ALGEBRAIC REDUCTION THEOREM FOR COMPLEX CODIMENSION ONE SINGULAR FOLIATIONS

DOMINIQUE CERVEAU, ALCIDES LINS-NETO, FRANK LORAY, JORGE VITORIO PEREIRA AND FRÉDÉRIC TOUZET

Abstract. Let M be a compact complex manifold equipped with $n = \dim(M)$ meromorphic vector fields that are independant at a generic point. The main theorem is the following. If M is not bimeromorphic to an algebraic manifold, then any codimension one complex foliation $\mathcal{F}$ with a codimension 2 singular set satisfies the following alternative: either $\mathcal{F}$ is the meromorphic pull-back of an algebraic foliation on a lower dimensional algebraic manifold, or $\mathcal{F}$ is transversely projective outside a compact hypersurface. The ingredients are essentially the Algebraic Reduction Theorem for M, Lie's classification of geometries on the line and algebraic manipulations with the (meromorphic) Godbillon-Vey sequences associated to the foliation. We also derive from our study (even in the case M algebraic) several sufficient conditions on the Godbillon-Vey sequence insuring such alternative. For instance, if there exists a finite Godbillon-Vey sequence or if the Godbillon-Vey invariant is zero, then either $\mathcal{F}$ is the pull-back of a foliation on a surface, or $\mathcal{F}$ is transversely projective. We illustrate our results with many examples.

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We thank the red crocodile for inspiring the main Theorem.

Contents

1. Introduction

Let M be a compact connected complex manifold of dimension $n \geq 2$. A (codimension 1 singular holomorphic) foliation $\mathcal{F}$ on M will be given by a covering of M by open subsets $(U_j)_{j\in J}$ and a collection of integrable holomorphic 1-forms ω_j on U_j, $\omega_j \wedge d\omega_j = 0$, having codimension ≥ 2 zero-set such that, on each non empty intersection $U_j \cap U_k$, we have

$$(*) \quad \omega_j = g_{jk} \cdot \omega_k, \quad \text{with} \quad g_{jk} \in \mathcal{O}^*(U_j \cap U_k).$$

Let $\mathrm{Sing}(\omega_j) = \{p \in U_j \ ; \ \omega_j(p) = 0\}$. Condition $(*)$ implies that $\mathrm{Sing}(\mathcal{F}) := \cup_{j\in J}\mathrm{Sing}(\omega_j)$ is a codimension ≥ 2 analytic subset of M. If ω is an integrable meromorphic 1-form on M, $\omega \wedge d\omega = 0$, then we can associate to ω a foliation $\mathcal{F}_\omega$ as above. Indeed, at the neighborhood of any point $p \in M$, one can write $\omega = f \cdot \tilde{\omega}$ with f meromorphic, sharing the same divisor with ω; therefore, $\tilde{\omega}$ is holomorphic with codimension ≥ 2 zero-set and defines $\mathcal{F}_\omega$ on the neighborhood of p.

The manifold M is called *pseudo-parallelizable*, if there exist n meromorphic vector fields $X_1, \ldots, X_n$ on M that are independent at a generic point. On such a manifold, differential calculus can be done likely as on an algebraic manifold and a foliation $\mathcal{F}$ is always defined by a global meromorphic 1-form ω (satisfying $\omega \wedge d\omega = 0$). Indeed, given a meromorphic vector field on M which is not identically tangent to $\mathcal{F}$, then ω is the unique meromorphic 1-form defining $\mathcal{F}$ and satisfying $\omega(X) \equiv 1$. We will denote $\mathcal{F} = \mathcal{F}_\omega$.

The notion of pseudo-parallelizable manifolds is invariant by bimeromorphic transformations; more generally, if $f : \tilde{M} \dashrightarrow M$ is meromorphic and generically étale, and if M is pseudo-parallelizable, then $\tilde{M}$ is also. Besides algebraic manifolds, one can find complex tori, Hopf manifolds and homogeneous spaces among examples of such manifolds. Of course, even among surfaces, there are manifolds which are not pseudo-parallelizable.

We say that $\mathcal{F}_\omega$ is *transversely projective* if there exist meromorphic 1-forms $\omega_0 = \omega$, ω_1 and ω_2 on M satisfying

$$(1) \qquad \begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases}$$

This means that, outside the polar and singular set of the ω_i's, the foliation $\mathcal{F}$ is (regular and) transversely projective in the classical sense (see [7] or section 2.2) and this projective structure has "reasonable singularities". When $\omega_2 = 0$ (i.e. $d\omega_1 = 0$) or $\omega_1 = 0$ (i.e. $d\omega_0 = 0$), we respectively say that $\mathcal{F}_\omega$ is actually *transversely affine* or *euclidian*.

Now, denote by $a(M)$ the algebraic dimension of M, that is the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(M)$ of meromorphic functions on M. The algebraic Reduction Theorem (see [22] or section 2.5) provides a meromorphic map $f : M \dashrightarrow N$ onto a projective manifold N of dimension $a(M)$ such that $\mathcal{M}(M)$ identifies with $f^*\mathcal{M}(N)$. In fact, the fibers of f are the maximal subvarieties on which every meromorphic function on M is constant. Of course, the map f is unique up to birational modifications of N. We will denote by $\mathrm{red} : M \dashrightarrow \mathrm{red}(M)$ this map. There exist pseudo-parallelizable manifolds M of arbitrary dimension $n \geq 2$ with arbitrary algebraic dimension $0 \leq a(M) \leq n$.

When M is not algebraic (up to a bimeromorphism), i.e. $a(M) < n$, our main result is the following "Foliated Reduction Theorem".

Theorem 1.1. *Let $\mathcal{F}$ be a complex codimension one singular foliation on a pseudo-parallelizable compact complex manifold M. Then*

- *either $\mathcal{F}$ is the pull-back by the reduction map $M \dashrightarrow red(M)$ of an algebraic codimension one foliation $\underline{\mathcal{F}}$ defined on $red(M)$,*
- *or $\mathcal{F}$ is transversely projective.*

More precisely, we are in the former case when the fibers of the map $M \dashrightarrow \mathrm{red}(M)$ are contained in the leaves of $\mathcal{F}$. In other words, Theorem 1.1 says that only algebraic foliations may have complicated transverse dynamics. In the case $a(M) = 0$ (i.e. $\mathcal{M}(M) = \mathbb{C}$) or $a(M) = 1$, we have no alternative (if $\mathcal{F}$ is the pull-back of a foliation by points on a curve, then it is automatically transversely euclidean)

Corollary 1.2. *Let $\mathcal{F}$ and M be as above and assume $a(M) = 0$ or 1. Then $\mathcal{F}$ is transversely projective.*

When M is simply connected and $a(M) = 0$, it follows that $\mathcal{F}$ necessarily admits an invariant hypersurface, that is the singular set of the projective structure. Indeed, if the projective structure were not singular, the development map of the structure would provide a non constant meromorphic function on M, thus contradicting $a(M) = 0$. In the case M is a surface, we get the more precise statement

Corollary 1.3. *Let $\mathcal{F}$ be a singular foliation on a pseudo-parallelizable compact surface S. If $a(M) < \dim(M)$, then $\mathcal{F}$ is transversely affine.*

In section 3.4, we also give a precised statement in the case of threefolds. One of the ingredients for the proof of Theorem 1.1 is the following algebraic version of Lie's Lemma for which we did not found any reference.

Lemma 1.4. *Let $\mathcal{L}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$ of characteristic 0. If $\mathcal{L}$ has a codimension one Lie subalgebra $\mathcal{L}'$, then there exists a non trivial morphism $\phi : \mathcal{L} \to sl(2, \mathbb{K})$ such that the kernel of ϕ is contained in $\mathcal{L}'$.*

We thank D. Arnal who gave us an alternate algebraic proof of this result in case $\mathbb{K} = \mathbb{C}$.

The other ingredient for the proof of Theorem 1.1 is the global differential calculus. We call *a Godbillon-Vey sequence for* $\mathcal{F}$ any sequence of meromorphic 1-forms $(\omega_0, \omega_1, \ldots, \omega_k, \ldots)$ on M such that ω_0 defines $\mathcal{F}$ and the formal 1-form

$$\Omega = dz + \sum_{k=0}^{\infty} \frac{z^k}{k!}\omega_k \,, \tag{2}$$

is integrable: $\Omega \wedge d\Omega = 0$. In this sense, Ω defines a formal development of $\mathcal{F}$ on the space $(\hat{\mathbb{C}}, 0) \times M$. This condition is equivalent to

$$d\omega_k = \omega_0 \wedge \omega_{k+1} + \sum_{l=1}^{k} \binom{l}{k} \omega_l \wedge \omega_{k+1-l} \,. \tag{3}$$

One can see that ω_{k+1} is well defined by $\omega_0, \omega_1, \ldots, \omega_k$ up to the addition by a meromorphic factor of ω_0. Conversally, $\omega_0, \omega_1, \ldots, \omega_k, \omega_{k+1} + f \cdot \omega_0$ is the begining of another Godbillon-Vey sequence for any $f \in \mathcal{M}(M)$.

