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Mathematical Programs with Complementarity Constraints: Regularity, Optimality Conditions, and Sensitivity

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Abstract—Various issues in the theory of optimization problems with complementarity constraints are examined. Along with a survey of well-known constraint qualifications and optimality conditions, a number of new results concerning second-order optimality conditions for such problems are given. These results are used in the core of this paper, which is devoted to the sensitivity theory for abnormal optimization problems and its applications to problems with complementarity constraints.

1. INTRODUCTION

It can be said without exaggeration that, in the last decade, enormous activity of experts has been directed at mathematical programs with equilibrium constraints, including complementarity constraints. Numerous examples of such problems arising in applications can be found in [1, 2] and the literature referenced therein. The most important applications are related to the so-called bilevel optimization problems [3]. In addition to its unquestionable applied value, this class of optimization problems is also of substantial mathematical interest. The matter is that the special structure of constraints makes these problems difficult to treat from the point of view of the conventional optimization theory; they are also resistant to efficient numerical solution. Naturally, these two aspects are closely connected with one another. This paper is devoted to the theoretical aspect. For the latest progress in the numerical solution of problems with complementarity constraints, see, for example, [4, 5].

2. SOME RESULTS ON THE SENSITIVITY OF OPTIMIZATION PROBLEMS

We first consider the "conventional" optimization problem, or the mathematical program (MP)

$$f(x) \longrightarrow \min, \quad x \in D, \tag{2.1}$$

where

$$D = \{x \in \mathbb{R}^n \mid F(x) = 0, G(x) \le 0\}. \tag{2.2}$$

Here, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ and $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are smooth mappings.

2.1. Constraint Qualifications and Optimality Conditions

We remind the reader of certain concepts and facts from the MP theory (e.g., see [6]). The Lagrangian function of problem (2.1), (2.2) is introduced by

$$L(x, \lambda, \mu) = f(x) + \langle \lambda, F(x) \rangle + \langle \mu, G(x) \rangle, \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^l, \quad \mu \in \mathbb{R}^m.$$

The *Karush–Kuhn–Tucker optimality condition* (KKT) is as follows. Let $\bar{x} \in \mathbb{R}^n$ be a local solution to problem (2.1), (2.2), and let a constraint qualification (see below) be satisfied at this solution. Then, there exist *multipliers* $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ such that

$$\frac{\partial L}{\partial x}(\bar{x}, \lambda, \mu) = 0, \quad \mu \ge 0, \quad \langle \mu, G(\bar{x}) \rangle = 0. \tag{2.3}$$

Denote by $\mathcal{M} = \mathcal{M}(\bar{x})$ the set composed of the multipliers associated with $\bar{x}$. Thus, $\mathcal{M}$ consists of pairs $(\lambda, \mu) \in \mathbb{R}^l \times \mathbb{R}^m$ satisfying condition (2.3). A point $\bar{x} \in D$ is called a *stationary point* of problem (2.1), (2.2) if $\mathcal{M} \neq \varnothing$. The *KKT system* of this problem is the following set of equations and inequalities:

$$\frac{\partial L}{\partial x}(x, \lambda, \mu) = 0, \quad F(x) = 0, \quad \mu \geq 0, \quad G(x) \leq 0, \quad \langle \mu, G(x) \rangle = 0,$$

This system specifies stationary points of problem (2.1), (2.2) and the corresponding multipliers.

The *Mangasarian–Fromovitz constraint qualification* (MFCQ) at $\bar{x}$ is

$$\operatorname{rank} F'(x) = l, \quad \exists \bar{\xi} \in \ker F'(x) : \langle G_i'(\bar{x}), \bar{\xi} \rangle < 0, \quad \forall i \in A.$$

Here, $A = A(\bar{x}) = \{i = 1, 2, \ldots, m \mid G_i(\bar{x}) = 0\}$ is the index set of inequality constraint active at $\bar{x}$. The *linear independence constraint qualification* (LICQ) is a stronger regularity condition; it requires that the rows of $F'(\bar{x})$ and the vectors $G_i'(\bar{x})$, $i \in A$ be linearly independent. The MFCQ guarantees that, at a local solution $\bar{x}$ to problem (2.1), (2.2), the set $\mathcal{M}$ is nonempty and bounded, whereas the LICQ ensures that this set consists of a single element $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^l \times \mathbb{R}^m$.

Define the cone

$$LD = LD(\bar{x}) = \{\xi \in \ker F'(\bar{x}) \mid \langle G_i'(\bar{x}), \xi \rangle \leq 0 \ \forall i \in A\}, \tag{2.4}$$

which is the first-order outer approximation of the tangent cone of D at $\bar{x} \in D$. The *critical cone* of problem (2.1), (2.2) at $\bar{x}$ is defined by

$$C = C(\bar{x}) = \{\xi \in LD \mid \langle f'(\bar{x}), \xi \rangle \leq 0\}. \tag{2.5}$$

If $\bar{x}$ is a stationary point of problem (2.1), (2.2), then, for each $(\lambda, \mu) \in \mathcal{M}$, it holds that

$$C = \{\xi \in LD \mid \mu_i \langle G_i'(\bar{x}), \xi \rangle = 0 \ \forall i \in A\}. \tag{2.6}$$

If the LICQ is fulfilled at $\bar{x}$, then we have the following *second-order necessary condition* (SONC) for the local optimality of $\bar{x}$ in problem (2.1), (2.2):

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\lambda}, \bar{\mu})[\xi, \xi] \geq 0 \quad \forall \xi \in C.$$

This necessary condition is associated with the *second-order sufficient condition* (SOSC)

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\lambda}, \bar{\mu})[\xi, \xi] > 0 \quad \forall \xi \in C \backslash \{0\}.$$

Furthermore, in the analysis below, we use the so-called *strong second-order sufficient condition* (SSOSC)

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\lambda}, \bar{\mu})[\xi, \xi] > 0 \quad \forall \xi \in C^+ \backslash \{0\},$$

where

$$C^+ = \{\xi \in \ker F'(x) \mid \mu_i \langle G_i'(\bar{x}), \xi \rangle = 0 \ \forall i \in A\}$$

(cf. (2.6)). Second-order necessary conditions for optimality in the case where the LICQ (and even the MFCQ) is not fulfilled can be found in [7]. Here, we limit ourselves to the corresponding sufficient optimality conditions that are not related to constraint qualifications. Indeed, it is these sufficient conditions that we need in what follows.

We introduce the cone

$$\begin{aligned} QD &= QD(\bar{x}) \\ &= \{\xi \in LD \mid \exists u \in \mathbb{R}^n : F'(x)u + F''(\bar{x})[\xi, \xi] = 0, \langle G_i'(\bar{x}), u \rangle + G_i''(\bar{x})[\xi, \xi] \leq 0, i \in A\}, \end{aligned} \tag{2.7}$$

which is the second-order outer approximation of the tangent cone of D at $\bar{x} \in D$ (cf. the cone T in [8]). It is shown in [8, Lemma 3] that if $\mathcal{M} = \varnothing$ (which is possible when the MFCQ is violated), then the following *first-order sufficient condition* (FOSC) is a natural sufficient condition for the local optimality of $\bar{x}$:

$$QD \cap C = \{0\}. \tag{2.8}$$

We emphasize that

$$QD \cap C = \{\xi \in QD \mid \langle f'(\bar{x}), \xi \rangle \le 0\}$$

and (2.8) is a first-order condition in the sense that it uses only the first derivative of the objective function. The FOSC is sufficient for the *linear growth condition* (LGC) to be fulfilled; the latter condition means that there exists a positive $\gamma > 0$ such that

$$f(x) \ge f(\bar{x}) + \gamma|x - \bar{x}| \quad \forall x \in D$$

in a neighborhood of $\bar{x}$. Note that $QD = LD$, provided that the MFCQ is fulfilled; hence, $QD \cap C = LD \cap C = C$, and the FOSC (2.8) takes the form

$$C = \{0\}. \tag{2.9}$$

(In particular, it becomes a true first-order condition). Moreover, under MFCQ this condition is necessary for the LGC (see [9, Proposition 6.2]).

If $\mathcal{M} \ne \varnothing$, then the following is a natural form of using the SOSC:

$$\forall \xi \in C\backslash\{0\} \;\; \exists (\lambda, \mu) \in \mathcal{M} : \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda, \mu)[\xi, \xi] > 0. \tag{2.10}$$

This is sufficient for the *quadratic growth condition* (QGC) to be fulfilled; the latter means that there exists a positive $\gamma > 0$ such that

$$f(x) \ge f(\bar{x}) + \gamma|x - \bar{x}|^2 \quad \forall x \in D$$

in a neighborhood of $\bar{x}$. By [8, Lemma 3], if $\mathcal{M} \ne \varnothing$, then the cone C in (2.10) can be replaced by $QD \cap C$ (the corresponding two conditions are equivalent). Moreover, if the MFCQ is fulfilled, then SOSC (2.10) is necessary for the QGC (see [9, Theorem 6.3]).

2.2. Sensitivity Analysis when an Estimate for the Distance to the Feasible Set of a Perturbed Problem is Available

Assume that the function f and the mappings F and G (and, hence, all the quantities they determine) depend on a parameter $\sigma \in \mathbb{R}^s$. In particular, with each value of this parameter, we associate the perturbed problem

$$f(\sigma, x) \longrightarrow \min, \quad x \in D(\sigma), \tag{2.11}$$

$$D(\sigma) = \{x \in \mathbb{R}^n \mid F(\sigma, x) = 0, G(\sigma, x) \le 0\}. \tag{2.12}$$

Let $\bar{x}$ be a local solution to problem (2.11), (2.12) associated with the base value $\sigma = \bar{\sigma} \in \mathbb{R}^s$. Throughout this paper, we assume that the set A is specified at $\bar{x}$ exactly for this base value of the parameter. Let $B = B(\bar{x})$ be a ball of a small radius centered at $\bar{x}$ (we mean that the radius of this ball is as small as required in all of the assertions below). For each $\sigma \in \mathbb{R}^s$, consider the restriction of problem (2.11), (2.12) to B:

$$f(\sigma, x) \longrightarrow \min, \quad x \in D(\sigma) \cap B.$$

Define the minimum value $\omega(\sigma)$ and the solution set $S(\sigma)$ of this problem:

$$\omega(\sigma) = \inf_{x \in D(\sigma) \cap B} f(\sigma, x), \quad S(\sigma) = \{x \in D(\sigma) \cap B \mid f(\sigma, x) = \omega(\sigma)\}.$$

The behavior of $\omega : \mathbb{R}^s \longrightarrow \mathbb{R}$ and the multifunction mapping $S : \mathbb{R}^s \longrightarrow 2^B$ is the subject of the local sensitivity theory.

In this section, we do not impose any constraint qualifications. Instead, we assume that the following estimate is fulfilled: for a certain $p \in (0, 1]$, it holds that

$$\operatorname{dist}(\bar{x}, D(\sigma)) = O(|\sigma - \bar{\sigma}|^p), \tag{2.13}$$

for $\sigma \in \mathbb{R}^s$. In particular, this assumption presupposes that $D(\sigma)$ is nonempty for all σ sufficiently close to $\bar{\sigma}$. According to Robinson's stability theorem [10, Corollary 1], estimate (2.13) is valid for $p = 1$, provided that the MFCQ is fulfilled at $\bar{x}$. Under the weaker conditions given in [11, Theorem 4], one can obtain estimate (2.13) with $p = 1/2$ and even, in certain special cases, with $p = 1$.

The following theorem (on the stability and an upper bound for the minimum value) can be proved by a standard argument.

Theorem 1. *Assume that, for a certain* $p \in (0, 1]$, *estimate* (2.13) *is fulfilled for* $\sigma \in \mathbb{R}^s$. *Then, the function* ω *is continuous at the point* $\bar{\sigma}$ *and* $S(\sigma) \neq \varnothing$ *for any* $\sigma \in \mathbb{R}^s$ *sufficiently close to* $\bar{\sigma}$. *Moreover,*

$$\sup_{x \in S(\sigma)} \operatorname{dist}(x, S(\bar{\sigma})) \longrightarrow 0 \ \text{ as } \ \sigma \longrightarrow \bar{\sigma}$$

and it holds that

$$\omega(\sigma) \le \omega(\bar{\sigma}) + O(|\sigma - \bar{\sigma}|^p). \tag{2.14}$$

In particular, if $\bar{x}$ *is a strict local solution to problem* (2.11), (2.12) *for* $\sigma = \bar{\sigma}$, *then*

$$\sup_{x \in S(\sigma)} |x - \bar{x}| \longrightarrow 0 \ \text{ при } \ \sigma \longrightarrow \bar{\sigma}.$$

The following two theorems yield estimates for solutions to perturbed problems and lower bounds for ω, provided that the FOSC and the SOSC are fulfilled, respectively.

Theorem 2. *Assume that sequences* $\{\sigma^k\} \subset \mathbb{R}^s$ *and* $\{x^k\} \subset \mathbb{R}^n$ *are such that* $\{\sigma^k\} \longrightarrow \bar{\sigma}$ *and* $\{x^k\} \longrightarrow \bar{x}$ *as* $k \longrightarrow \infty$, $x^k \in S(\sigma^k)$ *for* all k, *and*

$$\omega(\sigma^k) \le \omega(\bar{\sigma}) + O(|\sigma^k - \bar{\sigma}|^p) \tag{2.15}$$

for some $p \in (0, 1]$.

Then,

(a) *if* FOSC (2.8) *is fulfilled, we have*

$$|x^k - \bar{x}| = O(|\sigma^k - \bar{\sigma}|^{\min\{p, 1/2\}}),$$

$$\omega(\sigma^k) \ge \omega(\bar{\sigma}) + O(|\sigma^k - \bar{\sigma}|^{\min\{p, 1/2\}});$$

(b) *if* (2.9) *is fulfilled, we have*

$$|x^k - \bar{x}| = O(|\sigma^k - \bar{\sigma}|^p), \tag{2.16}$$

$$\omega(\sigma^k) \ge \omega(\bar{\sigma}) + O(|\sigma^k - \bar{\sigma}|^p); \tag{2.17}$$

(c) *if FOSC* (2.8) *is fulfilled and, for any sufficiently large k, there exists a* $u^k \in \mathbb{R}^n$ *such that*

$$\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})u^k - \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma^k - \bar{\sigma}) = 0, \tag{2.18}$$

$$\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), u^k \right\rangle - \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma^k - \bar{\sigma} \right\rangle \le 0 \quad \forall i \in A, \tag{2.19}$$

then estimates (2.16) *and* (2.17) *hold true.*

We singled out assertion (c), because this case is typical (in a certain sense) for problems with complementarity constraints (see below).

Proof. Assertions (a) and (b) can be justified by a small modification of the proof in [8, Theorem 5] or the proof of assertion (c) given below.

Suppose that (2.16) does not hold. Without loss of generality, we can assume that

$$|\sigma^k - \bar{\sigma}|^p / |x^k - \bar{x}| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{2.20}$$

Define $\xi^k = (x^k - \bar{x})/|x^k - \bar{x}|$. It can be assumed that the sequence $\{\xi^k\}$ converges to a certain $\xi \in \mathbb{R}^n \backslash \{0\}$; moreover,

$$0 = F(\sigma^k, x^k) = F(\sigma^k, x^k) - F(\bar{\sigma}, \bar{x}) = |x^k - \bar{x}| \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi^k + O(|\sigma^k - \bar{\sigma}|) + o(|x^k - \bar{x}|),$$

$$0 \ge G_i(\sigma^k, x^k) = G_i(\sigma^k, x^k) - G_i(\bar{\sigma}, \bar{x})$$

$$= |x^k - \bar{x}| \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \xi^k \right\rangle + O(|\sigma^k - \bar{\sigma}|) + o(|x^k - \bar{x}|) \quad \forall i \in A.$$

This relation, combined with (2.4) and (2.20), implies that $\xi \in LD$ (recall that $p \in (0, 1]$). Furthermore, in view of (2.18) and (2.19), we have

$$0 = \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(u^k + x^k - \bar{x}) + F(\sigma^k, x^k) - F(\bar{\sigma}, \bar{x}) - \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma^k - \bar{\sigma}) - \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) =$$

$$= \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(u^k + x^k - \bar{x}) + \frac{1}{2}|x^k - \bar{x}|^2 \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi^k, \xi^k] + O(|\sigma^k - \bar{\sigma}|^2)$$

$$+ O(|\sigma^k - \bar{\sigma}|^2) + O(|\sigma^k - \bar{\sigma}||x^k - \bar{x}|) + o(|x^k - \bar{x}|^2),$$

$$0 \ge \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), u^k + x^k - \bar{x} \right\rangle + G_i(\sigma^k, x^k) - G_i(\bar{\sigma}, \bar{x}) - \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma^k - \bar{\sigma} \right\rangle - \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), x^k - \bar{x} \right\rangle$$

$$= \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), u^k + x^k - \bar{x} \right\rangle + \frac{1}{2}|x^k - \bar{x}|^2 \frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi^k, \xi^k] + O(|\sigma^k - \bar{\sigma}|^2) + O(|\sigma^k - \bar{\sigma}||x^k - \bar{x}|) + o(|x^k - \bar{x}|^2)$$

$$\forall i \in A.$$

Now, taking into account (2.7), (2.20), and the fact that the image of a finite-dimensional linear operator is closed, one can easily show that $\xi \in QD$.

On the other hand, by virtue of (2.15), we have

$$O(|\sigma^k - \bar{\sigma}|^p) \ge \omega(\sigma^k) - \omega(\bar{\sigma}) = f(\sigma^k, x^k) - f(\bar{\sigma}, \bar{x}) = |x^k - \bar{x}| \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi^k \right\rangle + O(|\sigma^k - \bar{\sigma}|) + o(|x^k - \bar{x}|).$$

In view of (2.20), this implies that $\left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \le 0$. Therefore, by (2.5), $\xi \in (QD \cap C) \backslash \{0\}$, which contradicts FOSC (2.8).

Thus, estimate (2.16) is proved. Estimate (2.17) is an immediate implication of (2.16). The theorem is proved.

Theorem 3. *Assume that, under the hypotheses of Theorem* 2 $\mathcal{M} \neq \varnothing$, *SOSC* (2.10) *is fulfilled. Then, we have*

$$|x^k - \bar{x}| = O((\max\{0, \omega(\sigma^k) - \omega(\bar{\sigma})\})^{1/2}) + O(|\sigma^k - \bar{\sigma}|^{\min\{p, 1/2\}}) = O(|\sigma^k - \bar{\sigma}|^{p/2}),$$

$$\omega(\sigma^k) \ge \omega(\bar{\sigma}) + O(|\sigma^k - \bar{\sigma}|^{\min\{2p, 1\}}).$$

The proof of this theorem is essentially the same as that of Theorem 6 in [8].

If the MFCQ is assumed to hold, then one can give a subtler sensitivity analysis, which includes "quantitative" results for perturbations in a given direction in the parameter space. The relevant theory is presented in [9, 12, 13] and is used below.

3. FORMULATION OF THE MATHEMATICAL PROGRAM WITH COMPLEMENTARITY CONSTRAINTS

The basic (and generally accepted) formulation of a *mathematical program with complementarity constraints* (MPCC) is as follows:

$$f(x) \longrightarrow \min, \quad x \in D, \tag{3.1}$$

where

$$D = \{x \in \mathbb{R}^n \mid G(x) \geq 0, H(x) \geq 0, \langle G(x), H(x)\rangle = 0\}. \tag{3.2}$$

Here, $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function and $G, H: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ are smooth mappings. This is an important particular case of the *mathematical program with equilibrium constraints*, in which the feasible set is specified by a variational inequality. The constraints may also include "conventional" equalities and inequalities; however, this extension does not involve any additional fundamental difficulties and is not considered here. The meaning of the complementarity constraints is as follows: the feasible points are those where all components of G and H are nonnegative; moreover, in each pair of the corresponding components of G and H, at least one component is zero.

Example 1. Assume that $n = 2$, $m = 1$, $f(x) = a_1x_1 + a_2x_2$, $G(x) = x_1$, and $H(x) = x_2$, where $x \in \mathbb{R}^2$ and a_1 and a_2 are scalar parameters.

The feasible set is formed by two rays (the so-called branches; see below), namely, the nonnegative coordinate half-axes. If $a_1 \geq 0$ and $a_2 \geq 0$, then $\bar{x} = 0$ is a solution to MPCC (3.1), (3.2).

This model example is repeatedly used below.

We set $I = \{1, 2, \ldots, m\}$. Let $\bar{x} \in \mathbb{R}^n$ be a feasible point of problem (3.1), (3.2). Define the related index sets

$$I_G = I_G(\bar{x}) = \{i \in I \mid G_i(\bar{x}) = 0\}, \quad I_H = I_H(\bar{x}) = \{i \in I \mid H_i(\bar{x}) = 0\},$$
$$I_0 = I_0(\bar{x}) = I_G \cap I_H.$$

Necessarily, $I_G \cup I_H = I$,

$$G_i(\bar{x}) > 0 \quad \forall i \in I\backslash I_G = I_H\backslash I_G = I_H\backslash I_0, \tag{3.3}$$

$$H_i(\bar{x}) > 0 \quad \forall i \in I\backslash I_H = I_G\backslash I_H = I_G\backslash I_0. \tag{3.4}$$

The following interpretation of MPCC constraints can be helpful in local considerations: in the intersection with a neighborhood of $\bar{x}$,

$$D = \{x \in \mathbb{R}^n \mid G_i(x) = 0, i \in I_G\backslash I_H, H_i(x) = 0, i \in I_H\backslash I_G, G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0, i \in I_0\}. \tag{3.5}$$

The equality $I_0 = \varnothing$ is called the *strict complementarity condition* (SCC). If the SCC is fulfilled, then (3.5) implies that

$$D = \{x \in \mathbb{R}^n \mid G_i(x) = 0, i \in I_G, H_i(x) = 0, i \in I_H\},$$

i.e., the feasible set D near $\bar{x}$ is specified by smooth equalities, and the complementarity structure of constraints is lost. Therefore, in local considerations, the case where the SCC is fulfilled is simple and of little interest (in the sense that it is treated by conventional tools). It is more important, however, that the SCC at an MPCC *solution* is considered a too restrictive assumption. That is why this assumption is usually not imposed in the modern literature.

In Example 1, the SCC is fulfilled at any feasible point except $\bar{x} = 0$, where $I_G = I_H (= I = I_0)$. Note that, when $a_1 > 0$ and $a_2 > 0$ in this example, the point $\bar{x} = 0$ is a unique solution. If at least one of the numbers a_1 and a_2 is negative, then there are no solutions. This is an illustration of the fact that the SCC is often violated at an MPCC solution.

The nature of the fundamental difficulties inherent in the theoretical analysis and an efficient numerical solution of the MPCC is explained in the following section.

4. THE LACK OF CONVENTIONAL REGULARITY AND STATIONARITY CONCEPTS

The constraints in problem (3.1), (3.2) are irregular at any feasible point even if this point satisfies the SCC (this fact was, probably, first mentioned in [3]). Namely, if $I_G = I_H (= I_0 = I)$ (as at the point $\bar{x} = 0$ in Example 1), then the gradient of the equality constraint is equal to zero and the MFCQ cannot be fulfilled. Otherwise (as at feasible points distinct from zero in Example 1), this gradient is given by

$$\begin{aligned}(G'(x))^{\mathrm{T}}H(\bar{x}) + (H'(\bar{x}))^{\mathrm{T}}G(\bar{x}) &= \sum_{i \in I \setminus I_H} H_i(\bar{x})G_i'(\bar{x}) + \sum_{i \in I \setminus I_G} G_i(\bar{x})H_i'(\bar{x}) \\ &= \sum_{i \in I_G \setminus I_H} H_i(\bar{x})G_i'(\bar{x}) + \sum_{i \in I_H \setminus I_G} G_i(\bar{x})H_i'(\bar{x}),\end{aligned} \tag{4.1}$$

while the gradients of the active inequality constraints are

$$G_i'(\bar{x}), \quad i \in I_G, \quad H_i'(\bar{x}), \quad i \in I_H. \tag{4.2}$$

In this case, the MFCQ implies that there exists an element $\bar{\xi} \in \mathbb{R}^n$ such that

$$\begin{gathered}\langle G_i'(\bar{x}), \bar{\xi}\rangle > 0 \quad \forall i \in I_G, \quad \langle H_i'(\bar{x}), \bar{\xi}\rangle > 0 \quad \forall i \in I_H, \\ \sum_{i \in I_G \setminus I_H} H_i(\bar{x})\langle G_i'(\bar{x}), \bar{\xi}\rangle + \sum_{i \in I_H \setminus I_G} G_i(\bar{x})\langle H_i'(\bar{x}), \bar{\xi}\rangle = 0,\end{gathered} \tag{4.3}$$

which is impossible, since, according to (3.3), (3.4), and (4.3), all the terms in the sums on the left-hand side of this equality are positive. Thus, the MFCQ cannot be fulfilled; hence, the LICQ and other constraint qualifications that are stronger than the MFCQ cannot be fulfilled as well.

In general, the MFCQ implies the existence of feasible points at which all inequality constraints are fulfilled as strict inequalities. There are no such points in the case of the MPCC.

According to what has been said, the (local) optimality of $\bar{x}$ in problem (3.1), (3.2) does not, in general, entail the KKT optimality condition. Here, the KKT means the existence of multipliers $\mu_G, \mu_H \in \mathbb{R}^m$ and $\nu \in \mathbb{R}$ such that

$$\frac{\partial L}{\partial x}(\bar{x}, \mu_G, \mu_H, \nu) = 0, \tag{4.4}$$

$$\mu_G \geq 0, \quad \langle \mu_G, G(\bar{x})\rangle = 0, \quad \mu_H \geq 0, \quad \langle \mu_H, H(\bar{x})\rangle = 0, \tag{4.5}$$

where

$$L(x, \mu_G, \mu_H, \nu) = f(x) - \langle \mu_G, G(x)\rangle - \langle \mu_H, H(x)\rangle + \nu\langle G(x), H(x)\rangle,$$

$$x \in \mathbb{R}^n, \quad \mu_G, \mu_H \in \mathbb{R}^m, \quad \nu \in \mathbb{R},$$

is the Lagrangian function of problem (3.1), (3.2). Nevertheless, we show below that this is a quite reasonable stationarity concept for MPCC. From now on, we assume that $\mathcal{M}$ consists of triplets $(\mu_G, \mu_H, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ satisfying (4.4) and (4.5).

With a given feasible point $\bar{x}$ of problem (3.1), (3.2), we associate two conventional MPs. The *relaxed MP* (RMP) is

$$f(x) \longrightarrow \min, \quad x \in D_{\mathrm{RMP}}, \tag{4.6}$$

where

$$\begin{aligned}D_{\mathrm{RMP}} &= D_{\mathrm{RMP}}(\bar{x}) \\ &= \{x \in \mathbb{R}^n \mid G_i(x) = 0,\, i \in I_G \setminus I_H,\, H_i(x) = 0,\, i \in I_H \setminus I_G,\, G_i(x) \geq 0,\, H_i(x) \geq 0,\, i \in I_0\}.\end{aligned} \tag{4.7}$$

Its constraints are obtained by removing the equality constraints from (3.5).

The *tightened MP* (TMP) is

$$f(x) \longrightarrow \min, \quad x \in D_{\text{TMP}}, \tag{4.8}$$

where

$$D_{\text{TMP}} = D_{\text{TMP}}(\bar{x}) = \{x \in \mathbb{R}^n \mid G_i(x) = 0, i \in I_G, H_i(x) = 0, i \in I_H\}. \tag{4.9}$$

Here, the constraints are obtained by replacing all of the inequality constraints in (3.5) by equations (and by removing the equality constraints in (3.5) that become redundant).

It is obvious that, in the intersection with a neighborhood of $\bar{x}$, we have the inclusions

$$D_{\text{TMP}} \subset D \subset D_{\text{RMP}}, \tag{4.10}$$

moreover, $\bar{x}$ is a feasible point for all of these problems. We define the MPCC Lagrangian of problem (3.1), (3.2) by

$$\mathcal{L}(x, \lambda_G, \lambda_H) = f(x) - \langle \lambda_G, G(x) \rangle - \langle \lambda_H, H(x) \rangle, \quad x \in \mathbb{R}^n, \quad \lambda_G, \lambda_H \in \mathbb{R}^m.$$

It is clear that this is the conventional Lagrangian function for the TMP and RMP if the inequalities

$$G_i(x) \ge 0, \quad i \in I_H \backslash I_G, \quad H_i(x) \ge 0, \quad i \in I_G \backslash I_H, \tag{4.11}$$

which are inactive at $\bar{x}$, are formally added to the constraints in these problems. If the KKT optimality condition is fulfilled at $\bar{x}$ for the RMP (4.6), (4.7) (respectively, the TMP (4.8), (4.9)), then we say that $\bar{x}$ is a *strongly stationary* (respectively, *weakly stationary*) point for the original MPCC (see [14]). Thus, weak stationarity implies the existence of multipliers $\lambda_G, \lambda_H \in \mathbb{R}^m$ such that

$$\frac{\partial \mathcal{L}}{\partial x}(\bar{x}, \lambda_G, \lambda_H) = 0, \tag{4.12}$$

$$(\lambda_G)_i G_i(\bar{x}) = 0, \quad (\lambda_H)_i H_i(\bar{x}) = 0, \quad i \in I. \tag{4.13}$$

The strong stationarity means that, in addition,

$$(\lambda_G)_i \ge 0, \quad (\lambda_H)_i \ge 0, \quad i \in I_0. \tag{4.14}$$

According to the left inclusion in (4.10), the local optimality of $\bar{x}$ in the MPCC implies its local optimality in the TMP. In particular, weak stationarity is a necessary optimality condition for the MPCC if the *MPCC linear independence constrained qualification* (MPCC-LICQ) is fulfilled. The latter requires that the gradients in (4.2) be linearly independent (which is equivalent to the LICQ fulfilled at $\bar{x}$ for the TMP). However, in general, this concept of stationarity is too weak. It may appear that strong stationarity is, on the contrary, a too strong stationarity concept for the MPCC. However, this concept is considered quite reasonable and natural in the sensitivity analysis and in justifying Newton-type methods for the MPCC. The reason for this is as follows. If the MPCC-LICQ is fulfilled, then strong stationarity (as well as the weak one) is a necessary optimality condition for the MPCC (see [14, Theorem 4] and Theorem 4 below). At the same time, the MPCC-LICQ is a generic condition [15], in contrast to the SCC. We emphasize that the MPCC-LICQ is the same as the conventional LICQ for both TMP and RMP.

It can be directly verified (see [4, Proposition 4.1]) that strong stationarity is equivalent to the KKT optimality condition fulfilled for the original MPCC (3.1), (3.2). To be more exact, one can easily prove the following assertion.

Proposition 1. *If* $\bar{x} \in D$ *is a stationary point of MPCC* (3.1), (3.2) *in the sense that there exist multipliers* $\mu_G, \mu_H \in \mathbb{R}^m$ *and* $\nu \in \mathbb{R}$ *satisfying* (4.4) *and* (4.5), *then* $\bar{x}$ *is a strongly stationary point of this problem. Moreover, as multipliers satisfying* (4.12)–(4.14), *one can take*

$$(\lambda_G)_i = (\mu_G)_i - \nu H_i(\bar{x}), \quad (i \in I_G \backslash I_H, \quad (\lambda_H)_i) = (\mu_H)_i - \nu G_i(\bar{x}), \quad i \in I_H \backslash I_G, \tag{4.15}$$

$$(\lambda_G)_i = (\mu_G)_i, \quad i \in I_H, \quad (\lambda_H)_i = (\mu_H)_i, \quad i \in I_G. \tag{4.16}$$

Conversely, if $\bar{x}$ *is a strongly stationary point of MPCC* (3.1), (3.2) (*i.e., there exist multipliers* $\lambda_G, \lambda_H \in \mathbb{R}^m$ *satisfying* (4.12)–(4.14)), *then* $\bar{x}$ *is a stationary point of this problem. Moreover, any multipliers* μ_G, $\mu_H \in \mathbb{R}^m$ *and* $\nu \in \mathbb{R}$ *related by* (4.15), (4.16), *and the inequality*

$$\nu \geq \max\left\{\max\left\{-\frac{(\lambda_G)_i}{H_i(\bar{x})}\middle|\ i \in I_G \backslash I_H\right\}, \max\left\{-\frac{(\lambda_H)_i}{G_i(\bar{x})}\middle|\ i \in I_H \backslash I_G\right\}\right\} \tag{4.17}$$

satisfy (4.4) *and* (4.5). *In addition, for all* $\xi \in \mathbb{R}^n$, *we have*

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] = \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \lambda_G, \lambda_H)[\xi, \xi] + 2\nu \sum_{i \in I} \langle G_i'(\bar{x}), \xi\rangle \langle H_i'(\bar{x}), \xi\rangle. \tag{4.18}$$

However, the existence of multipliers μ_G, μ_H and ν satisfying (4.4) and (4.5) is insufficient for justifying Newton-type methods. To prove superlinear convergence, one must either modify conventional methods in a special way [5] or use very special assumptions and arguments [4]. Also, from a theoretical viewpoint, it is desirable to have as subtle optimality conditions as possible (i.e., subtler than strong stationarity), and results of this kind do exist (see the next section). It must be kept in mind though that, besides being subtle, the optimality conditions should also be sufficiently simple in the sense that they should be convenient to use (in, say, numerical methods).

The most lucid, simple, and natural approach to the MPCC (including the derivation of subtle and proper optimality conditions) is described in the following section. At the same time, this approach can be an illustration of the fact that the desire of maximum conceptual simplicity and the "correctness" of resulting optimality conditions may limit their practical utility.

5. PIECEWISE ANALYSIS

Denote by $\mathcal{I} = \mathcal{I}(\bar{x})$ the set composed of the partitions of I_0, i.e., pairs of index sets (I_1, I_2) such that $I_1 \cup I_2 = I_0$ and $I_1 \cap I_2 = \varnothing$. It is clear that $\mathcal{I}$ is a finite set and $|\mathcal{I}| = 2^{|I_0|}$. For each pair $(I_1, I_2) \in \mathcal{I}$, the *branch* $D_{(I_1, I_2)}$ of D is defined as

$$\begin{gathered} D_{(I_1, I_2)} = \{x \in \mathbb{R}^n \mid G_i(x) = 0, i \in I_G \backslash I_H, H_i(x) = 0, i \in I_H \backslash I_G, \\ G_i(x) = 0, H_i(x) \geq 0, i \in I_1, G_i(x) \geq 0, H_i(x) = 0, i \in I_2\}. \end{gathered} \tag{5.1}$$

It is obvious that, in the intersection with a neighborhood of $\bar{x}$, the set D splits into branches of this kind. More specifically, the inclusion chain (4.10) can be supplemented as follows: for all $(I_1, I_2) \in \mathcal{I}$,

$$D_{\mathrm{TMP}} = \bigcap_{(J_1, J_2) \in \mathcal{I}} D_{(J_1, J_2)} \subset D_{(I_1, I_2)} \subset D = \bigcup_{(J_1, J_2) \in \mathcal{I}} D_{(J_1, J_2)} \subset D_{\mathrm{RMP}}, \tag{5.2}$$

and $\bar{x}$ belongs to each branch.

The cone LD defined by (2.4) has the form

$$\begin{gathered} LD \\ = \{\xi \in \mathbb{R}^n \mid \langle G_i'(\bar{x}), \xi\rangle = 0, i \in I_G \backslash I_H, \langle H_i'(\bar{x}), \xi\rangle = 0, i \in I_H \backslash I_G, \langle G_i'(\bar{x}), \xi\rangle \geq 0, \langle H_i'(\bar{x}), \xi\rangle \geq 0, i \in I_0\}. \end{gathered} \tag{5.3}$$

We introduce the cone

$$BD = BD(\bar{x}) = \{\xi \in LD \mid \langle G_i'(\bar{x}), \xi\rangle \langle H_i'(\bar{x}), \xi\rangle = 0, i \in I_0\}, \tag{5.4}$$

which is obtained from (3.5) by linearizing G and H at $\bar{x}$. By simple calculations, one can verify that this cone and the cone QD defined by (2.7) satisfy $QD \subset BD$; moreover, if the MPCC-LICQ is fulfilled at $\bar{x}$, then we have the equality $QD = BD$.

A point $\bar{x}$ of the MPCC is called *B-stationary* (or *piecewise stationary*) if

$$\langle f'(\bar{x}), \xi\rangle \geq 0 \quad \forall \xi \in BD$$

(see [1, 14]; however, note that this concept of B-stationarity does not quite conform to that used for conventional optimization problems [16]). It is easy to see that

$$BD = \bigcup_{(I_1, I_2) \in \mathcal{I}} LD_{(I_1, I_2)}, \tag{5.5}$$

where, for each partition $(I_1, I_2) \in \mathcal{I}$, the cone

$$LD_{(I_1, I_2)} = \{\xi \in \mathbb{R}^n \mid \langle G_i'(\bar{x}), \xi\rangle = 0, i \in (I_G \backslash I_H) \cup I_1, \langle H_i'(\bar{x}), \xi\rangle = 0, i \in (I_H \backslash I_G) \cup I_2,$$
$$\langle G_i'(\bar{x}), \xi\rangle \geq 0, i \in I_2, \langle H_i'(\bar{x}), \xi\rangle \geq 0, i \in I_1\}$$

is again defined by (2.4). Hence, B-stationarity is equivalent to the KKT optimality condition fulfilled at $\bar{x}$ in the *piecewise problem*

$$f(x) \longrightarrow \min, \quad x \in D_{(I_1, I_2)} \tag{5.6}$$

for each partition $(I_1, I_2) \in \mathcal{I}$; i.e., there exist multipliers $\lambda_G, \lambda_H \in \mathbb{R}^m$ satisfying (4.12), (4.13), and the relations

$$(\lambda_G)_i \geq 0, \quad i \in I_2, \quad (\lambda_H)_i \geq 0, \quad i \in I_1. \tag{5.7}$$

This and the second inclusion in (5.2) imply that B-stationarity is a necessary optimality condition for the MPCC if, say, the *piecewise MFCQ* (i.e., the MFCQ for the constraints specifying each branch $D_{(I_1, I_2)}$ ($(I_1, I_2) \in \mathcal{I}$) is fulfilled at $\bar{x}$. Note that the MPCC-LICQ is equivalent to the LICQ fulfilled for each branch; hence, the MPCC-LICQ implies the piecewise MFCQ. Also, note that the strong stationarity of $\bar{x}$ in problem (3.1), (3.2) implies its B-stationarity, and B-stationarity entails the weak stationarity of this point.

Now assume that $\bar{x}$ is a B-stationary point of problem (3.1), (3.2) and the MPCC-LICQ is fulfilled at this point. Then, for each partition $(I_1, I_2) \in \mathcal{I}$, there exists a unique set of multipliers (λ_G, λ_H) satisfying (4.12), (4.13), and (5.7). Moreover, as noted above, the MPCC-LICQ is equivalent to the LICQ for the TMP, which implies that there exists only one set of multipliers satisfying (4.12) and (4.13). Thus, the sets of multipliers are the same for all piecewise problems, and they are identical with the unique set of multipliers for the TMP. Moreover, (5.7) implies that this set satisfies (4.14); i.e., $\bar{x}$ is a strongly stationary point.

Summarizing what has been said, we arrive at the following theorem (cf. [14, Theorem 4]), which describes the relation between different concepts of stationarity for the MPCC, provided that the MPCC-LICQ is fulfilled.

Theorem 4. *Let the MPCC-LICQ be fulfilled at $\bar{x} \in D$. If $\bar{x}$ is a local solution to problem* (3.1), (3.2), *then $\bar{x}$ is a B-stationary point of this problem; moreover, the B-stationarity of $\bar{x}$ is equivalent to its strong stationarity, and the corresponding pain of multipliers $(\bar{\lambda}_G, \bar{\lambda}_H)$ is uniquely determined. Furthermore, the weak stationarity condition and the stationarity condition for $\bar{x}$ in the piecewise problem* (5.6), (5.1) *for any partition $(I_1, I_2) \in \mathcal{I}$ can be fulfilled only with this pair of multipliers.*

Summarizing, we can say that B-stationarity is the most natural stationarity concept among those discussed above, because it takes into account the combinatorial character of the feasible set inherent in the MPCC. At the same time, it is this feature that makes B-stationarity difficult to use in practice. Indeed, to verify this condition, one needs to verify the KKT optimality conditions for $2^{|I_0|}$ "conventional" optimization problems, and this number is often enormous in applications. Various approaches that allow one to get rid of the combinatorial character of the B-stationarity condition while still keeping the resulting stationarity concept natural were examined in [16–19]. Necessary optimality conditions for the MPCC that use Mordukhovich's generalized derivatives were proposed in [20, 21]; however, these concepts are also of a combinatorial nature.

On the other hand, verifying weak or strong stationarity reduces to the verification of the KKT optimality conditions for a single conventional optimization problem, namely, for the TMP and the RMP, respectively. Furthermore, the analysis presented above shows that if the generic MPCC-LICQ condition is fulfilled, then strong stationarity is equivalent to B-stationarity, which makes the latter much easier to verify in these problems.

In Example 1, the MPCC-LICQ is fulfilled at $\bar{x} = 0$. For $a_1 \geq 0$ and $a_2 \geq 0$, this is a strongly stationary point with the unique multipliers $\lambda_G = \bar{\lambda}_G = a_1 \geq 0$ and $\lambda_H = \bar{\lambda}_H = a_2 \geq 0$. Note that this point satisfies the KKT optimality condition (4.4), (4.5) with the same multipliers $\mu_G = \bar{\lambda}_G$ and $\mu_H = \bar{\lambda}_H$ and an arbitrary ν. Also, note that BD is the union of nonnegative coordinate half-axes and B-stationarity is actually equivalent to strong stationarity in this example.

6. SECOND-ORDER NECESSARY CONDITION AND SUFFICIENT OPTIMALITY CONDITIONS

Using (2.5) and (5.3), we determine the critical cone C of the MPCC at a point $\bar{x} \in D$. We begin with second-order conditions, provided that the MPCC-LICQ is fulfilled. Recall that the MPCC-LICQ at $\bar{x}$ ensures that the LICQ is fulfilled at this point for each piecewise problem (5.6). This, combined with the SONC and SOSC (see Subsection 2.1), equality (5.5), and Theorem 4, implies the following *MPCC second-order necessary condition* (MPCC-SONC) and the corresponding *MPCC second-order sufficient condition* (MPCC-SOSC) for the local optimality of $\bar{x}$ in the MPCC (cf. [14, Theorem 7]).

Theorem 5. *Let the MPCC-LICQ be fulfilled at a point $\bar{x} \in D$. Then:*

(a) *if $\bar{x}$ is a local solution to problem* (3.1), (3.2), *then the pair of multipliers $(\bar{\lambda}_G, \bar{\lambda}_H)$ described in Theorem* 4 *satisfies the relation*

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)[\xi, \xi] \geq 0 \quad \forall \xi \in BD \cap C$$

(*the MPCC-SONC*);

(b) *if $\bar{x}$ is a strongly stationary point of problem* (3.1), (3.2) *and the corresponding pair of multipliers $(\bar{\lambda}_G, \bar{\lambda}_H)$ satisfies the relation*

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)[\xi, \xi] > 0 \quad \forall \xi \in (BD \cap C)\backslash\{0\} \tag{6.1}$$

(*the MPCC-SOSC*), *then $\bar{x}$ is a strict local solution of problem* (3.1), (3.2).

Note that, according to (5.5),

$$BD \cap C = \{\xi \in BD \mid \langle f'(\bar{x}), \xi\rangle \leq 0\} = \bigcup_{(I_1, I_2) \in \mathcal{I}} C_{(I_1, I_2)}, \tag{6.2}$$

where

$$C_{(I_1, I_2)} = C_{(I_1, I_2)}(\bar{x}) = \{\xi \in LD_{(I_1, I_2)} \mid \langle f'(\bar{x}), \xi\rangle \leq 0\}$$
$$= \{\xi \in LD_{(I_1, I_2)} \mid (\bar{\lambda}_G)_i\langle G_i'(\bar{x}), \xi\rangle = 0, i \in I_2, (\bar{\lambda}_H)_i\langle H_i'(\bar{x}), \xi\rangle = 0, i \in I_1\}$$

is the critical cone of the piecewise problem (5.6), (5.1) at $\bar{x}$ defined by (2.5). It is obvious that the MPCC-SOSC is equivalent to the SOSC

$$\left\langle \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)\xi, \xi \right\rangle > 0 \quad \forall \xi \in C_{(I_1, I_2)}\backslash\{0\} \tag{6.3}$$

fulfilled for the piecewise problem for any partition $(I_1, I_2) \in \mathcal{I}$. Note that, in general, the MPCC-SOSC does not imply the SOSC for the TMP even if the MPCC-LICQ is fulfilled (see [14]), though the critical cones of these problems do coincide.

In what follows, we say that the *MPCC strong second-order sufficient condition* (MPCC-SSOSC; see [22]) is fulfilled at $\bar{x}$ if the SSOSC

$$\left\langle \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)\xi, \xi \right\rangle > 0 \quad \forall \xi \in C^+_{(I_1, I_2)}\backslash\{0\} \tag{6.4}$$

is fulfilled for the piecewise problem (5.6), (5.1) for any partition $(I_1, I_2) \in \mathcal{I}$. Here,

$$C^+_{(I_1, I_2)} = C^+_{(I_1, I_2)}(\bar{x}) = \{\xi \in \mathbb{R}^n \mid \langle G'_i(\bar{x}), \xi\rangle = 0, i \in (I_G \backslash I_H) \cup I_1, \langle H'_i(\bar{x}), \xi\rangle = 0, i \in (I_H \backslash I_G) \cup I_2,$$
$$(\bar{\lambda}_G)_i \langle G'_i(\bar{x}), \xi\rangle = 0, i \in I_2, (\bar{\lambda}_H)_i \langle H'_i(\bar{x}), \xi\rangle = 0, i \in I_1\}.$$

If we do not assume that the MPCC-LICQ is fulfilled at $\bar{x}$, then it is natural to use the MPCC-SOSC in the form

$$\forall \xi \in (BD \cap C)\backslash\{0\} \ \ \exists(\mu_G, \mu_H, \nu) \in \mathcal{M} : \frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] > 0. \tag{6.5}$$

The following proposition shows, in particular, that this definition of the MPCC-SOSC conforms to the corresponding definition in Theorem 5, provided that the MPCC-LICQ is fulfilled.

Proposition 2. *Let the MPCC-LICQ be fulfilled at* $\bar{x} \in D$. *Assume that* $\bar{x}$ *is a strongly stationary point of problem* (3.1), (3.2) *and* $(\bar{\lambda}_G, \bar{\lambda}_H)$ *is the corresponding pair of multipliers.*

Then, (6.1) *is fulfilled if and only if* (6.5) *is fulfilled. Moreover, it holds that,*

(a) *for any* $\xi \in BD$, $\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi]$ *is independent of the choice of* $(\mu_G, \mu_H, \nu) \in \mathcal{M}$; *in particular, condition* (6.5) *is equivalent to*

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] > 0 \ \ \forall \xi \in (BD \cap C)\backslash\{0\}, \quad \forall(\mu_G, \mu_H, \nu) \in \mathcal{M};$$

(b) *condition* (6.5) *is equivalent to the existence of a universal triplet* $(\mu_G, \mu_H, \nu) \in \mathcal{M}$ *such that*

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] > 0 \ \ \forall \xi \in C\backslash\{0\}.$$

Proof. If $\xi \in LD$, then, by Proposition 1 (and, in particular, by formula (4.18)) and definition (5.3), we have

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] = \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)[\xi, \xi] + 2\nu \sum_{i \in I_0} \langle G'_i(\bar{x}), \xi\rangle \langle H'_i(\bar{x}), \xi\rangle \tag{6.6}$$

for any triplet of multipliers $(\mu_G, \mu_H, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ satisfying (4.15)–(4.17) with $\lambda_G = \bar{\lambda}_G$ and $\lambda_H = \bar{\lambda}_H$; moreover, the set $\mathcal{M}$ contains such and only such triplets of multipliers. In particular, if $\xi \in BD$, then (5.4) implies the equality

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] = \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)[\xi, \xi],$$

which proves the equivalence of (6.1) and (6.5), as well as assertion (a).

To prove (b), we fix $\varepsilon > 0$ such that

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)[\xi, \xi] > 0 \ \ \forall \xi \in U_\varepsilon,$$

where

$$U_\varepsilon = U_\varepsilon(\bar{x}) = \left\{\xi \in C \ \middle| \ \sum_{i \in I_0} \langle G'_i(\bar{x}), \xi\rangle \langle H'_i(\bar{x}), \xi\rangle < \varepsilon, |\xi| = 1\right\}.$$

(The existence of such a number can easily be derived from formulas (5.3) and (5.4) and condition (6.5), which is equivalent to (6.1), as proved above). Then, we choose a ν β> 0 satisfying (4.17) and the relation

$$2\nu\varepsilon > \left\| \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H) \right\|.$$

Next, we determine the multipliers μ_G and μ_H from (4.15) and (4.16). Then, according to (6.6), it holds that

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu_G, \mu_H, \nu)[\xi, \xi] \geq 2\nu\varepsilon - \left\| \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H) \right\| > 0$$

for any $\xi \in C \backslash U_\varepsilon$ such that $|\xi| = 1$. The required assertion is an immediate implication of this inequality and the facts established above. The proposition is proved.

Taking into account the inclusion $QD \subset BD$ and what was said in Subsection 2.1, we conclude that MPCC-SOSC (6.5) implies that the SOSC is fulfilled at $\bar{x}$ for the MPCC; hence, the QGC is also fulfilled.

We also introduce the *MPCC first-order sufficient condition* (MPCC-FOSC)

$$BD \cap C = \{0\}. \tag{6.7}$$

By (6.2), this condition is equivalent to the FOSC

$$C_{(I_1, I_2)} = \{0\}$$

fulfilled for the piecewise problem (5.6), (5.1) for any partition $(I_1, I_2) \in \mathcal{I}$.

From the inclusion $QD \subset BD$, we conclude that MPCC-FOSC (6.7) implies the fulfillment of FOSC (2.8) at $\bar{x}$ for the MPCC; hence, the LGC is also fulfilled.

7. SENSITIVITY ANALYSIS IN THE PRESENCE OF A REGULAR BRANCH

Now assume that f, G, and H (and, hence, all the quantities determined by f, G, and H) depend on a parameter $\sigma \in \mathbb{R}^s$ that describes perturbations. Note that we mean only perturbations in G and H rather than arbitrary perturbations of the problem (i.e., the perturbed problems retain the MPCC structure). In this and the next sections, we assume that $\bar{x}$ is a local solution to the MPCC associated with the base value $\sigma = \bar{\sigma} \in \mathbb{R}^s$ and the index sets I_G, I_H, and I_0 are specified at $\bar{x}$ for this base value of the parameter. Let the ball B, the local minimum value function $\omega : \mathbb{R}^s \longrightarrow \mathbb{R}$, and the point-to-set mapping $S : \mathbb{R}^s \longrightarrow 2^B$, which specifies the solutions to the perturbed MPCC restricted to B, be defined in accordance with Subsection 2.2.

In this section, we assume that, for at least one partition $(I_1, I_2) \in \mathcal{I}$, the MFCQ is fulfilled at $\bar{x}$ for the corresponding piecewise problem; i.e., the gradients

$$\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_G \backslash I_H) \cup I_1, \quad \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_H \backslash I_G) \cup I_2,$$

are linearly independent and

$$\exists \bar{\xi} \in \mathbb{R}^n : \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle = 0 \quad \forall i \in (I_G \backslash I_H) \cup I_1,$$

$$\left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle = 0 \quad \forall i \in (I_H \backslash I_G) \cup I_2, \tag{7.1}$$

$$\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle > 0 \quad \forall i \in I_2, \quad \left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle > 0 \quad \forall i \in I_1.$$

Then, the application of Robinson's stability theorem [10, Corollary 1] to the corresponding branch $D_{(I_1, I_2)}$ of the feasible set yields the estimate

$$\operatorname{dist}(\bar{x}, D_{(I_1, I_2)}(\sigma)) = O(|\sigma - \bar{\sigma}|).$$

Moreover, it is easy to verify that the second equality in (5.2) remains true under perturbations; i.e., for any

$\sigma \in \mathbb{R}^s$ sufficiently close to $\bar{\sigma}$, it holds that

$$D(\sigma) = \bigcup_{(I_1, I_2) \in \mathcal{I}} D_{(I_1, I_2)}(\sigma) \tag{7.2}$$

in the intersection with a neighborhood of $\bar{x}$. Hence, estimate (2.13) is valid with $p = 1$ for the feasible set of the perturbed MPCC. It follows that we can apply the sensitivity results stated in Subsection 2.2. In particular, Theorem 1 implies the following assertion.

Theorem 6. *Assume that, for at least one partition* $(I_1, I_2) \in \mathcal{I}$, *the gradients*

$$\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_G \backslash I_H) \cup I_1, \quad \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_H \backslash I_G) \cup I_2,$$

are linearly independent and (7.1) *is fulfilled. Then,* ω *is continuous at the point* $\bar{\sigma}$, *and* $S(\sigma) \neq \varnothing$ *for any* $\sigma \in \mathbb{R}^s$ *sufficiently close to* $\bar{\sigma}$. *Moreover,*

$$\sup_{x \in S(\sigma)} \operatorname{dist}(x, S(\bar{\sigma})) \longrightarrow 0 \ \ as \ \ \sigma \longrightarrow \bar{\sigma},$$

and it holds that

$$\omega(\sigma) \le \omega(\bar{\sigma}) + O(|\sigma - \bar{\sigma}|).$$

In particular, if $\bar{x}$ *is a strict local solution to the unperturbed problem* (3.1), (3.2), *then*

$$\sup_{x \in S(\sigma)} |x - \bar{x}| \longrightarrow 0 \ \ as \ \ \sigma \longrightarrow \bar{\sigma}.$$

This theorem and Theorems 2 and 3 yield the following estimates for solutions to the perturbed problems and lower bounds for ω, provided that the MPCC-FOSC and the MPCC-SOSC, respectively, are fulfilled.

Theorem 7. *Assume that, for at least one partition* $(I_1, I_2) \in \mathcal{I}$, *the gradients*

$$\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_G \backslash I_H) \cup I_1, \quad \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_H \backslash I_G) \cup I_2,$$

are linearly independent and (7.1) *is fulfilled. Let MPCC-FOSC* (6.7) *be fulfilled.*

Then, for $\sigma \in \mathbb{R}^s$, *we have*

$$\sup_{x \in S(\sigma)} |x - \bar{x}| = O(|\sigma - \bar{\sigma}|),$$

$$\omega(\sigma) \ge \omega(\bar{\sigma}) + O(|\sigma - \bar{\sigma}|).$$

Proof. According to assertion (c) in Theorem 2, it is sufficient to prove that, for any $\sigma \in \mathbb{R}^s$, there exists $u \in \mathbb{R}^n$ such that

$$\begin{aligned} &\sum_{i \in I_G \backslash I_H} H_i(\bar{\sigma}, \bar{x})\left(\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), u\right\rangle - \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma}\right\rangle\right) \\ &+ \sum_{i \in I_H \backslash I_G} G_i(\bar{\sigma}, \bar{x})\left(\left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), u\right\rangle - \left\langle \frac{\partial H_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma}\right\rangle\right) = 0, \end{aligned} \tag{7.3}$$

$$\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), u\right\rangle - \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma}\right\rangle \ge 0 \quad \forall i \in I_G, \tag{7.4}$$

$$\left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), u\right\rangle - \left\langle \frac{\partial H_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma}\right\rangle \ge 0 \quad \forall i \in I_H, \tag{7.5}$$

(see formulas (4.1) and (4.2) for the gradient of an equality constraint and the gradients of active inequality constraints).

The linear independence condition on the gradients

$$\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_G \backslash I_H) \cup I_1, \quad \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_H \backslash I_G) \cup I_2,$$

implies the existence of $\tilde{u} \in \mathbb{R}^n$ such that

$$\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \tilde{u} \right\rangle = \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma} \right\rangle \ \forall i \in (I_G \backslash I_H) \cup I_1,$$

$$\left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \tilde{u} \right\rangle = \left\langle \frac{\partial H_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), \sigma - \bar{\sigma} \right\rangle \ \forall i \in (I_H \backslash I_G) \cup I_2.$$

Then, (7.1) implies that, for any scalar t, the vector $u = \tilde{u} + t\bar{\xi}$ satisfies (7.3) and the inequalities (which are fulfilled as equalities) in (7.4) and (7.5) that correspond to $i \in (I_G \backslash I_H) \cup I_1$ and $i \in (I_H \backslash I_G) \cup I_2$, respectively. Moreover, the remaining inequalities in (7.4) and (7.5) can also be satisfied by choosing a sufficiently large $t > 0$. The theorem is proved.

Theorem 8. *Assume that, for at least one partitio*n $(I_1, I_2) \in \mathcal{I}$, *the gradients*

$$\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_G \backslash I_H) \cup I_1, \quad \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}), \quad i \in (I_H \backslash I_G) \cup I_2,$$

are linearly independent and (7.1) *is fulfilled. Let MPCC-SOSC* (6.5) *be fulfilled.*

Then, for $\sigma \in \mathbb{R}^s$, *we have*

$$\sup_{x \in S(\sigma)} |x - \bar{x}| = O(|\sigma - \bar{\sigma}|^{1/2}),$$

$$\omega(\sigma) \geq \omega(\bar{\sigma}) + O(|\sigma - \bar{\sigma}|).$$

The following example shows that the estimates in Theorem 8 cannot be improved even in the case where the problem is perturbed in a given direction in the parameter space (cf. [9, Example 4.3]).

Example 2. Assume that $s = 1$, $n = 4$, $m = 2$, $f(\sigma, x) = -x_2 + (x_3^2 + x_4^2)/2$, $G(\sigma, x) = (-x_2 - x_1^2 + \sigma, -x_2 + x_1^2)$, and $H(\sigma, x) = (x_3, x_4)$, where $\sigma \in \mathbb{R}$ and $x \in \mathbb{R}^4$.

It is easy to see that $\bar{x} = 0$ is a solution to the unperturbed MPCC for $\sigma = \bar{\sigma} = 0$. Moreover, for this solution, the MFCQ is fulfilled for the branch $D_{(\varnothing, I)}$ (and only for this branch). Furthermore,

$$\mathcal{M} = \{(\mu_G, 0, \nu) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \mid (\mu_G)_1 + (\mu_G)_2 = 1, (\mu_G)_1 \geq 0, (\mu_G)_2 \geq 0\},$$

$$BD \cup C = C = \{\xi \in \mathbb{R}^4 \mid \xi_2 = 0, \xi_3 \geq 0, \xi_4 \geq 0\}$$

and MPCC-SOSC (6.5) is fulfilled even with a universal multiplier (e.g., with $(\mu_G)_1 = 1$, $(\mu_G)_2 = 0$, and $\nu = 0$). Moreover, for $\sigma > 0$, we have

$$\omega(\sigma) = -\sigma/2, \quad S(\sigma) = \{(\pm(t/2)^{1/2}, t/2)\}.$$

8. SENSITIVITY ANALYSIS UNDER THE MPCC-LICQ

Assume now that the MPCC-LICQ is fulfilled at a solution $\bar{x}$ of the unperturbed MPCC. Then, the sensitivity results given above can be refined and amplified.

Recall that, under the assumptions made, there exist unique multipliers $\lambda_G = \bar{\lambda}_G$, $\lambda_H = \bar{\lambda}_H \in \mathbb{R}^m$ satisfying (4.12) and (4.13); they also satisfy (4.14). Moreover, for each piecewise problem, the LICQ is fulfilled at its local solution $\bar{x}$; hence, the conventional sensitivity theory for MPs (see [9, 12, 13]) is applicable to these problems.

For a partition $(I_1, I_2) \in \mathcal{I}$, the local minimum value function $\omega_{(I_1, I_2)} : \mathbb{R}^s \longrightarrow \mathbb{R}$ and the mapping $S_{(I_1, I_2)} : \mathbb{R}^s \longrightarrow 2^B$, which specifies solutions to the perturbed piecewise problem restricted to B, are defined in accordance with Subsection 2.2. It follows from [9, Proposition 4.3] that the upper bound

$$\omega_{(I_1, I_2)}(\bar{\sigma} + td) \le \omega_{(I_1, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t + o(t) \tag{8.1}$$

holds for any partition, any direction $d \in \mathbb{R}^s$, and for $t \ge 0$. (Recall that $\mathcal{L}$ is the Lagrangian for each piecewise problem and $(\bar{\lambda}_G, \bar{\lambda}_H)$ is the unique pair of Lagrange multipliers corresponding to the local solution $\bar{x}$ to this problem). In the derivation of this bound, we use the following linearization of the piecewise problem (with respect to both the variable and the parameter):

$$\langle f'(\bar{\sigma}, \bar{x}), (d, \xi) \rangle \longrightarrow \min, \quad \xi \in LD_{(I_1, I_2), d}. \tag{8.2}$$

Here,

$$\begin{gathered} LD_{(I_1, I_2), d} = LD_{(I_1, I_2), d}(\bar{\sigma}, \bar{x}) = \{\xi \in \mathbb{R}^n \mid \langle G_i'(\bar{\sigma}, \bar{x}), (d, \xi) \rangle = 0, i \in (I_G \backslash I_H) \cup I_1, \\ \langle H_i'(\bar{\sigma}, \bar{x}), (d, \xi) \rangle = 0, i \in (I_H \backslash I_G) \cup I_2, \langle G_i'(\bar{\sigma}, \bar{x}), (d, \xi) \rangle \ge 0, i \in I_2, \langle H_i'(\bar{\sigma}, \bar{x}), (d, \xi) \rangle \ge 0, i \in I_1 \}. \end{gathered} \tag{8.3}$$

Using the assumptions made above and the duality theory for linear programming, one can easily show that the minimum value in this problem is equal to $\left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle$; in particular, this value is independent of (I_1, I_2).

If $\bar{x}$ is a strict local solution to the unperturbed piecewise problem, then (8.1) is a sharp bound (see [9, Theorem 4.5]). This can be expressed by the following equality for the directional derivative of $\omega_{(I_1, I_2)}$ at the point $\bar{\sigma}$ in the direction d:

$$\omega'_{(I_1, I_2)}(\bar{\sigma}; d) = \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle. \tag{8.4}$$

In particular, this function is Gâteaux differentiable at $\bar{\sigma}$, and its derivative is independent of (I_1, I_2).

We use equality (7.2) to adapt these results to the original MPCC. It follows that

$$\omega(\sigma) = \min_{(I_1, I_2) \in \mathcal{I}} \omega_{(I_1, I_2)}(\sigma), \tag{8.5}$$

$$S(\sigma) = \bigcup_{\substack{(I_1, I_2) \in \mathcal{I} \\ \omega_{(I_1, I_2)}(\sigma) = \omega(\sigma)}} S_{(I_1, I_2)}(\sigma). \tag{8.6}$$

From what has been said above and equality (8.5), one can easily derive the following: the upper bound

$$\omega(\bar{\sigma} + td) \le \omega(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t + o(t) \tag{8.7}$$

holds for any direction $d \in \mathbb{R}^s$ and any $t \ge 0$; moreover, if $\bar{x}$ is a strict local solution to the unperturbed MPCC, then (8.7) is sharp; i.e., ω is Gâteaux differentiable at $\bar{\sigma}$, and its derivative is given by

$$\omega'(\bar{\sigma}) = \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H).$$

Here, we took into account the fact that the right-hand side in (8.4) is independent of (I_1, I_2).

The last assertion is the essence of Corollary 1 in [22]. (In addition, the authors prove the Lipschitz property of ω near $\bar{\sigma}$ and another property, which they call the strict differentiability of ω at $\bar{\sigma}$.) Similar results presented in terms of Mordukhovich's generalized derivatives are obtained in [23]. It is quite remarkable that, under the MPCC-LICQ, the formula for the first derivative of the minimum value function is in no way related to the combinatorial nature of the MPCC. However, a further analysis (dealing, say, with the calculation of the second directional derivatives [22]) inevitably reveals this combinatorial nature.

Indeed, for each partition $(I_1, I_2) \in \mathcal{I}$ and each direction $d \in \mathbb{R}^s$, denote by $SLP_{(I_1, I_2), d} = SLP_{(I_1, I_2), d}(\bar{\sigma}, \bar{x})$ the solution set of the linearized problem (8.2), (8.3). Consider the auxiliary problem

$$\left\langle \frac{\partial^2 \mathcal{L}}{\partial(\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle \longrightarrow \min, \quad h \in SLP_{(I_1, I_2), d}. \tag{8.8}$$

According to [9, Theorem 7.1], the upper bound (8.1) can be refined by using the minimum value in this problem: for $t \geq 0$,

$$\begin{aligned} \omega_{(I_1, I_2)}(\bar{\sigma} + td) &\leq \omega_{(I_1, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t \\ &+ \frac{1}{2} \inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial(\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle t^2 + o(t^2). \end{aligned} \tag{8.9}$$

This inequality, combined with (8.5), implies the following refinement of the upper bound (8.7):

$$\begin{aligned} \omega(\bar{\sigma} + td) &\leq \omega(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t \\ &+ \frac{1}{2} \min_{(I_1, I_2) \in \mathcal{I}} \inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial(\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle t^2 + o(t^2). \end{aligned} \tag{8.10}$$

Here, a combinatorial nature is explicitly present.

A further analysis requires that either growth conditions or sufficient optimality conditions are invoked. For instance, suppose that, for a given partition $(I_1, I_2) \in \mathcal{I}$, the corresponding unperturbed piecewise problem satisfies SOSC (6.3) at $\bar{x}$. Applying [9, Proposition 6.4] (or [13, Theorem 4.55]), we obtain the linear bound

$$\sup_{x \in S_{(I_1, I_2)}(\bar{\sigma} + td)} |x - \bar{x}| = O(t),$$

where $d \in \mathbb{R}^s$ is a given direction and $t \geq 0$. Then, applying [9, Theorem 7.2], we infer that (8.9) is a sharp bound; i.e.,

$$\begin{aligned} \omega_{(I_1, I_2)}(\bar{\sigma} + td) &= \omega_{(I_1, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t \\ &+ \frac{1}{2} \inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial(\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle t^2 + o(t^2). \end{aligned}$$

Moreover, the solution set of problem (8.8) is the same as the set of $h \in \mathbb{R}^n$ such that the perturbed piecewise problem has an $o(t^2)$-solution of the form $\bar{x} + th + o(t)$ at $\sigma = \bar{\sigma} + td$. Now, assume that the FOSC rather than the SOSC is fulfilled for the piecewise problem. Then, it is easy to verify that the linearized problem (8.2), (8.3) has a unique solution, problem (8.8) loses its optimization content, and the formulations become less cumbersome. We stress that the SOSC does not guarantee that problem (8.2), (8.3) is uniquely solvable (see Example 3 below).

Suppose that MPCC-SOSC (6.5) is fulfilled. Then, from what has been said above and equality (8.6), one can derive the following bound for the solution to the perturbed MPCC:

$$\sup_{x \in S(\bar{\sigma} + td)} |x - \bar{x}| = O(t).$$

Here, $d \in \mathbb{R}^s$ and $t \geq 0$. Moreover, (8.5) implies that (8.10) is a sharp bound; i.e.,

$$\omega(\bar{\sigma} + td) = \omega(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H), d \right\rangle t$$
$$+ \frac{1}{2} \min_{(I_1, I_2) \in \mathcal{I}} \inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial (\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle t^2 + o(t^2).$$

A similar bound was obtained in [22, Theorem 3]. Furthermore, the union of the solution sets of problem (8.8) over all partitions $(I_1, I_2) \in \mathcal{I}$ such that

$$\inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial (\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle$$
$$= \min_{(I_1, I_2) \in \mathcal{I}} \inf_{h \in SLP_{(I_1, I_2), d}} \left\langle \frac{\partial^2 \mathcal{L}}{\partial (\sigma, x)^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)(d, h), (d, h) \right\rangle,$$

is identical with the set of $h \in \mathbb{R}^n$ for which the perturbed MPCC has an $o(t^2)$-solution of the form $\bar{x} + th + o(t)$ at $\sigma = \bar{\sigma} + td$.

Finally, assume that, for a given partition $(I_1, I_2) \in \mathcal{I}$, the unperturbed piecewise problem satisfies SSOSC (6.4) at $\bar{x}$. According to [9, Proposition 5.4] (or [13, Proposition 5.38]), $(\bar{x}, \bar{\lambda}_G, \bar{\lambda}_H)$ is a *strongly regular* solution (in the sense of Robinson; see [24]) to the corresponding KKT system. This and Robinson's well-known result on perturbations of strongly regular solutions [9, Theorem 5.1] (or [13, Theorem 5.13]; see also Proposition 5.2 in [9], where this result is specialized for KKT systems) imply the following: for any $\sigma \in \mathbb{R}^s$ sufficiently close to $\bar{\sigma}$, the perturbed piecewise problem has a unique local solution $\bar{x}(\sigma)$ in a neighborhood of $\bar{x}$; this solution is associated with a unique pair of multipliers $(\bar{\lambda}_G(\sigma), \bar{\lambda}_H(\sigma))$; and the mapping $\sigma \longrightarrow (\bar{x}(\sigma), \bar{\lambda}_G(\sigma), \bar{\lambda}_H(\sigma)) : \mathbb{R}^s \longrightarrow \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ is Lipschitz continuous near $\bar{\sigma}$. Combining this and what has been said above about the Gäteaux differentiability of $\omega_{(I_1, I_2)}$, one can easily prove that this function is continuously differentiable near $\bar{\sigma}$.

If the MPCC-SSOSC is fulfilled, then the argument given above and equality (8.5) additionally imply that ω is a piecewise smooth function (see [22, Theorem 1]). Suppose that the *upper level strict complementarity condition* (ULSCC) is fulfilled; i.e.,

$$(\bar{\lambda}_G)_i > 0, \quad (\bar{\lambda}_H)_i > 0 \quad \forall i \in I_0$$

(see [14]). Then, one can guarantee that all perturbed piecewise problems have the same unique local solution in a neighborhood of $\bar{x}$; in that neighborhood, this is a unique local solution to the perturbed MPCC [14, Theorem 11].

The situation examined above does not seem to be typical for the MPCC (see [22]). However, let us get back to Example 1 with perturbations "in the right-hand sides." Assume that $s = 2$, $G(\sigma, x) = x_1 - \sigma_1$, and $H(\sigma, x) = x_2 - \sigma_2$, where $\sigma, x \in \mathbb{R}^2$ and $\bar{\sigma} = \bar{x} = 0$. If $a_1 > 0$ and $a_2 > 0$, then $\bar{x}$ is a strict local solution to the unperturbed MPCC (3.1), (3.2). Moreover, we have $\bar{\lambda}_G > 0$ and $\bar{\lambda}_H > 0$; i.e., the ULSCC is fulfilled. This implies that the MPCC-SSOSC is equivalent to the MPCC-SOSC. The latter is obviously fulfilled, because even the MPCC-FOSC holds. This agrees with the results presented above. Indeed, one can easily verify that the equalities

$$\omega(\sigma) = a_1\sigma_1 + a_2\sigma_2, \quad S(\sigma) = \{(\sigma_1, \sigma_2)\}$$

hold for any $\sigma \in \mathbb{R}^2$ in this example. In particular, the perturbed MPCC has a unique solution, which linearly depends on the parameter. The minimum value function is linear, and its gradient has the form

$$(a_1, a_2) = (\bar{\lambda}_G, \bar{\lambda}_H) = \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}_G, \bar{\lambda}_H).$$

Note that $h = d$ is a solution to the linearized problem (8.2), (8.3) for both possible partitions (I_1, I_2) of I_0 and for any $d \in \mathbb{R}^2$.

Example 3. Let the constraints be the same as in Example 1, but $f(x) = x_1 + x_2^2/2$, $x \in \mathbb{R}^2$. As above, we assume perturbations "in the right-hand sides" for G and H and set $\bar{\sigma} = \bar{x} = 0$.

Here, the MPCC-SSOSC (hence, MPCC-SOSC) is fulfilled, whereas the MPCC-FOSC and the ULSCC do not hold. It is easy to see that the equalities

$$\omega(\sigma) = \begin{cases} \sigma_1 + \sigma_2^2/2, & \text{if } \sigma_2 \geq 0, \\ \sigma_1, & \text{if } \sigma_2 \leq 0, \end{cases} \qquad S(\sigma) = \begin{cases} \{(\sigma_1, \sigma_2)\}, & \text{if } \sigma_2 \geq 0, \\ \{(\sigma_1, 0)\}, & \text{if } \sigma_2 \leq 0, \end{cases}$$

are valid for any $\sigma \in \mathbb{R}^2$, which is in agreement with the theory presented above. Note that, for $I_2 = \varnothing$ and any $d \in \mathbb{R}^2$, we have $SLP_{(I_1, I_2), d} = LD_{(I_1, I_2), d}$ and problem (8.8) is essential for determining the asymptotic behavior of solutions.

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