $\Lambda \subset \tilde{L}(f)$ be a compact invariant set exhibiting dominated splitting and $p \in \Lambda$ be a non-hyperbolic periodic point. Denote its period by n_p. Assume that p has a continuation on $\mathcal{V}$. Then, there exist a neighborhood U_p of p, a neighborhood V of Λ, and a neighbourhood $\mathcal{U}$ such that for any $g \in \mathcal{V} \cap \mathcal{U}$ and any g-periodic point q with period greater than $2n_p$ and whose orbit is contained in V and intersecting U_p is a hyperbolic periodic point of saddle type.*

Proof of Theorem 5.1

The proof goes along the same lines as the proof of Theorem 3.1, because all the estimatives involved in the proof of this Theorem holds for diffeomorphisms close to f contained in $\mathcal{V}$. To begin with, take V a small compact admissible neighbourhood of Λ and a neighbourhood $\mathcal{U}(f)$ such that for $g \in \mathcal{U}$, the set $\cap_n g^n(V)$ has dominated splitting. In the sequel we shall set $\Lambda_V(g) = \cap_n g^n(V)$. The following result can be found in [HPS].

Lemma 5.1.1 *Let $f \in Diff^2(M^2)$ and Λ a compact invariant set exhibiting dominated splitting. Then there exist $\mathcal{U}$ and a neighborhood V of Λ such that for any $g \in \mathcal{U}$ there exist two continuous functions $\phi_g^{cs} : \Lambda_V(g) \to Emb^2(I_1, M)$ and $\phi_g^{cu} : \Lambda_V(g) \to Emb^2(I_1, M)$ such that if define $W_\epsilon^{cs}(x,g) = \phi_g^{cs}(x)I_\epsilon$ and $W_\epsilon^{cu}(x,g) = \phi_g^{cu}(x)I_\epsilon$ the following properties hold:*

a) $T_x W_\epsilon^{cs}(x,g) = E_g(x)$ and $T_x W_\epsilon^{cu}(x,g) = F_g(x)$

b) for all $0 < \epsilon_1 < 1$ there exist ϵ_2 such that

$$g(W_{\epsilon_2}^{cs}(x,g)) \subset W_{\epsilon_1}^{cs}(g(x), g)$$

and

$$g^{-1}(W^{cu}_{\epsilon_2}(x,g)) \subset W^{cu}_{\epsilon_1}(g^{-1}(x),g).$$

Moreover, these central manifold depend continuously on g; i.e.: given $g_k \to g$, $x_k \in \Lambda_V(g_k)$ converging to $x \in \Lambda_V(g)$ follows that $\phi^{cs}_{g_k}(x_k) \to \phi^{cs}_g(x)$ and $\phi^{cu}_{g_k}(x_k) \to \phi^{cu}_g(x)$ (in $Emb^2(I_1,M)$).

From the previous result follows that a Lipschitz constant of $\log(Df)$ along the central manifolds can be chosen uniformly in a neighbourhood of f, and hence we get the following lemma.

Lemma 5.1.2 *Let $f \in Diff^2(M^2)$ and Λ a compact invariant set exhibiting dominated splitting. Then there exist K_0, $\mathcal{U}$ and a neighborhood V of Λ such that for any $g \in \mathcal{U}$ if $\mathcal{O}(x) \subset V$ and any arc $J \subset W^{cu}_\epsilon(x,g)$ such that, $g^{-j}(J) \subset W^{cu}_\epsilon(g^{-j}(x,g))$ for $0 \le j \le n$ then*

1. $\frac{\|Dg^{-n}_{/\tilde{F}_g(y)}\|}{\|Dg^{-n}_{/\tilde{F}_g(z)}\|} \le \exp(K_0 \sum_{j=0}^{n-1} \ell(g^{-j}(J)))$; $y,z \in J$, $\tilde{F}_g(y) = T_yJ$, $\tilde{F}_g(z) = T_zJ$.
2. $\|Dg^{-n}_{/\tilde{F}_g(y)}\| \le \frac{\ell(g^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{j=0}^{n-1} \ell(g^{-j}(J)))$.

Remark 5.1.1 *Let p be as in Theorem 5.1, having a continuation on $\mathcal{V}$. From now on, we fix the continuation $\pi : \mathcal{V} \to M$ of p. Moreover, after change of coordinates for $g \in \mathcal{V} \cap \mathcal{U}$ we may assume:*

1. $p(g) = p$;
2. $W^{cs}_\epsilon(p(g)) = W^{cs}_\epsilon(p)$;
3. $W^{cu}_\epsilon(p(g)) = W^{cu}_\epsilon(p)$.

Recall that for a non-hyperbolic periodic point p and for any positive numbers δ^s, δ^u small enough, we have defined the box $B_{(\delta^s,\delta^u)}(p)$ (see Definition 3.3.1). This is also a box for any $g \in \mathcal{V} \cap \mathcal{U}$ if $\mathcal{U}$ is small enough. Let $B^i_{(\delta^s,\delta^u)}(p)$ be a quadrant of the above box and V a compact neighbourhood of Λ. For $g \in \mathcal{V} \cap \mathcal{U}$ and $x \in B^i_{(\delta^s,\delta^u)}(p) \cap \Lambda_V(g)$ we define $J^{cu,i}_{\delta^u}(x,g)$ to be the connected component of $W^{cu}_{\epsilon_0}(x,g) \cap B^i_{(\delta^s,\delta^u)}(p)$ that contains x. Analogously, we define $J^{cs,i}_{\delta^s}(x,g)$.

The following lemma is a version of Lemma 3.3.1 for diffeomorphisms close to f. The proof is similar and we leave it to the lector.