When M is pseudo-parallelizable, then any foliation $\mathcal{F}$ admits a Godbillon-Vey sequence. Indeed, let X be a meromorphic vector field which is not identically tangent to $\mathcal{F}$ and ω be the meromorphic 1-form satisfying $\omega(X) = 1$ and defining the foliation $\mathcal{F}$. Then we define a Godbillon-Vey sequence for $\mathcal{F}$ by setting

$$\omega_k := L_X^{(k)} \omega, \tag{4}$$

where $L_X^{(k)}\omega$ denotes the k^{th} Lie derivative along X of the form ω (we have $d\Omega = \Omega \wedge \frac{\partial \Omega}{\partial z}$). In fact, Theorem 1.1 is still true when we can replace the assumption "pseudo-parallelizable" by the existence of a non zero meromorphic vector field X which is tangent to the fibers of the algebraic reduction.

Theorem 1.5. *Let $\mathcal{F}$ be a foliation on a pseudo-parallelizable compact complex manifold M. Assume that the meromorphic 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$ is zero for some Godbillon-Vey sequence associated to $\mathcal{F}$. Then*

- *either $\mathcal{F}$ is the pull-back by a meromorphic map $\Phi : M \dashrightarrow S$ of a foliation $\tilde{\mathcal{F}}$ on an algebraic surface S,*
- *or $\mathcal{F}$ is transversely affine.*

This result has some interest even in the case M is algebraic, although we do not know how to interpret geometrically the assumption. It is a well known and easy computation (see [7]) to show that the meromorphic 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$, considered up to the addition by an exact meromorphic 3-form, is closed and does not depend on the choice of the Godbillon-Vey sequence. We do not know if the conclusion of Theorem 1.5 still holds when $\omega_0 \wedge \omega_1 \wedge \omega_2$ is only exact.

We now define the *length* $l(\mathcal{F})$ of a foliation $\mathcal{F}$ as follows. If we write a Godbillon-Vey sequence as $\Omega = \{\omega_0, \omega_1, \ldots\}$, i.e. the 1-forms ω_i satisfy (3), then the *length* of Ω is the smallest positive integer N such that $\omega_i \equiv 0$ for all $i \geq N$. The *length of* $\mathcal{F}$ is therefore defined by

$$l(\mathcal{F}) = \inf_{\Omega}\{\text{length}(\Omega)\} \in \mathbb{N} \cup \{\infty\}\,,$$

where Ω runs over all possible Godbillon-Vey sequences for $\mathcal{F}$.

A foliation has length 0, 1 or 2 if, and only if, it is respectively transversely euclidian, affine or projective in the sense above. Also, consider an ordinary differential equation over a curve C

$$dz + \sum_{k=0}^{N} \omega_k z^k\,, \tag{5}$$

(where ω_k are meromorphic 1-forms defined on C). Then, the foliation defined on $C \times \mathbb{C}P(1)$ by equation (5) has length $\leq k$ (consider the Godbillon-Vey algorithm given by equation (4) with $X = \frac{\partial}{\partial y}$). Although it is expected that k is the actual length of the generic equation (5), this is clear only for the Riccati equations $k \leq 2$, for monodromy reasons. The study of foliations having finite length has been initiated by Camacho and Scárdua in [3] when the ambient space is $\mathbb{C}P(2)$. We generalize their main result in the

Theorem 1.6. *Let $\mathcal{F}$ be a foliation on a pseudo-parallelizable compact complex manifold M. If $3 \leq l(\mathcal{F}) < \infty$, then $\mathcal{F}$ is the pull-back by a meromorphic map $\Phi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation $\underline{\mathcal{F}}$ defined by an ordinary differential equation over a curve C like above.*

There are examples of foliations on $\mathbb{C}P(2)$ having length 0, 1 or 2 that are not pull-back of a Riccati equation (see [12] and [21]). Therefore, condition $3 \leq l(\mathcal{F})$ is necessary. Recall that the degree of a foliation $\mathcal{F}$ on $\mathbb{C}P(n)$ is the number d of tangencies with a generic projective line. At least, we prove the

Theorem 1.7. *Every foliation of degree 2 on the complex projective space $\mathbb{C}P(n)$ has length at most 3. This bound is sharp.*

In particular, Jouanolou examples (see [11]) have actually length 3. In the same spirit, we also derive from [13] the

Theorem 1.8. *If $\mathcal{F}$ is a germ of foliation at the origin of $\mathbb{C}^n$ defined by an holomorphic 1-form with a non zero linear part, then $l(\mathcal{F}) \leq 3$.*

From Theorems 1.6 and 1.7, we immediately retrieve the following result previously obtained by two of us in [5]:

Corollary 1.9. *A degree* 2 *foliation on* $\mathbb{C}P(n)$ *is either transversely projective, or the pull-back of a foliation on* $\mathbb{C}P(2)$ *by a rational map.*

We do not understand the strength of the assumption $l(\mathcal{F}) < \infty$ of Theorem 1.6. In fact, we still do not know any example of a foliation having length ≥ 4. It is not excluded that the generic foliation of degree 3 on $\mathbb{C}P(2)$ has infinite length.

The degree of a foliation $\mathcal{F}$ on $\mathbb{C}P(n)$ is also the smallest integer d such that $\mathcal{F}$ is defined in the affine chart $\mathbb{C}^n$ by a degree $d+1$ polynomial 1-form ω whose homogeneous component of degree $d+1$ is radial. Consider the projective space $\mathbb{C}P(N)$ of those 1-forms: $N = (d+n+1)\frac{(d+n-1)!}{d!(n-1)!} - 1$. Therefore, the set $\mathcal{F}(n,d)$ of degree d foliations on $\mathbb{C}P(n)$ identifies with the algebraic subset defined by integrability condition $\omega \wedge d\omega = 0$ minus the set of those 1-forms for which $\omega_{d+1} = 0$ and the degree d homogeneous component ω_d is radial. For $n \geq 3$, this algebraic subset is not trivial (not the whole of $\mathbb{C}P(N)$) and, up to now, all known irreducible components are essentially of two types (see [5]). Either the generic element is the pull-back of a generic foliation of degree d_1 on $\mathbb{C}P(2)$ by a generic rational map $\mathbb{C}P(n) \dashrightarrow \mathbb{C}P(2)$ of degree d_2, or the generic element is defined by a closed meromorphic 1-form. Observe that, in the pull-back case, the generic element is not transversely projective and has length ≥ 3. We construct in Section 4.3 a component of $\mathcal{F}(3,4)$ whose generic element is transversely projective but not transversely affine (in particular, not defined by a closed 1-form). We do not know if every foliation of this component is a pull-back (by a non-generic rational map) or not of a foliation on $\mathbb{C}P(2)$.

In section 4.4, we give an example of a transversely projective foliation $\mathcal{H}$ in $\mathbb{C}P(3)$ (with explicit equations) which is not the pull-back of a foliation in $\mathbb{C}P(2)$ by a rational map. In fact, $\mathcal{H}$ is the suspension (see section 2.3) of one of the "Hilbert modular foliations" on $\mathbb{C}P(2)$ studied in [15].

Finally, since our arguments are mainly of algebraic nature, it is natural to ask what remains from our work in the positive characteristic. In this direction, we prove in the last section the

Theorem 1.10. *Let* M *be a smooth projective variety defined over a field* K *of characteristic* $p > 0$ *and* ω *be a rational* 1*-form. If* ω *is integrable, then there exist a rational function* $F \in K(M)$ *such that* $F\omega$ *is closed. In this sense, the associated foliation has length* 1.

2. Background and first steps

2.1. **Godbillon-Vey sequences.** We introduce Godbillon-Vey sequences for a codimension one foliation $\mathcal{F}$ and describe basic properties. Let ω be a differential 1-form defining $\mathcal{F}$ and X be a vector field satisfying $\omega(X) = 1$. Then, the integrability condition of ω is equivalent to

$$\omega \wedge d\omega = 0 \quad \Leftrightarrow \quad d\omega = \omega \wedge L_X\omega. \tag{6}$$

Indeed, from $L_X\omega = d(\omega(X)) + d\omega(X, .) = d\omega(X, .)$, we derive

$$0 = \omega \wedge d\omega(X, ., .) = \omega(X) \cdot d\omega - \omega \wedge (d\omega(X, .)) = d\omega - \omega \wedge L_X\omega$$

(the converse is obvious). Applying this identity to the formal 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + \frac{z^2}{2}\omega_2 + \cdots + \frac{z^k}{k!}\omega_k + \cdots$$

together with the vector field $X = \partial_z$, we derive

$$\Omega \wedge d\Omega = 0 \quad \Leftrightarrow \quad \sum_{k=0}^{\infty} \frac{z^k}{k!} d\omega_k = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\omega_k \right) \wedge \left(\sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!}\omega_k \right).$$

We therefore obtain the full integrability condition (3) for Ω:

$$\begin{array}{rcl}
d\omega_0 & = & \omega_0 \wedge \omega_1 \\
d\omega_1 & = & \omega_0 \wedge \omega_2 \\
d\omega_2 & = & \omega_0 \wedge \omega_3 + \ \ \omega_1 \wedge \omega_2 \\
d\omega_3 & = & \omega_0 \wedge \omega_4 + 2\omega_1 \wedge \omega_3 \\
 & \vdots & \\
d\omega_k & = & \omega_0 \wedge \omega_{k+1} + \sum_{l=1}^{k} \binom{l}{k} \omega_l \wedge \omega_{k+1-l} \\
 & \vdots &
\end{array}$$

