Power Crust for Noisy Data

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1 Introduction

Surface reconstruction is an important problem in Geometric Modeling. It has attracted a lot of attention recently because of the development of 3D scanner technology and its wide applications in different areas, such as: reverse engineering, product design, medical applications, and archeology.

Several algorithms have been proposed for surface reconstruction with theoretical guarantees [1–6]; they all produce a piecewise linear manifold interpolating a finite set of samples lying on a smooth surface and thus cannot handle noisy data. Up to now, there is only one algorithm for surface reconstruction in presence of noise with theoretical guarantees: the pioneer work of Dey and Goswami [9]. For curve reconstruction from noisy data there is a recent algorithm by Chen et al. [8], which provides theoretical guarantees.

In this paper, we extend the power crust algorithm of Amenta et al. [4] — and its theoretical guarantees — to handle noisy data. We start with a brief outline of the paper, discussing the noise model we use in the light of [4] and [9], and our main results. The precise definitions and result statements appear in subsequent sections. Some 2D examples, meant to show the geometry of our results, are shown in Section 8. Some of the proofs appear in the appendix; a complete presentation is available [10].

2 Outline of paper

This work is inspired in the power crust algorithm [4]; we extend most of the results in [4] to noisy data. The other work closely related to ours is [9]; we use their proposed noisy sample model, the *noisy* (r, k)*-sample*, in a more general setting. As originally defined in [9], a noisy (r, k)-sample of a surface S must satisfy three conditions (see Definition 1), which can be paraphrased as follows: (i) the projection of the sample set onto S is dense (that is, is an r-sample for some r [1]); (ii) the samples are close to the surface S; (iii) the sampling is locally uniform. Condition (iii) prohibits the samples to be arbitrarily close to S, which is very restrictive and unlikely to be met in practice. Thus we propose keep conditions (i) and (ii) and give up condition (iii), obtaining a less restrictive sample model which we call *noisy* r*-sample* and which is enough to prove all the necessary lemmas.

The algorithm in [9] is based in the observation that, under the assumed noise model, some Delaunay balls are large. Dey and Goswami [9] show that the boundary of the union of the inner (outer) large Delaunay balls is homeomorphic to the sampled surface. The key lemma in [9], which justifies the role of these large Delaunay balls, is their Lemma 5, which can be interpreted as follows: for every point $x \in S$ there is a Delaunay ball which contains the center m of the inner (outer) medial ball B_m touching the surface S at x and whose boundary contains a sample point p within distance $O(\sqrt{r})\,\mathrm{lfs}(x)$ from x.

We ask ourselves if a (inner or outer) polar ball B' that corresponds to a sample point p contains the medial axis point m which is the center of the medial ball B_m; if $m \notin B'$, we also ask how close m is to the boundary of B'. The answers to these questions are given by the following lemma, which we propose as an extension of Lemma 5 from Dey and Goswami [9] to handle polar balls:

Lemma *Given a point $x \in S$, let B_m be the inner (outer) medial ball centered at m and tangent to S at x. Then, there exists a sample point p with $d(p, x) = O(\sqrt{r})\,\mathrm{lfs}(x)$ such that its associated inner (outer)* *polar ball B' satisfies either:*

- *The medial axis point m is inside B' ($m \in B'$).*
- *The medial axis point m is not inside B' ($m \notin B'$), the segment xm intersects $\partial B'$ in at least a points, let p_2 be the closest one to m, then $d(m, p_2) \leq O(\sqrt[4]{r}) \min\{\rho, \rho_m\}$ for small enough r, the numbers ρ and ρ_m are the radius of the B' and B_m respectively.*

This lemma states that, if the inner (outer) polar ball B' of the sample p does not contains m, then the medial axis point m is within a distance $O(\sqrt[4]{r}) \min\{\rho, \rho_m\}$ to the boundary of B'. As a consequence of this lemma, instead of using this large set of large Delaunay balls proposed in [9], we use certain subsets $\mathbf{B}_I$ of the inner polar balls and $\mathbf{B}_O$ of the outer polar balls (see Section 4).

The first main result of this paper is that S_I, the boundary of the union of balls in $\mathbf{B}_I$, is close and isotopic to the sampled surface S. An analogous result holds for the outer balls. These results are stated in Lemma 12 of Section 4 and Lemma 19 of Section 6. The proof of Lemma 19 is based on a recent isotopy condition by Chazal and Cohen-Steiner [7], which is repeated here as Theorem 1 in Section 6. Lemma 12 is a consequence of Lemma 11.

The other main result is that the power crust of the set of polar balls in $\mathbf{B}_I \cup \mathbf{B}_O$ is isotopic to the sampled surface. This is Theorem 2 of Section 7, and it also follows from the isotopy condition of Chazal and Cohen-Steiner [7]. Lemmas 30, 22 and 23 of Section 7 prove that the power crust satisfies the conditions of Theorem 1: it is homeomorphic to S; it is contained in a topological thickening N_ϵ of S; and it separates the sides of N_ϵ. That the power crust is inside N_ϵ, as stated in Lemma 22, is a consequence of Lemma 12.

3 Basic definitions

We start by reviewing the geometric concepts of *power diagram* and *power distance*. Denote by $B_{c,\rho}$ the ball with center c and radius ρ; a point $x \in \mathbf{R}^3$ corresponds to the ball $B_{x,0}$. The *power distance* between two balls B_{c_1,ρ_1} and B_{c_2,ρ_2} is $d_{pow}(B_{c_1,\rho_1}, B_{c_2,\rho_2}) = d^2(c_1, c_2) - \rho_1^2 - \rho_2^2$. The *power diagram* $\text{Pow}(\mathbf{B})$ of a set of balls $\mathbf{B}$ is the weighted Voronoi diagram which assigns an (unweighted) point $x = B_{x,0}$ in space to the cell of the ball $B \in \mathbf{B}$ which minimizes $d_{pow}(x, B)$. The *power cell* of a ball $B \in \mathbf{B}$ is the cell of B in $\text{Pow}(\mathbf{B})$ and is denoted by $\text{Pcell}(B)$. Unlike Voronoi cells, a power cell can be empty, the center of the ball associated to the cell can be outside of the cell and the ball can even have an empty intersection with its power cell.

Throughout the paper we shall use the following notation. For any set $X \subset \mathbf{R}^3$, $\text{int}(X)$, X^c and ∂X denote respectively the interior of X, the complement of X and the boundary of X. Given a set X and a point $y \in \mathbf{R}^3$, let $d(y, X)$ denote the Euclidean distance of y from X, that is, $d(y, X) = \inf_{x \in X} d(y, x)$.

A *smooth surface* is a two-dimensional, compact, C^2 manifold without boundary. We want to reconstruct an unknown smooth surface S from a noisy sample. We shall assume that S is contained in an open, bounded domain Ω (e.g., a large bounding sphere). The surface S divides Ω into two open solids: the *inside* (inner region) and the *outside* (outer region) of S, which are disjoint. A ball $B \subset \Omega$ is *empty* (with respect to S) if $\text{int}(B) \cap S = \emptyset$. A *medial ball* is a maximal empty ball, that is, there is no other ball $B_1 \subset \Omega$ with $\text{int}(B_1) \cap S = \emptyset$ and $B \subsetneq B_1$. The *medial axis* M of S is the set of centers of the medial axis balls. The medial axis is bounded because it is contained in the bounded domain Ω; therefore exits an upper bound Δ for the radius of the medial balls. The medial axis M is divided in two parts: the inner and outer medial axis, according to whether it is in the inner or outer region of the surface S.

The following function is fundamental to the reconstruction problem: $\text{lfs} : S \to \mathbf{R}$ where $\text{lfs}(x) = d(x, M)$. It is easy to see that lfs is Lipschitz function (and hence continuous). Define $\text{lfs}(S) = \inf_{x \in S} \text{lfs}(x) > 0$. Since S is compact and lfs is continuous, we have $\text{lfs}(S) = \inf_{x \in S} \text{lfs}(x) = \min_{x \in S} \text{lfs}(x) > 0$; see [5].

A finite set of points P is an *r-sample* of the surface S if $P \subset S$ and for any $x \in S$ there is a point $p \in P$ with distance to x smaller or equal to $r\,\mathrm{lfs}(x)$.

We shall consider a noisy sample set P of S, which may not be a subset of S. Let $c : \mathbf{R}^3 \to S$ be the function which assigns to a point $x \in \mathbf{R}^3$ its closest point $\tilde{x}$ on S. Let $\tilde{P} = \{\tilde{p} \;:\; \tilde{p} = c(p),\; p \in P\;\}$ be the projection of the set P onto the surface S and $\tilde{p} = c(p)$ the projection of a sample $p \in P$ onto the surface S. Dey and Goswami in [9] introduced the notion of *noisy* (k, r)*-sample*:

Definition 1 (Noisy (k, r)-sample) *A finite set of points P is a* noisy (k, r)-sample *if the following conditions hold:*

(i) $\tilde{P}$ is a r-sample of S.

(ii) For every $p \in P$ we have $d(p, \tilde{p}) \leq k_1\, r\; \mathrm{lfs}(\tilde{p})$ for some constant k_1.

(iii) For every $p \in P$ we have $d(p, q) \geq k_2\, r\, \mathrm{lfs}(\tilde{p})$, where q is the k^{th} nearest sample to p.

We shall use a different definition of noisy sample, which we call *noisy r-sample*. Essentially we shall use conditions (i) and (ii) above but give up condition (iii), which aims is to produce a locally uniform sampling.

Definition 2 (Noisy r-sample) *A finite set of points P is a noisy r-sample if the following two conditions hold:*

(i) $\tilde{P}$ is a r-sample of S.

(ii) For every $p \in P$ we have $d(p, \tilde{p}) \leq k_1\, r\; \mathrm{lfs}(\tilde{p})$ for some constant k_1.

As in the case of a *noisy (r, k)-sample*, the first condition says that the projection of the samples onto the surface is an *r-sample*, and the second condition says that all the samples are close to the surface S.

4 Unions of polar balls

Given a noisy sample set P, to avoid to deal with infinite Voronoi cell, we follow [4] and add to the sample set a set of eight points, the vertices of a large box containing Ω. The concept of *poles* was defined by Amenta and Bern [1]. The *poles* p_i and p_o of a sample $p \in P$ are the two vertices of its Voronoi cell farthest from p, one on either side of the surface. The Voronoi balls $B_{p_i,\rho_{p_i}}$, $B_{p_o,\rho_{p_o}}$ are the polar balls with radius $\rho_{p_i} = d(p_i, p)$ and $\rho_{p_o} = d(p_o, p)$, respectively.

Notice that, given a noisy sample set, a Voronoi cell could have either a unique pole or both poles the inner and outer poles. A Voronoi ball B_{v,ρ_v} is classified as an inner (outer) Voronoi ball if its center (Voronoi vertex) is inside the inner (outer) region of $\mathbf{R}^3 \setminus S$. We denote by V_p^I and V_p^O the set of inner and outer Voronoi balls centered at Voronoi vertices corresponding to the Voronoi cell of the sample p. We denote by $\mathbf{P}_I$ and $\mathbf{P}_O$ the set of all inner and outer polar balls, respectively. We define the following two polar ball subsets:

$$\begin{aligned}
\mathbf{B}_I &= \{\; B_{c,\rho_c} \in \mathbf{P}_I \;:\; \exists\, p \in P \text{ where } c \text{ is an inner pole of } p \text{ and } \exists\, B_v \;\in V_p^I \text{ with } B_v \cap M \neq \emptyset \;\} \\
\mathbf{B}_O &= \{\; B_{c,\rho_c} \in \mathbf{P}_O \;:\; \exists\, p \in P \text{ where } c \text{ is an outer pole of } p \text{ and } \exists\, B_v \;\in V_p^O \text{ with } B_v \cap M \neq \emptyset \;\}
\end{aligned}$$

Their boundaries are: $S_I = \partial(\bigcup_{B \in \mathbf{B}_I} B)$ and $S_O = \partial(\bigcup_{B \in \mathbf{B}_O} B)$. Our goal is to prove that the boundary sets S_I and S_O are close and isotopic to the surface S. Moreover, we shall show that a subset Crust($\mathbf{B}_I \cup \mathbf{B}_O$) of the two-dimensional faces of Pow($\mathbf{B}_I \cup \mathbf{B}_O$) is isotopic to the surface S.

The next lemma is from Dey and Goswami [9] and will be used in this section for several proofs. We present this lemma in a slight different form: instead of taking x in the offset surface, we take x in the sampled surface. The proof is exactly the same as in the original lemma.

Lemma 2 *For any point $x \in S$, let B_m be the medial ball tangent to x. Then there is a sample p and an inner (outer) Voronoi ball B associated with p such that $m \in B$ and $d(x,p) = O(\sqrt{r})\,\mathrm{lfs}(x)$.*

The following lemma from Amenta and Bern [1] estimates the angle between the normals to the surface at two close points.

Lemma 3 *For any two points p and q on S with $d(p,q) \leq r\ \min\{\mathrm{lfs}(p), \mathrm{lfs}(q)\}$, for any $r \leq \frac{1}{3}$, the angle between the normals to S at p and q is at most $\frac{r}{1-3r}$.*

The next lemma is an immediate consequence of the sampling requirements and its proof can be found in Lemma 4 of [10].

Lemma 4 *Given P a noisy r-sample of S, let D be a ball with int$(D) \cap P = \emptyset$ and $D \cap S \neq \emptyset$, let x be a point in $D \cap S$. If $B(x, \rho_x) \subset D$, then $\rho_x \leq O(r)\,\mathrm{lfs}(x)$.*

Given a sample p and a point $x \in \mathrm{Vor}(p)$, we can estimate the angle between the vector $\vec{xp}$ and the surface normal $\vec{n}_{\tilde{p}}$ at the projection of the sample p onto S. The following lemma states that when x is far away from p, this angle must be small.

Lemma 5 *Given a sample $p \in P$ such that there exists a point x on the inner (outer) region of the Voronoi cell of p with distance ρ_x between x and p satisfying the inequality $\rho_x \geq \frac{\mathrm{lfs}(\tilde{p})}{c}$ for some constant c. Then the angle between the vector $\vec{xp}$ and the oriented outward (inward) surface normal $\vec{n}_{\tilde{p}}$ is $O(\sqrt{r})$.*

As a consequence of this lemma, we have

Lemma 6 *Given an inner (outer) medial axis point m, let t be a sample such that $m \in$ Vor(t) and let p be the inner (outer) pole of t. Then $m \in$ int$(B_{p,d(p,t)})$ and $B_{p,d(p,t)} \in \mathbf{B}_I$ $(B_{p,d(p,t)} \in \mathbf{B}_O)$.*

The above lemma implies that the inner and outer part of the medial axis M is inside the set $\mathrm{int}(\bigcup_{B \in \mathbf{B}_I} B)$ and $\mathrm{int}(\bigcup_{B \in \mathbf{B}_O} B)$ respectively. The next four lemmas play an important role in the proof of Lemma 11 and 12. Their proof can be seen in Lemma 7, 8, 9 and 10 respectively of [10].

Lemma 7 *Given any ball $B_{c,\rho} \subset \mathbf{R}^3$, let $x \notin B_{c,\rho}$ be a point with $d(x, \partial B_{c,\rho}) = O(\sqrt{r})\rho$ and let l_t be a ray passing through x and tangent to $B_{c,\rho}$ at the point t, then the angle ϕ between the vectors $\vec{xt}$ and $\vec{xc}$ is bigger than $\frac{\pi}{6}$, for small enough r.*

Lemma 8 *Given any ball $B_{c,\rho} \subset \mathbf{R}^3$, let $x \in \mathbf{R}^3$ and p be a point on $\partial B_{c,\rho}$ such that $d(x,p) \leq O(\sqrt{r})\rho$, then the angle ϕ between the vectors $\vec{cp}$ and $\vec{cx}$ is $O(\sqrt{r})$.*

Lemma 9 *Given any ball $B_{c,\rho} \subset \mathbf{R}^3$, let $x \notin B_{c,\rho}$ and $y \in \mathbf{R}^3$ be points such that the segment xy intersects $\partial B_{c,\rho}$, let p be the closest intersection point to x. If $d(x, \partial B_{c,\rho}) \leq O(\sqrt{r})\rho$, then $d(x,p) = O(\sqrt[4]{r})\rho$ and $\sqrt{d_{pow}(x, B_{c,\rho})} = O(\sqrt[4]{r})\rho$.*

Lemma 10 *Let B_{c,ρ_c} and B_{v,ρ_v} be two balls such that $B_{c,\rho_c} \cap B_{v,\rho_v} \neq \emptyset$ and $\rho_c \geq \rho_v$, let $p \in \partial B_{c,\rho_c} \cap \partial B_{v,\rho_v}$ and $m \in B_{v,\rho_v} \setminus B_{c,\rho_c}$ such that $\angle(\vec{cp}, \vec{vp}) = O(t)$, then $d(m, \partial B_{c,\rho_c}) = O(t)\rho_c$.*

Using the previous lemmas we can state the following lemma 11, which can be seen as an extension of Lemma 2 of Dey and Goswami to handle polar balls. In this case we have not guarantees that the polar ball corresponding to p in Lemma 2 contains the medial axis points m which is the center of the inner (outer) medial ball tangent to S at $\tilde{p}$, instead of that we can prove that if this polar ball not contains m, then m is close to the boundary of the polar ball.

Lemma 11 *Given a point $x \in S$, let B_{m,ρ_m} be the inner (outer) medial ball centered at m tangent to S at x and let g be the segment joining x and m. Then, there exists a sample p with $d(p,x) = O(\sqrt{r})\,\mathrm{lfs}(x)$ such that its associated inner (outer) polar ball B_{c_2,ρ_2} which is in $\mathbf{B}_I$ ($\mathbf{B}_O$) and satisfies either:*

- *The medial axis point m is inside B_{c_2,ρ_2}.*
- *The medial axis point m is not inside B_{c_2,ρ_2}, the segment g intersects $\partial B_{c_2,\rho_2}$ in at least a points, let p_2 be the closest one to m, then $d(m,p_2) \leq O(\sqrt{r})\min\{\rho_2,\rho_m\}$ and $\rho_m \geq \frac{\rho}{2}$ for small enough r.*

From the Lemma 11 we obtain the following corollary, the proof can be found in [10], corollary 1.

Corollary 1 *Given a point $x \in S$, let B_{m,ρ_m} be the inner (outer) medial ball centered at m tangent to x, then there exists a sample p with $d(p,x) = O(\sqrt{r})\,\mathrm{lfs}(x)$ such that the inner (outer) polar ball $B_{c,\rho}$ associated to p satisfies either $B_{c,\rho}$ contains m, or the distance of m to $\partial B_{c,\rho}$ is smaller than $O(\sqrt{r})\min\{\rho_2,\rho_m\}$.*

Now we shall show that the surfaces S_I and S_O are near to the surface S. It is a consequence of the Lemma 11.

Lemma 12 *Given a point $u \in S_I$ ($u \in S_O$), the distance to its closest point on S is $O(\sqrt[4]{r})$.*

The next lemma states that any ball in $\mathbf{B}_I$ ($\mathbf{B}_O$) has a medial axis point inside it or close to its boundary. The proof of this lemma can be found in [10], Lemma 13.

Lemma 13 *Let $B_{c,\rho} \in \mathbf{B}_I$ ($B_{c,\rho} \in \mathbf{B}_O$), then there exists an inner (outer) medial axis point m such that, either $m \in B_{c,\rho}$, or $m \notin B_{c,\rho}$ with $d(m,\partial B_{c,\rho}) = O(\sqrt{r})\rho$.*

5 Angle between polar balls

We start by presenting a lemma that will be used to analyze the intersection angle between a polar ball in $\mathbf{B}_I$ and a polar ball in $\mathbf{B}_O$.

Lemma 14 . *Let $B_{c_1,\rho_1} \in \mathbf{B}_I$ and $B_{c_2,\rho_2} \in \mathbf{B}_O$ with $B_{c_1,\rho_1} \cap B_{c_2,\rho_2} \neq \emptyset$. Let x be a point on the surface S inside the segment c_1c_2 and $\tilde{p}$ is the closest projected sample in P to x, then $d(x,\tilde{p}) \geq \max\{d(x,y) - O(r)\rho_1, d(x,y) - O(r)\rho_2\}$ where y is a point on the circle $C = \partial B_{c_1,\rho_1} \cap \partial B_{c_2,\rho_2}$.*

Proof. See proof in [10], Lemma 14. $\square$

Next we extend the Lemma [16] of Amenta, Choi and Kollury [4] to the context of our sampling assumptions. In order to prove this lemma, we shall follow the proof of Amenta, Choi and Kollury [4]. We mean by the angle of intersection between two balls, the angle formed by the tangent planes at any intersection point.

Lemma 15 *The angle of intersection between a polar ball $B_{c_1,\rho_1} \in \mathbf{B}_I$ and a polar ball $B_{c_2,\rho_2} \in \mathbf{B}_O$ with $B_{c_2,\rho_2} \cap B_{c_1,\rho_1} \neq \emptyset$ is $O(r)$.*

Now we extend the lemma [23] of Amenta, Choi and Kollury [4] to our context. its proof can be found in [10], Lemma 17

Lemma 16 *Given a point $u \in S_I$ ($u \in S_O$) and a ball $B_{c,\rho} \in \mathbf{B}_I$ ($B_{c,\rho} \in \mathbf{B}_O$) such that $u \in \partial B_{c,\rho}$, then the angle between the vector $\vec{cu}$ and the outward (inward) normal $\vec{n}_{\tilde{u}}$ is $O(\sqrt[8]{r})$.*

From the above lemma we can bound the intersection angle between two polar balls in $\mathbf{B}_I$ ($\mathbf{B}_O$) when their boundaries have a common intersection with S_I (S_O), its proof can be found in [10], Corollary 2.

Corollary 2 *The angle of intersection between two balls $B_{c_1,\rho_1} \in \mathbf{B}_I$ and $B_{c_2,\rho_2} \in \mathbf{B}_I$ ($B_{c_1,\rho_1} \in \mathbf{B}_O$ and $B_{c_2,\rho_2} \in \mathbf{B}_O$) such as $B_{c_1,\rho_1} \cap B_{c_2,\rho_2} \cap S_I \neq \emptyset$ ($B_{c_1,\rho_1} \cap B_{c_2,\rho_2} \cap S_O \neq \emptyset$), is $\pi - O(\sqrt[8]{r})$.*

6 Homeomorphism and isotopy

In this section we prove that the sets S_I and S_O are isotopic to the surface S. Let $\mathrm{d} : \mathbf{R}^3 \to S$ be the function such that assigns to $x \in \mathbf{R}^3$ its closest point on S. The restriction of this function to the sets $S' = S_I$ ($S' = S_O$) is a homeomorphism as following lemma states.

Lemma 17 *The distance function d is a homeomorphism between S' and S.*

Proof. It follows as consequence of Lemmas 12 and 16, its proof is the same as Lemma 25 proved in Amenta, Choi and Kolluri. [4]. □

In order to prove the existence of an isotopy between S' and S we shall use the following theorems of Chazal and Cohen-Steiner [7]. Also, We took from [7] the definition of topological thickening of a surface S.

Definition 3 (Chazal and Cohen-Steiner) *A topological thickening of a surface S is the set $N \subset \mathbf{R}^3$ such that there exists a homeomorphism $F : S \times [0,1] \to N$ satisfying $F(S \times \{1/2\}) = S$.*

Theorem 1 (Chazal and Cohen-Steiner) *Suppose that:*
i) S' is homeomorphic to S.
ii) S' is included in the topological thickening N of S.
iii) S' separates the sides of S.
Then S' is isotopic to S in N.

Given a set $N \subset \mathbf{R}^3$, we shall say that S and S' are isotopic in N, if there exists an isotopy between S and S' whose image is included in N. The term "separates" in the condition iii) of this theorem means that one cannot go from one side of N to the other one without passing through S. To apply Lemma 1 we have to find an appropriated topological thickening N_ϵ with $\epsilon < \mathrm{lfs}(S)$. That is $N_\epsilon = \{\ x \in \mathbf{R}^3\ :\ d(x,\tilde{x}) \leq \epsilon\ \}$.

Lemma 18 *The set N_ϵ is a topological thickening for $\epsilon < \mathrm{lfs}(S)$.*

For $\epsilon < \mathrm{lfs}(S)$ the boundary ∂N_ϵ of the set N_ϵ is the union of the two surfaces $F(S,0) = S_{-\epsilon} = \{x \in \mathbf{R}^3\ :\ x = \tilde{x} - \epsilon\,\vec{n}_{\tilde{x}}\}$ and $F(S,1) = S_\epsilon = \{x \in \mathbf{R}^3\ :\ x = \tilde{x} + \epsilon\,\vec{n}_{\tilde{x}}\}$ which will be referred as the sides of N_ϵ. One can easily see from Lemma 18 that this sets are isotopic to S for $\epsilon < \mathrm{lfs}(S)$. Note that $N_\epsilon \subset \Omega$ for $\epsilon < \mathrm{lfs}(S)$. We denote by $\tilde{d}_H(X,Y) = \sup_{x\in X} \inf_{y \in Y} d(x,y)$ the "half Hausdorff distance". The Lemma 12 implies that $\tilde{d}_H(S_I,S) = O(\sqrt[4]{r})$ and $\tilde{d}_H(S_O,S) = O(\sqrt[4]{r})$. Now we shall show that the sets S_I and S_O are isotopic to the surface S. Given a point $x \in S$ let B_{m_x} and $B_{m'_x}$ be the inner (outer) medial balls tangent to $x \in S$ and m_x and m'_x their respective medial axis centers

Lemma 19 *The Surface S_I and S_O are isotopic to the surface S.*

7 Power Crust

The *power crust* was defined by Amenta, Choi and Kolluri. [4] as the set of faces in the *power diagram* of the polars balls separating inner ones from outer ones. We shall now prove, under the sampling requirements, that the set of faces in $\text{Pow}(\mathbf{B}_I \cup \mathbf{B}_O)$ separating cells of balls in $\mathbf{B}_I$ from cells of balls in $\mathbf{B}_O$ is a polyhedral surface isotopic to S. This set of faces will be called *power crust* too and it will be denoted by $\text{Crust}(\mathbf{B}_I \cup \mathbf{B}_O)$. We recall that for $\epsilon < \text{lfs}(S)$ there exists a topological thickening N_ϵ such as $S_I \cup S_O \subset \text{int}(N_\epsilon)$ and $\partial N_\epsilon = S_\epsilon \cup S_{-\epsilon}$ with $\epsilon = 2\max\{\tilde{d}_H(S_I, S), \tilde{d}_H(S_O, S)\} = O(\sqrt[4]{r})$ for r small enough. We start proving that $\text{Pow}(\mathbf{B}_I \cup \mathbf{B}_O)$ is inside N_ϵ and separates the sides S_ϵ and $S_{-\epsilon}$ of N_ϵ. The proofs of the next two lemmas can be found in Lemmas 21 and 22 of [10].

Lemma 20 *The set* $(\bigcup_{B \in \mathbf{B}_I \bigcup \mathbf{B}_O} B)^c \subset N_\epsilon$ *is inside the topological thickening* N_ϵ.

Lemma 21 *Let* $B_1 \in \mathbf{B}_I$ *and* $B_2 \in \mathbf{B}_O$. *If* $B_1 \cap B_2 \neq \emptyset$ *then* $B_1 \cap B_2 \subset N_\epsilon$.

Lemma 22 *The power crust is inside the topological thickening* N_ϵ.

Lemma 23 *The power crust separates the sides of the topological thickening* N_ϵ.

The goal of the next lemma is to bound the *power distance* between an arbitrary point u on the topological thickening N_ϵ to the set of balls $\mathbf{B}_I \cup \mathbf{B}_O$. The proof can be found in [10], Lemma 25

Lemma 24 *Given* $u \in N_\epsilon$ *and* $B_u \in \mathbf{B}_I \cup \mathbf{B}_O$ *a ball minimizing* $d_{pow}(u, B)$ *with* $B \in \mathbf{B}_I \cup \mathbf{B}_O$, *then* $\sqrt{d_{pow}(u, B_u)} = O(\sqrt[4]{r})\rho$.

Denote by the segment $s_{-\epsilon}s_\epsilon = m_x m'_x \cap N_\epsilon$ where m_x and m'_x are the medial axis point centers of the inner and outer medial balls tangent to S at x. We have that $s_{-\epsilon} \in S_{-\epsilon}$ and $s_\epsilon \in S_\epsilon$. Moreover, the segment $m_x m'_x$ is orthogonal to S at x. We have the following lemma whose proof can be found in [10], Lemma 26.

Lemma 25 *Given a point* $x \in S$ *and* m_x (m'_x) *is the inner (outer) medial axis point such that the inner (outer) medial ball* $B_{m_x, \rho_{m_x}}$ $(B_{m'_x, \rho_{m'_x}})$ *is tangent to* S *at* x. *Denote* $\rho'_{m_x} = \rho_{m_x} - \epsilon$ $(\rho'_{m'_x} = \rho_{m'_x} - \epsilon)$. *Then* $int(B_{m_x, \rho'_{m_x}}) \cap S_{-\epsilon} = \emptyset$ $(int(B_{m'_x, \rho'_{m'_x}}) \cap S_\epsilon = \emptyset)$.

Lemma 26 *Given a point* $u \in N_\epsilon$ *and a ball* $B_{c,\rho} \in \mathbf{B}_I$ *such that* $d_{pow}(u, B_{c,\rho}) = \min_{B \in \mathbf{B}_I} d_{pow}(u, B)$, *then the angle between the vector* $\vec{cu}$ *and the normal* $\vec{n}_{\tilde{u}}$ *is* $O(\sqrt[8]{r})$.

The next lemma gives an estimative of the angle between two two-dimensional faces of the power crust sharing a common edge.

Lemma 27 *The angle* α *between two faces* a *and* b *in the power crust sharing a common edge is* $\pi - O(\sqrt[8]{r})$.

Proof. See proof of Lemma 29 in [10] □

From Lemma 27 we can conclude that for r small enough the dihedral angle between two two-dimensional faces of the *power crust* sharing an edge is bigger than 120 degree, showing that any edge in the power crust has at most two adjacent power crust faces.

Define by $f_I(x) = \min_{B \in \mathbf{B}_I} d_{pow}(x, B)$ and $f_O(x) = \min_{B \in \mathbf{B}_O} d_{pow}(x, B)$ the functions which return the minimum *power distance* from x to the sets $\mathbf{B}_I$ and $\mathbf{B}_O$ respectively, it is easy to see that these functions are continuous. By Lemma 26, the following Lemma 28 from Amenta, Choi and Kolluri. [4] is also valid under our sampling assumption and for our particulars polar ball sets $\mathbf{B}_I$ and $\mathbf{B}_O$.

Lemma 28 *Given a point $s \in S$ the restrictions of the functions f_I and f_O to the segment $s_{-\epsilon}s_\epsilon$ is monotone increasing and decreasing respectively.*

Lemma 29 *For any point $x \in S$, the segment $s_{-\epsilon}s_\epsilon$ has an unique intersection point with the power crust.*

Lemma 30 *The power crust is homeomorphic to the surface S.*

From Lemma 30 we conclude that $\text{Crust}(\mathbf{B}_O \cup \mathbf{B}_I)$ is a polyhedral compact surface without boundary, and from Lemma 27 we have that the *power crust* is a smooth polyhedral surface without sharp features for small value of the sampling parameter r.

Theorem 2 *Crust$(\mathbf{B}_O \cup \mathbf{B}_I)$ is isotopic to the surface S.*

Proof. Lemmas 30, Lemma 22 and Lemma 23 imply conditions (i), (ii), and (iii) of Theorem 1. □

8 Algorithm

In order to compute the *power crust* we need to determine the sets $\mathbf{B}_O$ and $\mathbf{B}_I$. Since each ball $B_{c,\rho}$ in $\mathbf{B}_O \cup \mathbf{B}_I$ either contains a medial axis point m_c or there exists a medial axis point m_c with distance $d(m_c, \partial B_{c,\rho}) = O(\sqrt[4]{r})$ such that $B_{c,\rho}$ touch some sample point p with distance to $\tilde{p} \in S$ equal to $O(r)\text{lfs}(\tilde{p})$. Then we have that $d(\tilde{p}, p) + 2\rho + d(m_c, \partial B_{c,\rho}) > d(\tilde{p}, m) > \text{lfs}(\tilde{p})$. That is, $2\rho \geq \text{lfs}(\tilde{p}) - O(r)\,\text{lfs}(\tilde{p}) - O(\sqrt[4]{r})$, hence $\rho \geq \text{lfs}(\tilde{p})/2 \geq \text{lfs}(S)/2$ for r small enough.

We augment the set of samples P with the set Z of eight vertices of a box surrounding the domain Ω. From this new set $P \cup Z$ we determine the set of polar balls $\mathbf{P}$ in the same way as in [1]. Based on the lower bound $\rho \geq \text{lfs}(S)/2$ for the radius of the balls in $\mathbf{B}_O \cup \mathbf{B}_I$ we filter out the set of polar balls $\mathbf{P}$ by removing from $\mathbf{P}$ the balls with radius small than an user defined parameter $\Gamma \approx \text{lfs}(S)/2$. This result in a new set $\mathbf{P}'$ very close to $\mathbf{B}_O \cup \mathbf{B}_I$.

Once we have determined the set $\mathbf{P}'$ we need to label each balls in $\mathbf{P}'$ as outer or inners ball. We start labeling as outer the polar balls centered at the Voronoi vertex of the cells associated to points in Z touching samples in P. We classify as outer any ball in $\mathbf{P}'$ that intersects an already classified outer ball at an angle bigger than a predefined parameter γ, a typical value for γ is 0.6π. This process is repeated until there is not a new ball that can be classified as outer, obtaining the set $\mathbf{B}_O$. The remaining balls $\mathbf{P}' \setminus \mathbf{B}_O$ are classified as inner.

Figure 1 shows four curve reconstructions using this method.

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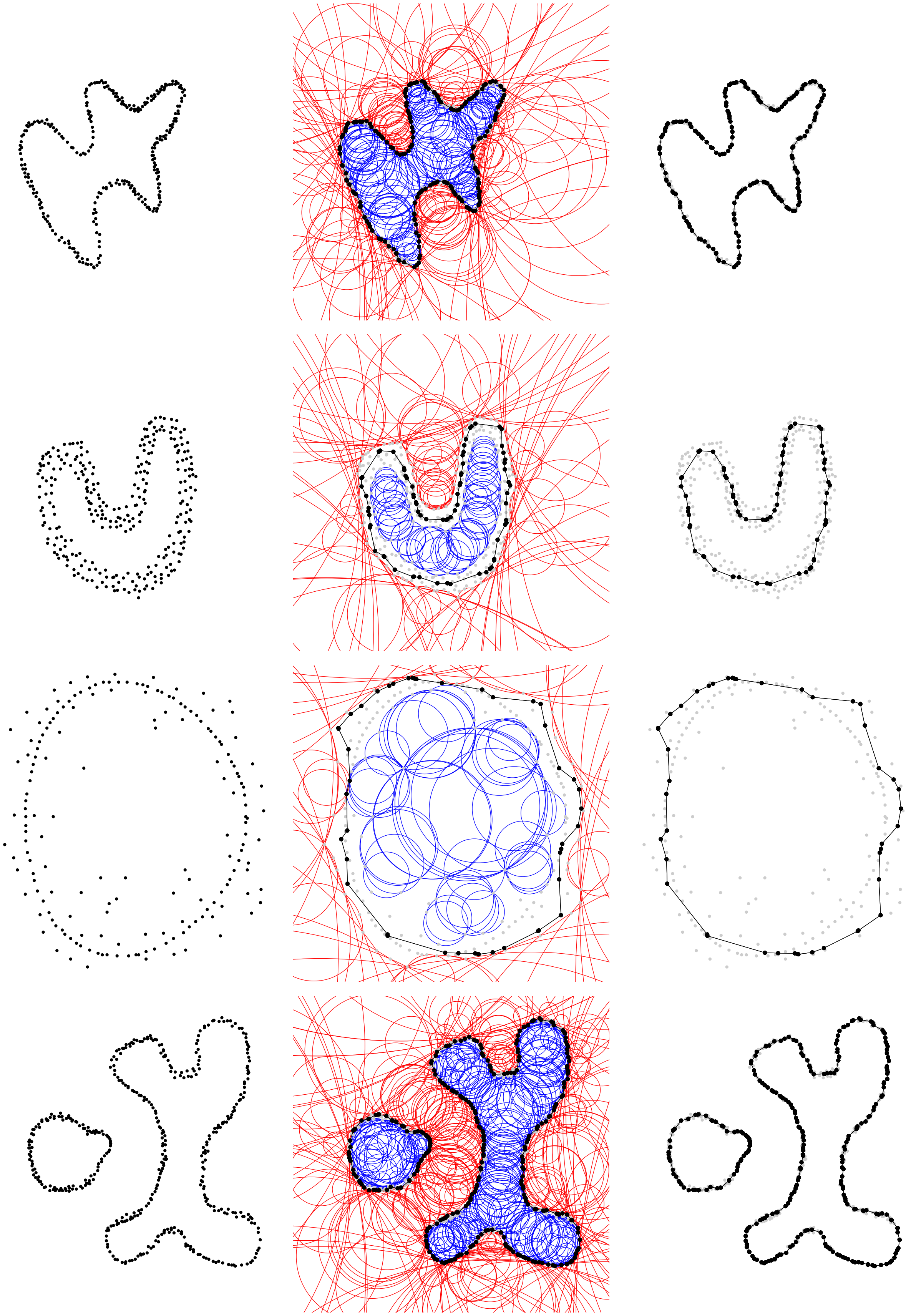

Figure 1: Some examples: noisy data (left); power balls (middle); reconstruction (right).

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Appendix: Some proofs

Proof of Lemma 5

Denote by B_{m,ρ_m} the outer (inner) medial ball tangent to the surface S at $\tilde{p}$. Let B_{x,ρ_x} be the ball centered at x with radius $\rho_x = d(x,p)$. Since x is in the Voronoi cell of p we have $\mathrm{int}(B_{x,\rho_x}) \cap P = \emptyset$.

Let xmp be the triangle with vertices x, m and p. Let t be the projection of the vertex p onto the segment xm, that is, the segment pt is orthogonal to xm. Since B_{m,ρ_m} is tangent to S at $\tilde{p}$, the three points p, $\tilde{p}$ and m are collinear and the segment $m\tilde{p}$ and mp are parallel to $\vec{n}_{\tilde{p}}$. From these observations we get that the angle between $\vec{xp}$ and $\vec{n}_{\tilde{p}}$ is the same as the angle between the vectors $\vec{xp}$ and $\vec{mp}$ which is the sum $\angle(t,x,p) + \angle(t,m,p)$. Our aim will be to find upper bounds for the angles $\angle(t,x,p)$ and $\angle(t,m,p)$, respectively. Since $d(x,t) < d(x,p) = \rho_x$ we get that $t \in B_{x,\rho_x}$ and the following situations either $t \in B_{m,\rho_m} \cap B_{x,\rho_x}$ or $t \in B^c_{m,\rho_m} \cap B_{x,\rho_x}$ are possibles.

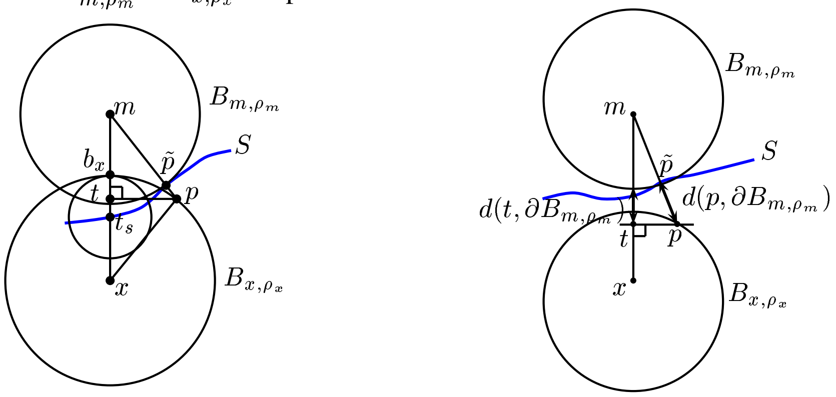

Figure 2: Left the $t \in B_{m,\rho_m} \cap B_{x,\rho_x}$. Right $t \in B^c_{m,\rho_m} \cap B_{x,\rho_x}$.

Consider $t \in B_{m,\rho_m} \cap B_{x,\rho_x}$, see figure 2 left. Since $t \in B_{m,\rho_m}$ we have that t is on the outer (inner) region of $\Omega \setminus S$. The segment xt intersects the surface at the point t_s, and we get that $t_s \in B_{x,\rho_x}$ due to the segment $xt \subset B_{x,\rho_x}$. Using that $t_s \in B_{x,\rho_x}$ we have, for small enough r, the following inequality that will be useful later.

$$\begin{aligned} \mathrm{lfs}(t_s) &\le d(t_s,\tilde{p}) + \mathrm{lfs}(\tilde{p}) \le d(t_s,x) + d(x,\tilde{p}) + \mathrm{lfs}(\tilde{p}) \le \rho_x + d(x,p) + d(p,\tilde{p}) + \mathrm{lfs}(\tilde{p}) \\ &\le 2\rho_x + (1+O(r))\,\mathrm{lfs}(x) \le 2\rho_x + (1+O(r))\,c\,\rho_x \le (2+2c)\rho_x = k_c\,\rho_x \end{aligned} \tag{1}$$

Let b_x be the intersection point between $\partial B_{x,\rho_x}$ and the segment xm. Because the points t_s and b_x are inside the ball B_{x,ρ_x} we have that $B_{t_s,d(t_s,b_x)} \subset B_{x,\rho_x}$. Since $\mathrm{int}(B_{x,\rho_x}) \cap S = \emptyset$ because ρ_x is the distance of x to its closest point in P, we have that $\mathrm{int}(B_{t_s,d(t_s,b_x)}) \cap S = \emptyset$, consequently, by Lemma 4, we get $d(t_s,b_x) \le r\ \mathrm{lfs}(t_s)$. From this last equation together with equation 1 and the with fact that t is inside the segment $b_x t_s$ we obtain the following inequalities

$$d(t,b_x) \le d(t_s,b_x) \le O(r)\,\mathrm{lfs}(t_s) \le O(r)\rho_x \quad \text{and} \quad d(t,t_s) \le d(t_s,b_x) \le O(r)\,\mathrm{lfs}(t_s) \le O(r)\rho_x \tag{2}$$

Consequently, by 2 we have $d(t,x) = \rho_x - d(t,b_x) \ge (1-O(r))\rho_x$, hence

$$d(p,t) = \sqrt{d(p,x)^2 - d(x,t)^2} = \sqrt{\rho_x^2 - (1-O(r))^2\rho_x^2} = O(\sqrt{r})\rho_x \tag{3}$$

so, the angle $\angle(t,x,p)$ is bounded by

$$\angle(t,x,p) = \arcsin\left(\frac{d(p,t)}{d(x,p)}\right) \le \arcsin\left(\frac{O(\sqrt{r})\rho_x}{\rho_x}\right) = O(\sqrt{r}) \tag{4}$$

On the other hand, since $t \in B_{m,\rho_m}$, then we have that $\mathrm{lfs}(t_s) < d(t_s,t) + d(t,m) \le O(r)\,\mathrm{lfs}(t_s) + \rho_m$ obtaining that $\mathrm{lfs}(t_s) < \frac{\rho_m}{1-O(r)}$. Because $t \in B_{m,\rho_m}$ and $t_s \notin B_{m,\rho_m}$ we obtain the following lower bound for the distance between t and m

$$d(t,m) \ge \rho_m - d(t,t_s) \ge \rho_m - O(r)\,\mathrm{lfs}(t_s) > (1-O(r))\rho_m$$

Since $\mathrm{lfs}(\tilde{p}) < \rho_m$ and by the sampling conditions, we get that $d(m,p) < d(m,\tilde{p}) + d(\tilde{p},p) \le (1+O(r))\rho_m$, consequently

$$d(p,t) = \sqrt{d(m,p)^2 - d(t,m)^2} \le \sqrt{(1+O(r))^2\rho^2{}_m - (1-O(r))^2\rho_m^2} = O(\sqrt{r})\rho_m \tag{5}$$

We have that $\rho_m = d(m,\tilde{p}) \le d(m,p) + d(p,\tilde{p})$, so using that $\mathrm{lfs}(\tilde{p}) < \rho_m$ we have $d(m,p) \ge \rho_m - d(\tilde{p},p) \ge (1-O(r))\rho_m$. From this equation and equation 5 we can bound the angle $\angle(t,m,p)$ as follow

$$\angle(t,m,p) = \arcsin\left(\frac{d(p,t)}{d(m,p)}\right) \le \arcsin\left(\frac{O(\sqrt{r})\rho_m}{(1-O(r))\rho_m}\right) = O(\sqrt{r}) \tag{6}$$

Therefore, from 4 and 6 we have that $\angle(t,x,p) + \angle(t,m,p)$ is $O(\sqrt{r})$.

Now we shall consider that $t \in B^c_{m,\rho_m} \cap B_{x,\rho_x}$ (Note that this case implies that $B_{m,\rho_m} \cap B_{x,\rho_x} = \emptyset$). Since $t \notin B_{m,\rho_m}$ and $d(p,m) \ge d(t,m)$ we obtain that $p \notin B_{m,\rho_m}$, consequently we have that $d(t,\partial B_{m,\rho_m}) \le d(p,\partial B_{m,\rho_m}) = d(p,\tilde{p}) \le O(r)\,\mathrm{lfs}(\tilde{p})$, see figure 2 b). From this inequality and due to that $t \in B_{x,\rho_x}$ we get

$$d(t,x) \ge \rho_x - d(t,\partial B_{m,\rho_m}) \ge \rho_x - d(p,\partial B_{m,\rho_m}) \ge \rho_x - O(r)\,\mathrm{lfs}(\tilde{p}) \tag{7}$$

Since $\mathrm{lfs}(\tilde{p}) \le d(\tilde{p},p) + d(p,x) \le O(r)\,\mathrm{lfs}(\tilde{p}) + \rho_x$ we get $\mathrm{lfs}(\tilde{p}) \le \frac{\rho_x}{1-O(r)}$, consequently the equation 7 can be rewritten in terms of ρ_x, that is $d(t,x) \ge \rho_x - O(r)\,\mathrm{lfs}(\tilde{p}) \ge (1-O(r))\rho_x$. Hence we have

$$d(p,t) = \sqrt{d(p,x)^2 - d(t,x)^2} \le \sqrt{\rho_x^2 - (1-O(r))^2\rho_x^2} = O(\sqrt{r})\rho_x$$

Therefore, we obtain $\angle(t,x,p) = \arcsin\left(\frac{d(p,t)}{d(x,p)}\right) \le \arcsin\left(\frac{O(\sqrt{r})\rho_x}{\rho_x}\right) = O(\sqrt{r})$.

On the other hand, since $t \notin B_{m,\rho_m}$ we get $d(t,m) > \rho_m$. This implies that $d(p,m) \le d(p,\tilde{p}) + d(\tilde{p},m) \le O(r)\,\mathrm{lfs}(\tilde{p}) + \rho_m = (1+O(r))\rho_m$. Hence

$$d(p,t) = \sqrt{d(p,m)^2 - d(t,m)^2} \le \sqrt{(1+O(r))^2\rho_m^2 - \rho_m^2} \le O(\sqrt{r})\rho_m$$

and the angle $\angle(t,m,p) = \arcsin\left(\frac{d(p,t)}{d(p,m)}\right) \le \arcsin\left(\frac{O(\sqrt{r})\rho_m}{\rho_m}\right) = O(\sqrt{r})$, concluding that $\angle(t,x,p) + \angle(t,m,p)$ is $O(\sqrt{r})$. $\square$

Proof of Lemma 6

By the definition of local feature size we have $d(m,\tilde{t}) \ge \mathrm{lfs}(\tilde{t})$. By triangular inequality we have $d(m,t) + d(t,\tilde{t}) \ge d(m,\tilde{t})$, so we have $d(m,t) \ge d(m,\tilde{t}) - d(t,\tilde{t}) \ge \mathrm{lfs}(\tilde{t}) - O(r)\,\mathrm{lfs}(\tilde{t})$, taking r small enough we get $d(m,t) \ge \frac{\mathrm{lfs}(\tilde{t})}{2}$. From this fact the lemma 5 implies that $\angle(\vec{mt},\vec{n}_{\tilde{t}}) = O(\sqrt{r})$, using the same argument since $d(p,t) \ge d(m,t)$ we obtain $\angle(\vec{pt},\vec{n}_{\tilde{t}}) = O(\sqrt{r})$. Hence we obtain that $\angle(\vec{pt},\vec{mt}) = O(\sqrt{r})$.

Taking r small enough such that $\angle(\vec{pt},\vec{mt}) \le \frac{\pi}{4}$ and due to $d(m,t) \le d(p,t)$ we obtain that the m is inside the interior of the inner (outer) polar ball $B_{p,d(p,t)}$. Hence, by the definition of $\mathbf{B}_I$ ($\mathbf{B}_O$) we have that $B_{p,d(p,t)} \in \mathbf{B}_I$ ($B_{p,d(p,t)} \in \mathbf{B}_O$) proving the lemma. $\square$

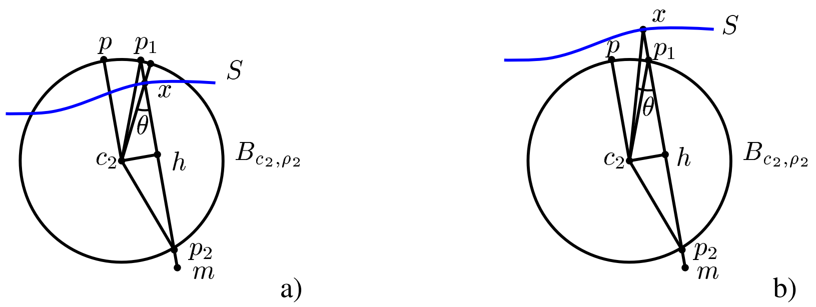

Figure 3: a) The point $x \in S$ is inside the ball B_{c_2,ρ_2}, b) $x \in S$ is outside the ball B_{c_2,ρ_2}

Proof of Lemma 11

The Lemma 2 implies that there exists a sample p with $d(p,x) = O(\sqrt{r})\,\mathrm{lfs}(x)$ and an inner (outer) Voronoi ball B_{c_1,ρ_1} corresponding to p with $m \in B_{c_1,\rho_1}$. Let B_{c_2,ρ_2} be the inner (outer) polar ball associated to p. First we start determining upper bounds for $\mathrm{lfs}(x)$ and $\mathrm{lfs}(\tilde{p})$ in term of ρ_1 and ρ_2. Since $m \in B_{c_1,\rho_1}$ we have that $\mathrm{lfs}(x) \leq d(x,p) + d(p,m) \leq O(\sqrt{r})\,\mathrm{lfs}(x) + 2\rho_1$ therefore, making some algebraic manipulations, for r small enough we obtain

$$\mathrm{lfs}(x) \leq \frac{2\rho_1}{1 - O(\sqrt{r})} \leq 4\rho_1 \leq 4\rho_2 \tag{8}$$

Due to the sampling requirements $\mathrm{lfs}(\tilde{p}) \leq d(\tilde{p},p) + d(p,x) + \mathrm{lfs}(x)$ with $d(\tilde{p},p) = O(r)\,\mathrm{lfs}(\tilde{p})$ and $d(x,p) = O(\sqrt{r})\,\mathrm{lfs}(x)$ we have the equation 9 for some constants c_1 and $c_2 = 4c_1$ for r small enough.

$$\mathrm{lfs}(\tilde{p}) \leq \frac{1 + O(\sqrt{r})}{1 - O(\sqrt{r})}\,\mathrm{lfs}(x) \leq c_1\,\mathrm{lfs}(x) \leq c_2\rho_1 \leq c_2\rho_2 \tag{9}$$

Since $d(c_2,p) = \rho_2 \geq \frac{\mathrm{lfs}(\tilde{p})}{c_2}$, then the Lemma 5 states that the angle $\angle(\vec{c_2p}, \vec{n}_{\tilde{p}})$ between the vector $\vec{c_2p}$ and the normal $\vec{n}_{\tilde{p}}$ is $O(\sqrt{r})$. The Lemma 3 states that the angle $\angle(\vec{n}_{\tilde{p}}, \vec{n}_{\tilde{x}})$ between the normals $\vec{n}_{\tilde{p}}$ and $\vec{n}_{\tilde{x}}$ is $O(\sqrt{r})$, therefore the angle $\angle(\vec{c_2p}, \vec{n}_{\tilde{x}})$ is smaller than $\angle(\vec{c_2p}, \vec{n}_{\tilde{p}}) + \angle(\vec{n}_{\tilde{p}}, \vec{n}_{\tilde{x}}) = O(\sqrt{r})$.

From $d(x,p) \leq O(\sqrt{r})\,\mathrm{lfs}(x) \leq O(\sqrt{r})\rho_2$ the Lemma 8 implies that the angle $\angle(\vec{c_2x}, \vec{c_2p})$ between the vectors $\vec{c_2p}$ and $\vec{c_2x}$ is $O(\sqrt{r})$. Using that $\angle(\vec{c_2p}, \vec{n}_{\tilde{x}}) = O(\sqrt{r})$ we have $\angle(\vec{c_2x}, \vec{n}_{\tilde{x}}) \leq \angle(\vec{c_2x}, \vec{c_2p}) + \angle(\vec{c_2p}, \vec{n}_{\tilde{x}}) = O(\sqrt{r})$, consequently for r small enough, the Lemma 7 implies that the segment $g = xm$ intersects the ball B_{c_2,ρ_2} (it is enough that $\angle(\vec{c_2x}, \vec{n}_{\tilde{x}}) = \angle(\vec{xc_2}, \vec{xm}) \leq \pi/6$). We have two possibilities either $m \in B_{c_2,\rho_2}$ or $m \notin B_{c_2,\rho_2}$. When $m \in B_{c_2,\rho_2}$ then the theorem follows, so we consider $m \notin B_{c_2,\rho_2}$.

Let p_1 and p_2 be the intersection points of the line containing the segment xm with $\partial B_{c_2,\rho_2}$, assume that p_2 is the closest one to m. The triangle $T_1 = p_2c_2p_1$ is isosceles, let h be the midpoint of the segment p_1p_2, then the segment c_2h is orthogonal to the segment p_1p_2.

We have two cases either $x \in B_{c_2,\rho_2}$, see figure 3 a) or $x \notin B_{c_2,\rho_2}$, see figure 3 b). When $x \in B_{c_2,\rho_2}$, we have that $x \in p_1h$ due to $\theta = \angle(c_2x, \vec{n}_{\tilde{x}}) \leq 90$ for small enough r. Our goal is to obtain an upper bound for the distance between the points p_2 and m. We have that $d(m,c_2) + d(c_2,h) \geq d(p_2,m)$, equivalently, we have $d(m, \partial B_{c_2,\rho_2}) + \rho + d(c_2,h) \geq d(h,p_2) + d(p_2,m)$, so we can determine an upper bound for the distance $d(m,p_2)$

$$d(m,p_2) \leq d(m, \partial B_{c_2,\rho_2}) + \rho_2 + d(c_2,h) - d(p_2,h) \tag{10}$$

Now we shall determine an upper bounds for $d(m, \partial B_{c_2,\rho_2})$ and $d(h,c_2)$ respectively and a lower bound for $d(c_2,h)$. Because $d(c_2,p) \geq d(c_1,p) \geq \frac{\mathrm{lfs}(\tilde{p})}{c_2}$, the Lemma 5 implies that $\angle(\vec{c_2p}, \vec{n}_{\tilde{p}}) = O(\sqrt{r})$ and

$\angle(\vec{c_1p}, \vec{n}_{\tilde{p}}) = O(\sqrt{r})$. Hence $\angle(\vec{c_1p}, \vec{c_2p}) = O(\sqrt{r})$. From this and the fact that $m \in B_{c_1,\rho_1} \setminus B_{c_2,\rho_2}$ with $\rho_2 \geq \rho_1$ the Lemma 10 establish that

$$d(m, \partial B_{c_2,\rho_2}) = O(\sqrt{r})\rho_2 \tag{11}$$

We have that $d(c_2, h) = d(c_2, x)\sin(\theta)$. If $x \in B_{c_2,\rho_2}$ we have that $d(c_2, x) \leq \rho_2$ and otherwise when $x \notin B_{c_2,\rho_2}$ we have $d(x, c_2) = d(x, \partial B_{c_2,\rho_2}) + \rho_2$. Therefore, due to $d(x, \partial B_{c_2,\rho_2}) \leq d(x, p)$ we get $d(c_2, x) \leq \rho_2 + d(x, p)$, consequently by equation 8 we obtain $d(x, c_2) = (1 + O(\sqrt{r}))\rho_2$. From these upper bounds we get

$$d(c_2, h) \leq d(c_2, x)\sin(\theta) = O(\sqrt{r})\rho_2 \tag{12}$$

By other hand we have $d(h, p_2) + d(c_2, h) \geq d(c_2, p_2)$, then

$$d(h, p_2) \geq d(c_2, p_2) - d(c_2, h) \geq (1 - O(\sqrt{r}))\rho_2 \tag{13}$$

Substituting the equations 11, 12 and 13 into 10 we have obtain

$$d(m, p_2) \leq O(\sqrt{r})\rho_2 + \rho_2 + O(\sqrt{r})\rho_2 - (1 - O(\sqrt{r}))\rho_2 = O(\sqrt{r})\rho_2 \tag{14}$$

Finally, since $\rho_m = d(m, x) \geq d(h, p_2)$, then $\rho_m \geq (1 - O(\sqrt{r}))\rho_2$. Consequently for r small enough $\rho_m \geq \rho_2/2$. $\square$

Proof of Lemma 12

We shall prove that the lemma for the case $u \in S_I$ the case $u \in S_O$ can be proved similarly. Consider u on the outer region of $\mathbf{R}^3 \setminus S$, let $B_{c,\rho} \in \mathbf{B}_I$ such that $u \in \partial B_{c,\rho}$, then the segment uc intersects the surface S at a point x. Since the ball $B_{x,d(x,u)} \subset B_{c,\rho}$ and $B_{c,\rho}$ is a polar ball empty of samples we have by Lemma 4 that $d(x, u) = O(r)\,\mathrm{lfs}(x)$. Because lfs is a continuous function on a compact manifold, then there exists a constant $\Delta_1 = \max_{x\in S} \mathrm{lfs}(x)$, hence $d(u, \tilde{u}) \leq d(x, u) \leq O(r)\Delta_1 = O(r)$.

Take u on the inner region of $\mathbf{R}^3 \setminus S$, let $\tilde{u} \in S$ be the closest point of u on S. By Lemma 11 there exists a point q with distance $d(\tilde{u}, q) = O(\sqrt{r})\,\mathrm{lfs}(\tilde{u})$ and an inner (outer) polar ball $B_{c_2,\rho_2} \in \mathbf{B}_I$ touching q such that the medial axis point m, center of the medial ball B_{m,ρ_m} tangent to S at $\tilde{u}$, satisfies either the medial axis point m is inside B_{c_2,ρ_2}, or $m \notin B_{c_2,\rho_2}$ and the segment $\tilde{u}m$ intersects $\partial B_{c_2,\rho_2}$ in a point p_2 with $d(p_2, m) \leq O(\sqrt{r})\min\{\rho_2, \rho_m\}$. By the proof of Lemma 11 there exists an inner (outer) Voronoi ball B_{c_1,ρ_1} corresponding to p such that $m \in B_{c_1,\rho_1}$. Because the closest point to u on S is $\tilde{u}$ we have that the segment $u\tilde{u}$ is orthogonal to S and the points $\tilde{u}$, u and m are collinear, beside that, we have that u is inside the segment $\tilde{u}m$, otherwise $B_{m,\rho_m} \subset B_{u,d(u,\tilde{u})}$ with $\mathrm{int}(B_{u,d(u,\tilde{u})}) \cap S = \emptyset$, which is a contradiction because B_{m,ρ_m} is a maximal empty ball.

Consider that $m \notin B_{c_2,\rho_2}$ (the case $m \in B_{c_2,\rho_2}$ will be analyzed later) and $\tilde{u} \notin B_{c_2,\rho_2}$, then the segment $\tilde{u}m$ intersects $\partial B_{c_2,\rho_2}$ at two points p_1 and p_2 with $d(p_2, m) \leq O(\sqrt{r})\min\{\rho_2, \rho_m\}$. The segment $\tilde{u}m$ can be decomposed in three segments: $\tilde{u}p_1$, p_1p_2 and p_2m, hence we have three possibilities:

1) $u \in \tilde{u}p_1$. Since $d(\tilde{u}, \partial B_{c_2,\rho_2}) \leq d(\tilde{u}, p) = O(\sqrt{r})\,\mathrm{lfs}(\tilde{u}) \leq O(r)\rho_m$, then applying Lemma 9 we have $d(\tilde{u}, p_1) \leq O(\sqrt[4]{r})\rho_m \leq O(\sqrt[4]{r})\Delta = O(\sqrt[4]{r})$.

2) $u \in p_1p_2$. We have that $u \in \mathrm{int}(B_{c_2,\rho_2}) \subset \mathrm{int}(\bigcup_{B\in\mathbf{B}_I} B)$ and $u \in S_I$ which is a contradiction.

3) $u \in p_2m$. By Lemma 11 we have $d(m, p_2) \leq O(\sqrt{r})\rho_m$, therefore $d(u, m) \leq O(\sqrt{r})\rho_m = O(\sqrt{r})$. Let h be the sample such that m is inside inner (outer) part of the Voronoi cell of h, let p_h be the inner (outer) pole of h. By the sampling requirements we have that $d(m, h) \geq d(m, \tilde{h}) - d(h, \tilde{h}) \geq \mathrm{lfs}(\tilde{h}) - O(r)\,\mathrm{lfs}(\tilde{h}) \geq \frac{\mathrm{lfs}(\tilde{h})}{2}$. Define by $r_m = d(p_h, h) - d(m, h)$, Then the ball B_{m,r_m} is inside the ball $B_{p_h,d(p_h,h)}$.

By Lemma 6 we obtain that m is inside the interior of the polar ball $B_{p_h,d(p_h,h)} \in \mathbf{B}_I$. We shall prove that $u \in \text{int}(B_{p_h,d(p_h,h)})$ obtaining a contradiction with the fact that $u \in S_I$. In order to prove this, it is enough to prove that $u \in \text{int}(B_{m,r_m})$.

Since $d(m,p_h) \geq d(p_h,h) - d(m,h)$ and $d(m,h) \geq \frac{\text{lfs}(\tilde{h})}{2}$, then we obtain $d(m,p_h) \geq d(p_h,h) - \frac{\text{lfs}(\tilde{h})}{2}$. Hence, $r_m \geq \frac{\text{lfs}(\tilde{h})}{2}$. Since $d(m,u) = O(\sqrt{r})$ and $r_m \geq \frac{\text{lfs}(\tilde{h})}{2}$, then $u \in \text{int}(B_{m,r_m})$.

The case $m \notin B_{c_2,\rho_2}$ and $\tilde{u} \in B_{c_2,\rho_2}$ implies that $u \in \tilde{u}p_2$ or $u \in p_2 m$ which can be reduced to 2) and 3) of the above case respectively. The case $m \in B_{c_2,\rho_2}$ and $\tilde{u} \notin B_{c_2,\rho_2}$ implies that $u \in \tilde{u}p_1$ or $u \in p_1 m$ which can be reduced to 1) and 2) of the above case respectively. Finally, the case $m \in B_{c_2,\rho_2}$ and $\tilde{u} \in B_{c_2,\rho_2}$ can be reduced to 2) of the above case. $\square$

Proof of Lemma 15

We have that the center c_1 and c_2 of B_{c_2,ρ_2} are in different sides of S, thus the segment c_1c_2 intersects S at a point x. Denote by $B_{c,\rho}$ any ball of B_{c_1,ρ_1} and B_{c_2,ρ_2} which contains x and let $B_{c',\rho'}$ be the remaining ball. By Lemma 13 there exist m and m' such that $d(m, \partial B_{c,\rho}) = O(\sqrt{r})\rho$ and $d(m', \partial B_{c',\rho'}) = O(\sqrt{r})\rho'$ respectively. Hence for r small enough we have $\text{lfs}(x) \leq d(x,c) + d(c,m) \leq d(x,c) + \rho + d(m, \partial B_{c,\rho}) \leq 2\rho + O(\sqrt{r})\rho \leq 3\rho$. Let $\{v\} = c_1c_2 \cap \partial B_{c,\rho}$, then, since $B_{x,d(x,v)} \subset B_{c,\rho}$ the Lemma 4 states that $d(x,v) \leq O(\sqrt{r}) \,\text{lfs}(x)$, since $d(x, \partial B_{c',\rho'}) \leq d(x,v)$ we have

$$\text{lfs}(x) \leq d(x,c') + d(c',m') \leq d(x, \partial B_{c',\rho'}) + 2\rho' + d(m', \partial B_{c',\rho'}) \leq 2\rho' + O(\sqrt{r}) \,\text{lfs}(x) + O(\sqrt{r})\rho' \quad (15)$$

Hence $\text{lfs}(x) \leq \frac{2+O(\sqrt{r})}{1-O(\sqrt{r})}\rho' \leq 5\rho'$, consequently $\text{lfs}(x) \leq 5\min\{\rho,\rho'\}$. The Pythagoras theorem implies that $d(x,y) = \sqrt{\lambda^2 + d^2(x,z)} > d(x,y) > \lambda$ and by the sampling condition there exists a sample p such that $d(x,\tilde{p}) \leq r \,\text{lfs}(x)$. As a consequence of these facts the Lemma 14 implies

$$\lambda - O(r)\min\{\rho,\rho'\} \leq d(x,y) - O(r)\min\{\rho,\rho'\} \leq d(x,\tilde{p}) \leq r\,\text{lfs}(x) \leq 4r \,\min\{\rho,\rho'\} \qquad (16)$$

Hence, $\lambda \leq O(r)\min\{\rho,\rho'\}$ and consequently the angles α_1 and α_2 are smaller than

$$\max\{\arcsin\left(\frac{O(r)\rho'}{\rho'}\right), \arcsin\left(\frac{O(r)\rho}{\rho}\right)\} = O(r).$$

$\square$

Proof of Lemma 18

Define the function $F(\cdot,\cdot) : S \times [0,1] \to N_\epsilon$ where $F(x,t) = x + (2t-1)\,\epsilon\,\vec{n}_x$. It is easy to see that F is an onto and a continuous function. Suppose that is not one to one, then there exist $x_1, x_2 \in S$ and $t_1, t_2 \in [0,1]$ such as $F(x_1,t_1) = F(x_2,t_2)$, that is $x_1 + (2t_1 - 1)\,\epsilon\,\vec{n}_{x_1} = x_2 + (2t_2 - 1)\,\epsilon\,\vec{n}_{x_2}$. The Lemma 7 in [5] implies that $\max\{|(2t_1-1)|\,\epsilon, |(2t_2-1)|\,\epsilon\} \geq \text{lfs}(S)$. This is a contradiction because $|2t-1| < 1$ when $t \in [0,1]$ and $\epsilon < \text{lfs}(S)$. Hence, the function F is a homeomorphism. $\square$

Proof of Lemma 19

As we know the set N_ϵ is a topological thickening of S when $\epsilon < \text{lfs}(S)$. For r small enough such that $\epsilon = 2\,\max\{\tilde{d}_H(S_I,S), \tilde{d}_H(S_O,S)\} < \text{lfs}(S)$, the Lemma 12 states that $S_I \subset \text{int}(N_\epsilon)$ and $(S_O \subset \text{int}(N_\epsilon))$.

By Lemma 6 we have that S_I and S_O are homeomorphic to S. So, we only need to prove that S_I (S_O) separates the sides of M in order to apply theorem 1.

Given a point $x \in S$, the segment $m_x m'_x$ is orthogonal to S at x. The Lemma 6 implies that there exists polar balls $B \in \mathbf{B}_I$ and $B' \in \mathbf{B}_O$ such that $m_x \in \text{int}(B)$ and $m'_x \in \text{int}(B')$. Since $m_x \in \text{int}(B) \subset \bigcup_{B \in \mathbf{B}_I} B$ and $m'_x \in \text{int}(B) \subset \bigcup_{B \in \mathbf{B}_O} B$ we have that this segment intersects S_I (S_O) at a point s_x which is inside the segment $a_x b_x = m_x m'_x \cap N_\epsilon$ with $a_x \in S_\epsilon$ and $b_x \in S_{-\epsilon}$, consequently S_I and (S_O) separates the sides of N_ϵ. We have that all the conditions of the theorem 1 are valid obtaining that S_I and (S_O) are isotopic to S. □

Proof of Lemma 22

Let $f \in \text{Pow}(\mathbf{B}_I \cup \mathbf{B}_O)$ the common facet of the power cells $\text{Pcell}(B_1)$ and $\text{Pcell}(B_2)$ where $B_1 \in \mathbf{B}_I$ and $B_2 \in \mathbf{B}_O$ and let x be a point on f, two cases are possible:

First, $x \in f \cap B_1$. This implies that $d_{pow}(x, B_2) = d_{pow}(x, B_1) \leq 0$. From $d_{pow}(x, B_2) \leq 0$ we obtain that $x \in B_2$, thus $x \in B_1 \cap B_2$, by Lemma 21 we have that $x \in B_1 \cap B_2 \subset N_\epsilon$.

Second, $x \in f$ and $x \notin B_1$. Then $d_{pow}(x, B_2) = d_{pow}(x, B_1) = \min_{B \in \mathbf{B}_I \cup \mathbf{B}_O} d(x, B) > 0$ and consequently $x \notin \bigcup_{B \in \mathbf{B}_I \bigcup \mathbf{B}_O} B$. By Lemma 20 we have $(\bigcup_{B \in \mathbf{B}_I \bigcup \mathbf{B}_O} B)^c \subset N_\epsilon$ therefore we obtain $x \in (\bigcup_{B \in \mathbf{B}_I \bigcup \mathbf{B}_O} B)^c \subset N_\epsilon$. □

Proof of Lemma 23

Since $\epsilon > \max\{\tilde{d}_H(S_I, S), \tilde{d}_H(S_O, S)\}$ then $S_I \subset N_\epsilon$ and $S_O \subset N_\epsilon$. Hence $S_{-\epsilon} \subset \text{int}(\bigcup_{B \in \mathbf{B}_I} B)$ and $S_\epsilon \subset \text{int}(\bigcup_{B \in \mathbf{B}_O} B)$. Given $x \in S$, let m_x and m'_x the medial axis points such that their respective medial balls are tangent to S at x. Denote by x_1 and x_2 the sides of the segment $x_1 x_2 = m_x m'_x \cap N_\epsilon$, where $x_1 \in S_{-\epsilon} \subset \text{int}(\bigcup_{B \in \mathbf{B}_I} B)$ and $x_2 \in S_\epsilon \subset \text{int}(\bigcup_{B \in \mathbf{B}_O} B)$.

Since $x_1 \in S_{-\epsilon} \subset \text{int}(\bigcup_{B \in \mathbf{B}_I} B)$ we have that $d_{pow}(x_1, B') < 0$ for some $B' \in \mathbf{B}_I$ and since $S_{-\epsilon} \cap \bigcup_{B \in \mathbf{B}_O} B = \emptyset$ we have that $d_{pow}(x_1, B) > 0$ for all balls $B \in \mathbf{B}_O$. Similarly, since $x_2 \in S_\epsilon \subset \text{int}(\bigcup_{B \in \mathbf{B}_O} B)$ we obtain that $d_{pow}(x_2, B') < 0$ for some $B' \in \mathbf{B}_O$ and since $S_\epsilon \cap \bigcup_{B \in \mathbf{B}_I} B = \emptyset$ we have that $d_{pow}(x, B) > 0$ for all balls $B \in \mathbf{B}_I$. These two facts imply that x_1 is inside the power cell of a ball in $\mathbf{B}_I$ and x_2 is inside the power cell of a ball in $\mathbf{B}_O$. Moving a point m along the segment $x_1 x_2$ from x_1 to x_2 the point m meet a face of the *power crust*, otherwise m always remains inside the cells of balls in $\mathbf{B}_I$, which is false. □

Proof of Lemma 26

Looking at the figure 4. Let B'_{m,ρ'_m} be a contraction of the outer medial ball B_{m,ρ_m} tangent to the point $\tilde{u} \in S$ where $\rho'_m = \rho_m - \epsilon = \rho_m - O(\sqrt[4]{r})$. By Lemma 25 we have that $\text{int}(B'_{m,\rho'_m}) \cap S_\epsilon = \emptyset$, consequently $\text{int}(B'_{m,\rho'_m}) \cap N_\epsilon = \emptyset$.

The intersection between the ball $B_{c,\rho}$ and B'_{m,ρ'_m} is empty, due to the ball $B_{c,\rho}$ is inside the interior of the inner region of S_ϵ and the ball B'_{m,ρ'_m} is inside the interior of the outer region of S_ϵ, see figure 4. The angle between the vectors $\vec{cu}$ and $\vec{n}_{\tilde{u}}$ is the sum $\angle(m, c, u) + \angle(c, m, u)$, see figure 4. Our goal is to bound each angles $\angle(m, c, u)$ and $\angle(c, m, u)$. First case: Suppose that the point $t \notin B'_{m,\rho'_m} \cup B_{c,\rho}$. Since the distance $d_{pow}(u, B_{c,\rho}) = O(\sqrt[4]{r})\rho$ by Lemma 24, we have that $d(u, c) = \rho + d(u, \partial B_{c,\rho}) \leq \rho + O(\sqrt[4]{r})\rho$.

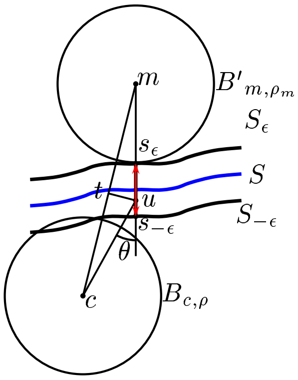

Figure 4: The angle θ between the vector $\vec{cu}$ and the vector $\vec{s_{-\epsilon}s_\epsilon}$
.

Hence

$$d(u,t) < \sqrt{d(u,c)^2 - \rho^2} = \sqrt{(\rho + O(\sqrt[4]{r})\rho)^2 - \rho^2} = O(\sqrt[8]{r})\rho$$

and the angle

$$\angle(m,c,u) \leq \arcsin\left(\frac{d(u,t)}{\rho}\right) \leq \arcsin\left(\frac{O(\sqrt[8]{r})\rho}{\rho}\right) = O(\sqrt[8]{r})$$

Using that $t \notin B'_{m,\rho_m}$ and $u \notin B'_{m,\rho'_m}$ we have that

$$d(t,m) \geq \rho'_m = \rho_m - \epsilon \geq \rho_m - d(s_{-\epsilon}, s_\epsilon) \quad \text{and} \quad d(u,m) \leq \rho'_m + d(s_{-\epsilon}, s_\epsilon) \leq \rho_m + d(s_{-\epsilon}, s_\epsilon) \tag{17}$$

from 17 and $d(s_{-\epsilon}, s_\epsilon) = 2\epsilon = O(\sqrt[4]{r})$ we obtain

$$\begin{aligned} d(u,t) = \sqrt{d(u,m)^2 - d(t,m)^2} &\leq \sqrt{(\rho_m + d(s_{-\epsilon}, s_\epsilon))^2 - (\rho_m - d(s_{-\epsilon}, s_\epsilon))^2} \\ &\leq \sqrt{4d(s_{-\epsilon}, s_\epsilon)\rho_m} = O(\sqrt[8]{r})\sqrt{\rho_m} \end{aligned} \tag{18}$$

Because ρ_m is lower bounded by $\mathrm{lfs}(S) > 0$ and using 18 we can bound the angle $\angle(c,m,u)$ as follows

$$\angle(c,m,u) \leq \arcsin\left(\frac{d(u,t)}{\rho_m}\right) \leq \arcsin\left(\frac{O(\sqrt[8]{r})}{\sqrt{\rho_m}}\right) = O(\sqrt[8]{r}) \tag{19}$$

Therefore, our target angle $\angle(m,c,u) + \angle(c,m,u)$ is $O(\sqrt[8]{r})$.

Second case: $t \notin B'_{m,\rho'_m}$ and $t \in B_{c,\rho}$. Since $t \notin B'_{m,\rho'_m}$ and $u \notin B'_{m,\rho'_m}$ we have that the equations 18 is valid which implies 19, by other side we have $d(u, \partial B'_{m,\rho'_m}) = d(u, s_\varepsilon) = \epsilon = O(\sqrt[4]{r})$, see figure 4. Since $t \notin B_{m,\rho'_m}$ and $d(t,m) \leq d(u,m)$ we have that $u \in B_{m,\rho'_m}$ and $d(t, \partial B'_{m,\rho'_m}) \leq d(u, \partial B'_{m,\rho'_m})$ which implies

$$d(t,c) \geq \rho - d(t, \partial B'_{m,\rho'_m}) \geq \rho - d(u, \partial B'_{m,\rho'_m}) \geq \rho - O(\sqrt[4]{r})$$

Because $d(u,c) \leq \rho$ when $u \in B_{c,\rho}$, and $d(u,c) \leq \rho + d_{pow}(p, B_{c,\rho}) \leq \rho + O(\sqrt[4]{r})\rho$ when $u \notin B_{c,\rho}$ we have the following bound for the distance between the points c and t

$$d(u,t) = \sqrt{d(u,c)^2 - d(t,c)^2} \leq \sqrt{(\rho + O(\sqrt[4]{r})\rho)^2 - (\rho - O(\sqrt{r}))^2} = \sqrt{O(\sqrt[4]{r})\rho + O(\sqrt{r})\rho^2 - O(r)} \tag{20}$$

By definition of the set $\mathbf{B}_I$ we have $\rho \geq \text{lfs}(S)$, therefore we obtain the following bound for the angle

$$\angle(m,c,u) = \arcsin\left(\frac{d(u,t)}{d(u,c)}\right) \leq \arcsin\left(\frac{\sqrt{O(\sqrt[4]{r})\rho + O(\sqrt{r})\rho^2 - O(r)}}{\rho}\right) = O(\sqrt[8]{r})$$

Concluding that the angle $\angle(m,c,u) + \angle(c,u,m)$ is $O(\sqrt[8]{r})$. The third case $t \notin B_{c.\rho}$ and $t \in B'_m$ is similar to second case. □

Proof of Lemma 28

Let B_{c_x,ρ_x} be the ball such that the distance $d_{pow}(x, B_{c_x,\rho_x}) = \min_{B \in \mathbf{B}_I} d_{pow}(x, B)$ with $x \in s_{-\epsilon}s_\epsilon$.

Given a point $x \in s_{-\epsilon}s_\epsilon$, the Lemma 26 establish that the angle α_x between the vector $\vec{c_x x}$ and $\vec{s_{-\epsilon}s_\epsilon}$ is $O(\sqrt[8]{r})$. Consequently for r small enough the angle α_x is always strictly smaller than 90 degree. From this fact, when x moves along the segment g from $s_{-\epsilon}$ to s_ϵ the function $f_I(x)$ increase its value. The proof that the the function f_O is decreasing is similar. □

Proof of Lemma 29

By Lemma 23 given any point $x \in S$ the segment $s_{-\epsilon}s_\epsilon$ intersects the power crust in at least a point. Since $s_{-\epsilon} \in \text{int}(\bigcup_{B \in \mathbf{B}_I} B) \cap S_{-\epsilon}$ we have $f_I(s_{-\epsilon}) \cdot f_O(s_{-\epsilon}) < 0$ and since $s_\epsilon \in \text{int}(\bigcup_{B \in \mathbf{B}_O} B) \cap S_{-\epsilon}$ we have $f_I(s_\epsilon) \cdot f_O(s_\epsilon) < 0$.

The Lemma 28 states that the function f_I is decreasing along the segment $s_{-\epsilon}s_\epsilon$ and f_O is increasing along $s_{-\epsilon}s_\epsilon$. From these two facts and the continuity of f_I and f_O there exists an unique point $p \in s_{-\epsilon}s_\epsilon$ such that $f_I(p) = f_O(p)$. Hence, we have $p \in \text{Crust}(\mathbf{B}_I \cup \mathbf{B}_O)$ and it is unique. □

Proof of Lemma 30

We shall prove that the restriction function d_r of the function $d : \Omega \to S$ mapping a point x to its closest point in S to the Crust($\mathbf{B}_O \cup \mathbf{B}_I$), is a homeomorphism. Since S is compact, it is sufficient to prove that d_r is a continuous bijection. Since the discontinuity set of d is the medial axis M, then the restriction d_r of d to Crust($\mathbf{B}_O \cup \mathbf{B}_I$) $\subset N_\epsilon$ is a continuous function. One to one: Suppose that there are two points $\{p_1, p_2\} \in$ Crust($\mathbf{B}_I \cup \mathbf{B}_O$) such as $d(p_1) = d(p_2) = s'$, then the two points p_1 and p_2 are in the segment $s'_{-\epsilon}s'_\epsilon$. This lead to a contradiction because Lemma 29 states that there exist an unique intersection point of the segment $s'_{-\epsilon}s'_\epsilon$ with Crust($\mathbf{B}_I \cup \mathbf{B}_O$). Onto: By Lemma 23 for any point $s \in S$ the segment $s_{-\epsilon}s_\epsilon$ intersects Crust($\mathbf{B}_O \cup \mathbf{B}_I$), therefore the function d_r is onto. □