Lemma 5.1.3 *Let $\Lambda \subset \tilde{L}(f)$ be a set having dominated splitting without closed curves supporting irrational rotations and let $p \in \Lambda$ be a non-hyperbolic periodic point. Let $B_{(\delta^s,\delta^u)}(p)$ be a small box and let $B^i_{(\delta^s,\delta^u)}(p)$ be a non-isolated quadrant for f. Then, for any ϵ, $0 < \epsilon < \delta_0$, there exist $\delta^u_* = \delta^u_*(\epsilon) < \delta^u$, $\mathcal{U} = \mathcal{U}(\epsilon)$ and* *a neighborhood V of Λ such that for any $g \in \mathcal{V} \cap \mathcal{U}$ and $x \in B^i_{(\delta^s,\delta^u_*)}(p) \cap \Lambda_V(g)$ different from p we get that*

$$g^{-n}(J^{cu,i}_{\delta^u_*}(x,g)) \subset W^{cu}_\epsilon(g^{-n}(x),g)$$

holds for any $n \geq 0$.

A similar statement also holds for the central stable manifolds, more precisely: there is $\delta^s_* = \delta^s_*(\epsilon) < \delta^s$ such that for any $x \in B^i_{(\delta^s_*,\delta^u)}(p) \cap \Lambda_V(g)$ different from p we get that

$$g^{n}(J^{cs,i}_{\delta^s_*}(x,g)) \subset W^{cs}_\epsilon(g^{n}(x),g)$$

holds for any $n \geq 0$.

In order to prove Theorem 5.1 we shall argue by contradiction. So, let us assume that the conclusion of the theorem is false, that is, there exist a sequence of open neighbourhoods V_n of Λ, $\cap_n V_n = \Lambda$, a sequence of diffeomorphisms $g_n \in \mathcal{V} \cap \mathcal{U}$ converging to f and a sequence of periodic points $\{q_n\}$ of g_n with $\mathcal{O}(q_n) \subset V_n$ accumulating at p with periods greater than $2n_p$ (and hence increasing to infinity) such that they are not hyperbolic of saddle type, i.e., either

$$\|Dg_n^{-m_n}{}_{|F_{g_n}(q_n)}\| \geq 1 \text{ or } \|Dg_n^{m_n}{}_{|E_{g_n}(q_n)}\| \geq 1$$

where m_n is the g_n-period of q_n. We may (and will) assume that

$$\|Dg_n^{-m_n}{}_{|F_{g_n}(q_n)}\| \geq 1$$

holds for any n. We will show that is not the case for sufficient large n, leading to a contradiction.

This sequence of g_n-periodic points accumulates (taking a subsequence if necessary) through a quadrant $B^i_{(\delta^s,\delta^u)}(p)$. It is not difficult to see that this is a non-isolated quadrant for f. Denote by $W^{cs,+}_{\delta^s}(p)$ and $W^{cu,+}_{\delta^s}(p)$ the branches that bounds this quadrant and recall that they are contained in the stable and unstable manifold respectively of p (with respect to f). It follows (recall remark 5.1.1) that $g_n(W^{cs,+}_{\delta^s}(p)) \subset W^{cs,+}_{\delta^s}(p)$ and $g_n^{-1}(W^{cu,+}_{\delta^s}(p)) \subset W^{cu,+}_{\delta^s}(p)$. A (maximal) interval in $W^{cs,+}_{\delta^s}(p)$ such the forward iterates under g_n are pairwise disjoint will be called a fundamental domain of g_n in $W^{cs,+}_{\delta^s}(p)$. In a similar way, we define fundamental domain of g_n in $W^{cu,+}_{\delta^s}(p)$. Take points $x \in W^{cs,+}_{\delta}(p) \cap \overline{\{\mathcal{O}(q_n) : n \geq 0\}}$ and $y \in W^{cu,+}_{\delta}(p) \cap \overline{\{\mathcal{O}(q_n) : n \geq 0\}}$ It follows that these points belongs Λ.

By corollary 4.4.1, the points x and y are accumulated by hyperbolic f-periodic points whose orbits are arbitrarily close to Λ. As in the proof of lemma 3.4.1, one way take hyperbolic periodic points p_1, p_2, p_3 such that the box $B(x)$ determined by the arcs $W^{cs,+}_{\delta^s}(p)$, $W^{cs}_\epsilon(p_1); W^{cu}_\epsilon(p_2)$ and $W^{cu}_\epsilon(p_3)$ is a well *cu*-adapted box such that

1. x belongs to a component of $\partial^{cu}(B(x))$ which is also contained in a fundamental domain of $W^{cs,+}_{\delta^s}(p)$.

2. For any large n, the g_n-orbits $\mathcal{O}(q_n)$ have nonempty intersection with $B(x)$.

Since the points p_1, p_2, p_3 have an analytic continuation in some neighbourhood of f and also p has an analytic continuation $p(g)$ in $\mathcal{V}$, we can define the box $B_g(x)$ bounded by the respective local stable and unstable manifolds of $p(g)$, $p_1(g)$, $p_2(g)$ and $p_3(g)$ as the box $B(x) = B_f(x)$. These boxes are also well cu-adapted and satisfy

1. x belongs to a component of $\partial^{cu}(B_g(x))$ which is also contained in a fundamental of g_n in $W^{cs,+}_{\delta^s}(p)$.

2. For any large n, the g_n-orbits $\mathcal{O}(q_n)$ have nonempty intersection with $B_g(x)$.

In a similar way we construct cu-adapted boxes $B_g(y)$ satisfying

1. y belongs to a component of $\partial^{cs}(B_g(y))$ which is also contained in a fundamental of g_n in $W^{cu,+}_{\delta^u}(p)$.

2. For any large n, the g_n-orbits $\mathcal{O}(q_n)$ have nonempty intersection with $B_g(y)$.

Definition 5.1.2 Boxes III *Let $g \in \mathcal{V} \cap \mathcal{U}$. Set $B^g_0 = B_g(y)$ and for $k \geq 1$ we define B^g_k as the connected component of $g^{-1}(B^g_{k-1}) \cap B_{(\delta^s,\delta^u)}(p(g))$ that contains $g^{-k}(y)$.*

Moreover, given the point $\hat{q}_n$ of the g_n-orbit of q_n accumulating at y and belonging to $B_g(y)$, let $k_n = \min\{k \geq 0 : g_n^{-k_n}(\hat{q}_n) \in B_g(x)\}$. We define $B^{x,g}_{k_n}$ as the component of $B^g_{k_n} \cap B_g(x)$ that contains $g_n^{-k_n}(\hat{q}_n)$ (the boxes B^g_k defined as above).

Proposition 5.1 *Given $r > 0$, there exists $s > 0$ and $\mathcal{U}$ such that if $g_n \in \mathcal{V} \cap \mathcal{U}$ and $dist(\hat{q}_n, y) < s$ the following hold:*

1. *$B^{x,g}_{k_n}$ is a r-cu-adapted cs-subbox in $B_g(x)$.*

2. *If r is small enough, then any return to $B^{x,g}_{k_n}$ is a hyperbolic return. Indeed, for any return $\psi \in \mathcal{R}^{cu}(B^{x,g}_{k_n}, \Lambda_{V_n}(g))$, $\psi' < \frac{1}{2}$ holds.*

Theorem 5.1 follows from this proposition. Indeed, take n large enough so that there is a point $\hat{q}_n \in B(y)$ in the g_n-orbit of q_n close enough to y so that the above proposition applies. Set $q_n = g_n^{k_n}(\hat{q}_n) \in B^{x,g_n}_{k_n}$ and assume that the period is m. Let $0 < m_1 < m_2 < ... < m_l = m$ be the successive returns of the point q_n to $B^{x,g_n}_{k_n}$ until return to itself. Then,

$$\|Dg_n{}^{-m}_{/F(q_n)}\| \leq \left(\frac{1}{2}\right)^l < 1.$$

These contradicts the assumption made over the points q_n.

To prove Proposition 5.1, again, we have to deal with arguments involving distortion and summability.

Lemma 5.1.4 *Let $C_1 > 0$. Then, there exists $K_1 = K_1(C) > 0$ such that for any cu-adapted cs-subbox B'_g of $B_g(x)$ (previously defined) having distortion C_1 and any $z \in B' \cap \Lambda$ we have that*

$$\sum_{j=0}^{n} \ell(g^{-j}(J^{cu}_{B'_g}(z))) \leq K_1$$

whenever $g^{-j}(z) \notin B'_g$, $1 \leq j \leq n$.

Proof: It is a straightforward adaptation of Lemma 3.4.4, since all the constants involved in the proof can be chosen uniformly on g.

■

For each $g \in \mathcal{V} \cap \mathcal{U}$, we may choose a foliation on $B_g(y)$ such that the g-distortion is bounded uniformly on g by some C.

Lemma 5.1.5 *Let $C > 0$ be such that the box $B_g(y)$ has g-distortion C for any $g \in \mathcal{V} \cap \mathcal{U}$. Then,*

1. *There exists $K = K(C) > 0$ such that for any cu-subbox $B'_g \subset B_g(y)$ and $m > 0$ such that $g^{-k}(B') \subset B^g_k$ for $0 \leq k \leq m$ then*

   $$\sum_{i=0}^{m} \ell(g^{-k}(J)) \leq K$$

   holds for any arc $J \subset B'_g$ transversal to the E_g-direction with endpoints in $\partial^{cu}(B')$. A similar result holds for a cs-subbox in $B_g(x)$

2. *there exists $C_1 = C_1(C)$ such that $B^{x,g_n}_{k_n}$ is a cu-adapted cs-subbox in $B_{g_n}(x)$ having distortion C_1.*

Proof: It is a straightforward adaptation of the proof of Lemma 3.4.3.

■

Finally, Proposition 5.1 can be proved in the same way as Proposition 3.1. This completes the proof of Theorem 5.1.

Remark 5.1.2 *Notice that if the point p in the statement of Theorem 5.1 is not of saddle-node type, i.e., it is of saddle type, then it* has a continuation on a neighbourhood of f. *In particular, Theorem 5.1 can be restated without mentioning the set $\mathcal{V}$.*

5.2 The set Λ_g

From now on let Λ be as in Theorem C, that is, a basic piece of $\tilde{L}(f)$. In this section we will define the set $\mathcal{V}$ and the candidate Λ_g that will satisfy the thesis of Theorem C.

We shall assume, replacing f by f^2, that the eigenvalue of a non-hyperbolic periodic point with modulus one is 1.

Lemma 5.2.1 *Let p be a non-hyperbolic periodic point in a basic piece of $\tilde{L}(f)$. Then, there exist a connected open set $\mathcal{V}(p,f) \subset Diff^2(M)$ and a continuous map $\pi : \overline{\mathcal{V}(p,f)} \to M$ such that*

1. *$f \in \overline{\mathcal{V}(p,f)}$ and $\pi(f) = p$.*
2. *For $g \in \mathcal{V}(p,f)$, $\pi(g)$ is a hyperbolic periodic point of saddle type of the same period as p.*
3. *$\pi(g)$ is the unique periodic point with the same period as p in a neighbourhood of p which is not $\Omega\backslash P$-isolated.*

The point $\pi(g)$ will be called the hyperbolic continuation of p and will be denoted by $p(g)$.

Proof: It is not difficult to see that there is a one parameter family $f_t, f_0 = f$ such that for $t > 0$ there is a unique non-isolated hyperbolic periodic point p_t with the same period as p, $p_t \to_{t\to 0} p$. For each f_t consider an open neighbourhood $\mathcal{V}_t$ such that p_t has an analytic continuation. For the set $\mathcal{V} = \cup_{t>0}\mathcal{V}_t$ it is straightforward to show the lemma.

■

Let p be a non-hyperbolic periodic point and let $\mathcal{V}(p,f)$ be the maximal open connected set such that p has a hyperbolic continuation (that exists by the previous lemma). Given any neighbourhood $\mathcal{U} = \mathcal{U}(f)$ denote by

$$\mathcal{V}(f,p,\mathcal{U}) = \mathcal{V}(f,p) \cap \mathcal{U}(f)$$

Moreover, consider $\{p_1, p_2, ..., p_n\}$ the set of non-hyperbolic periodic points in Λ. Denote by

$$\mathcal{V}(f,\Lambda,\mathcal{U}) = \cap_i \mathcal{V}(f,p_i,\mathcal{U}).$$

Notice that $\mathcal{V}(f,\Lambda,\mathcal{U})$ is non-empty.

Consider V a compact admissible neighbourhood of Λ. Let $\mathcal{U}(f)$ be such that, for $g \in \mathcal{U}(f)$, any g-invariant compact set in V has dominated splitting. For

$g \in \mathcal{V}(f, \Lambda, \mathcal{U})$ consider $\Lambda_g = \cap g^n(V) \cap \tilde{L}(g)$. We shall prove that this Λ_g satisfies the thesis of Theorem C. This will be done in two steps: first we will show that Λ_g is hyperbolic and secondly that Λ_g is homeomorphic to Λ and $f_{/\Lambda}$ is conjugated to $g_{/\Lambda_g}$.

5.3 Hyperbolicity of Λ_g

By Theorem 2.1, to show that Λ_g is hyperbolic it is enough to show that the periodic points in Λ_g are hyperbolic.

Let $\{p_1, ..., p_n\}$ be the non-hyperbolic periodic points in Λ.. For each non-hyperbolic point, take $\mathcal{U}_i$, N_i, V_i and U_i given by Theorem 5.1 for $\Lambda = \Lambda$. On the other hand, take a compact neighborhood $V \subset \cap_i V_i$ of Λ such that $\cap_n f^n(V \setminus \cup_i U_i)$ is hyperbolic (by Theorem 2.1). Let $\mathcal{U} \subset \cap_i \mathcal{U}_i$ be such that for $g \in \mathcal{U}$, the set $\cap_n g^n(V \setminus \cup_i U_i)$ is also hyperbolic. Let q be any g-periodic point in $\Lambda_g, g \in \mathcal{V}(f, \Lambda, \mathcal{U})$. Either the orbit $\mathcal{O}(q)$ intersect some U_i or not. If not, then $q \in \cap_n g^n(V \setminus \cup_i U_i)$ and so q is hyperbolic. In the other case, if it intersects some U_i, either we get that the period of q is greater than the double of the period of p and so it is hyperbolic by Theorem 5.1, or it is the point $p(g)$ which is hyperbolic. This shows that Λ_g is hyperbolic.

5.4 Conjugacy between $f_{/\Lambda}$ and $g_{/\Lambda_g}$

The idea to construct the conjugacy is to use the shadowing property. Indeed, if g is near f, every g-orbit in Λ_g project to a f-pseudo orbit in Λ and hence can be shadowed by an f-orbit in Λ. Moreover, by the expansivity of $f_{/\Lambda}$ the f-orbit that shadows is unique. Hence, there is a map $h : \Lambda_g \to \Lambda$ such that $h(x)$ is the point such that its f-orbit shadows the g-orbit through x. By the uniqueness, it follows that $f \circ h = h \circ g$, and the map h is continuous. Since these last arguments are quite standard, we shall omit the details.

However, we have to prove that h is a homeomorphism. For this, is enough to prove that h is injective and surjective. The injectiveness of h follows from the fact that the expansivity constant of $g_{/\Lambda_g}$ can be chosen uniformly on g. We need the following results. The first one is a uniform version of Theorem 4.1 for Λ_g. The proof follows from the uniformity of the Denjoy's property (Theorem 2.3) and that the g-periodic points in Λ_g are of saddle type.

Theorem 5.2 *Let Λ be a basic piece of f, and let $p_1, ...p_r$ be the F-saddle-node type non-hyperbolic periodic points for f in Λ, and let $q_1, ..., q_t$ be the E-saddle-node ones. Then, there exist $\mathcal{U}(f)$ and an admissible neighourbourhood V such that for any $g \in \mathcal{V}(f, \Lambda, \mathcal{U})$ the following holds: given $\epsilon < \delta_0$ there exist $\delta^u = \delta^u(\epsilon), \delta^s = \delta^s(\epsilon)$ and $\gamma = \gamma(\delta^u, \delta^s, \epsilon)$ such that for any $x \in \Lambda_g$ we have that*

1. *If $x \notin \cup_{i=1}^{r} B_{(\delta^s, \delta^u)}(p_i(g))$ then $W^{cu}_{\gamma}(x, g) \subset W^u_{\epsilon}(x, g)$.*

2. *If* $x \in B^{+}_{(\delta^s,\delta^u)}(p_i(g))$ *then* $J^{cu,+}_{B^+}(x,g) \subset W^u_\epsilon(x,g)$

3. *If* $x \notin \cup_{i=1}^{t} B_{(\delta^s,\delta^u)}(q_i(g))$ *then* $W^{cs}_\gamma(x,g) \subset W^s_\epsilon(x,g)$.

4. *If* $x \in B^{+}_{(\delta^s,\delta^u)}(q_i(g))$ *then* $J^{cs,+}_{B^+}(x,g) \subset W^s_\epsilon(x,g)$.

Proof: The proof is quite similar to the proof of Theorem 4.1 so we just give an outline of it. We shall prove just item 1.

Assume that $\epsilon \leq \delta_0$. Take $\delta^s = \delta^s_*(\epsilon)$ and $\delta^u = \delta^u_*(\epsilon)$ from Lemma 5.1.3. Assume that they are small enough such that $B_{(\delta^s,\delta^u)}(i) \cap B_{(\delta^s,\delta^u)}(j) = \emptyset$ for $i,j = p_1,...,p_r,q_1,...,q_t$.

Let $x \in \Lambda_g$ and assume that $x \notin \cup_{i=1}^{r} B_{(\delta^s,\delta^u)}(p_i(g))$. Let show first that there is some γ such that

$$\ell(g^{-n}(W^{cu}_\gamma(x))) \leq \delta, \quad n \geq 0.$$

If such γ does not exist, then there are sequences $x_n \notin \cup_{i=1}^{r} B_{(\delta^s,\delta^u)}(p_i)$, $\gamma_n \to 0$ and $m_n \to \infty$ such that, for $0 \leq j \leq m_n$,

$$\ell(f^{-j}(W^{cu}_{\gamma_n}(x_n))) \leq \delta$$

and

$$\ell(g^{-m_n}(W^{cu}_{\gamma_n}(x_n))) = \delta$$

Letting $I_n = g^{-m_n}(W^{cu}_{\gamma_n}(x_n))$ we can assume (taking a subsequence if necessary) that $I_n \to I$ and $g^{-m_n}(x_n) \to z$, $z \in \Lambda_g$, $z \in \overline{I}$ (the closure of I).

Now, we have that $\ell(g^n(I)) \leq \delta \leq \delta_0$ for all positive n, and since $I \subset W^{cu}_\epsilon(z)$, we conclude that I is a δ-E-interval. Thus, $\omega(z)$ is a periodic orbit p because $z \in \overline{I}$. Since $z \in \Lambda_g$ we conclude that $p \in \Lambda_g$ and hence is hyperbolic of saddle type. Hence $z \in W^s(p)$.

Then one of the components of $W^u(p) - \{p\}$ has length less than ϵ. Thus, in case

$$g^{-m_n}(W^{cu}_{\gamma_n}(x_n)) \cap W^s(p) \neq \emptyset$$

we get a contradiction with the inclination lemma (or λ-lemma, see [P]) because this intersection is transversal and

$$\ell(g^{m_n}(g^{-m_n}(W^{cu}_{\gamma_n}(x_n)))) = \ell(W^{cu}_{\gamma_n}(x_n)) \to 0.$$

On the other hand, if

$$g^{-m_n}(W^{cu}_{\gamma_n}(x)) \cap W^s(p) = \emptyset$$

it follows, for sufficiently large n, that $\omega(g^{-m_n}(x_n))$ is the other endpoint (say q) of the component of $W^u(p) - \{p\}$ having length less than δ. By the lemma 3.3.1 of [PS1], it is a sink or a non-hyperbolic periodic point. This implies that $\omega(f^{-m_n}(x_n)) = \omega(x_n) = q$ for n large enough and $q \in \Lambda_g$, acontradiction.

To finish the proof of item 1) it remains to prove that

$$\ell(g^{-n}(W^{cu}_\gamma(x))) \to 0$$

for $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i(g))$. Arguing by contradiction, assume that this is not the case. Then, there exist $\eta > 0$ and a sequence $n_k \to \infty$ such that

$$\ell(g^{-n_k}(W^{cu}_\gamma(x))) > \eta$$

for some $x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i(g))$. Letting $I_{n_k} = g^{-n_k}(W^{cu}_\gamma(x))$ we can assume that $I_{n_k} \to I$ and $f^{-n_k}(x) \to z \in \overline{I}, z \in \Lambda$. As we did above, we get that I is a δ_0-E-interval, and so $\omega(z)$ is a hyperbolic periodic point $p \in \Lambda_g$. If $z \in int(I)$, then, since I is transversal to $W^s(p)$, it follows, by the inclination lemma, that $\ell(W^u(p)) \leq \delta$ and hence the endpoints q_1, q_2 of $W^u(p)$ are not hyperbolic periodic points of saddle type. Therefore, $p \in \Lambda_g$ is $\Omega\backslash P$-isolated, a contradiction with the definiton of Λ_g.

On the other hand, if $z \notin int(I)$, again, the inclination lemma implies that one of the components of $W^u(p) - \{p\}$ has length less than δ. As we did above, the case

$$g^{-n_k}(W^{cu}_\gamma(x)) \cap W^s(p) = \emptyset$$

leads to a contradiction. So

$$g^{-n_k}(W^{cu}_\gamma(x)) \cap W^s(p) \neq \emptyset.$$

Using the inclination lemma, the fact that $\ell(g^j(g^{-n_k}(W^{cu}_\gamma(x)))) \leq \delta,\ 0 \leq j \leq n_k$ together with $g^{-n_k}(x) \to z$ imply that $x \in W^u_{loc}(p)$.moreover, p should be one of the $p_i(g)$. Hence $x \in B_{(\delta^s,\delta^u)}(p_i(g))$, a contradiction.

■

Lemma 5.4.1 *Let Λ be a basic piece of f, and let $p_1, ...p_r$ be the F-saddle-node type non-hyperbolic periodic points for f in Λ, and let $q_1, ..., q_t$ be the E-saddle-node ones. Then, there exist $\mathcal{U}(f)$ such that for any $g \in \mathcal{V}(f, \Lambda, \mathcal{U})$ the following holds: given $\gamma > 0, \delta^s, \delta^u$ there exists β such that if $x, y \in \Lambda_g$ and*

$$x \notin \cup_{i=1}^r B_{(\delta^s,\delta^u)}(p_i(g)) \bigcup \cup_{i=1}^t B_{(\delta^s,\delta^u)}(q_i(g))$$

and $d(g^j(y), g^j(x)) \leq \beta,\ j \geq 0$ (respec. $j \leq 0$) then $y \in W^{cs}_\gamma(x, g)$ (respec. $y \in W^{cu}_\gamma(x, g)$).

Proof: The same proof of Theorem 4.2 items (3) works without major modifications.

■

Now, we are ready to prove that the expansivity constant is uniform. Arguing by contradiction, assume that this is not the case, that is, there are sequences $g_n \in \mathcal{V}(f, \Lambda, \mathcal{U})$ converging to f, $\alpha_n \to 0$ and $x_n, y_n \in \Lambda_{g_n}$, $x_n \neq y_n$ such that $d(g_n^j(x_n), g_n^j(y_n)) \leq \alpha_n \; \forall j \in \mathbb{Z}$.

Recall that there are only finitely many nonhyperbolic f-periodic points in Λ. For each nonhyperbolic f-periodic point, there is a small neighbourhood such that, since $g \in \mathcal{V}(f, \Lambda, \mathcal{U})$, there is only one g_n-periodic orbit in Λ_{g_n} that remains in this neighbourhood, namely the hyperbolic continuation, and any other g_n-orbit in Λ_{g_n} must leave this neighbourhood in the future or in the past. Therefore, we may assume that x_n is not one of these periodic points, and there is no loss of generality if we assume that

$$x_n \notin \cup_{i=1}^{r} B_{(\delta^s, \delta^u)}(p_i(g_n)) \bigcup \cup_{i=1}^{t} B_{(\delta^s, \delta^u)}(q_i(g_n))$$

where p_i and q_i are as in the previous lemma. Let β be from the previous lemma. Then, for $\alpha_n < \beta$ we conclude that

$$y_n \in W_{\gamma}^{cs}(x_n, g_n) \cap W_{\gamma}^{cu}(x_n, g_n) = x_n,$$

a contradiction. This proves the uniformity of the expansivity constant.

The injectiveness of the map h follows now by standard arguments. It remains to prove that h is surjective. A classical argument for this is that the shadowing property holds uniformly. However, we shall pursue a different argument. For this it is enough to show that $h(\Lambda_g)$ is dense in Λ. Recall that we have a Markov partition of arbitrary small size for Λ (and so contained in the admissible neighbourhood V). The rectangles R_i of this Markov partition are bounded by compact arcs of (central) stable and unstable manifolds of finitely many periodic points (see [PT]). Hence this rectangles have a continuation $R_i(g)$ (in fact, they form a Markov partition for Λ_g). On the other hand, the f-periodic points are dense in Λ, each one is in some R_i and they persist for g (by Theorem 5.1). Furthermore, this continuations can not cross the boundary of the rectangles $R_i(g)$ and thus they are in Λ_g. In particular $h(\Lambda_g)$ is dense in Λ and so h is surjective.

Furthermore, Λ_g is hyperbolic and has local product structure (by the conjugacy), hence it is maximal invariant and a basic set.

Remark 5.4.1 *If the basic piece Λ does not contain saddle-node periodic points (in particular it is maximal invariant), it can be proved with similar methods the following: there is a neighbourhood $\mathcal{U}$ and a nieghbourhood V of Λ such that $\Lambda_g = \cap_n g^n(V)$ is a g-compact invariant set and there is a semiconjugacy $h : \Lambda_g \to \Lambda$ between g and f.*

5.5 Proof of Theorem C'

Let M be the 2-torus. We claim that there exists$f : M \to M$ such that

1. M has dominated splitting, $TM = E \oplus F$.

2. f has just one fixed point and nonhyperbolic periodic point p and $Df_{/F(p)} = 1$. Any other periodic point is hyperbolic.

3. f is conjugated to an Anosovo linear diffeomorphism.

Indeed, it can be constructed a one parameter family f_μ such that f_μ is Anosov for $\mu < 0$ and f_μ is a derived from Ansosov (DA) for $\mu > 0$. This is done trough the bifurcation of the fixed point p (see for instance [R]). $f = f_0$ satisfyies our claim. Since f is conjugated to an Anosov diffeomorphism, there is a sequence of periodic points p_n such that their orbits spend most of the time near p. It follows, denotin by m_n the period of p_n, that

$$\left(\|Df^{m_n}_{/F(p_n)}\|\right)^{\frac{1}{m_n}} \to_n 1.$$

In other words, p_n has a normalized eigenvalue arbitrarily close to 1.

Since p is of saddle-type, it has a continuation in a neighbourhood $\mathcal{V} = \mathcal{U}_1(f)$ of f. Let U_p and $\mathcal{U} \subset \mathcal{U}_1$ be as in Theorem 5.1. Since the maximal invariant set in $M \backslash U_p$ is hyperbolic for f we may assume that the same holds for any $g \in \mathcal{U}$.

Finally let see that f and $\mathcal{U}$ satisfyies the conclusion of Theorem C'. Let $g \in \mathcal{U}$ and let q be a g-periodic point of period ≥ 2. If the g-orbit of q does not intersects U_p then it is hyperbolic. On the other hand, if the orbit of q intersects U_p, since the period of q is ≥ 2 (indeed, the period will be very large) it follows from Theorem 5.1 that q is hyperbolic.

5.6 On the imposibility of a C^2 Frank's lemma

In [F], Franks proved the following simple yet powerful lemma:

Franks' Lemma: *Let θ be a finite set of points in M, let $Q = \oplus_{x\in\theta} TM_x$ and let $Q' = \oplus_{x\in\theta} TM_{f(x)}$. If ϵ is small (independent of θ) and $G : Q \to Q'$ is an ismorphism such that*
$G - df\| < \frac{\epsilon}{10} = \delta$ then there exists a diffeomorphism $g : M \to M$, ϵ-close to f in the C^1 topology, such that $dg_x = G_/TM_x$ for any $x \in]theta$ (and $g = f$ in θ). Moreover if R is a compact subseto of M disjoint from θ we can require $f(x) = g(x)$ for $x \in R$.

When θ is a periodic orbit, the previous lemma implies that we can perturb the tangent map of f along the periodic orbit and find a diffeomorphism g close to f having the same periodic orbit and the tangent map of g along this orbit realize that perturbation. One may ask if the above lemma can be proven within the C^2 topology (without any requirement on the support of the perturbation) for some $\delta > 0$ (much smaller than $\epsilon/10$). Theorem C' implies that this is imposible.

Indeed, if such a statement could be proved, then it would be possible to perturb a diffeomorphism f having a periodic orbit with (normalized) eigenvalue close to 1 to obtain a diffeomorphism close to f having this periodic orbit non-hyperbolic.

6 Proof of Theorem D and E

Theorem D is similar to the proof of the Ω-stability theorem for Axiom A systems satisfying the no-cycle condition. The classical argument is through the construction of a filtration (see [Sh]). In our case we can not build a filtration in the classical sense. For this reason, we push another argument (that also works in the hyperbolic case).

Recall from the Main Theorem that if $L(f)$ has dominated decomposition, then

$$L(f) = \Gamma_1 \cup ... \cup \Gamma_r \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_l \cup \Lambda_1 \cup ... \cup \Lambda_n$$

where $\Gamma_i, i = 1, ..., r$ is a periodic point or it is contained in a normally hyperbolic periodic arc or closed curve (containing periodic points), $\mathcal{C}_i, i = 1, ..., l$ is a normally hyperbolic curve supporting an irrational rotation and $\Lambda_i, i = 1, ..., n$ is a basic piece of $\tilde{L}(f)$.

Definition 6.0.1 Saddle sets *We will consider the following sets:*

1. *the Γ_i sets which consist of a single periodic orbit and it is either of saddle-type or saddle-node type.*
2. *the endpoints of the arcs Γ_i which are periodic points of saddle-type of saddle-node type.*
3. *the basic pieces of $\tilde{L}(f)$ which are neither attractor nor repellers.*

We shall denote these sets by $K_i, i = 1, ..., s$ and denote by $\mathcal{P}$ its collection.

For a periodic point p of saddle type, we denote by $W^{ss}(p)$ the stable manifold of p. For a saddle-node periodic point p we denote by $W^{ss}(p)$ the strong stable manifold. And for basic set Λ whose F-saddle nodes $p_1, ..., p_r$ we define

$$W^{ss}(\Lambda) = \cup_{x \in \Lambda \setminus \{p_1, ..., p_r\}} W^s(x) \bigcup \cup_{i=1}^{r} W^{ss}(p_i).$$

In a similar form, define W^{uu}.

Definition 6.0.2 Strong cycles
For two pieces of $\mathcal{P}$, K_i, K_j we say that $K_i \ll K_j$ if and only if

$$\overline{W^{uu}(K_i) - K_i} \cap \overline{W^{ss}(K_j) - K_j} \neq \emptyset.$$

We say that there is a (strong) cycle in $L(f)$, if there are $i_1, i_2, ..., i_t$ such that $K_{i_1} \ll K_{i_2} \ll ... \ll K_{i_t} \ll K_{i_1}$.

Let $p_1, ..., p_m$ be the non-hyperbolic periodic points in some K_i in $\mathcal{P}$. Give a neighbourhood $\mathcal{U}$ of f consider

$$\mathcal{V}(f,\mathcal{U}) = \cap_{i=1}^{m} \mathcal{V}(f, p_i, \mathcal{U})$$

as in section 5.2.

Now, assuming that there is no (strong) cycle, we shall prove Theorem D for $\mathcal{V}(f,\mathcal{U})$ for some $\mathcal{U}$ small enough, that is, for $g \in \mathcal{V}(f,\mathcal{U})$ we get that $\tilde{L}(g)$ is hyperbolic and $f_{/\tilde{L}(f)}$ is conjugated to $g_{/\tilde{L}(g)}$.

Theorem 6.1 *Let $f \in Diff^2(M)$ such that $L(f)$ has dominated splitting and without strong cycles. Let V be a neighborhood of $\tilde{L}(f)$. Then, there exists $\mathcal{U}$ such that if $g \in \mathcal{V}(f,\mathcal{U})$ we get that $\tilde{L}(g) \subset V$.*

Lets show that Theorem 6.1 imply Theorem D. In fact, by the Main Theorem, we get that $\tilde{L}(f)$ can be decomposed in a finite number of disjoint basic pieces $\Lambda_1 ....\Lambda_n$. Let Λ_i be one of these basic pieces and consider V_i a neighbourhood of Λ_i as in (the proof of) Theorem C. Recall that there exists $\mathcal{U}_i$ such that if $g \in \mathcal{V}(f, \Lambda_i, \mathcal{U}_i)$ then the set

$$\Lambda_{i,g} = \cap_{n\in\mathbb{Z}} g^n(V) \bigcap \tilde{L}(g)$$

is hyperbolic for g and f/Λ_i and $g/\Lambda_{i,g}$ are conjugated.

Let $V = V_1 \cup .. \cup V_n$ and take $\mathcal{U}$ from the above theorem such that $\mathcal{U} \subset \cap_i \mathcal{U}_i$. For $g \in \mathcal{V}(f,\mathcal{U})$ it follows that

$$\tilde{L}(g) = \Lambda_{1,g} \cup ... \cup \Lambda_{n,g}$$

and Theorem D follows.

Now lets prove Theorem 6.1. We shall argue by contradiction, i.e., if the statement is false then we will be able to show the existence of a strong cycle in $L(f)$. So, let V be a neighbourhood of $\tilde{L}(f)$ and assume that there exist a decreasing sequence of neighbourhood $\mathcal{U}_n$ such that $\cap_n \mathcal{U}_n = \{f\}$ and $g_n \in \mathcal{V}(f,\mathcal{U}_n)$ such that $\tilde{L}(g_n) \nsubseteq V$.

For each n take $y_n \in \tilde{L}(g_n) \backslash V$. We may assume that $y_n \to z$ for some z. There is no loss of generality if we assume that y_n belongs to $\omega(x_n, g_n)$ for some point x_n and also that $x_n \to z$. We need the following lemma.

Lemma 6.0.1 *Let $w \notin \tilde{L}(f)$ be such that either $\omega(w,f) \cap \mathcal{P} = \emptyset$ or $w \in W^s(p) \backslash W^{ss}(p)$ where p is a saddle node in some set of $\mathcal{P}$. Then there exist $\mathcal{U}(f)$ and $U(w)$ such that for any $x \in U(w)$ and $g \in \mathcal{V}(f,\mathcal{U})$, $\omega(x,g) \cap U(w) = \emptyset$ holds.*

Proof:

Notice that if $\omega(w,f)\cap\mathcal{P}=\emptyset$ then it is a subset of:

1. an attractor basic piece of $\tilde{L}(f)$)
2. the interior of an (E,P_ϵ)-arc or an (F,P_ϵ)-arc.
3. a closed curve normally hyperbolic.

In each of these cases, the proof of the lemma is straightforward. On the other hand, if $w\in W^s(p)\backslash W^{ss}(p)$, since $g\in\mathcal{V}(f,\mathcal{U})$ for $\mathcal{U}$ small, then w belongs to the basin of attraction of a sink (of g) and the lemma follows.

■

Continuing with the proof of Theorem 6.1, from the last lemma we conclude that $\omega(z,f)\cap\mathcal{P}\neq\emptyset$, and z does not belongs to $W^s(p)\backslash W^{ss}(p)$ where p is a saddle node in some set of $\mathcal{P}$. That is, $\omega(z,f)\subset K_{i_1}$ and $z\in W^{ss}(K_{i_1})$. We need another lemma.

Lemma 6.0.2 *Let $w\notin\tilde{L}(f)$ such that $\omega(w,f)\subset K_i$ for some K_i in $\mathcal{P}$. Let U_i be a small neighbourhood of K_i such that $U_i\cap K_j=\emptyset, j\neq i$, K_j in $\mathcal{P}$ and let $\mathcal{U}$ be a neighbourhood of f. Assume that there are sequences $u_n\to w$, $\mathcal{V}(f,\underline{\mathcal{U}) \ni g_n\to f}$ such that $\omega(u_n,g_n)\not\subseteq U_i$. Then there exists $w_1\notin U_i$ such that $w_1\in\overline{\cup_n\omega(u_n,g_n)}$ and moreover $w_1\in W^{uu}(K_i)$.*

Proof: Notice that exists m_0 such that $f^m(w)\in U_i$ $\forall m\geq m_0$. Then, there is n_0 such that for $n\geq n_0$, $g_n^{m_0}(u_n)\in U_i$. For each $n\geq n_0$ consider $m_n=\min\{m\geq m_0 : g_n^m(u_n)\notin U_i\}$. Let w_1 be an accumulation point of $g_n^{m_n}(u_n)$. It follows that $f^{-n}(w_1)\in U_i$ for $n\geq 0$. Thus, applying lemma 6.0.1 for f^{-1}, we get that $\alpha(w_1,f)\cap\mathcal{P}\neq\emptyset$, so $\alpha(w_1,f)\subset K_i$ and $w_1\in W^{uu}(K_i)$.

■

Now, continuing with the proof of Theorem 6.1, apply the previous lemma to $z=w$ and take $z_1=w_1$. Now, repeating the previous arguments to $z_1=z$ we conclude that there is K_{i_2} in $\mathcal{P}$ such that $\omega(z_1,f)\subset K_{i_2}$. Therefore

$$K_{i_1}\ll K_{i_2}.$$

Inductively we get $K_{i_1}\ll K_{i_2}\ll K_{i_3}\ll ...$ Since there are finitely many set K_i we conclude that it must be a (strong) cycle, a contradiction. This completes the proof of Theorem 6.1 and Theorem D.

Let us consider a diffeomorphism f such that $\Omega(f)$ has dominated splitting. In particular $L(f)\subset\Omega(f)$ has dominated splitting. Assume that there is $x\in$

$\Omega(f)\backslash L(f)$ and it is outside of a neighbourhood of $L(f)$. Then, by Theorem 4.1 $W_{\gamma}^{cu}(x) \subset W^u(x)$ and $W_{\gamma}^{cs}(x) \subset W^s(x)$ and they intersects transversally at x. Arguing as in the proof of Theorem 6.1 we may construct a cylce in $L(f)$. This observation and following *exactly* the proof of Theorem 1 in [NP] yield the following corollary:

Corollary 6.0.1 *Let $f \in Diff^2(M^2)$ and assume that $\Omega(f)$ has dominated splitting. Then $\Omega(f) = L(f)$.*

6.1 Proof of Theorem E

We first recall the definition of topological entropy (see for example [M])

Given a metric space X and a transformation $T : X \to X$, we say that a subset $S \subset X$ is an $(n, \epsilon)-$generator if for every $x \in X$ there is $y \in S$ such that $d(T^j(x), T^j(y)) \leq \epsilon$ for all $0 \leq j \leq n$. Let $r(n, \epsilon) = min\{\text{cardinal}(S) : S \text{ is a } (n, \epsilon) - \text{generator}\}$.

We define the **topological entropy** $h_{top}(T)$ as

$$h_{top}(T) = lim_{\epsilon \to 0} limsup_{n \to \infty} \frac{1}{n} log[r(n, \epsilon)].$$

Lets prove Theorem E. The map $f \to h_{top}(f)$ is continuous in the C^∞-topology (see [N2]). On the other hand, it is well known that $h_{top}(f) = h_{top}(f_{/\Omega(f)})$. If $\Omega(f)$ has dominated splitting, it follows that

$$h_{top}(f) = h_{top}(f_{/\Omega(f)}) = h_{top}(f_{/L(f)}) = h_{top}(f_{/\tilde{L}(f)}).$$

Now, if $\Omega(f)$ has dominated splitting for $f \in \mathcal{U}$, it follows from Theorem D that there exist a open set $\mathcal{V}_f \subset \mathcal{U}$ such that for $g \in \mathcal{V}$, $\tilde{L}(g)$ is conjugated to $\tilde{L}(f)$ and hence $h_{top}(g) = h_{top}(f)$. Therefore, if the topological entropy is not constant in $\mathcal{U}$, the image of the map $\mathcal{U} \ni f \to h_{top}(f)$ contains an interval $[a, b]$. For each $t \in [a, b]$ let $f \in \mathcal{U}$ such that $h_{top}(f) = t$. It follows that there is an open set $\mathcal{V}_t$ such that $h_{top}(g) = t$ for $g \in \mathcal{V}_t$. Thus, the collection $\mathcal{V}_t$, $t \in [a, b]$ is not enumerable and $\mathcal{V}_t \cap \mathcal{V}_{t'} = \emptyset$ if $t \neq t'$. This contradicts that $Diff^\infty(M^2)$ is a separable space.

Remark 6.1.1 *Theorem E remains valid if we assume for $f \in \mathcal{U} \subset Diff^\infty(M^2)$ that $L(f)$ has dominated splitting. In fact, if $L(f)$ has dominated splitting for f in a open set, it can be prove that $L(f) = \Omega(f)$ with similar arguments as in [NP].*

Remark 6.1.2 *Theorem E applies for example to figure a. of previous section when the saddle-node is destroyed: although many new bifurcation (of periodic points) could appear, the topolgical entropy remains constant.*

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Enrique R. Pujals
(enrique@impa.br)
Instituto de Matemática, Universidad Federal do Rio de Janeiro
C.P. 68.530, CEP 21.945-970, Rio de Janeiro, R.J., Brazil

Martín Sambarino
(samba@fing.edu.uy)
IMERL-Fac. de Ingenieria, CC 30, Montevideo-Uruguay.