For instance, if we start with ω integrable and X satisfying $\omega(X) = 1$, then the iterated Lie derivatives $\omega_k := L_X^{(k)}\omega$ define a Godbillon-Vey sequence for $\mathcal{F}_\omega$. Indeed, from the formula $(L_X\omega)(X) = d\omega(X, X) = 0$, we have

$$\omega_0(X) = 1 \quad \text{and} \quad \omega_k(X) = 0 \text{ for all } k > 0;$$

therefore, $\Omega(X) = 1$ and integrability condition comes from

$$\Omega \wedge L_X\Omega = \left(dz + \sum_{k=0}^{\infty} \frac{z^k}{k!}\omega_k \right) \wedge \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\omega_{k+1} \right) = d\Omega.$$

From a given Godbillon-Vey sequence, we derive many other ones. For instance, given any non zero meromorphic function $f \in \mathcal{M}(M)$, after applying the formal change of variable $z = f \cdot t$ to integrable 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + \frac{z^2}{2}\omega_2 + \cdots + \frac{z^k}{k!}\omega_k + \cdots ,$$

we derive the new integrable 1-form

$$\frac{\Omega}{f} = dt + \frac{\omega_0}{f} + t(\omega_1 + \frac{df}{f}) + \frac{t^2}{2}(f\omega_2) + \frac{t^3}{3!}(f^2\omega_3) + \cdots + \frac{t^k}{k!}(f^{k-1}\omega_k) + \cdots$$

In other words, we obtain a new Godbillon-Vey sequence $(\tilde{\omega}_k)$ by setting

$$\left\{\begin{array}{ccc} \tilde{\omega}_0 & = & \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 & = & \omega_1 + \frac{df}{f} \\ \tilde{\omega}_2 & = & f \cdot \omega_2 \\ & \vdots & \\ \tilde{\omega}_{k+1} & = & f^k \cdot \omega_{k+1} \\ & \vdots & \end{array}\right. \tag{7}$$

By the same way, we can apply to Ω the formal change of variable $z = t + f \cdot t^{k+1}$, $k = 1, 2, \ldots$, and successively derive new Godbillon-Vey sequences

$$\left\{\begin{array}{ccc} \tilde{\omega}_0 & = & \omega_0 \\ \tilde{\omega}_1 & = & \omega_1 + f\omega_0 \\ \tilde{\omega}_2 & = & \omega_2 + f\omega_1 - df \\ & \vdots & \end{array}\right. \qquad \left\{\begin{array}{ccc} \tilde{\omega}_0 & = & \omega_0 \\ \tilde{\omega}_1 & = & \omega_1 \\ \tilde{\omega}_2 & = & \omega_2 + f\omega_0 \\ & \vdots & \end{array}\right. \quad \text{etc.\,.\,.}$$

Conversally, we easily see from integrability condition (3) that ω_{k+1} is well defined by $\omega_0, \omega_1, \ldots, \omega_k$ up to the addition by a meromorphic factor of ω_0. In fact, every Godbillon-Vey sequence can be deduced from a given one after applying to the 1-form Ω a formal transformation belonging to the following group

$$G = \left\{ (p, z) \mapsto \left(p, \sum_{k=1}^{\infty} f_k(p) \cdot z^k \right),\ f_k \in \mathcal{M}(M),\ f_1 \not\equiv 0 \right\}.$$

In particular, the so-called Godbillon-Vey invariant $\omega_0 \wedge \omega_1 \wedge \omega_2 = -\omega_1 \wedge d\omega_1$ is closed and is well defined up to the addition by an exact meromorphic 3-form of the form

$$\frac{df}{f} \wedge \omega_0 \wedge \omega_2 = \frac{df}{f} \wedge d\omega_1 \quad \text{or} \quad df \wedge \omega_0 \wedge \omega_1 = df \wedge d\omega_0$$

for some meromorphic function $f \in \mathcal{M}(M)$.

Remark 2.1. A natural Godbillon-Vey sequence for the formal foliation $\mathcal{F}_\Omega$ defined by Ω is given by

$$\Omega_k = L_{\partial_z}^{(k)}\Omega = \sum_{l=k}^{\infty} \frac{z^{l-k}}{(l-k)!}\omega_l, \quad k > 0$$

or equivalently by the formal integrable 1-form

$$d(t+z) + \omega_0 + (t+z)\omega_1 + \frac{(t+z)^2}{2}\omega_2 + \cdots$$

$$= dt + \Omega_0 + t\Omega_1 + \frac{t^2}{2}\Omega_2 + \cdots$$

In fact, this remark also applies to the case where the ω_k are meromorphic 1-forms on a complex curve C. The so-called "ordinary differential equation" defined by

$$\Omega = dz + \sum_{k=1}^{N} \frac{z^k}{k!}\omega_k\,,$$

defines a foliation $\mathcal{F}$ on $C \times \mathbb{C}P(1)$ (integrability conditions (3) are trivial in dimension 1). This foliation admits a natural Godbillon-Vey sequence of length N given by $L_{\partial_z}^{(k)}\Omega$ (or by replacing z by $z+t$) as above.

Remark 2.2. When $\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \neq 0$ (where $n = \dim(M)$ is the dimension of the ambient space), it follows from relations (3) that all subfamilies $\omega_0, \omega_1, \ldots, \omega_{i-1}$ are Frobenius integrable for $i = 1, 2, \ldots, n-1$, thus defining a codimension i foliation $\mathcal{F}_i$:

$$\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge d\omega_{j-1} = 0 \quad \text{for all } j = 1, 2, \ldots, i.$$

Therefore, we obtain an "integrable flag":

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{n-1}$$

(the tangents spaces $T_p\mathcal{F}_i$ define is a flag at a generic point $p \in M$).

We end-up the section with preliminary lemmas about finite Godbillon-Vey sequences.

Lemma 2.3. *Let $\omega_0, \omega_1, \ldots, \omega_N$ be a Godbillon-Vey sequence of finite length N. Then $\omega_k \wedge \omega_l = 0$ for all $k, l \geq 2$ and integrability conditions become*

$$d\omega_k = \omega_0 \wedge \omega_{k+1} + (k-1)\omega_1 \wedge \omega_k \quad k = 0, 1, \ldots, N.$$

In particular, the condition $\omega_{N+1} = 0$ in a Godbillon-Vey sequence is not sufficient to conclude that the truncated sequence

$$\omega_0, \omega_1, \ldots, \omega_N, 0, 0, \ldots$$

provides a finite Godbillon-Vey sequence, except when $N = 0, 1$ or 2.

Proof. We assume $\omega_N \neq 0$ with $N \geq 2$, otherwise we have done. The integrability conditions (3)

$$\begin{array}{rcl} d\omega_0 & = & \omega_0 \wedge \omega_1 \\ d\omega_1 & = & \omega_0 \wedge \omega_2 \\ & \vdots & \\ d\omega_N & = & \sum_{l=1}^{N} \binom{l}{N} \omega_l \wedge \omega_{N+1-l} \\ 0 = \quad d\omega_{N+1} & = & \sum_{l=2}^{N} \binom{l}{N+1} \omega_l \wedge \omega_{N+2-l} \\ & \vdots & \\ 0 = \quad d\omega_{2N-2} & = & \frac{1}{N}\binom{N-1}{2N-2} \omega_{N-1} \wedge \omega_N \end{array}$$

Examining the line of index $k = 2N-2$, we deduce that $\omega_{N-1} \wedge \omega_N \equiv 0$. Futhermore, by descendent induction, we also deduce from the line of index $k + N - 1$ that $\omega_k \wedge \omega_N \equiv 0$ for every $k \geq 2$. Therefore, the remining N first lines of integrability conditions are as in the statement. □

Corollary 2.4. *Let ω_0, ω_1 and ω_2 be differential 1-forms satisfying conditions (3) for $k = 0, 1, 2$ with $d\omega_1 \neq 0$. Then, there exists at most one finite Godbillon-Vey sequence $\omega_0, \ldots, \omega_N$ completing this triple.*

Proof. The assumption $d\omega_1 = \omega_0 \wedge \omega_2 \neq 0$ implies in particular that $\omega_2 \neq 0$. If $\omega_0, \omega_1, \ldots, \omega_N$ is a finite sequence, then we recursively see from integrability conditions of Lemma 2.3 that the line of index k determines ω_k, $k = 3, \ldots, N$, up to a meromorphic factor of ω_0. But since ω_k is tangent to ω_2 but ω_0 is not, we deduce that ω_k is actually completely determined by the line of index k. □

2.2. **Transversely projective foliations: the classical case** [7, 20]**.** A **regular** codimension one foliation $\mathcal{F}$ on a manifold M is transversely projective if there exists an atlas of submersions $f_i : U_i \to \mathbb{C}P(1)$ on M satisfying the cocycle condition:

$$f_i = \frac{a_{ij} f_j + b_{ij}}{c_{ij} f_j + d_{ij}}, \quad \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in PGL(2, \mathbb{C}).$$

on any intersection $U_i \cap U_j$. Any two such atlases $(f_i : U_i \to \mathbb{C}P(1))_i$ and $(g_k : V_k \to \mathbb{C}P(1))_k$ define the same projective structure if the union of them is again a projective structure, i.e. satisfying the cocycle condition $f_i = \frac{a_{ik} g_k + b_{ik}}{c_{ik} g_k + d_{ik}}$ on $U_i \cap V_k$.

Starting from one of the local submersions $f : U \to \mathbb{C}P(1)$ above, one can step-by-step modify the other charts so that they glue with f and define an analytic continuation for f. Of course, doing this along an element $\gamma \in \pi_1(M)$ of the fundamental group, we obtain monodromy $f(\gamma \cdot p) = A_\gamma \cdot f(p)$ for some $A_\gamma \in PGL(2, \mathbb{C})$. By this way, we define the *monodromy representation* of the structure, that is a homomorphism

$$\rho : \pi_1(M) \to PGL(2, \mathbb{C}); \gamma \mapsto A_\gamma,$$

as well as the *developing map*, that is the full analytic continuation of f on the universal covering $\widetilde{M}$ of M

$$\tilde{f} : \tilde{M} \to \mathbb{C}P(1).$$

By construction, $\tilde{f}$ is a global submersion on $\tilde{M}$ whose determinations $f_i : U_i \to \mathbb{C}P(1)$ on simply connected subsets $U_i \subset M$ define unambiguously the foliation $\mathcal{F}$ and the projective structure. In fact, the map $\tilde{f}$ is ρ-equivariant

$$f(\gamma \cdot p) = \rho(\gamma) \cdot f(p), \quad \forall \gamma \in \pi_1(M). \tag{8}$$

Finally, we obtain

Proposition 2.5. *A regular foliation $\mathcal{F}$ on M is transversely projective if, and only if, there exist*

- *a representation $\rho : \pi_1(M) \to PGL(2, \mathbb{C})$ and*
- *a submersion $\tilde{f} : \tilde{M} \to \mathbb{C}P(1)$ defining $\mathcal{F}$ and satisfying (8).*

Any other pair $(\rho', \tilde{f}')$ will define the same structure if, and only if, we have $\rho'(\gamma) = A \cdot \rho(\gamma) \cdot A^{-1}$ and $\tilde{f}' = A \cdot \tilde{f}$. for some $A \in PGL(2, \mathbb{C})$.

Remark 2.6. If M is simply connected, then any transversely projective foliation $\mathcal{F}$ on M actually admits a global meromorphic first integral $\tilde{f} : M \to \mathbb{C}P(1)$.

Example 2.7 (*Suspension of a representation)***.** Given a representation $\rho : \pi_1(M) \to PGL(2,\mathbb{C})$ of the fundamental group of a manifold M into the projective group, we derive the following representation into the group of diffeomorphisms of the product $\tilde{M} \times \mathbb{C}P(1)$

$$\tilde{\rho} : \pi_1(M) \to \mathrm{Aut}(\tilde{M} \times \mathbb{C}P(1))\ ;\ (p, z) \mapsto (\gamma \cdot p, \rho(\gamma) \cdot z)$$

($\tilde{M}$ is the universal covering of M and $p \mapsto \gamma \cdot p$, the Galois action of $\gamma \in \pi_1(M)$). The image $\tilde{G}$ of this representation acts freely, properly and discontinuously on the product $\tilde{M} \times \mathbb{C}P(1)$ since its restriction to the first factor does. Moreover, $\tilde{G}$ preserves the horizontal foliation $\mathcal{H}$ defined by dz as well as the vertical $\mathbb{C}P(1)$-fibration defined by the projection $\pi : \tilde{M} \times \mathbb{C}P(1) \to \tilde{M}$ onto the first factor. In fact, we have $\pi(\tilde{\rho}(\gamma) \cdot p) = \rho(\gamma) \cdot \tilde{\pi}(p)$ for all $p \in \tilde{M}$ and $\gamma \in \pi_1(M)$. Therefore, the quotient $N := \tilde{M} \times \mathbb{C}P(1)/\tilde{G}$ is a manifold equipped with a locally trivial $\mathbb{C}P(1)$-fibration given by the projection $\pi : N \to M$ as well as a codimension one foliation $\mathcal{H}$ transversal to π. In fact, the foliation $\mathcal{H}$ is transversely projective with monodromy representation $\rho \circ \pi_* : \pi_1(N) \to PGL(2,\mathbb{C})$ (π induces an isomorphism $\pi_* : \pi_1(N) \to \pi_1(M)$) and developing map $\tilde{M} \times \mathbb{C}P(1) \to \mathbb{C}P(1); (p, z) \mapsto z$ ($\tilde{M} \times \mathbb{C}P(1)$ is the universal covering of N).

Conversely, a codimension one foliation $\mathcal{H}$ transversal to a $\mathbb{C}P(1)$-fibration $\pi : N \to M$ is actually the suspension of a representation $\rho : \pi_1(M) \to PGL(2,\mathbb{C})$. In particular, $\mathcal{H}$ is transversely projective and uniquely defined by its monodromy ρ.

Now, given a transversely projective foliation $\mathcal{F}$ on M, we construct *the suspension of* $\mathcal{F}$ as follows. We first construct the suspension of the monodromy representation $\rho : \pi_1(M) \to PGL(2,\mathbb{C})$ of $\mathcal{F}$ as above and consider the graph

$$\tilde{\Gamma} = \{(p, z) \in \tilde{M} \times \mathbb{C}P(1)\ ;\ z = \tilde{f}(p)\}$$

of the developing map $\tilde{f} : \tilde{M} \to \mathbb{C}P(1)$. Since $\tilde{f}$ is ρ-equivariant, its graph $\tilde{\Gamma}$ is invariant under the group $\tilde{G}$ and defines a smooth cross-section $f : M \hookrightarrow N$ to the $\mathbb{C}P(1)$-fibration $\pi : N \to M$. By construction, its image $\Gamma = f(M)$ is also transversal to the "horizontal foliation" $\mathcal{H}$ and the transversely projective foliation induced by $\mathcal{H}$ on Γ actually coincides (via f or π) with the initial foliation $\mathcal{F}$ on M.

Proposition 2.8. *A regular foliation $\mathcal{F}$ on M is transversely projective if, and only if, there exist*

- *a locally trivial $\mathbb{C}P(1)$-fibration $\pi : N \to M$ over M,*
- *a codimension one foliation $\mathcal{H}$ on N transversal to π and*
- *a section $f : M \to N$ transversal to $\mathcal{H}$ such that the foliation induced by $\mathcal{H}$ on $f(M)$ coincides via f with $\mathcal{F}$.*

Any other triple $(\pi' : N' \to M, \mathcal{H}', f')$ will define the same structure if, and only if, there exists a diffeomorphism $\Phi : N' \to N$ such that $\pi' = \pi \circ \Phi$, $f = \Phi \circ f'$ and $\mathcal{H}' = \Phi^*\mathcal{H}$.

Over any sufficiently small open subset $U \subset M$, the $\mathbb{C}P(1)$-fibration is trivial and one can choose trivializing coordinates $(p, z) \in U \times \mathbb{C}P(1)$ such that $f : U \to \pi^{-1}(U)$ coincides with the zero-section $\{z = 0\}$. The foliation $\mathcal{H}$ is defined by a unique differential 1-form of the type

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

where ω_0, ω_1 and ω_2 are holomorphic 1-forms defined on U. The integrability condition $\Omega \wedge d\Omega = 0$ reads

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 2\omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases} \tag{9}$$

Now, any change of trivializing coordinates preserving the zero-section takes the form $(\tilde{p}, \tilde{z}) = (p, f_0 \cdot z/(1 + f_1 \cdot z))$ where $f_0 : U \to \mathbb{C}^*$ and $f_1 : U \to \mathbb{C}$ are holomorphic. The foliation $\mathcal{H}$ is therefore defined by

$$\tilde{\Omega} := \frac{(f_0 - f_1\tilde{z})^2}{f_0}\Omega = d\tilde{z} + \tilde{\omega}_0 + \tilde{z}\tilde{\omega}_1 + \tilde{z}^2\tilde{\omega}_2$$

where the new triple $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ is given by

$$\begin{cases} \tilde{\omega}_0 &= f_0\omega_0 \\ \tilde{\omega}_1 &= \omega_1 - 2f_1\omega_0 - \frac{df_0}{f_0} \\ \tilde{\omega}_2 &= \frac{1}{f_0}\left(\omega_2 - f_1\omega_1 + f_1^2\omega_0 + df_1\right) \end{cases} \tag{10}$$

Proposition 2.9. *A regular foliation $\mathcal{F}$ on M is transversely projective if, and only if, there exists an atlas of charts U_i equipped with 1-forms $(\omega_0^i, \omega_1^i, \omega_2^i)$ satisfying (9) and related to each other by (10) on $U_i \cap U_j$.*

Example 2.10. Consider

$$SL(2,\mathbb{C}) = \{\begin{pmatrix} x & u \\ y & v \end{pmatrix} \ ; \ xv - yu = 1\}.$$

The meromorphic function defined by

$$f : SL(2,\mathbb{C}) \to \mathbb{C}P(1) \ ; \ \begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto \frac{x}{y}$$

is a global submersion defining a transversely projective foliation $\mathcal{F}$ on $SL(2,\mathbb{C})$. The leaves are the right cosets for the “affine” subgroup

$$\mathbb{A} = \{\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \ ; \ a \neq 0\}.$$

Indeed, we have for any $z \in \mathbb{C}$

$$\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbb{A} = \{\begin{pmatrix} az & bz - \frac{1}{a} \\ a & b \end{pmatrix} \ ; \ a \neq 0\} = \{f = z\}$$

and for any $w = 1/z \in \mathbb{C}$

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \cdot \mathbb{A} = \{\begin{pmatrix} a & b \\ aw & bw + \frac{1}{a} \end{pmatrix} \ ; \ a \neq 0\} = \{f = 1/w\}.$$

In fact, if we consider the projective action of a matrix $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$ on $(z : 1) \in \mathbb{C}P(1)$, then f is nothing but the image of the direction $(1 : 0)$ (i.e. $z = \infty$) by the matrix and $\{f = \infty\}$ coincides with the affine subgroup $\mathbb{A}$ fixing $z = \infty$.

A global holomorphic triple $(\omega_0, \omega_1, \omega_2)$ for $\mathcal{F}$ can be constructed as follows. Consider the Maurer-Cartan form

$$\mathcal{M} := \begin{pmatrix} x & u \\ y & v \end{pmatrix}^{-1} d\begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} vdx - udy & vdu - udv \\ xdy - ydx & xdv - ydu \end{pmatrix}.$$

The matrix $\mathcal{M}$ is a differential 1-form on $SL(2,\mathbb{C})$ taking values in the Lie algebra $sl(2,\mathbb{C})$ ($\text{trace}(\mathcal{M}) = d(xv - yu) = 0$) and its coefficients form a basis for the left-invariant 1-forms on $SL(2,\mathbb{C})$. If we set

$$\mathcal{M} = \begin{pmatrix} -\frac{\omega_1}{2} & -\omega_2 \\ \omega_0 & \frac{\omega_1}{2} \end{pmatrix},$$

then Maurer-Cartan formula $d\mathcal{M} + \mathcal{M} \wedge \mathcal{M} = 0$ is equivalent to integrability conditions (9) for the triple $(\omega_0, \omega_1, \omega_2)$. In fact, the “meromorphic triple”

$$(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2) = (df, 0, 0)$$

is derived by setting $f_0 = -\frac{1}{y^2}$ and $f_1 = -\frac{v}{y}$ in formula (10).

A left-invariant 1-form $\omega = \alpha\omega_0+\beta\omega_1+\gamma\omega_2$, $\alpha,\beta,\gamma \in \mathbb{C}$, is integrable, $\omega \wedge d\omega = 0$, if, and only if, $\alpha\gamma = \beta^2$. The right translations act transitively on the set of integrable left-invariant 1-forms and thus on the corresponding foliations. For instance, if we denote by T_z the right translation

$$T_z : SL(2,\mathbb{C}) \to SL(2,\mathbb{C}) \ ; \ \begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto \begin{pmatrix} x & u \\ y & v \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{C},$$

then we have $T_z^*\omega_0 = z^2\omega_0 + z\omega_1 + \omega_2$ and the corresponding foliation $\mathcal{F}_z$ is actually defined by the global submersion

$$f \circ T_z : SL(2,\mathbb{C}) \to \mathbb{C}P(1) \ ; \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}.$$

The leaf $\{f \circ T_z = w\}$ of $\mathcal{F}_z$ is the set of matrices sending the direction $(z:1)$ onto $(w:1)$.

Remark 2.11. Let $(\omega_0,\omega_1,\omega_2)$ be a triple of holomorphic 1-forms on a manifold M satisfying integrability condition (9). The differential equation

$$dz + \omega_0 + z\omega_1 + z^2\omega_2 = 0$$

defined on the trivial projective bundle $M \times \mathbb{C}P(1)$ can be lifted as an integrable differential $sl(2,\mathbb{C})$-system defined on the rank 2 vector bundle $M \times \mathbb{C}^2$ by

$$\begin{cases} dz_1 &= -\frac{\omega_1}{2} z_1 - \omega_2 z_2 \\ dz_2 &= \omega_0 z_1 + \frac{\omega_1}{2} z_2 \end{cases}$$

which can be shortly written as

$$dZ = A \cdot Z \quad \text{where} \quad A = \begin{pmatrix} -\frac{\omega_1}{2} & -\omega_2 \\ \omega_0 & \frac{\omega_1}{2} \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

The matrix A may be thought as a differential 1-form on M taking values in the Lie algebra $sl(2,\mathbb{C})$ satisfying integrability condition $dA + A \wedge A = 0$. Then, Darboux Theorem (see [7], III, 2.8, iv, p.230) asserts that there exists, on any simply connected open subset $U \subset M$, an holomorphic map

$$\Phi : U \to SL(2,\mathbb{C}) \quad \text{such that} \quad A = \Phi^*\mathcal{M}$$

where $\mathcal{M}$ is the Maurer-Cartan 1-form on $SL(2,\mathbb{C})$ (see example 2.10). Moreover, the map Φ is unique up to composition by a translation of $SL(2,\mathbb{C})$.

Example 2.12. Consider the quotient $M := \Gamma\diagdown SL(2,\mathbb{C})$ by a co-compact lattice $\Gamma \subset SL(2,\mathbb{C})$. The left-invariant 1-forms $(\omega_0,\omega_1,\omega_2)$ defined in example 2.10 are well-defined on M and M is parallelizable. Following [10], there is no non constant meromorphic function on M (i.e. the algebraic dimension of M is $a(M) = 0$). Therefore, any foliation $\mathcal{F}$ on M is defined by a global meromorphic 1-form

$$\omega = \alpha\omega_0 + \beta\omega_1 + \gamma\omega_2$$

and the coefficients are actually constants $\alpha,\beta,\gamma \in \mathbb{C}$.

Corollary 2.13. *Any foliation $\mathcal{F}$ on a quotient $M := \Gamma\diagdown SL(2,\mathbb{C})$ by a co-compact lattice Γ is actually defined by a left-invariant 1-form. In particular, $\mathcal{F}$ is regular, transversely projective and minimal: any leaf of $\mathcal{F}$ is dense in M. The set of foliations on M is a rational curve.*

A foliation $\mathcal{F}$ is *transversely euclidean* if there exists an atlas of submersions $f_i : U_i \to \mathbb{C}$ on M defining $\mathcal{F}$ such that on any $U_i \cap U_j$ we have

$$f_i = f_j + a_{ij}, \quad a_{ij} \in \mathbb{C}\,.$$

Of course, we can *glue* the df_i and produce a global closed holomorphic 1-form ω_0 inducing $\mathcal{F}$. In particular $l(\mathcal{F}) = 0$. By the same way, $\mathcal{F}$ is *transversely linear* when it can be defined by submersions $f_i : U_i \to \mathbb{C}^*$ satisfying the cocycle condition:

$$f_i = \lambda_{ij} \cdot f_j, \quad \lambda_{ij} \in \mathbb{C}^*\,.$$

Again, we can *glue* the $\frac{df_i}{f_i}$ and produce a global closed holomorphic 1-form inducing $\mathcal{F}$ and we have $l(\mathcal{F}) = 0$. Via the exponential map, this notion is equivalent to the previous one (in the complex setting).

Finally, a foliation $\mathcal{F}$ is *transversely affine* when it can be defined by submersions $f_i : U_i \to \mathbb{C}$ satisfying the cocycle condition:

$$f_i = a_{ij} f_j + b_{ij}, \quad a_{ij} \in \mathbb{C}^*,\, b_{ij} \in \mathbb{C}.$$

Equivalently, an affine structure is locally defined by a pair of holomorphic 1-forms (ω_0,ω_1) satisfying

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 0 \end{cases} \quad \text{up to modification} \quad \begin{cases} \tilde{\omega}_0 &= f \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 - \frac{df}{f} \end{cases}$$

2.3. **Transversely projective foliations: the singular case** [18]. A singular foliation $\mathcal{F}$ on a complex manifold M will be said *transversely projective* if it admits a Godbillon-Vey sequence of length 2, i.e. if there exist meromorphic 1-forms ω_0, ω_1 and ω_2 on M satisfying $\mathcal{F} = \mathcal{F}_{\omega_0}$ and

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 2\omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases}$$

The foliation $\mathcal{F}$ is actually regular and transversely projective in the classical sense of §2.2 on the Zariski open subset $U = M \setminus ((\Omega)_\infty \cup \mathcal{Z}_0)$ complementary to the set $(\Omega)_\infty$ of poles for ω_0, ω_1 and ω_2 and the set $\mathcal{Z}_0$ of zeroes for ω_0 that are not in $(\Omega)_\infty$. In fact, $(\omega_0, \omega_1, \omega_2)$ is a regular projective triple on U. Another triple $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ defines the same projective structure (on a Zariski open subset) if it is obtained from the previous one by a combination of

$$\begin{cases} \tilde{\omega}_0 &= \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + \frac{df}{f} \\ \tilde{\omega}_2 &= f \cdot \omega_2 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\omega}_0 &= \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + g \cdot \omega_0 \\ \tilde{\omega}_2 &= \omega_2 + g \cdot \omega_1 + g^2 \cdot \omega_0 - dg \end{cases} \tag{11}$$

where f, g denote meromorphic functions on M.

We note that any pair (ω_0, ω_1) satisfying $d\omega_0 = \omega_0 \wedge \omega_1$ can be completed into a triple subjacent to the projective struture in an unique way. It follows that, in the pseudo-parallelizable case, a projective transverse structure is always defined by a global meromorphic triple.

We say that $\mathcal{F}$ is *transversely affine* if it admits a Godbillon-Vey sequence of length 1, i.e. meromorphic 1-forms ω_0 and ω_1 satisfying

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 0 \end{cases}$$

Another pair $(\tilde{\omega}_0, \tilde{\omega}_1)$ will define the same affine structure if we have

$$\begin{cases} \tilde{\omega}_0 &= \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + \frac{df}{f} \end{cases}$$

for a meromorphic function f. Finally, we say that $\mathcal{F}$ is *transversely euclidean* (resp. *transversely trivial*) if it is defined by a closed meromorphic 1-form ω_0 (resp. by an exact 1-form $\omega_0 = df$, $f \in \mathcal{M}(M)$).

The foliation $\mathcal{H}$ defined on $M \times \mathbb{C}P(1)$ by the integrable 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

coincides over U with the suspension of the projective structure, and will be still called *suspension of* $\mathcal{F}$. In fact, the vertical hypersurface $(\Omega)_\infty \times \mathbb{C}P(1)$ is invariant by the foliation $\mathcal{H}$. Outside of this vertical invariant set, the foliation $\mathcal{H}$ is transversal to the vertical $\mathbb{C}P(1)$-fibration. Along $\mathcal{Z}_0$, the foliation $\mathcal{H}$ is tangent to the zero-section $M \times \{z = 0\}$ and the projective structure ramifies: it is locally defined by an holomorphic map $f_i : U_i \mapsto \mathbb{C}P(1)$ up to composition by an element of $PGL(2,\mathbb{C})$. This ramification set $\mathcal{Z}_0$ is invariant for $\mathcal{F}$ (union of leaves and singular points). As in the regular case, one can define the monodromy representation

$$\rho : \pi_1(M \setminus (\Omega)_\infty) \to PGL(2,\mathbb{C})$$

(ramification points $\mathcal{Z}_0$ have no monodromy).

In contrast with the regular case, the suspension $\mathcal{H}$ is well-defined only up to a bimeromorphic transformation preserving the generic vertical fibres $\{p\} \times \mathbb{C}P(1)$ and the zero-section $M \times \{z = 0\}$

$$\Phi : M \times \mathbb{C}P(1) \dashrightarrow M \times \mathbb{C}P(1) \ ; \ (p, z) \mapsto (p, f(p)z/(1 - g(p)z)),$$

where $f, g \in \mathcal{M}(M)$ are meromorphic. Note that some irreducible components of $(\Omega)_\infty$ may disappear after such a transformation Φ. For instance, one can show that any irreducible component of $(\Omega)_\infty$ which is not $\mathcal{F}$-invariant may be deleted by a change of triple. Only the remaining persistent components can generate non trivial local monodromy for the representation ρ. This leads to the following

Proposition 2.14. *Let $\mathcal{F}$ be a (singular) transversely projective (resp. affine) foliation on a simply connected manifold M. If $(\Omega)_\infty$ has no persistent component, then $\mathcal{F}$ admits a meromorphic (resp. holomorphic) first integral.*

Proof. The assumption just means that there exists a covering U_i of M by Zariski open subset on which the projective structure can be defined by an holomorphic triple. Therefore, like in Remark 2.6 the developing map provides a well-defined meromorphic first integral $f : M \to \mathbb{C}P(1)$ (possibly with ramifications). □

Corollary 2.15. *Let $\mathcal{F}$ be a transversely projective (resp. affine) foliation on a simply connected manifold M. Then*

- *either $\mathcal{F}$ has a meromorphic (resp. holomorphic) first integral,*
- *or $\mathcal{F}$ admits an invariant hypersurface.*

Remark 2.16. A transversely projective foliation $\mathcal{F}$ on M with suspension $\mathcal{H}$ on $M \times \mathbb{C}P(1)$ is actually transversely affine if, and only if, there is a section $g : M \to M \times \mathbb{C}P(1)$ which is invariant by $\mathcal{H}$. Indeed, after change of coordinate $\tilde{z} = z/(1 - \frac{z}{g})$ on $M \times \mathbb{C}P(1)$, we have sent the invariant hypersurface $g(M)$ onto $\{z = \infty\}$ which means that $\omega_2 = 0$. In the regular case, this is still true after replacing $M \times \mathbb{C}P(1)$ by the locally trivial $\mathbb{C}P(1)$-bundle $\pi : N \to M$ (see Proposition 2.8) and if we ask moreover that the section $g : M \to N$ has no intersection with the section $f : M \to N$ providing the projective structure.

Example 2.17 (The Riccati equation over a curve)**.** Given meromorphic 1-forms α, β, γ on a curve C, the Riccati differential equation

$$dz + \alpha + \beta z + \gamma z^2 = 0$$

defines a transversely projective foliation $\mathcal{H}$ on $C \times \mathbb{C}P(1)$ with meromorphic projective triple

$$\begin{cases} \omega_0 &= dz + \alpha + \beta z + \gamma z^2 \\ \omega_1 &= \beta + 2\gamma z \\ \omega_2 &= \gamma \end{cases}$$

The polar set $(\Omega)_\infty$ is the union of the vertical lines over the poles of α, β, γ and the horizontal line $L_\infty = \{z = \infty\}$. In the chart $w = 1/z$, the alternate triple

$$\begin{cases} \tilde{\omega}_0 &= -dw + \alpha w^2 + \beta w + \gamma \\ \tilde{\omega}_1 &= -\beta - 2\alpha w \\ \tilde{\omega}_2 &= -\alpha \end{cases}$$

(obtained by setting successively $f = 1/w^2$ and $g = -2/w$ in (11)) shows that L_∞ is not a persistent pole for the projective structure. When $\gamma = 0$, the foliation $\mathcal{H}$ is transversely affine with poles like above, but additionally L_∞ is a persistent zero for the affine structure (the transverse affine coordinate has a pole along L_∞).

The Riccati foliation above can be thought as the suspension of a singular projective structure on the curve C (i.e. a dimension 0 transversely projective foliation on C).

In the spirit of Theorem 1.6, one can find in [18] the following

Proposition 2.18 (Scardua)**.** *Let $\mathcal{F}$ be a transversely projective foliation defined by a global meromorphic triple $(\omega_0, \omega_1, \omega_2)$ on M. Assume that the foliation $\mathcal{G}$ defined by ω_2 admits a meromorphic first integral $f \in \mathcal{M}(M)$. Then, $\mathcal{F}$ is the pull-back by a meromorphic map $\Phi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation $\mathcal{H}$ defined by a Riccati equation on a curve C.*

Proof. One can assume that $\omega_2 = df$. Integrability conditions yield

$$\begin{cases} 0 = d\omega_2 & = & \omega_1 \wedge \omega_2 \\ \quad d\omega_1 & = & 2\omega_0 \wedge \omega_2 \\ \quad d\omega_0 & = & \omega_0 \wedge \omega_1 \end{cases} \Rightarrow \quad \begin{cases} \omega_1 & = & gdf \\ \omega_0 & = & \frac{1}{2}dg + hdf \\ \ 0 & = & d(h - g^2) \wedge df \end{cases}$$

for meromorphic functions g, h on M. It follows from Stein Factorization Theorem that there exists some holomophic map $\phi : M \mapsto C$ onto a curve C through which we can factorize $h - g^2 = \tilde{h}(\phi)$ and $f = \tilde{f}(\phi)$. Therefore

$$\omega_0 = \frac{1}{2}dg + \{\tilde{h}(\phi) + g^2\}\phi^* d\tilde{f}$$

and $\mathcal{F}$ is the pull-back via the map $\Phi = (\phi, g)$ of the foliation defined by the Riccati equation $dz + \tilde{h}d\tilde{f} + z^2 d\tilde{f}$. $\square$

Lemma 2.19. *If a foliation $\mathcal{F}$ admits 2 distinct projective (resp. affine, euclidean) structures, then it is actually transversely affine (resp. euclidean, trivial).*

Proof. Assume we have 2 projective triples $(\omega_0, \omega_1, \omega_2)$ and $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ that are not related by a composition of the admissible changes above: after the admissible change setting $\tilde{\omega}_0 = \omega_0$ and $\tilde{\omega}_1 = \omega_1$, we have $\tilde{\omega}_2 \neq \omega_2$. Therefore, by comparing the second line of integrability conditions for both triples, we see that $\tilde{\omega}_2 = \omega_2 + f\omega_0$ for a meromorphic function $f \in \mathcal{M}(M)$. Then, by comparing the third condition, we obtain

$$d(f\omega_0) = \omega_1 \wedge (f\omega_0) \quad \text{and thus} \quad \omega_0 \wedge \omega_1 = \omega_0 \wedge \frac{df}{2f}$$

which proves that the pair $(\tilde{\omega}_0, \tilde{\omega}_1) := (\omega_0, \frac{df}{2f})$ is an affine structure for $\mathcal{F}$. Notice that $\frac{\omega_0}{\sqrt{f}}$ is closed: $\mathcal{F}$ becomes transversely euclidean on a 2-fold ramified covering of M. By the same way, if (ω_0, ω_1) and $(\tilde{\omega}_0, \tilde{\omega}_1)$ are 2 distinct affine structures, then we may assume $\tilde{\omega}_0 = \omega_0$ and $\tilde{\omega}_1 = \omega_1 + f\omega_0$ with $d\omega_1 = d(f\omega_0) = 0$ and conclude that $\mathcal{F}$ is actually defined by the closed meromorphic 1-form $f\omega_0$. Finally, if ω_0 and $f\omega_0$ are 2 closed meromorphic 1-forms defining $\mathcal{F}$, then f is a meromorphic first integral for $\mathcal{F}$. $\square$

The present singular notion of transversely projective foliation is clearly stable under bimeromorphic transformations. Moreover, the main result of [4] permits to derive

Theorem 2.20. *Let $\phi : \tilde{M} \dashrightarrow M$ be a dominant meromorphic map between pseudo-parallelizable compact manifolds and let $\mathcal{F}$ be a foliation on M. Then, $\tilde{\mathcal{F}} = \phi^*\mathcal{F}$ is transversely projective (resp. affine) if, and only if, $\mathcal{F}$ is so.*

The analogous result for transversely euclidean foliations is false: one can find in [12] an example of a transversely affine foliation which becomes transversely euclidean on a finite covering (a linear foliation on a torus). The assumption dominant is necessary since there are examples of non transversely projective foliations which become transversely affine in restriction to certain non tangent hypersurface (see section 3.5).

Proof. Since a Godbillon-Vey sequence can be pulled-back by any non constant meromorphic map, we just have to prove that projective (resp. affine) structure can be pushed-down under the assumptions above. In the case ϕ is a finite ramified covering, then the statement is equivalent to Theorem 1.6 (resp. 1.4) in [4].

In the case ϕ is holomorphic with connected generic fibre, then choose meromorphic 1-forms ω_0 defining $\mathcal{F}$ and ω_1 satisfying $d\omega_0 = \omega_0 \wedge \omega_1$ on M and consider their pull-back $\tilde{\omega}_0$ and $\tilde{\omega}_1$ on $\tilde{M}$. Then, there is a unique meromorphic 1-form $\tilde{\omega}_2$ completing the previous ones into a projective triple compatible with the structure of $\tilde{\mathcal{F}}$. On the other hand, reasonning as in Lemma 2.22 at the neighborhood $\tilde{U} = \phi^{-1}(U)$ of a generic fibre $\phi^{-1}(p)$, we see that the foliation $\tilde{\mathcal{F}}$ is defined by a submersion $\tilde{f} : \tilde{U} \mapsto \mathbb{C}P(1)$ defining the projective structure and can be pushed-down into a submersion $f : U \mapsto \mathbb{C}P(1)$. This latter one defines a projective structure transverse to $\mathcal{F}$ on U. There exists a unique meromorphic 1-form ω_2 on U completing ω_0 and ω_1 into a compatible projective triple. By construction, $\tilde{\omega}_2$ must coincide with $\phi^*\omega_2$ on $\tilde{U}$. Therefore, $\tilde{\omega}_2$ is tangent to the fibration given by ϕ on $\tilde{U}$, and thus everywhere on $\tilde{M}$. By connexity of the fibres, $\tilde{\omega}_2$ is actually the pull-back of a global meromorphic 1-form ω_2 on M (which extends the one previously defined on U).

Finally, by Stein Factorization Theorem, the statement reduces to the two cases above. $\square$

2.4. **Algebraic Reduction Theorem.** We now state the Algebraic Reduction Theorem in the form we need. Let M be a compact connected complex manifold and consider a subfield $K \subset \mathcal{M}(M)$ of the field of meromorphic functions on M. The transcendence degree (or algebraic dimension) $n = a(K)$ of K over $\mathbb{C}$ is the maximal number of elements $f_1, \ldots, f_n \in K$ satisfying

$$df_1 \wedge \cdots \wedge df_n \neq 0.$$

Following [19], K is *integrally closed* in $\mathcal{M}(M)$ if, and only if, given a transcendence basis as above, we have

$$K = \{f \in \mathcal{M}(M) \ / \ df \wedge df_1 \wedge \cdots \wedge df_n \equiv 0\}.$$

For instance, $\mathcal{M}(M)$ is trivially closed in itself and its transcendence degree $a(M)$ is called the *algebraic dimension of* M. Recall that $a(M) = \dim(M)$ if, and only if, M is bimeromorphically equivalent to an algebraic manifold.

Another important example for us is the following. Given meromorphic 1-forms $\omega_1, \ldots, \omega_n$ on M, the field of meromorphic first integrals

$$\mathcal{M}(\omega_1, \ldots, \omega_n) = \{f \in \mathcal{M}(M) \ / \ df \wedge \omega_1 \wedge \cdots \wedge \omega_n \equiv 0\}.$$

is integrally closed and $0 \leq a\left(\mathcal{M}(\omega_1, \ldots, \omega_n)\right) \leq n$.

Theorem 2.21 (Algebraic Reduction Theorem)**.** *Let M be a compact connected complex manifold. Let $K \subset \mathcal{M}(M)$ be an integrally closed subfield having transcendence degree n. There exist*

1. *a bimeromorphic modification $\Psi : \tilde{M} \to M$,*
2. *an holomorphic projection $\pi : \tilde{M} \to N_K$ with connected fibers onto a n-dimensional algebraic manifold N_K*

such that $\Psi^* K = \pi^* \mathcal{M}(N_K)$.

We will denote by red_K the meromorphic map $\pi \circ \Psi^{-1}$. In particular, any integrally closed subfield $K \subset \mathcal{M}(M)$ is *a posteriori* the field of first integrals $\mathcal{M}(\mathcal{G}_K)$ of a codimension n singular foliation $\mathcal{G}_K$, namely the foliation by fibers of red_K. When $a(M) < \dim(M)$, i.e. M is not bimeromorphic to an algebraic manifold $\tilde{M}$, M is equipped with the canonical codimension $a(M)$ fibration $\mathcal{G}$ induced by $K = \mathcal{M}(M)$.

The space $\mathcal{X}(M)$ of meromorphic vector fields over M acts by derivation on $\mathcal{M}(M)$ and, in this sense, preserves the fibration $\mathcal{G}$. Precisely, given any $X \in \mathcal{X}(M)$, the pseudo-flow of X sends fibers to fibers at the neighborhood of any point $p \in M$ where X and $\mathcal{G}$ are regular. In other words, any vector field X on M is a lifting of some vector field Y on the reduction $N = \mathrm{red}(M)$. This can be seen also directly from the fact that a derivation on $\mathcal{M}(M) := \mathrm{red}^*\mathcal{M}(N)$ is actually a derivation on $\mathcal{M}(N)$. The kernel $\mathcal{X}_0(M) = \{X \in \mathcal{X}(M) | X(f) = 0, \forall f \in \mathcal{M}(M)\}$ coincides with the subspace of those vector fields that are tangent to the fibration $\mathcal{G}$. The space $\mathcal{X}(M)$ is a Lie algebra over $\mathbb{C}$, having infinite dimension as soon as $a(M) \neq 0$, and $\mathcal{X}_0(M)$ is an ideal: $[\mathcal{X}_0(M), \mathcal{X}(M)] \subset \mathcal{X}_0(M)$. Observe that $\mathcal{X}_0(M)$ is also a Lie algebra over the field $\mathcal{M}(M)$, having dimension $\leq \dim(M) - a(M)$. We take care that the space of meromorphic vector fields $\mathcal{X}(F)$ on a given fiber F can actually be much bigger than the restriction $\mathcal{X}_0(M)|_F$: except in the case $a(M) = 0$, some of the fibers could carry non constant meromorphic functions (even, all fibers could be algebraic).[1] Given a foliation $\mathcal{F}$ on M, we will distinguish between the case where $\mathcal{F}$ is tangent to the fibration $\mathcal{G}$ and the case where they are transversal at a generic point. The latter case will be studied in Section 3. The former case is completely understood by means of

Lemma 2.22. *Let $\mathcal{F}$ be a foliation on a complex manifold M. Let $\pi : M \to N$ be a surjective holomorphic map whose fibers are connected and tangent to $\mathcal{F}$, that is, contained in the leaves of $\mathcal{F}$. Then, $\mathcal{F}$ is the pull-back by π of a foliation $\tilde{\mathcal{F}}$ on N.*

Proof. In a small connected neighborhood $U \subset M$ of a generic point $p \in M$, the foliation $\mathcal{F}$ is regular, defined by a local submersion $f : U \to \mathbb{C}$. Since f is contant along the fibers of π in U, we can factorize $f = \tilde{f} \circ \pi$ for an holomorphic function $\tilde{f} : \pi(U) \to \mathbb{C}$. In particular, the function $\tilde{f}$ defines a codimension one singular foliation $\tilde{\mathcal{F}}$ on the open set $\pi(U)$. Of course, $\tilde{\mathcal{F}}$ does not depend on the choice of f. Moreover, since $f = \tilde{f} \circ \pi$, the function f extends to the whole tube $T := \pi^{-1}(\pi(U))$. By connectivity of U and the fibers of π, the tube T is connected and the foliation $\mathcal{F}$ is actually defined by f on the whole of T, coinciding with $\pi^*(\tilde{\mathcal{F}})$ on T. In this way, we can define a foliation $\tilde{\mathcal{F}}$ on $N \setminus S$, where $S = \{p \in N \ ; \ \pi^{-1}(p) \subset \mathrm{Sing}(\mathcal{F})\}$ such that $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$. We note that S has codimension ≥ 2 in N; therefore, $\tilde{\mathcal{F}}$ extends on N by Levy's Extension Theorem. $\square$

[1] Un exemple ?

2.5. **First consequences.** The first and easy alternative of Theorem 1.1 immediately follows

Corollary 2.23. *Let $\mathcal{F}$ be a foliation on a compact manifold M. If the fibers of the algebraic reduction $red : M \dashrightarrow red(M)$ are tangent to $\mathcal{F}$, then $\mathcal{F}$ is actually the pull-back of an algebraic foliation $\tilde{\mathcal{F}}$ on $red(M)$.*

In particular, even if M was not pseudo-parallelizable, $\mathcal{F}$ is *a posteriori* defined by a global meromorphic 1-form, namely the pull-back of any rational 1-form defining $\tilde{\mathcal{F}}$ on the algebraic manifold $\mathrm{red}(M)$.

Let us apply Lemma 2.22 to another situation. Let ω_0 be an integrable 1-form on a compact manifold M, $\omega_0 \wedge d\omega_0 = 0$, and $\mathcal{F}$ be the associated foliation. If ω_0 is not closed, then the 2-form $d\omega_0$ defines a codimension 2 singular foliation $\mathcal{G}$ on M whose leaves are contained in those of $\mathcal{F}$. Denote by $\mathcal{M}(d\omega_0) \subset \mathcal{M}(M)$ the corresponding (integrally closed) subfield of $\mathcal{M}(M)$:

$$\mathcal{M}(d\omega_0) = \{f \in \mathcal{M}(M) \ / \ \ df \wedge d\omega_0 = 0\}\,.$$

The transcendence degree of $\mathcal{M}(d\omega_0)$ satisfies $0 \le a(\mathcal{M}(d\omega_0)) \le 2$. If $\dim(M) = 2$, then $\mathcal{M}(d\omega_0) = \mathcal{M}(M)$ and $a(\mathcal{M}(d\omega_0)) = 2$. Conversally, if $a(\mathcal{M}(d\omega_0)) = 2$, then we have

Lemma 2.24. *Let $\mathcal{F}$ be a foliation on a compact manifold M. If there exists a meromorphic 1-form ω_0 defining $\mathcal{F}$ such that $a(\mathcal{M}(d\omega_0)) = 2$, then $\mathcal{F}$ is the pull-back of a foliation $\tilde{\mathcal{F}}$ on an algebraic surface S by a meromorphic map $\phi : M \dashrightarrow S$.*

Proof. Let $K = \mathcal{M}(d\omega_0)$ and $\tilde{M}$, Ψ, π, $S = N_K$ be produced by the Algebraic Reduction Theorem applied to K. We loose no generality by supposing that $M = \tilde{M}$. Therefore, the codimension 2 foliation defined by $d\omega_0$ is tangent to $\mathcal{F}$ and actually coincides with the fibration defined by π. We conclude by Lemma 2.22. $\square$

In the case $a(\mathcal{M}(d\omega_0)) = 1$ and M is pseudo-parallelizable, we have

Lemma 2.25. *Let $\mathcal{F}$ be a foliation on a compact pseudo-parallelizable manifold M. If there exists a meromorphic 1-form ω_0 defining $\mathcal{F}$ such that $a(\mathcal{M}(d\omega_0)) = 1$, then $\mathcal{F}$ is transversely affine.*

Proof. Let f be a non constant element of $\mathcal{M}(d\omega_0)$. Since M is pseudo-parallelizable, there exists a meromorphic 1-form ω_1 satisfying $d\omega_0 = \omega_0 \wedge \omega_1$ (see introduction). Therefore, one can write

$$df = f_0\omega_0 + f_1\omega_1$$

for meromorphic functions f_0 and f_1 on M. After multiplication by ω_0, we derive

$$\omega_0 \wedge df = f_1 d\omega_0$$

After derivation, we deduce that $f_1 \in \mathcal{M}(d\omega_0)$. Finally, we have

$$d\omega_0 = \omega_0 \wedge \frac{df}{f_1} \quad \text{with} \quad d(\frac{df}{f_1}) = \frac{df \wedge df_1}{{f_1}^2} = 0.$$

In other words, the Godbillon-Vey sequence $(\tilde{\omega}_0, \tilde{\omega}_1) = (\omega_0, \frac{df}{f_1})$ provides an affine structure for $\mathcal{F}$. $\square$

From those two Lemmas, we deduce the

Proof of Theorem 1.5. Let $\mathcal{F}$ be a foliation on a compact manifold M admitting a Godbillon-Vey sequence satisfying $\omega_0 \wedge \omega_1 \wedge \omega_2 \equiv 0$. If $\omega_0 \wedge \omega_1 = 0$, then we conclude that ω_0 is closed and that $\mathcal{F}$ is actually transversely euclidean. Otherwise, there exist meromorphic functions f_0 and f_1 such that

$$\omega_2 = f_0 \cdot \omega_0 + f_1 \cdot \omega_1.$$

From the Godbillon-Vey algorithm, we deduce that

$$d\omega_1 = \omega_0 \wedge \omega_2 = f_1 \cdot d\omega_0$$

and by differentiation that $df_1 \wedge d\omega_0 = 0$. Thus, we have that $f_1 \in \mathcal{M}(d\omega_0)$. If $f_1 \in \mathbb{C}$ is contant, then $\tilde{\omega}_1 := \omega_1 - f_1 \cdot \omega_0$ is closed and satisfies $d\omega_0 = \omega_0 \wedge \tilde{\omega}_1$: $\mathcal{F}$ is actually transversely affine for the new Godbillon-Vey sequence $(\omega_0, \tilde{\omega}_1)$. If f_1 is not constant, then we deduce that $a(\mathcal{M}(d\omega_0)) > 0$ and we conclude with Lemmas 2.24 and 2.25. $\square$

Questions. Given a Godbillon-Vey sequence $(\omega_0, \omega_1, \ldots)$ for a foliation $\mathcal{F}$ on a compact manifold M, what can be said, similarly to Theorem 1.5, when $\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{k-1} = 0$ for some $k < \dim(M)$? Also, when $\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \neq 0$ for $n = \dim(M)$, what can be said when $a(\mathcal{M}(\mathcal{F}_k)) > 0$ for the codimension k foliation $\mathcal{F}_k$ defined by $\omega_0, \omega_1, \ldots, \omega_{k-1}$?

We end-up this section with a weaker version of Theorem 1.6.

Theorem 2.26. *Let $\mathcal{F}$ be a foliation on a compact pseudo-parallelizable manifold M. If length$(\mathcal{F}) < \infty$, then we have the following alternative:*
1. *either $\mathcal{F}$ is the pull-back of a foliation $\underline{\mathcal{F}}$ on an algebraic surface S by a meromorphic map $\phi : M \dashrightarrow S$,*
2. *or $\mathcal{F}$ is transversely projective, i.e. $l(\mathcal{F}) \leq 2$.*

Proof. Let $(\omega_0, \omega_1, \ldots, \omega_N)$ be a Godbillon-Vey sequence for $\mathcal{F}$ with $\omega_N \neq 0$, $N \geq 3$ and ω_1, ω_2, ω_3 both non zero (otherwise we are in the second alternative of the statement). Following Lemma 2.3, there exist meromorphic functions f_k such that $\omega_k = f_k \cdot \omega_2$. Observe that $f_3 \neq 0$ since $\omega_3 \neq 0$. Recall that $\{\omega_k\}$ is a Godbillon-Vey sequence if, and only if, the 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + \cdots + \frac{z^N}{N!}\omega_N \,,$$

is integrable. Applying to Ω the change of variables $z = t/f_3$ (see Section 2.1), we derive a new Godbillon-Vey sequence of length N satisfying $\omega_2 = \omega_3$. Therefore

$$\begin{cases} d\omega_2 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 = \omega_0 \wedge \omega_2 + \omega_1 \wedge \omega_2 \\ d\omega_3 = \omega_0 \wedge \omega_4 + 2 \cdot \omega_1 \wedge \omega_3 = f_4 \cdot \omega_0 \wedge \omega_2 + 2 \cdot \omega_1 \wedge \omega_2 \end{cases}$$

In particular $(1 - f_4)\omega_0 \wedge \omega_2 = \omega_1 \wedge \omega_2$ implying that $\omega_0 \wedge \omega_1 \wedge \omega_2 \equiv 0$. We conclude with Theorem 1.5. □

Proof that length$(\underline{\mathcal{F}})$ = length$(\mathcal{F})$ in Theorem 2.26. Since a Godbillon-Vey sequence for $\underline{\mathcal{F}}$ induces, by pull-back by ϕ, a sequence for ω_0, it follows that $l(\underline{\mathcal{F}}) \geq l(\mathcal{F}) = N$. Let $\underline{\omega}_0$ be the meromorphic 1-form on S such that $\phi^*\underline{\omega}_0 = \omega_0$. From the equality $0 = \omega_0 \wedge \omega_1 \wedge \omega_2 = \omega_1 \wedge d\omega_1$, we see that ω_1 is integrable. Writing down the equations in local coordinates we also see that the fibers of ