DYNAMICAL PROPERTIES OF SINGULAR HYPERBOLIC ATTRACTORS

AUBIN ARROYO   ENRIQUE R. PUJALS

Abstract. Singular hyperbolicity is a weaker form of hyperbolicity that is found on any C^1-robust transitive set with singularities of a flow on a three-manifold, like the Lorenz Attractor, [MPP]. In this work we are concerned in the dynamical properties of such invariant sets. For instance, we obtain that if the attractor is singular hyperbolic and transitive, the set of periodic orbits is dense. Also we prove that it is the closure of a unique homoclinic class of some periodic orbit. A corollary of the first property is the existence of an SRB measure supported on the attractor. These properties are consequences of a theorem of existence of unstable manifolds for transitive singular hyperbolic attractors, not for the whole set but for a subset which is visited infinitely many times by a resiudal subset of the attractor. Here we give a complete proof of this theorem, in a slightly more general context. A consequence of these techniques is that they provide a sufficient condition for the C^1-robust transitivity.

1. Introduction

Hyperbolicity is the paradigm of stability for diffeomorphisms and flows without singularities; where some conditions on the behavior of the derivative constrain the dynamic to a robust scenario. They are so called Axiom A or Uniform Hyperbolic systems and they are well understood after the work of Smale [Sm] and others. Such systems are structurally stable, the set of periodic orbits is dense in the non-wandering set, and they have a spectral decomposition on a finite union of homoclinic classes. When we focus on flows defined on a closed manifold M the accumulation of regular orbits on fixed points (singularities) rule out such hyperbolic structure; however, it may produce a new phenomena which in some cases behaves very like the uniform hyperbolic systems. A striking example of this is the Lorenz Attractor [Lo]: an invariant set of a flow given by the solutions of the following polynomial vector field in $\mathbb{R}^3$.

$$X(x,y,z) = \begin{cases} \dot{x} = -\alpha x + \alpha y \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\gamma z + xy\,, \end{cases} \tag{1}$$

Numerical experiments performed by Lorenz around the mid-sixties suggested the existence of a strange attractor for some real parameters close to $\alpha = 10$, $\beta = 28$ and $\gamma = 8/3$; a set which trap the positive orbit of all points in a full neighborhood of it. Although such example is not hyperbolic, the attractor seemed to be robust: it can not be destroyed by small perturbations of the parameters.

Given a vector field $X \in \mathcal{X}^1(M)$ there is a one parameter family of diffeomorphisms on M: $\Phi_t : M \to M$, obtained by integration of the vector field. Such family is what is called a flow on M, that is, it satisfy that $\Phi_0 \equiv \mathrm{Id}$ and $\Phi_t \circ \Phi_s = \Phi_{s+t}$, for any $s, t \in \mathbb{R}$. An attractor set of Φ_t is an invariant set Λ such that there is a neighborhood $U \supset \Lambda$ that $\Lambda = \cap_{t>0} \Phi_t(U)$. Such attractor is *transitive* if there exist a dense orbit on it. And is C^1*-robustly transitive* if there is a neighborhood $\mathcal{U}$ of X for which the set $\cap_{t\in\mathbb{R}} \Phi(Y)_t(U)$ is transitive for all $Y \in \mathcal{U}$. Here and in what follows, given a vector field Y, the flow obtained by integration of Y will be denoted by $\Phi_t(Y)$.

To understand the dynamics behind the Lorenz Attractor a geometrical model was introduced in [Gu] and also in [ABS] in the mid-seventies. With that model they showed the existence of a non-trivial transitive attractor with singularities in a robust way. Notably, only three and a half decades after the remarkable discover of Lorenz, it was proved by Tucker in [Tu] that the solutions of (1) behave in the same way as the geometrical model for values α, β and γ near the ones considered by Lorenz. Moreover, such a model allowed to isolate important properties always present among the C^1-robustly transitive sets with singularities, at least in dimension 3. In fact, in [MPP2] they prove that any C^1-robustly transitive set with singularities on closed 3-manifold verifies some weaker form of hyperbolicity which now is called *Singular Hyperbolicity* and roughly speaking means that:

(1) It is either a proper attractor or a proper repeller;
(2) Eigenvalues at all singularities satisfy the same inequalities as the in the Lorenz geometrical model;
(3) They are partially hyperbolic and the central direction is volume expanding.

We define precisely this notion latter. For now let us consider a natural question: *Is it possible to develop a complete theory for singular hyperbolic systems as the one that was constructed for the uniform hyperbolic ones?*

In this work we provide a dynamical description for transitive singular hyperbolic sets on three-manifolds, and is intended to give a positive answer to the last question. In order to do so, we shall use stable and unstable manifolds of uniform size as a corner stone, in the same fashion they are used in the context of uniform hyperbolicity. After that, a small extra effort will give us also a sufficient condition to obtain C^1-robustly transitivity on singular hyperbolic attractors.

More precisely, Main Theorem below assert that even in the presence of equilibrium points in a singular hyperbolic attractor there are local unstable manifolds of uniform size, not for the whole attractor but for a subset of it which is visited infinitely many times by points in a residual subset of it. As a first consequence we obtain the following theorem about the set of periodic orbits. Denote by $\mathrm{Per}(\Lambda)$ the set of periodic orbits of Φ_t in Λ.

Theorem B. *Let Λ be a transitive singular hyperbolic attractor for the flow Φ_t, then $cl(Per(\Lambda)) = \Lambda$.*

Recall that there are examples of singular hyperbolic sets which do not have any periodic orbit at all (see [MoII]). However, such sets are not attractors neither robust.

The second consequence is about a spectral decomposition. It is known that a general singular hyperbolic system do not allow a spectral decomposition into a finite number of disjoint homoclinic classes; as it is proved in [BMP]. Also there are examples of singular hyperbolic transitive attractors with the property that they have a periodic orbit in the closure of an homoclinic class which is not homoclinically related. Such systems are also examples of transitive singular hyperbolic attractors that they are not C^1-robustly transitive sets; see [MP]. The following theorem states that this is the more general situation. Denote by $H(p)$ the homoclinic class of a periodic point $p \in \Lambda$.

Theorem C. *Let Λ be a transitive singular hyperbolic attractor for the flow Φ_t, then there is a periodic orbit p such that $cl(H(p)) = \Lambda$.*

It has been proved recently that any transitive singular hyperbolic attractor has at least one periodic orbit; see [BM]. However, all these questions remain open in dimension greater than three.

In [Co] he prove the following: *if Λ is a singular hyperbolic transitive attractor of a $C^{1+\alpha}$ vector field, $\alpha > 0$, with dense periodic orbits then it has an SRB measure.* As a consequence of this and Theorem B we obtain then

Corollary: *If Λ is a singular hyperbolic transitive attractor of a $C^{1+\alpha}$ vector field, $\alpha > 0$, it has an SRB measure.*

A deeper consequence of the theorem of existence of unstable manifolds is that it allows us to find a sufficient condition to obtain C^1-robustly transitive sets with singularities. Such condition is (H*), below, which roughly speaking states that there are some neighborhoods of Sing(Λ) where the maximal invariant set of the complement in U of such neighborhoods is a basic piece. If we denote by $\text{Sing}(\Lambda) := \{p \in \Lambda | X(p) = 0\}$, we can state precisely this theorem:

Theorem A. *Let Λ be a transitive singular hyperbolic attractor for $X \in \mathcal{X}^1(M)$ that verifies the following hypothesis:*

(H*) *There is $\delta_0 = \delta_0(X)$ such that for any $\delta < \delta_0$ there is $\delta_1 < \delta$ that $\cap_{t>0}\Phi_t(U - B_{\delta_1}(Sing(\Lambda)))$ is a basic piece.*

Then Λ is C^1-robustly transitive.

In order to do so, we need to control the recurrence between equilibrium points with a weaker notion of transitivity. In section 3 we define the notion of having complete recurrence with the aid of an oriented graph. Such notion is robust in the C^1 topology and, of course, is implied by the usual notion of transitivity. This will be done in subsection 3.

Let us state precisely the definition of a singular hyperbolic splitting.

Definition: *A compact invariant set $\Lambda \subset M$ of Φ_t is singular hyperbolic if there is $\lambda < 0$ and a constant $C > 0$, and if the tangent bundle of Λ splits into two invariant sub-bundles: $T_\Lambda M = E^s \oplus E^{cu}$ where the following holds either for Φ_t or Φ_{-t}:*

1. *Any $\sigma \in Sing(\Lambda)$ is hyperbolic.*
2. *The splitting $E^s \oplus E^{cu}$ is dominated by λ.*
3. *$||D\Phi_t|_{E^s}|| < C\exp(\lambda t)$; that is, E^s is uniformly contracting.*
4. *$det(D\Phi_t|_{E^{cu}}) > C\exp(-\lambda t)$; that is E^{cu} is volume expanding.*

Recall the definition of a dominated splitting. The splitting $T_\Lambda M = E \oplus F$ is dominated by $\lambda < 0$ if for any $t > 0$ we have that:

$$\frac{|D\Phi_t|_E|}{|D\Phi_t|_F|} < \lambda^t$$

The class of singular hyperbolic systems contain the Axiom A systems, the geometric Lorenz attractors and the singular horseshoes in [LP], among other systems. Consider a singular hyperbolic invariant set on a closed three manifold M. Once we set that the singular hyperbolic splitting hold for positive time t, the integrability of the strong stable bundle allow us to find stable manifolds of uniform size $\varepsilon_s > 0$ on any point of U; that is, for any $x \in U$ there is a C^1-interval say $W^s_{\varepsilon_s}(x)$ that any point y on it happens that $d(\Phi_t(x), \Phi_t(y)) \to 0$, when $t \to \infty$. In the case of a singularity it correspond to the stable manifold associated to the strongest contracting eigenvalue: $W^{ss}_{\text{loc}}(\sigma)$. This affirmation is contained in Lemma 1

For unstable manifolds such construction may be impossible on regular points. However, associated to any regular point $x \in \Lambda$ there is a family of 2-dimensional sections N_{x_t} whose size depends on the point x_t and which are transversal to the flow; here $x_t := \Phi_t(x)$.

This sections shall be defined precisely on Section 3. According to them we can write the family of holonomy maps between these transversal sections, for any $t \in \mathbb{R}$:

$$G^t_x : \text{Dom}(G^t_x) \subset N_x \longrightarrow N_{x_t}$$

We shall define such maps in a precise way in section 3 using the Implicit Function Theorem.

Consider some $\varepsilon > 0$. The unstable manifold of size ε of a regular point $x \in \Lambda$ is

$$\tilde{W}^u_\varepsilon(x) = \{y \in M | y \in \text{Dom}(G^{-t}_x) \text{ and } \text{dist}(G^{-t}_x(y), x_{-t}) \to 0, t \to \infty\}$$

Of course, $\tilde{W}^u_\varepsilon(x) \subset N_x$. On the other hand, there is some $\tilde{\varepsilon} > 0$ for which the central unstable manifold exists on any point x of a singular hyperbolic set Λ; according to [HPS] as we shall see in Section 3. Denote these central manifolds by $W^{cu}_{\tilde{\varepsilon}}(x)$. Also denote by $L(Y)$ the limit set of Y. The main goal of this paper is the following

Main Theorem. *Let $X \in \mathcal{X}^1(M)$, $\Lambda \subset M$ be a transitive singular hyperbolic attractor and $U \supset \Lambda$ an open neighborhood contained in its basin of attraction. Then there is a neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ such that for all $Y \in \mathcal{U}(X)$ there is a subset $K(Y) \subset \Lambda_Y := \cap_{t \geqslant 0} Y_t(U)$, $\varepsilon_u > 0$ and $\lambda_u < 0$ that*

1. *For any $y \in K(Y)$, we have that $W^{cu}_{\varepsilon_u}(y) \cap N_y = \tilde{W}^u_{\varepsilon_u}(y)$.*
2. *For any $y \in \hat{W}^u_{\varepsilon_u}(x)$ there is a not bounded sequence $t_i > 0$ such that*
$$dist(G^{-t_i}_x(y), x_{-t_i}) < C \exp(t_i \lambda_u).$$
3. *$\bigcup_{t > T_0} \bigcup_{y \in K} \Phi_{-t}(y)$ is an open and dense set in $\Lambda_Y \cap L(Y)$, for any $T_0 > 0$.*

In order to prove Main Theorem we shall find first a system of transversal sections associated to the passage through a neighborhood of each equilibrium points. Among the systems of transversal sections, some of them present an induced map with a kind of markovian property and uniform expansion. However, to guarantee the existence of such transversal sections it is needed a deep analysis of the dynamic inside a neighborhood of singularities and also the combinatorics between them. More precisely, we have to study the local holonomy maps between transversal sections in general. In section 2 we shall prove Theorem B and Theorem C assuming Main Theorem. In section 3 we shall study the holonomy maps, both locally and globally. In Section 4 we shall reduce the proof of Main Theorem to a Main Lemma about an induced map to certain system of transversal sections and then we prove Main Lemma. Finally, in section 5 we prove Theorem A.

2. Dynamical properties

In this section we exhibit some dynamical description of transitive singular hyperbolic attractors; that is, the content of Theorem B and Theorem C as a consequence of Main Theorem.

Proof of Theorem B. Let Λ be a transitive singular hyperbolic attractor for the flow Φ_t and let $\varepsilon_u > 0$ and K be as in Main Theorem. Denote by $D \subset \Lambda$ the set of points whose positive orbit is dense in Λ. This set is residual in Λ. Also recall $\lambda < 0$ from the Main Theorem. Notice that Main Theorem assert that for any $T_0 > 0$ the set

$$\tilde{K}_{T_0} = \bigcup_{t>T_0} \bigcup_{y\in K} \Phi_{-t}(y)$$

is residual in Λ, and hence also $\tilde{K} = \cap \tilde{K}_T$. Therefore there is $z \in D \cap \tilde{K}$.

Take any regular point $x \in \Lambda$. We shall show it is accumulated by periodic orbits. For that, we only have to care about regular points since any not isolated $\sigma \in \mathrm{Sing}(\Lambda)$ is accumulated by regular orbits and hence a diagonal selection process shall give us what we want.

Recall that, in order to simplify the notation we denote by $y_t := \Phi_t(y)$ for any point $y \in M$ and $t \in \mathbb{R}$. Observe that since $z \in D$, there is an increasing sequence of positive real numbers j_i that $z_{j_i} \to x$ as $i \to \infty$. On the other hand, there is another sequence r_i that $z_{r_i} \in K$. We can choose this sequence that r_i is not bounded since $z \in \tilde{K}$, such that $j_i \leqslant r_i \leqslant j_{i+1}$, for all $i \in \mathbb{N}$, and that z_{r_i} is a Cauchy sequence on M.

Take a positive integer n_0 such that $d(z_i, z_j) < \frac{1}{3}\min\{\varepsilon_s, \varepsilon_u\}$, for any $i,\ j \geqslant n_0$. Fix some $i > n_0$ and for it denote by $Z = z_{r_i}$. Now, for any $n > i$ we have that $\Phi_{-t_n}(\Phi_{r_n}(z)) = Z$, where $t_n = r_n - r_i$. Hence, as a consequence of Main Theorem we know that

$$W^{cu}_{\varepsilon_u}(z_{r_n}) \cap N_{z_{r_n}} \subset \tilde{W}^u_{\varepsilon_u}(z_{r_n})$$

On the other hand, there is a transversal section Σ, foliated by local stable manifolds, containing z_{r_n} and Z and C^1-close to $N_{z_{r_n}}$. Notice that $J := W^{cu}_{\varepsilon}(z_{r_n}) \cap \Sigma$ is an interval, and the family of holonomy maps associated to Σ behaves the same as the one for $N_{z_{r_n}}$. Now let

$$\tilde{J} = G^{-t_n}_{z_{r_n}}(W^{cu}_{\varepsilon_u}(z_{r_n})) \cap \Sigma.$$

Finally choose $n_1 > n_0$ such that for any $n > n_1$ verifies that $\exp(t_n\lambda) = \exp((r_n - r_i)\lambda)\varepsilon_1 < \varepsilon/4$. This is possible since the sequence t_n is not bounded and $\lambda < 0$.

Consider now the 2-dimensional transversal section made up with local stable manifolds $B = \bigcup_{y \in J} W^s_{\varepsilon_s}(y) \subset \Sigma$. If $n > n_1$ then $\tilde{J} \subset B$. This allow us to define a function $f : J \longrightarrow J$ by

$$y \mapsto \pi_s(G^{-t_n}_{z_{r_n}}(y))$$

where π_s denotes the projection along stable leaves. Since f is continuous then there exists $y \in J$ such that $f(y) = y$. This means that $\Phi_\tau(y) \in W^s_{loc}(y)$, for some $\tau \in \mathbb{R}$. This implies the existence of a periodic point $P \in B$ that y and y_τ, both belong to $W^s_{\mathrm{loc}}(P)$; see Lemma 4.3 of [AR].

Hence, for any r_i and $n > n_1$ we have constructed a sequence of periodic points P^i_n, such that P^i_n converges to z_{r_i} as $n \to \infty$. On the other hand,

$$\Phi_{j_i - r_i}(P^i_n) \to_{n \to \infty} \Phi_{j_i}(z) \to_{n \to \infty} x$$

and we are done. □

Proof of Theorem C. Let Λ be a transitive singular hyperbolic attractor for the flow Φ_t. Consider some $x \in \Lambda$ such their α and ω-limits are dense. As we saw in the proof of Theorem B, we can assume that $x \in K(X) = K$ and the set of positive times R that $\Phi_r(x) \in K$ is not bounded. Hence, it is enough to prove that there is some periodic point $p \in \Lambda$ such that for any r_i that $\Phi_{r_i}(x) \in K$, then it is accumulated by points in the homoclinic class of p.

For that, consider an accumulation point of $\Phi_{r_i}(x)$. Since the points belong to K they have large local unstable manifold and arguing as in previous corollary we get the existence of the periodic point p.

Moreover we have that:

$$\hat{W}^u_{\varepsilon_u}(\Phi_{r_i}(x)) \cap W^s_{\mathrm{loc}}(p) = \{\tilde{q}\}$$

$$W^u_{\varepsilon_u}(p) \cap W^s_{\mathrm{loc}}(\Phi_{r_i}(x)) = \{q\}$$

Since $\alpha(x)$ is dense we get that there is a sequence $j_k \in \mathbb{R}^+$ such that $\Phi_{-j_k}(x) \to \Phi_{r_i}(x)$, as $k \to \infty$. Using that $\Phi_{r_i}(x) \in K$, and hence it has large unstable manifold, it imply that there are segments γ^s_k contained in the local stable manifold of p such that $G^{-j_k - r_i}_x(\gamma^s_k)$ accumulate in the local stable manifold of $\Phi_{r_i}(x)$. Hence, there are points of intersection of the stable manifold of p and the local unstable manifold of p, say q_k, accumulating on q.

Again, using that there is $n_k \to +\infty$ that $\Phi_{n_k}(x)$ accumulates on $\Phi_{r_i}(x)$ follows that $\Phi_{n_k - r_i}(q)$ accumulates on $\Phi_{r_i}(x)$ and so, there is a sequence of points in the homoclinic class of p accumulating in $\Phi_{r_i}(x)$, as we wanted. □

3. Combinatorics, transversal sections and transitions

Now we start working to obtain Main Theorem. Let us denote by Φ_t a flow defined on a closed riemaniann three-manifold, obtained by the integration of a C^1 vector field X.

Consider $\Lambda \subset M$ a transitive attractor which is singular hyperbolic. By definition, there is an open set U containing Λ that $\Lambda = \bigcap_{t \geqslant 0} \Phi_t(U)$. From now on we shall assume that $\mathrm{Sing}(\Lambda)$ is not empty, otherwise Λ is hyperbolic; indeed we denote by $k = \#\mathrm{Sing}(\Lambda)$.

Perhaps considering an equivalent riemannian metric in M, for instance the adapted metric we can assume that $||D\Phi_t|_E)|| < \exp(t\lambda)$ and $\det(D\Phi_{-t}|_F) < \exp(t\lambda)$, for any $t \geqslant 0$; for the same $\lambda < 0$; that is, it sets the constant $C = 1$.

On the open set $\tilde{U} = U \setminus \mathrm{Sing}(\Lambda)$ it is well defined the normal bundle $\mathcal{N}$, consisting of the orthogonal spaces to $[X(p)]$ in $T_{\tilde{U}}M$. Each fiber $\mathcal{N}_x$ is two dimensional and it has a coordinate system induced by the singular splitting in the following way:

$$\hat{E}^s_x := \Pi(E^s_x) \text{ and } \hat{E}^u_x := E^{cu}_x \cap \mathcal{N}_x$$

where $\Pi : TM \to \mathcal{N}$ denotes the orthogonal projection. We assume that for the metric on $T_{\tilde{U}}M$ the splitting $\hat{E}^s \oplus \hat{E}^u = \mathcal{N}$ is orthogonal, since the singular splitting is continuous and the angle between them is uniformly bounded away from zero.

Notice that there is a strictly positive function $\eta : \tilde{U} \longrightarrow \mathbb{R}^+$ that the application:

$$\exp_p : \mathcal{N}_p(\eta(p)) \longrightarrow M$$

is an isometry. Here $\mathcal{N}_p(\eta)$ stands for the standard rectangle $(-\eta, \eta) \times (-\eta, \eta) \subset \mathcal{N}_p$, according to the previously defined coordinates. If one consider some $\delta > 0$, there is a uniform lower bound $\eta^* = \eta^*(\delta) > 0$ that $\eta(p) > \eta^*$ for all p such that $d(p, \mathrm{Sing}(\Lambda)) > \delta$. After this, denote by $\eta^* := \eta^*(\delta) : \tilde{U} \longrightarrow \mathbb{R}^+$ as $\eta^*(p) = \min\{\eta(p), \eta^*\}$. Therefore, denote by $N_p = \exp(\mathcal{N}_p(\eta^*(p))) \subset M$, which is a two dimensional transversal section to X of size $\eta^*(p)$.

3.1. **Transitions.** Consider some point $p \in \tilde{U}$ and $t \in \mathbb{R}$. There is a continuous function $\tau : D \subset N_p \longrightarrow \mathbb{R}$ such that $\tau(p) = t$ and $\Phi_{\tau(x)}(x) \in N_{p_t}$, defined in certain domain D by the Implicit Function Theorem. This function defines the *transition* map between N_p and N_{p_t} as follows:

$$G^t_p : \mathrm{dom}(G^t_p) \subset N_p \longrightarrow N_{p_t}$$
$$G^t_p(x) = \Phi_{\tau(x)}(x) \in N_{p_t}$$

The map G^t_p represents the holonomy between N_p and N_{p_t}. Assuming $t > 0$, notice that the set $C(p, t) = \{\Phi_s(x) | x \in \mathrm{dom}(G^t_p) \text{ and } s \in [0, \tau(x)]\}$ is foliated by transversal sections N_{p_s}, for $s \in [0, t]$, and if we do not consider the caps contained in N_p and N_{p_t}, it is an open set. Also consider the set $D(p, t)$ defined by the connected component that contains p of the intersection $\cap_{s\in[0,t]}\mathrm{dom}(G^s_p)$. These are actually the domains we are going to use. In particular, for any point $x \in D(p, t)$ there are $s \in [0, \tau(x)]$ and $\tilde{s} \in [0, t]$ that $x_s \in N_{p_{\tilde{s}}}(\eta^*(p))$. All these constructions are still valid for $t < 0$, considering the opposite time intervals.

On the other hand, the singular hyperbolic splitting on an invariant set fits into the work of [HPS] and allow us to prove the following lemma that guarantees the existence of central manifolds. In order to state it denote by $D^s_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbb{R}$ and D^{cu}_ε the 2 dimensional disc of radius ε for some $\varepsilon > 0$. Also denote by $\mathrm{Emb}^1(D^{s,cu}_\varepsilon, M)$ the set of C^1-embeddings on M of D^s_ε, and D^{cu}_ε, respectively.

Lemma 1. *If Λ is a singular hyperbolic invariant set for the flow Φ_t then there exists two continuous functions:*

$$\Psi^{s,cu} : U \to Emb^1(D^{s,cu}_1, M)$$

that, if we denote by $W^{s,cu}_{\varepsilon}(x) := \Psi^{s,cu}(x)(D^{s,cu}_{\varepsilon})$*, respectively, for any* $x \in U$*; then these sets verify that* $T_x(W^s_{\varepsilon}(x)) = E^s(x)$ *and* $W^{cu}_{\varepsilon}(x) = E^{cu}(x)$*. Also, if we denote by* G^t_x *the corresponding transition for* $t > 0$ *then we have that:*

(1) *There is* $\varepsilon_s > 0$ *and* $\lambda_s < 0$ *such that* $G^t_x(W^s_{\varepsilon_s}(x)) \subset W^s_{\lambda^t_s \varepsilon_s}(x_t)$.
(2) *For any* $\varepsilon_1 > 0$ *there is* $\varepsilon_2 > 0$ *such that if* $y \in W^{cu}_{\varepsilon_2}(x) \cap N_x$ *then* $G^{-t}_x(y) \in W^{cu}_{\varepsilon_1}(x_{-t}) \cap N_{x_{-t}}$.

Item 1 states the existence of local stable manifolds of uniform size ε_s since the definition of singular hyperbolic set guarantees exponential contraction on E^s. However, on E^{cu} we only obtain central unstable manifolds which are not dynamically defined. Nevertheless, this lemma induce two families of curves, transversal to each other, on each transversal section N_p, called $\mathcal{F}^s$ and $\mathcal{F}^{cu}$, respectively.

For a sharper study of holonomy maps we need to consider smaller boxes contained in N_p, for points $p \in U$. A set $B \subset N_p$ is a box if it is the image under $\exp_p$ of some rectangle in $\mathcal{N}_p$ containing 0. A box has two lengths, the stable and the unstable one, and these numbers are defined by the length of the image under the corresponding projections along the leaves of $\mathcal{F}^s$ and $\mathcal{F}^{cu}$, say $\pi_{s,u} : \mathcal{N}_p \longrightarrow \hat{E}^{s,u}_p$, respectively, in a way that:

$$|B|_{s,u} = \text{length}[\pi_{s,u}(\exp_p^{-1}(B))]$$

To avoid confusing notation we shall not make any difference between the boxes in M and their image by $\exp^{-1}$ in $\mathcal{N}$. A sub-box $\tilde{B} \subset B$ is a set that contains p and that $\pi_s(\tilde{B}) = \pi_s(B)$ and $\pi_u(\tilde{B}) \subset \pi_u(B)$. The boundary of a box is divided into two subsets: $\partial^u B \subset \mathcal{F}^s$ and its complement: $\partial^s B$.

When it is necessary to be explicit on the dimensions of a box B, we shall denote first its unstable length followed by the stable one; that is $B(\varepsilon_u, \varepsilon_s)$, for some $\varepsilon_s, \varepsilon_u > 0$. Sometimes it will be only important the unstable length and then we shall denote only by $B(\varepsilon_u)$. A semi-box on p is a box containing p in the unstable boundary.

Denote by $V_\delta := U \setminus \bigcup \text{cl}(B_\delta(\sigma))$, where the union takes values on all $\sigma \in \text{Sing}(\Lambda)$ and $B_\delta(\sigma)$ is the open ball of radius δ around σ. In the following subsection we are going to prove the next two properties for singular hyperbolic attractors:

Property 1. *There is* $\lambda_u < 0$ *and* $\delta > 0$ *such that if* p *and* p_t *both belong to* V_δ *for some* $t > 0$*, and if a box* $B \subset D(p,t)$ *we have that*

$$|G^t_p(B)|_u > \exp(-\lambda_u t)|B|_u$$

Property 2. *Take* $t > 0$*. If* p *and* p_t *both belong to* V_δ *for some* $\delta > 0$ *and if we denote the corresponding transition:*

$$G^t_p : D(p,t) \subset N_p \longrightarrow N_{p_t}$$

then for any two boxes $B_p(\varepsilon_1) \subset N_p$ *and* $B_{p_t}(\varepsilon'_1) \subset N_{p_t}$ *of unstable length* ε_1 *and* ε'_1*, respectively, we have that one of the following holds:*

(1) $B_p \subset D(p,t)$ *and* $G^t_p(B_p) \subset B_{p_t}$*;*
(2) *There is a sub-box* $\tilde{B} \subset B_p$ *that* $G^t_p(\tilde{B})$ *covers* B_{p_t}*, that is* $\pi_s(G^t_p(\tilde{B})) = \pi_s(B_{p_t})$*;*

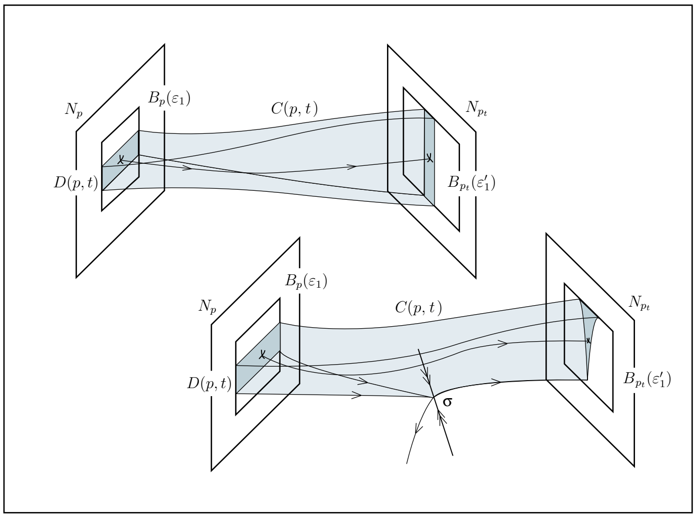

Figure 3.1 Items 2 & 3 of Property 2.

(3) *There is a point* $y \in B_p$ *such that* $\Phi_t(y) \in W^s_{loc}(\sigma)$ *for some* $t > 0$ *and* $\sigma \in Sing(\Lambda)$. *And more, the semi-box* $B^* \subset B_p$, *bounded by* $W^s_{loc}(x)$ *and* $W^s_{loc}(y)$ *is such that* $B^* \subset D(p,t)$.

However, to obtain such properties we have to study very carefully the behavior of transitions and its domains; whereas it passes through a neighborhood of the singular points and when it travels far away from them.

3.2. **Local analysis of equilibrium points.** Since Λ is singular hyperbolic then any $\sigma \in \mathrm{Sing}(\Lambda)$ is hyperbolic. More than that, in [MPP] they prove that their eigenvalues satisfy the following inequalities:

$$\lambda^{ss}_\sigma < \lambda^s_\sigma < 0 < -\lambda^s_\sigma < \lambda^u_\sigma$$

Also there they prove that $W^{ss}(\sigma) \cap \Lambda = \emptyset$, for any singular point $\sigma \in \mathrm{Sing}(\Lambda)$. Take O_σ some linearizing neighborhood on σ, on it, the vector field has the following expression (in such local coordinates):

$$X(x,y,z) = \begin{cases} \dot{x} = \lambda^{ss}_i\, x \\ \dot{y} = \lambda^u_i\, y + \mathrm{h.o.t}(x,y,z) \\ \dot{z} = \lambda^s_i\, z\, + \mathrm{h.o.t}(x,z), \end{cases}$$

Where h.o.t denote higher order terms in the variables referred. Hence, $W^s_{loc}(\sigma) = [y = 0]$, $W^{ss}_{loc}(\sigma) = [y = z = 0]$, $W^u_{loc}(\sigma) = [x = z = 0]$; and the flow is given by:

$$\Phi_t(x_0, y_0, z_0) = (x_0 e^{t\lambda^{ss}}, y_0 e^{t\lambda^u} + \mathrm{h.o.t}(x,y,z), z_0 e^{t\lambda^s} + \mathrm{h.o.t}(x,z)) \tag{2}$$

according to the eigenvalues of σ. To define a *dynamical neighborhood* $H(\sigma)$ for $\sigma \in \mathrm{Sing}(\Lambda)$ we have to look inside O_σ in terms of their linearizing coordinates, as follows. First notice that there are two positive numbers c_z and c_y that the planes $[z = \pm c_z]$ and $[y = \pm c_y]$ are transversal to the flow, on the former the vector field points inwards

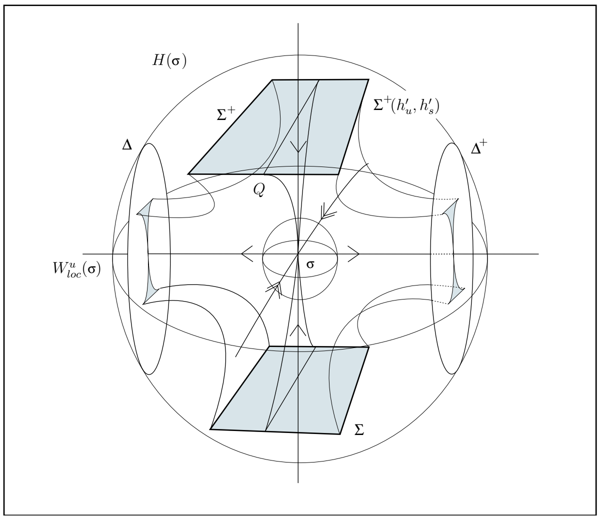

Figure 3.2 Transversal sections on a neighborhood of $\sigma \in \mathrm{Sing}(\Lambda)$

to the region containing the origin and the latter pointing outwards. Notice also that $W^u_{\mathrm{loc}}(\sigma) \cap [y = \pm cy] = \{p^\pm\}$. Considering the equations (2), we can find for any two small positive numbers h_s and h_u, and hence the rectangles

$$\Sigma^+(h_u, h_s) = \{(x, y, c_z) | y \in (-h_u, h_u),\, x \in (-h_s, h_s)\}$$

and the same for $\Sigma^-(h'_u, h'_s) \subset [z = -c_z]$ are such that there exists two disks $\Delta^\pm \subset [y = \pm c_y]$ that for any point $x \in \Sigma^\pm(h_u, h_s)$ there is $t > 0$ that $\Phi_t(x) \in \Delta^+ \cup \Delta^-$, and $\Phi_s(x)$ do not escapes from the region bounded by the previously defined planes and $[x = \pm h_s]$. Notice that the vector field also point inwards in the planes $[x = \pm h_s]$. This region is what we call a dynamical region $H(\sigma)$. Observe also that if $\varepsilon > 0$ is small enough and if $x \in \Sigma^+ \cup \Sigma^-$, we can assume that $N_x(\varepsilon) \subset [z = c_z] \cup [z = -c_z]$, since the actual transversal section is C^1 close to these planes.

Lemma 2. *If c_z is small enough then for any point $x \in \tilde{\Lambda}$ that its ω-limit contains σ then the orbit of x passes through Σ^+ or Σ^-.*

Proof. This statement is obtained after Proposition 2.4 in [MoP], since there asserts that $W^{ss}(\sigma) \cap \Lambda = \emptyset$, for all $\sigma \in \mathrm{Sing}(\Lambda)$, which corresponds to the x-axis. □

For each $\sigma \in \mathrm{Sing}(\Lambda)$ there is $\delta_\sigma > 0$ that $B_{\delta_\sigma} \subset H(\sigma)$ and more, if $x \in B_{\delta_\sigma}$ then there are two real numbers t_- and t_+ that $\Phi_{t_-}(x) \in \Sigma^+ \cup \Sigma^-$ and $\Phi_{t_+}(x) \in \Delta^+ \cup \Delta^-$. Fix, once and for all $\delta = \min\{\delta_\sigma\}$, and hence, the set V_δ.

After constructing dynamical neighborhoods around each point in $\mathrm{Sing}(\Lambda)$ we obtain $2k$ transversal sections $\Sigma^\pm$ as above; two by each singular point. Now consider only those that intersect Λ; that is the set $\Sigma := \{\Sigma_i | \Sigma_i \cap \Lambda \neq \emptyset\}$. After Lemma 2 this set is not empty, and hence the index i ranges in $\{1, \dots, \tilde{k}\}$. On each one of them, the interval $Q_i := \Sigma_i \cap W^s_{loc}(\sigma)$ (where σ is the corresponding singular point) splits Σ_i into two semi-boxes labeled: Σ_i^- and Σ_i^+. Given a pair of positive numbers $(\varepsilon_i^-, \varepsilon_i^+)$, we can define a subsection $\Sigma_i(\varepsilon_i^-, \varepsilon_i^+) \subset \Sigma_i$ of unstable length $\varepsilon_i^-, \varepsilon_i^+$; that is,

$$\pi_s(\Sigma_i(\varepsilon_i^-, \varepsilon_i^+)) = (\pi_s(x_i) - \varepsilon_i^-, \pi_s(x_i) + \varepsilon_i^+).$$

Any choice of $E := \{\varepsilon_i^-, \varepsilon_i^+ | \, i = 1 \dots \tilde{k}\}$ induce what we call a system of transversal sections $\Sigma(E) := \cup_i^{\tilde{k}} \Sigma_i(\varepsilon_i^-, \varepsilon_i^+)$ contained in Σ. Denote by $Q := \cup_{j=1}^{\tilde{k}} Q_i$, by $\hat{\Sigma}(E) := \Sigma(E) \setminus Q$, and by $\hat{\Sigma} := \Sigma \setminus Q$.

Definition. *A connected subset B in Σ is a band if it intersects both connected components of the stable boundary of Σ and $B \cap \Lambda \neq \emptyset$.*

To guarantee that under our hypothesis do we have points of Λ on both sides of $\Sigma_j(E)$ for any j, we have to study the combinatorics of the attractor. This is what we are going to do in the next subsection.

3.3. **Complete recurrence.** Consider a singular hyperbolic attractor $\Lambda \supset U$. If we take $\delta > 0$ small enough we can assume that $B_\delta(\sigma) \subset U$ and more $B_\delta(\sigma) \subset O_\sigma$, for any $\sigma \in \mathrm{Sing}(\Lambda)$. For a fixed $\delta > 0$ we have that around each equilibrium point $\sigma \in \mathrm{Sing}(\Lambda)$ there are two hemispheres $\partial B_\delta(\sigma) \setminus W^s_{\mathrm{loc}}(\sigma)$ called $h(\sigma)_+$ and $h(\sigma)_-$ respectively. Such hemispheres define the $2k$ vertices of an oriented graph $P_\delta(\Sigma)$; that is, $\mathcal{V} := \{h(\sigma)_+, h(\sigma)_- | \sigma \in \mathrm{Sing}(\Lambda)\}$. The edges of $P_\delta(\Lambda)$ come in pairs according to the following rule: given some $\varsigma \in \mathcal{V}$ associated to some $\sigma \in \mathrm{Sing}(\Lambda)$ there are two oriented edges $(\varsigma, h(\sigma')_+)$ and $(\varsigma, h(\sigma')_-)$ if there is a connected open set $V \subset \varsigma \setminus W^{cu}_{\mathrm{loc}}(p)$, where $p = \varsigma \cap W^u_{\mathrm{loc}}(\sigma')$ and $p \in \partial V$ and there is a positive continuous function $\tau : V \to \mathbb{R}^+$ such that:

(1) $\Phi_{\tau(x)}(x) \in \partial B_\delta(\sigma')$.
(2) $\Phi_s(x) \notin \bigcup_{\sigma \in \mathrm{Sing}(\Lambda)} B_\delta(\sigma)$, for $0 < s < \tau(x)$.
(3) and if we denote by $\tilde{V} = \bigcup_{x \in V} \Phi_{\tau(x)}(x)$, then $\tilde{V} \cap W^s_{\mathrm{loc}}(\sigma') \neq \emptyset$.

Notice that if there is a singular cycle in a first return to $B_\delta(\mathrm{Sing}(\Lambda))$; that is, a non singular point $x \in W^u_{\mathrm{loc}}(\sigma)$ that $\Phi_{\tau(x)}(x) \cap W^s_{\mathrm{loc}}(\sigma')$, for some σ and σ' in $\mathrm{Sing}(\Lambda)$, then no edge with starting point $h(\sigma)_\pm$ appears, since V can not contain any point of the unstable separatrix. However it is possible that there is a singular cycle on further iteration. By definition the number of oriented edges starting from a fixed vertex is even.

Definition. *A singular hyperbolic attractor Λ has complete recurrence if there is some $\delta > 0$ for which the oriented graph P_δ has a path connecting any two vertices of it.*

Notice that if the set Λ is transitive then it has complete recurrence; since all unstable separatrices are contained in Λ. Also the intersection of $\tilde{V}$ and $W^s_{\mathrm{loc}}(\sigma)$ is transversal, and hence the graph is stable under small perturbations of X. This argument proves the next lemma:

Lemma 3. *Given a singular hyperbolic attractor Λ which has complete recurrence for X, there is an open neighborhood $\mathcal{U} := \mathcal{U}(X) \subset \mathcal{X}^1(M)$ that for any $Y \in \mathcal{U}$ we have that $P_\delta(\Lambda_X) = P_\delta(\Lambda_Y)$.*

In particular, if Λ is transitive then there is a neighborhood $\mathcal{U}$ of X for which any $Y \in \mathcal{U}$ the set $\Lambda_Y = \cap_{t>0}\Phi(Y)_t(U)$ has complete recurrence. The notion of complete recurrence allow us to find points of Λ on both sides of each connected component of $\hat{\Sigma}$ for any small enough system of transversal sections. This is the statement of the next Lemma.

Lemma 4. *If Λ has complete recurrence, then for any small $\varepsilon > 0$ the set Λ intersect both connected components of $\hat{\Sigma}_j(E)$ for any $j \in \{1, \dots, \tilde{k}\}$ and for any E that* $\max E < \varepsilon$.

Proof. If Λ has complete recurrence for some $\delta > 0$ and if we take the linearizing neighborhoods we are considering to define the system of transversal sections on the scale of δ, then by the Implicit Function Theorem there are points of each connected component of $\hat{\Sigma}(E)$ that return to the same connected component. Recall that the size $\max(E)$ is for the height of the transversal section fixed on the sphere of radius δ. So, we are done. □

3.4. **Analysis of Transition maps.** The following lemma translates the hyperbolic properties of the singular splitting to the local action of transitions between normal sections. Recall $\lambda_s < 0$ of Lemma 1.

Lemma 5. *Consider some point $p \in \tilde{U}$ and some $t > 0$. Denote by $G = G_p^t$ we have that for any $x \in D(p,t)$ then*

$$DG(\hat{E}_x^s) = \hat{E}_{G(x)}^s \text{ and } DG(\hat{E}_x^u) = \hat{E}_{G(x)}^u$$

and also there is $\lambda_1 < 0$ such that

$$|DG|_{\hat{E}_x^s}| < \exp(\lambda_s \tau(x))$$

where $\tau(x)$ is such that $\Phi_{\tau(x)}(x) \in N_{p_t}$. Moreover, if $x \in D(p,t)$ then $W_{loc}^s(x) \cap N_p \subset D(p,t)$; and so, $D(p,t)$ is a box foliated by local stable manifolds.

Proof. The invariance of the splitting is immediate from its definition. The contraction property is a consequence from Lemma 1; that is, the local stable foliation on U. □

This lemma can be restated for boxes. Recall that $\eta^*(p)$ is the size of the normal section on p.

Corollary 1. *Given any $p \in \tilde{U}$ and $t > 0$. If $B_p(\varepsilon_s) \subset D(p,t) \subset N_p$ then we have that*

$$G_p^t(B_p) \subset B_{p_t}(\exp(\lambda_s t)\varepsilon_s, \eta^*(p_t))$$

However, if we take a piece of orbit which extremal points are far away from $\mathrm{Sing}(\Lambda)$ we obtain more information. Denote by $C_x^u(\gamma)$ the cone in $\mathcal{N}_x$ around $\hat{E}_x^u$ of angle $\gamma > 0$.

Lemma 6. *According the notation of the previous lemma, there are $\lambda_u < 0$ and $\gamma > 0$ that if p and $p_t \in V_\delta$, then for any point $x \in D(p,t)$ and any vector $v \in C_x^u(\gamma)$ we have that*

$$|DG_p^t(v)| > \exp(-\lambda_u \tau(x))$$

where $G_p^t(x) = \Phi_{\tau(x)}(x) \in N_{p_t}$, for some $\tau(x) \in \mathbb{R}^+$.

Proof. Denote by $G = G_p^t$ and by $\lambda_u < 0$ the rate of volume expansion in the definition of singular hyperbolicity. To show that $|DG(\hat{E}_x^u)| > \exp(-\tau(x)\lambda_1)$ consider the following two bases of unitary vectors: The fist one is for E_x^{cu}, called $\mathcal{B}$, and is formed by two vectors, one in the direction of $X(x)$ and the other $v_x \in \hat{E}_x^u$. The second basis, say $\tilde{\mathcal{B}}$, is for $E_{x_{t_x}}^{cu}$ and is formed by one vector in the direction of $X(x_{t_x})$ and the other $\tilde{v} \in \hat{E}_{x_{t_x}}^u$. Since both, p and $p_t \in V_\delta$, there exists a constant $K = K(\delta, \eta^*)$ such that:

$$K^{-1} < ||X(\Phi_t(x))|| < K; \text{ for } t = 0 \text{ and } t = t_x.$$

Now observe that with respect to these bases, the derivative of $\Phi_{\tau(x)}(x)$ can be written as:

$$D\Phi_{\tau(x)}(x) = \begin{pmatrix} \frac{||X(x)||}{||X(\Phi_{t_x}(x))||} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

Now, since $D\Phi_t$ expand volume along the central unstable direction, we know that $|\det D\Phi_{\tau(x)}(x)| > \exp(-\lambda_u \tau(x))$. So we have,

$$a_{22} > \exp(-t_x\lambda_u)\frac{||X(x_{t_x})||}{||X(x)||}$$

And hence, according to the bases we use we obtain then $DG_p^t(x)(v) > a_{22}$, and we are done. □

Lemma 7. *Let $p, p_t \in V_\delta$, then we have the following properties:*

(1) *For all $s < t$, we have that $G_p^s(D(p,t))$ is contained in the interior of N_{p_s} and so $D(p,t) \subset D(p,r)$;*
(2) *If we also assume that $p_s \in V_\delta$ for any $s \in (0,t)$, then $\partial^u G^t(D(p,t)) \subset \partial^u N_{p_t}$.*

Proof. If a point $x \in D(p,t)$ hits the boundary of N_{p_s} for some $s \in (0,t)$ then either the orbit of x escapes from $C(p,t)$ if we flow it a little, and hence $x \notin D(p,t)$ or the orbit of x travels through the boundary of the remaining sections, and hence hits the boundary of N_{p_t}. This is a contradiction since the sets $D(p,t)$ are open, by definition. So we conclude the first item of the lemma.

To get the second, let us assume that there is $x \in D = D(p,t)$ such that for some $s < t$ one gets that $\Phi_{s_x}(x) \in \partial^u N_{p_s}(\varepsilon)$ and $\Phi_{t_x}(x) \in N_{p_t}(\varepsilon)$. Let γ be a curve in D that connects p with some point in the local stable manifold of x and such that the arc is contained in the unstable cone $C^u(\gamma)$ of Lemma 6. Let $\gamma_t = G_p^t(\gamma) \subset N_{p_t}$ and $\gamma_s = G_p^s(\gamma) \subset N_{p_s}$. On one hand, we observe that one of the extremal point of γ_s is contained in the boundary of $N_{p_s}(\varepsilon)$ and γ_t is properly contained in N_{p_t}. So we can say that the length of γ_s is larger than the length of γ_t. On the other hand, observe that γ_t and γ_s are contained in the unstable cone, and hence Lemma 6 imply that the length of γ_t is exponentially larger than the length of γ_s which is a contradiction. □

Corollary 2. *There is some fixed constant $C > 0$, depending on δ, that if $p_s \in V_\delta$ for any $s \in [0,t]$ then for any box $B \subset D(p,t)$ we have that*

$$|G_p^t(B)|_u > C\exp(-\lambda_u t)|B|_u$$

Proof. The same argument of the proof of the item 2 of the previous lemma, together with the fact that the size of transversal sections is bounded from above by η^*, and hence the unstable length of any box contained in. □

On the other hand, if we start from a point $p \in V_\delta$ and for some $s \in (0,t)$ the point $p_s \in B_\delta(\sigma)$, then there are two real numbers t_- and t_+ that the orbit $p_s \in H(\sigma)$ for all $s \in (t_-, t_+)$, $p_{t_-} \in \Sigma$ and $p_{t_+} \in \Delta$; as we have shown in the previous subsection. Moreover, $G_p^{t_-}(D(p,t_-)) \subset \Sigma$, and more, $G_p^{t_-}(D(p,t_+))$ do not intersect $W^s_{\text{loc}}(\sigma)$, since in such a case the intersection point must exit $H(\sigma)$ through some $\Delta^\pm$; fact that is impossible, since it belongs to the stable manifold of σ.

Lemma 8. *In this case, for any box $B \subset D(p,t_+)$ we have that*

$$|G_p^{t_+}(B)|_u > C\exp(-\lambda_u t_+)|B|_u$$

Proof. Assuming that $p_s \in V_\delta$ for $s \in (0,t_-)$ then Corollary 2 assert that the statement is true until t_-; that is

$$|G_p^{t_-}(B)|_u > C\exp(-\lambda_u t_-)|B|_u$$

Hence, we have to care about its passage through the corresponding dynamical neighborhood $H(\sigma)$. However, since $\lambda^s_\sigma + \lambda^u_\sigma > 0$ and since

$$d(p_s, W^s_{\text{loc}}(\sigma) \cup W^u_{\text{loc}}(\sigma)) > d(p_{t_-}, W^s_{\text{loc}}(\sigma) \cup W^u_{\text{loc}}(\sigma))$$

for any $s \in (t_-, t_+)$ we obtain the desired property for the time t_+. □

With all we have done until now, we can prove Properties 1 and 2.

Proof of Property 1: We have only to apply inductively a finite number of times both Corollary 2 and Lemma 8, depending on the region the orbit passes through, if it stays in V_δ or not. And hence, we are done. □

Proof of Property 2: We only have to take care about the unstable boundary of the box $B_p(\varepsilon)$ since after Lemma 5 the stable boundary do not escapes from $C(p,t)$. If $B_p \subset D(p,t)$ we can reparametrize the flow in $C(p,t)$ by a strictly positive function $\varphi : C \to \mathbb{R}^+$ in order to obtain that $\hat{\Phi}_s(B_p) \subset N_{p_s}$, and $\hat{\Phi}_s|_{B_p} \equiv G^s_p|_{B_p}$, for all $s \in [0,t]$. We have now two options. If $G^t_p(B_p) \subset B_{p_t}$, setting $\tilde{B} = B_p$ we are done. The second option is that $G^t_p(B_p) \cap \partial^u(B_{p_t}) \neq \emptyset$. For now we shall assume this intersection happens in both connected components of $\partial^u(B_{p_t})$, since after Property 1 we know that that the expansion is uniform around the orbit of p. Then take y_1 and y_2 one point on each connected component of the intersection. There are l_1 and l_2 two positive times that $\Phi_{-l_i}(y_i) \in \text{cl}(B_p)$, for $i = 1, 2$. Then the local stable manifold of both y_i bounds a sub-box $\tilde{B} \subset B_p$ that $\tilde{B} \subset D(p,t)$ and $G^t_p(\tilde{B}) \subset B_{p_t}$, which is the statement of item 2.

On the case that B_p is not contained in $D(p,t)$ then there is a point $\tilde{x} \in B_p$ that $\tilde{x} \notin D(p,t)$. Since the set $D(p,t)$ is a non empty open set we can choose $x \in \partial^u D(p,t)$ that any point in the semi-box B^* bounded by $W^s_{loc}(x)$ and $W^s_{loc}(p)$ is that $B^* \subset D(p,t)$. Now consider a C^1-curve $\beta : [0,1] \to B^*$ such that $\beta(0) \in W^s_{loc}(p)$ and $\beta(1) = x$, and that $\beta'(r) \in C^u_{\beta(r)}(\gamma)$; $\gamma > 0$ from Lemma 6. Notice that

$$G^t_p \circ \beta : [0,1) \to N_{p_t}$$

is a well defined continuous curve. On the other hand, we can define a continuous function $\zeta : [0,1) \to \mathbb{R}^+$ that $G^t_p \circ \beta(s) = \Phi_{\zeta(s)}(\beta(s))$, for any $s \in [0,1)$. Either ζ is a bounded function or not. If it is bounded, there is a limit $l = \lim_{s\to 1} \zeta(s)$ that $\Phi_l(x) \in \mathrm{cl}(N_{p_t})$, and hence $\Phi_l(x) \in \partial^u(N_{p_t})$, since $x \notin D(p,t)$. Notice that after Lemma 7, the orbit of x can not escape from $C(p,t)$. Hence, considering y_1 and $y_2 \in G^t_p(B^*) \cap \partial^u B_{p_t}$ as in the previous case we obtain the desired item 2. If ζ is not bounded, take any sequence $s_i \to 1$. The points $x_i := \beta(s_i)$ converge to x and $\zeta_i := \zeta(s_i) \to \infty$ as $i \to \infty$. Moreover, for any i, the piece of the orbit

$$\mathcal{O}_i = \bigcup_{l\in(0,\zeta_i)} \Phi_l(x_i) \subset \mathrm{interior}(C(p,t))$$

Hence there is $q_i \in \mathcal{O}_i$ that $||X(q_i)|| \to 0$ as $i \to \infty$, since the closure of $C(p,t)$ is compact and $\zeta_i \to \infty$. May be considering only a sub-sequence, we have that $q_i \to \sigma$ for some $\sigma \in \mathrm{Sing}(\Lambda) \cap C(p,t)$. Therefore $x \in W^s(\sigma)$. If

$$\lim_{s\to 1} G^t_p \circ \beta(s) \in \mathrm{cl}(B_{p_t}) \tag{3}$$

then the semi-box B^* bounded by $W^s_{loc}(x)$ and $W^s_{loc}(p)$ satisfy item 3. Otherwise, the limit point in (3) belongs to $N_{p_t} \setminus \mathrm{cl}(B_{p_t})$, and hence there is $s^* \in (0,1)$ that $G^t_p \circ \beta(s^*) \in \partial^u B_{p_t}$, and then the semi-box B^* bounded by the local stable manifolds of $\beta(s^*)$ and p is that $B^* \subset D(p,t)$ and $G^t_p(B^*) \subset B_{p_t}$. Arguing in the same way in the other side of $B_p \setminus W^s_{loc}(p)$ either there is a semi-box $\tilde{B}^*$ that satisfy item 3 or the union $B^* \cup \tilde{B}^* \cup W^s_{loc}(p)$ is a sub-box that satisfy item 2, and we are done. $\square$

4. Markovian induced map

The key property that allow us to prove the existence of unstable manifolds is that some induced maps defined on certain system of transversal sections nearby the singular points is in fact markovian, see Corollary 3.

Let us assume that Λ is a singular hyperbolic attractor with complete recurrence, for certain flow Φ_t.

In fact, to get the markovian properties of the induced map we shall prove the following Main Lemma.

Main Lemma. *If Λ has complete recurrence, then there is a system of transversal sections $\Sigma(E)$ such that for any band B there is a sub-band $\tilde{B} \subset B$ and a transition $G : \tilde{B} \longrightarrow \Sigma$ such that $G(\tilde{B})$ covers one Σ^+_j or Σ^-_j, for some $j \in \{1, \dots, \tilde{k}\}$.*

The Main Lemma implies that there is a induced map, of returns to equilibrium points, which is Markovian.

Corollary 3. *If Λ has complete recurrence, then there is a system of transversal sections $\Sigma(E)$ such that for any band $B \subset \hat{\Sigma}(E)$ there is a sub-band $\tilde{B} \subset B$ and a transition $G : \tilde{B} \longrightarrow \Sigma$ that $\pi_s(G(\tilde{B})) = \pi_s(\Sigma_j(E))$, for some $j \in \{1, \dots, \tilde{k}\}$.*

Proof of Corollary 3. Observe that we can choose a family of sub-bands:

$$\mathcal{A} = \{A_1, A_2, \dots, A_{2k}\}$$

and for each $l \in \{1, \dots, \tilde{k}\}$ a continuous function

$$G_l \colon A_l \longrightarrow \Sigma(E)$$

that $G_l(A_l)$ covers one connected component of $\hat{\Sigma}(E)$. For that we only have to apply Main Lemma on each band $\Sigma_j^-(E)$ and $\Sigma_j^+(E)$. Notice that this selection may be not unique.

Now for each $j \in \{1, \dots, \tilde{k}\}$ denote the set of members of $\mathcal{A}$ that their image hits precisely $\Sigma_j(E)$:

$$\mathcal{A}_j = \{A \in \mathcal{A} | G_A(A) \cap \Sigma_j(E) \neq \emptyset\}$$

Notice that $\mathcal{A}_j$ defines a transversal section containing Q_j and contained in $\Sigma_j(E)$. In fact, there are $\hat{\varepsilon}_j^{\pm} > 0$ that $Q_j \subset \Sigma(\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+) \subseteq \Sigma_j(E)$ and

$$\bigcap_{A \in \mathcal{A}_j} \pi_s(G_A(A)) = (\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+) = \pi_s(\Sigma(\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+))$$

Moreover, for any $A \in \mathcal{A}_j$ there is a sub-band $\tilde{A}$ that $\pi_s(G_A(\tilde{A})) = (\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+)$.

Set $E^* = \{(\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+) | j \in \{1, \dots, \tilde{k}\}\}$; if for some j happens that $\mathcal{A}_j = \emptyset$ set $(\hat{\varepsilon}_j^-, \hat{\varepsilon}_j^+) = (\varepsilon_j^-, \varepsilon_j^+)$. The system of transversal sections $\Sigma(E^*)$ satisfy what we claim: Consider any band $B \subset \hat{\Sigma}(E^*)$. Since $\Sigma(E^*) \subset \Sigma(E)$, Main Lemma imply that there is a sub-band $\tilde{B} \subset B$ for which

$$G_{\tilde{B}} \colon \tilde{B} \longrightarrow \Sigma_j(E)$$

is continuous, for some j, and $G_{\tilde{B}}(B)$ cover one side of $\hat{\Sigma}_j(E)$. Hence, $G_{\tilde{B}}(\tilde{B}) \cap A \neq \emptyset$ for some $A \in \mathcal{A}$. This non-empty intersection defines a sub-band $\tilde{\tilde{B}} \subset \tilde{B} \subset B$, $\tilde{\tilde{B}} = G_{\tilde{B}}^{-1}(G_{\tilde{B}}(\tilde{B}) \cap A)$. On the other hand, the function

$$G = G_A \circ G_{\tilde{B}} \colon \tilde{\tilde{B}} \longrightarrow \Sigma(E^*)$$

is continuous and cover all $\Sigma_{j'}(E^*)$, for some j'. □

4.1. **Proof of Main Theorem.** Assuming Main Lemma which will be proved in the next subsection, and therefore assuming also Corollary 3, we can give a proof of Main Theorem. For that we shall first prove a unstable manifold theorem for a singular hyperbolic attractor with complete recurrence of a vector field $X \in \mathcal{X}^1(M)$; then it will be easy to conclude Main Theorem.

Theorem 1. *If Λ is a singular hyperbolic attractor with complete recurrence then there exists $K \subset \Lambda$, and $\varepsilon_u > 0$ and $\lambda_u < 0$ that*

(1) *For any $y \in K$, we have that $W_{\varepsilon_u}^{cu}(y) \cap N_y = \tilde{W}_{\varepsilon_u}^u(y)$.*
(2) *There is a not bounded sequence $t_i > 0$ such that $dist(G_x^{-t_i}(y), x_{-t_i}) < C \exp(t_i \lambda_u)$; for any $y \in \hat{W}_{\varepsilon_u}^u(x)$.*
(3) *$\bigcup_{t > T_0} \bigcup_{y \in K} \Phi_{-t}(y)$ is an open and dense set in $\Lambda \cap L(X)$, for any $T_0 > 0$.*

Proof. Consider Λ a singular hyperbolic attractor with complete recurrence. Take the system of transversal sections $\Sigma(E)$ given by Corollary 3. The α-limit of any point $x \in \Lambda$ must be contained in Λ; it may contain some singular point or not. This splits $\Lambda = \Lambda_S \cup \Lambda_H$, where $\Lambda_H = \{x \in \Lambda | \alpha(x) \cap \mathrm{Sing}(\Lambda) = \emptyset\}$ and $\Lambda_S = \Lambda \setminus \Lambda_H$. Notice that $\Lambda_S \neq \emptyset$, otherwise Λ must be uniform hyperbolic. First we care about points in Λ_S. Consider $\gamma = \min\{\varepsilon_j^-, \varepsilon_j^+ \in E\}$ and set

$$\tilde{E} = \{\varepsilon_j^- - \gamma/3, \varepsilon_j^+ - \gamma/3 | j \in \{1, \dots \tilde{k}\}\}$$

Call $\varepsilon = \gamma/4$. Notice then that $\Sigma(\tilde{E}) \subset \Sigma(E)$ and also the interval $J_x = W^{cu}_\varepsilon(x) \cap \Sigma$ is contained in $\Sigma(E)$. Now call:

$$K_\Sigma = \hat{\Sigma}(\tilde{E}) \cap \Lambda_S \setminus \cup_{\sigma \in \mathrm{Sing}(\Lambda)} W^s(\sigma)$$

Such a set is not empty since for any point in $\Lambda_S \setminus \cup_\sigma W^s(\sigma)$ there is a not bounded sequence of positive numbers t_i such that $x_{-t_i} \to Q_j$, for some $j \in \{1, \dots, \tilde{k}\}$. First we shall prove that $K_\Sigma \subset K$, that is, if $x \in K_\Sigma$ then $J_x = \hat{W}^u_\varepsilon(x)$ and satisfy item 1 and 2. To prove this claim consider some $x \in K_\Sigma$.

Notice that Corollary 3 implies that there is a band around x, named $B_1 \subset \hat{\Sigma}(E)$, and a transition $G_1 : B_1 \to \Sigma(E)$ such that $G_1(B_1)$ covers $\Sigma_j(E)$, for some $j \in \{1, \dots \tilde{k}\}$ and $x \in G_1(B_1)$. Hence $J_x \subset G_1(B_1)$, and more, the extremal points of J_x do not belong to the boundary of $\Sigma(E)$, since $\gamma/4 < \gamma/3$. Notice also that $G_1|_{B_1} \equiv G^{t_1}_{x_{-t_1}}$, for some $t_1 > 0$, and $x_{-t_1} \in B_1$. Hence, since $J_x \in \mathrm{image}(G^{t_1}_{x_{-t_1}})$ then $J_x \subset \mathrm{Dom}(G^{-t_1}_x)$. Applying Property 1 to $G^{t_1}_{x_{-t_1}}$ we obtain that:

$$|G^{t_1}_{x_{-t_1}}(B_1)|_u > \exp(-\lambda_u t_1)|B_1|_u$$

and so, if we set $C = \max_j\{\varepsilon_j^- + \varepsilon_j^+\}$ then

$$|G^{-t_1}_x(J_x)| < \exp(\lambda_u t_1)|B_1|_u < C\exp(\lambda_u t_1) \tag{4}$$

since $G^{-t_1}_x(J_x) \subset B_1$ and the stable foliation is invariant. Recall that x_{-t_1} and x_1 both belong to V_δ.

Now we proceed inductively. Assume that we find the point $x_{-t_i} \in \Sigma(E)$, which also belongs to Σ_S, then again Corollary 3 imply that there is a band around x_{-t_i} called B_{i+1} and a transition $G_{i+1} : B_{i+1} \to \Sigma(E)$ such that $x_{-t_i} \in G_{i+1}(B_{i+1})$, and more: $G^{-t_i}_x(J_x) \subset G_{i+1}(B_{i+1})$. In fact, $G_{i+1}(B_{i+1})$ cover some connected component of $\Sigma(E)$. Also the extremal points of $G^{-t_i}_x(J_x)$ do not belong to $Q = \cup_j Q_j$. Therefore there is $s > 0$ that if we set $t_{i+1} = t_i + s$ then $G_{i+1} \equiv G^s_{x_{-t_{i+1}}}$. Now, since $G^{-t_i}_x(J_x) \subset \mathrm{image}(G^s_{-t_{i+1}})$ then $J_x \subset \mathrm{Dom}(G^{-t_{i+1}}_x)$. Also Property 1 imply the equation in (4) replacing t_{i+1} instead of t_1.

Here we prove that $J_x \subset \mathrm{Dom}(G^{-t_i}_x)$ for all $i \in \mathbb{N}$. This clearly implies that J_x is contained in the domain of G^{-t}_x for all $t \geqslant 0$. And hence, we get the claim. Actually we prove that there is $C > 0$ for which

$$|G^{-t_i}_x(J_x)| < C\exp(\lambda_u t_i)$$

for the provided sequence t_i, which is precisely the statement on item 2 for points in Λ_S.

Now we have to consider points in Λ_H. However, the negative orbit of such points accumulates in some uniform hyperbolic set. Hence, after flowing some negative time they stay in a subset K_H which is uniformly away from $\Sigma(E)$. Those points in K_H they have some actual unstable manifold of size ε' and then, considering the $\varepsilon_u = \min\{\varepsilon, \varepsilon'\}$, we obtain the set $K = K_H \cup K_\Sigma$ of points that satisfy item 1 and 2 of the theorem.

To get the other item fix any $T_0 > 0$, and consider:

$$K_{T_0} = \bigcup_{t > T_0} \Phi_{-t}(K)$$

This set is open in Λ and $K_H \subset K_{T_0}$. In order to prove it is dense in Λ consider any point $x \in \Lambda$. If $x \in \Lambda_H$ then $x \in K_{t_0}$ since $\Phi_{-t_0}(K_H) \subset K_H$. On the other hand if $x \in \Lambda_S$ there is $\tau \geqslant T_0$ that $\Phi_{-\tau}(x) \in \hat{\Sigma}(E) \cap \Lambda$. Now take an arbitrarily small $\beta > 0$, then $B(\Phi_{-\tau}(x), \beta)$ is contained in some connected component of $\hat{\Sigma}_j(E)$; for some j. Corollary 3 imply there is a sub-band $\tilde{B} \subset B$ and a transition $G : \tilde{B} \longrightarrow \Sigma(E)$ that $G(\tilde{B})$ covers some $\Sigma_{j'}(E)$. Therefore, there are points $\Phi_t(z) \in K_\Sigma$ that $z \in B(\Phi_{-\tau}(x), \beta)$. Hence, there is $\tilde{\tau} > 0$ that $\Phi_{-\tilde{\tau}}(z) \in K_{T_0}$ and since β is arbitrarily small then $\Phi_{-\tilde{\tau}}(z)$ and x are close, and we are done: K_{T_0} is an open dense set of Λ, and we are done.

□

Proof of Main Theorem: Let $\Lambda \subset U$ be a transitive singular hyperbolic attractor for the flow defined by $X \in \mathcal{X}^1(M)$. Notice that Λ has complete recurrence. Therefore there is a neighborhood $\mathcal{U}$ of X where if $\Lambda_Y = \bigcap_{t \geqslant 0} \Phi(Y)_t(U)$ is a non trivial invariant set, then there is $\delta > 0$ for which $P_\delta(\Lambda) = P_\delta(\Lambda_Y)$ according to Lemma 3. Consider perhaps a smaller neighborhood, we can assume that the constants of the singular hyperbolic splitting are the same for all $Y \in \mathcal{U}$. Hence, the previous theorem apply to any $Y \in \mathcal{U}$. So, we are done. □

4.2. **Proof of Main Lemma.** Now we shall concentrate in the proof of Main Lemma. Before we start the proof we need spent some time studying transition maps between boxes, associated to some attractor set with complete recurrence. Given a box B and $\varepsilon_x > 0$, denote by $B(x, \varepsilon_x) \subset B$ a sub-box such that $\pi(B(x, \varepsilon_x)) = (\pi(x) - \varepsilon_x, \pi(x) + \varepsilon_x)$.

Remark: If B is a band of $\hat{\Sigma}$ then, if $y \in W^s(\sigma)$, for some $\sigma \in \mathrm{Sing}(\Lambda)$ then $W^s_{\mathrm{loc}}(y) \subset W^s(\sigma)$, and more, if $y' \in W^s_{\mathrm{loc}}(y)$ and $\Phi_t(y) \in Q_j$ then $\Phi_t(y') \in Q_j$, since Q_j is an actual stable manifold. This fact implies that the set

$$R = \{(z, t) \in \pi_s(B) \times \mathbb{R}^+ | \Phi_t(\pi_s^{-1}(z)) \subset Q_j \text{ for some } j\}$$

is well ordered according to the relation: $(z, t) < (z', t')$ if and only if $t < t'$; and hence if $R \neq \emptyset$, there is a first element of B that hits Σ_j.

Lemma 9. *Given $K > 2$, there exist $\varepsilon > 0$ and $\varepsilon^* = \varepsilon^*(K) > 0$ that for any system of transversal sections defined by E such that $max(E) < \varepsilon^*$ we have that, for any band $B \subset \hat{\Sigma}(E)$ one of the following happens:*

(1) *There is a point $x \in B \cap \Lambda$ that cl($\mathcal{O}^+(x)) \cap$ Sing($\Lambda) \neq \emptyset$, $t > 0$, and a sub-band $\tilde{B} \subset B$, containing x, that the corresponding transition*

$$G\colon \tilde{B} \longrightarrow B(\Phi_t(x), \varepsilon)$$

is continuous and such that $\pi_s \circ G|_{\tilde{B}}$ is onto the interval $\pi_s(B(\Phi_t(x), \varepsilon)$.

(2) *There is a point $y \in B$ and $t > 0$ such that $\Phi_t(y) \in Q_j$ for some $j \in \{1, \ldots, k\}$; and the transition $G\colon B \longrightarrow \Sigma_j$ is continuous and $|G(B)|_u > K|B|_u$.*

Proof. First consider any positive $\varepsilon < \eta^*$ and that $2\varepsilon < |\Sigma_j|_u$ for any $j \in \{1, \ldots, k\}$. Now fix a constant $K > 2$, and denote by $T_0 = \ln K / \ln \lambda_u$. In order to obtain $\varepsilon^* > 0$ consider the following: Given any system of transversal sections $\Sigma(E)$, for any point $x \in \Sigma(E)$ there is a number $t_x > 0$, for which $\Phi_{t_x}(x) \in \Sigma(E)$; it may happen that $t_x = +\infty$, if the point do not returns. However if $\varepsilon^* = \max(E) \to 0$, then $t_x \to +\infty$, uniformly in x. Then set $\varepsilon^* > 0$ that $t_x > T_0$ for any point $x \in \Sigma(E)$, and also that $\min\{|\Sigma_j|_u\} - \varepsilon^* > \varepsilon$. Recall that always there is a point x on each side of Q_j for which $t_x < \infty$; see Lemma 4.

Consider any system of transversal sections $\Sigma(E)$, that $\max(E) < \varepsilon^*$. For any band $B \subset \hat{\Sigma}(E)$ we shall do the following analysis:

Let us assume first that:

$$B \cap \left(\cup_{\sigma \in \mathrm{Sing}(\Lambda)} W^s(\sigma)\right) = \emptyset \tag{5}$$

Choose a point $x \in B \cap \Lambda$ which $\mathrm{cl}(\mathcal{O}^+(x)) \cap \mathrm{Sing}(\Lambda) \neq \emptyset$, and a positive ε_x that $B(x, \varepsilon_x) \subset B$. Now set $t > 0$ that $\lambda_u^t(2\varepsilon_x) > \varepsilon$ and that $x_t = \Phi_t(x) \in V_\delta$ [see Section 3]. Consider the transition map

$$G\colon D \subset B(x, \varepsilon_x) \longrightarrow B(x_t, \varepsilon)$$

Since we are assuming (5) it is impossible that item 3 of Property 2 holds. Hence, in the case of $B(x, \varepsilon_x) \subset D$ then by Property 1 we have that

$$|G(B(x, \varepsilon_x))|_u > \lambda_u^t(2\varepsilon_x) > \varepsilon,$$

and hence $\pi_s \circ G$ is onto on the interval $\pi_s(B(x_t, \varepsilon))$. In such a case $\tilde{B} = B(x, \varepsilon_x)$ on the statement. However, if $B(x, \varepsilon_x)$ is not contained in D, then there is a sub-band $\tilde{B} \subset B(x, \varepsilon_x)$ for which the function

$$\pi_s \circ G\colon \tilde{B} \longrightarrow \pi_s(B(x_t, \varepsilon))$$

is onto, and we are done.

Now, when condition (5) do not holds, it implies that the set R of the previous Remark is not empty; and hence, there are $y \in B$ and $t_y > 0$ that $(\pi_s(y), t_y)$ is the first element of R. For such a point we have then that,

$$\mathrm{interior}[C(B, y, t_y)] \cap Q = \emptyset \tag{6}$$

Otherwise, we find a pair $(y', t_{y'}) < (y, t_y)$, which is a contradiction. Now we shall study the corresponding transition map from B to the complete section Σ_j:

$$G\colon D \subset B \longrightarrow \Sigma_j$$

that $G(y) = \Phi_{t_y}(y) \in Q_j$. Since y and $\Phi_{t_y}(y)$ both belong to V_δ, Property 2 holds. However, item 3 can not holds since (6). Hence, if $B \subset D$ then since $y \in \Sigma(E)$ and $\Phi_{t_y}(y) \in \Sigma(E)$, then $t_y > T_0$; and therefore Property 1 implies that

$$|G(B)|_u > K|B|_u$$

as we claim. On the other hand, there is a sub-band $C \subset B$ containing y, where $C \subset D$ and for which $G(C)$ covers completely Σ_j; that is

$$\pi_s(G(C)) = \pi_s(\Sigma_j)$$

Notice that in this case, for any point $\tilde{x} \in G(C) \cap \Sigma(E) \cap \Lambda$ we have that $B(\tilde{x}, \varepsilon) \subset \Sigma_j$, since $|\Sigma_j|_u - \varepsilon^* > \varepsilon$. Take a point $\tilde{x}$ such that also its positive orbit accumulates on $\mathrm{Sing}(\Lambda)$. Hence, there is a sub-band $\tilde{B} = G^{-1}(B(\tilde{x}, \varepsilon) \cap G(C)) \subset C \subset B$ that the transition

$$G\colon \tilde{B} \longrightarrow B(\tilde{x}, \varepsilon)$$

is as we claim. □

Now we go into the proof of the Main Lemma.

Proof of Main Lemma. Take some $K > 2$. Consider a system of transversal sections defined by E that $\max(E) < \varepsilon^*$, from Lemma 9. Take any band $B \subset \hat{\Sigma}(E)$. First we deal with the following situation:

(◇) There is a point $x \in B \cap \Lambda$, a sub-band $B_0 \subset B$ containing x, and $t > 0$ such that the associated transition is continuous:

$$G_1 := G_x^t\colon B_0 \longrightarrow B(\Phi_t(x), \varepsilon)$$

Call $x_1 = \Phi_t(x)$. There exists some $t_1 \geqslant 0$ such that $\Phi_{t_1}(x_1) \in \Sigma_j(E)$ for some $j \in \{1, \ldots, k\}$ for the first time (that is $\Phi_r(x_1) \notin \Sigma(E)$ if $0 \leqslant r < t_1$; unless $x_1 \in \Sigma(E)$, and $t_1 = 0$), since the orbit of x accumulates on $\mathrm{Sing}(\Lambda)$. Now consider the transition associated to (x_1, t_1):

$$G_2\colon B(x_1, \varepsilon) \longrightarrow \Sigma_j$$

Notice that Property 2 states that either $B(x_1, \varepsilon)$ is completely contained in the domain of G_2, and hence $G_2(B(x, \varepsilon))$ covers all $\Sigma_j(E)$, since

$$|G_2(B(x_1, \varepsilon))| \geqslant 2\varepsilon > \varepsilon_j^- + \varepsilon_j^+$$

and $G_2(x_1) \in \Sigma_j(E)$. Therefore, if we set $\tilde{B} = B_0$ then

$$G = G_2 \circ G_1\colon \tilde{B} \longrightarrow \Sigma_j$$

covers all $\Sigma_j(E)$.

Or, there is a sub-box $B_1 \subset B(x_1, \varepsilon)$, containing x_1 where $G_2 : B_1 \longrightarrow \Sigma_j$ is such that $\pi_s(G_2(B_1)) = \pi_s(\Sigma_j)$, and hence $\tilde{B} = G_1^{-1}(B_1) \subset B_0$ is such that

$$G = G_2 \circ G_1\colon \tilde{B} \longrightarrow \Sigma_j$$

covers all $\Sigma_j(E)$.

Or finally, there is $y \in B(x_1, \varepsilon)$ and $t_y > 0$ such that $\Phi_{t_y}(y) \in Q_{j'}$ for some $j' \in \{1, \dots, k\}$; as we saw in the previous Remark, we can take (y, t_y) the first element of the corresponding set R.

If $j' = j$, the transition $G_2 : B(x_1, \varepsilon) \longrightarrow \Sigma_j$ is such that $G_2(B(x_1, \varepsilon)) \cap Q_j \neq \emptyset$ and hence, it covers the side of $\Sigma_j(E) \setminus W^s_{\mathrm{loc}}(\Phi_{t_y}(y))$ that contains $\Phi_t(x)$. In fact, notice that the box B_1 defined by the side of $B(x_1, \varepsilon) \setminus W^s_{\mathrm{loc}}(\Phi_{t_y}(y))$ that contains x_1 is contained in the domain of continuity of G_2, since the box is not previously splitted by any Q_i. Hence, $\tilde{B} = G_1^{-1}(B_1)$ is such that

$$G = G_2 \circ G_1 \colon \tilde{B} \longrightarrow \Sigma_j$$

covers the side of $\Sigma_j(E)$ that contains $\Phi_{t_1}(x_1)$.

Nevertheless, if $j' \neq j$, denote by B_1 the box defined by the side of $B(x_1, \varepsilon) \setminus W^s_{\mathrm{loc}}(\Phi_{t_y}(y))$ that contains x_1. Denote by $G_2 : B_1 \longrightarrow \Sigma_{j'}$ the corresponding transition that $G_2(y_1) \in Q_{j'}$. Then $G_2(x_1) \notin \Sigma_{j'}(E)$, since $j' \neq j$, hence the band $\tilde{B} = G_1^{-1}(B_1)$ is such that

$$G = G_2 \circ G_1 \colon \tilde{B} \longrightarrow \Sigma_j$$

covers one side of $\Sigma_{j'}(E)$.

Now let us assume that $(\diamondsuit)$ do not hold. Hence, there is a point $y \in B$ and $t > 0$ such that $\Phi_t(y) \in Q_j$ for some $j \in \{1, \dots, k\}$, and the corresponding transition $G_0 \colon B \longrightarrow \Sigma_j$ is continuous in B and $|G(B)|_u > K|B|_u$.

If $G_0(B)$ do not cover one side of $\hat{\Sigma}_j(E)$ then notice that at least one connected component of $G_0(B) \setminus W^s_{\mathrm{loc}}(\Phi_t(y))$ has it unstable length $\geqslant \nu|B|_u$, where $\nu = K/2 > 1$. Then, there is a sub-band $\tilde{B}_0 \subset B$ such that $G_0(\tilde{B}_0)$ is this connected component.

Now choose $B_1 \subset \Sigma_j(E)$ such that $\pi_s(B_1) = \pi_s(G_0(\tilde{B}_0))$. Of course $|B_1|_u > \nu|B|_u$. Observe that we can repeat this argument for several steps while in each step the band B_n do not satisfy $(\diamondsuit)$. If $n_0 \in \mathbb{N}$ is such that $\nu^{n_0} \geqslant \max(E)$ then we can conclude that there is a transition

$$G = G_{n_0} \circ \cdots \circ G_0 \colon \tilde{B}_{n_0} \longrightarrow \Sigma_{j'}(E)$$

for some $j' \in \{1, \dots, k\}$ and such that

$$|G(\tilde{B}_{n_0})|_u > \nu|B_{n_0-1}|_u = \nu^2|B_{n_0-2}|_u = \nu^{n_0}|B| > \max(E)$$

Therefore, $G(\tilde{B}_{n_0})$ covers one side of $\Sigma_{j'}(E)$, as we require. Nevertheless, if in some step the band B_n satisfy the condition $(\diamondsuit)$, the first part of the proof gives the Main Lemma. $\square$

5. Robustly transitive sets with singularities

In this section we give a proof for Theorem A, stated in the Introduction.

Theorem 2. *Any singular hyperbolic attractor Λ with complete recurrence and for which is valid condition (H*) then it is transitive.*

As a corollary of this theorem we obtain the proof of Theorem A.

Proof of Theorem A: Since by hypothesis Λ is transitive, then it has complete recurrence. Moreover, all non-trivial attractors Λ_Y for $Y \in \mathcal{U}$ have complete recurrence. Hence, Theorem 2 finishes the proof of Theorem A. $\square$

To get a proof of Theorem 2 we need to prove some lemmas. The following one is on the context of the construction of the markovian induced map, and we continue using the notation therein.

Lemma 10. *For each connected component of $\Sigma(E)$ there is a subset $\Gamma \subset \Sigma_j(E)$, containing Q_j and that $\Sigma_j(E) \setminus \Gamma$ is made up with two connected components and that there are two periodic points $p^{\pm} \in \Sigma_j(E) \setminus \Gamma$, on each side, such that $\mathcal{O}(p^{\pm}) \cap \Gamma = \emptyset$.*

Proof. Take any connected component of $\Sigma(E)$ and consider a sub-box around Q_j, say $\Gamma \subset \Sigma_j(E)$, that $\Sigma_j(E) \setminus \Gamma$ is made up two connected components. Since the behavior on each side of Q_j is independent, we can treat them in the same way; perhaps Γ is not symmetric with respect to Q_j. Take any of them and call it by A. As A gets thinner, because Γ grows, we can assume there is a point $x \in A$ and $t > 0$ arbitrarily large that $x_t \in A$. This is true since Lemma 4 states that there are recurrence of both unstable separatrices of all singular points. Now, there corresponding transition maps satisfies: $G^t_x(D)$ cover $\Sigma_j(E)$ and hence there is a point $z \in D$ and some $\tau > 0$ that $z_\tau \in W^s(z)$. Hence, Lemma 4.3 of [AR] gives the required periodic orbit. $\square$

Notice that hypothesis (H*) allow us to take some $\delta_1 > 0$ small enough that the set $K_0 = \cap X_t(U \setminus B_{\delta_1}(\mathrm{Sing}(\Lambda)))$ is a basic piece and contain both $p^{\pm}$. Hence, there are two transitions $G_{\pm}$ related to $p^{\pm}$ which are actual return maps, since the points are periodic; and whose domains $D_{\pm} \subset \Sigma_j(E)$, and $G_{\pm}(D_{\pm})$ cover $\Sigma_j(E)$ on the unstable direction. On the other hand, the same result is valid for all $Y \in \mathcal{U}$ of Main Theorem.

Now, the next lemma is an application of Property 1.

Lemma 11. *Assuming hypothesis (H*), for any $x \in K$ there is a band $B \subset N_x(\varepsilon_u)$ and a transition $G^l_x : D(x,l) \to N_{x_l}$ such that $B \subset D(x,l)$ and*

$$G^l_x(W^u_{\varepsilon_u}(x) \cap B) \cap W^{ss}_{loc}(K_0) \neq \emptyset. \tag{7}$$

Proof. Take a point $x \in K$ and consider its ω-limit. If $\omega(x) \cap \cup_j \Gamma_j = \emptyset$ then $\omega(x) \subset K_0$. Since the last set is a basic piece by hypothesis, then there is $T > 0$ that if $t > T$ then x_t belongs to the local stable manifold of K_0, and we are done. On the other hand, there is some $t > 0$ such that $x_t \in \Gamma_j$, for some j. It may be possible that $G^t_x(D(x,t))$ cover all $\Sigma_j(E)$ and in such a case we are done, since it follows that there is a point $y \in \hat{W}^u_{\varepsilon_u}(x)$, and certain $\tau > 0$ that $y_\tau \in W^s_{loc}(K_0)$. Recall the set K_0 passes through $\Sigma_j(E) \setminus \Gamma_j$. However t may be not large enough to obtain this. So, consider one of the semi-boxes on x, say $B^* \subset D(x,t)$, bounded by $W^s_{\mathrm{loc}}(x)$. Now $G^t_x(B^*)$ intersects some connected component of the domain of the induced map. Hence there is a transition $G : D \to \Sigma(E)$ that $G(D)$ cover some connected component $\Sigma_{j'}$, and we are done. $\square$

Notice that this Lemma is also true for small C^1 perturbations of the vector field X. That is, for any $Y \in \mathcal{U}$ (perhaps a smaller neighborhood $\tilde{\mathcal{U}}$) and any $x \in K(Y)$ the intersection in (7) is still not empty.

Proof of Theorem 2. Let us prove that Λ is transitive. For that, let us fix two arbitrary non empty open sets A_1, A_2 of Λ, and denote by $\varepsilon_u > 0$ the size of local unstable manifolds for points in the set K, obtained by the Main Theorem. Since the sets K_T are residual for any $T > 0$, we know that there is $x \in K$ and $t > T$ such that $x_{-t} \in A_1$. Moreover, if we set T sufficiently large we can assert that for the transition

$$G^t_{x_{-t}} : D(x_{-t}, t) \to N_x(\varepsilon_u)$$

there is a box R containing x in its interior such that $R \subset D(x_{-t}, t) \cap A_1$ and that $\partial^u R \subset \partial^u D(x_{-t}, t)$; that is, in terms of the unstable direction the set R works the same as the domain of the transition. And more, $G^t_{x_{-t}}(R)$ covers $N_x(\varepsilon_u)$.

Now Lemma 11 states that: there is a band $B \subset N_x(\varepsilon_u)$ and a transition $G^l_x : D(x, l) \to N_{x_l}$ such that $B \subset D(x, l)$ and

$$G^l_x(J) \cap W^{ss}_{\mathrm{loc}}(K_0) \neq \emptyset$$

where $J = W^u_{\varepsilon_u}(x) \cap B$. Recall that K_0 is a basic piece, by hypothesis, and so, there is a point $q \in K_0$ whose positive orbit is dense. This implies that there is some $\tau_1 \in \mathbb{R}$ for which there is a point $z \in W^s_{\mathrm{loc}}(q_{\tau_1}) \cap W^u_{\varepsilon_u}(x_l)$. Consider some small $\tilde{\varepsilon} > 0$ that for all $p \in W^u_{\tilde{\varepsilon}}(q_{\tau_1})$ we have that $W^s_{\mathrm{loc}}(p) \cap J \neq \emptyset$.

On the other hand, take a point $\tilde{y} \in A_2 \cap \Lambda$. If its α-limit $\alpha(\tilde{y}) \subset U \setminus B_{\delta_0}(\mathrm{Sing}(\Lambda))$ then $\alpha(\tilde{y}) \subset K_0$ and since it is a basic piece we know that

$$\bigcup_{t>0} \Phi_{-t}(\tilde{y}) \subset W^u(K_0)$$

and hence, there is some $t > 0$ that $\Phi_{-t}(\tilde{y}) \in W^u_{\tilde{\varepsilon}}(\tilde{q})$, for some $\tilde{q} \in K_0$. Anyway, for any point in $W^s_{\mathrm{loc}}(\tilde{y}_{-t})$, its orbit passes through A_2.

However, there is $\tau_2 > 0$ that $q_{\tau_1+\tau_2}$ is arbitrarily close to $\tilde{q}$, since we can choose τ_2 as big as we need, in order to guarantee the existence of a point $p \in W^u_{\tilde{\varepsilon}}(q_{\tau_1+\tau_2}) \cap W^s_{\mathrm{loc}}(\tilde{y}_{-t})$.

Now, this implies the the point $p_{-\tau_2} \in W^u(q_{t_1})$ and hence there is $z \in W^s_{\mathrm{loc}}(p_{-\tau_2}) \cap W^u_{\varepsilon_u}(x_l)$. For such a point there is some $r_1 > 0$ that $z_{-r_1} \in R \subset A_1$. On the other hand, there is $r_2 > 0$ such that $z_{r_2} \in W^s_{\mathrm{loc}}(\tilde{y})$; and more precisely $z_{r_2} \in A_2$ as we wanted.

Now if it is the case that $\alpha(\tilde{y}) \cap \mathrm{Sing}(X)$ we have to proceed in a different manner. First there is some $n > 0$ that $\Phi_{-n}(\tilde{y}) \in \Gamma \subset \Sigma_j(E)$ for some j. Such Γ is defined in Lemma 10.

Hence, there is a point $z \in W^u_{\varepsilon_u}(x_l) \cap W^s_{\mathrm{loc}}(\tilde{y}_{-n})$ which in some time of its negative orbit belongs to A_1 and in the positive orbit belong to A_2, concluding the proof of Theorem 2. □

References

[ABS] V. Afraimovich, V. Bykov, and L. Shil'nikov. On the appearance and structure of the Lorenz attractor. *Dokl. Acad. Sci. USSR*, **234** 336–339, 1977.

[AR] A. Arroyo, F. Rodriguez-Hertz. Homoclinic bifurcations and uniforma hyperbolicity for three-dimensional flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**, (2003), no. 5, 805–841.

[Co] W. Colmenares. PhD Thesis UFRJ (Universidade Federal do Rio de Janeiro), 2002.

[BM] S. Bautista, C. Morales. *Existence of periodic orbits for singular-hyperbolic attractors*, Preprint IMPA.

[BMP] S. Bautista, C. A. Morales, M. J. Pacífico. There is no spectral decomposition for singular-hyperbolic flows. *Preprint IMPA.* (2004)

[Gu] J. Guckenheimer. A strange, strange attractor. In J.E. Marsden and M.McCracken, editors, *The Hopf bifurcation theorem and its applications*, pages 368–381. Springer Verlag, 1976.

[HPS] M. Hirsch, C. Pugh, M. Shub, Invariant manifolds, *Springer Lecture Notes in Math.*, **583** (1977).

[Ko] M. Komuro, Expansive properties of Lorenz attractors. *The theory of dynamical systems and its applications to nonlinear problems (Kyoto, 1984)*, 4–26, World Sci. Publishing, Singapore, 1984.

[Lo] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmosph. Sci.*, **20** 130–141, 1963.

[LP] R. Labarca, M.J. Pacifico. Stability of singular horseshoes *Topology*, **25** 337–352, 1986.

[Mo] C. A. Morales, Lorenz attractors through Saddle-node bifurcations *Ann. Inst. henri Poincare (An. nonlin)*, **13** 589-617, 1996.

[MoII] C. A. Morales. A note on periodic orbits for singular-hyperbolic flows. Preprint.

[MP] C. A. Morales, E. R. Pujals. Singular strange attractors on the boundary of Morse-Smale systems. *Ann. Sci. École Norm. Sup.* **4** 30 (1997), no. 6, 693–717.

[MoP] C. A. Morales, M. J. Pacifico, Attractors and singularities robustly accumulated by periodic orbits. International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 64–67, World Sci. Publishing, River Edge, NJ, 2000

[MPP] C. A. Morales, M. J. Pacifico,E.R. Pujals. Singular hyperbolic systems. *Proc. Amer. Math. Soc.* **127** (1999), no. 11, 3393–3401.

[MPP1] C. A. Morales, M. J. Pacifico, E. R. Pujals, Strange attractors across the boundary of hyperbolic systems *Comm. Math. Phys.*, **211**, 527-558, 2000.

[MPP2] C. A. Morales, M. J. Pacifico, E. R. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, *Annals of mathematics*, to appear.

[Sm] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** 1967 747-817.

[Tu] W. Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris*, **328** 1197–1202, 1999.

A. Arroyo. [aubin@matcuer.unam.mx].
IM-UNAM. Universidad Nacional Autónoma de México.
A.P. 273, Admón. de correos #3. C.P. 62251. Cuernavaca, Morelos, Mexico.

E. R. Pujals. [enrique@impa.br].
IMPA - Instituto de Matemática Pura e Aplicada.
Est. D. Castorina 110, 22460-320. Rio de Janeiro, Brazil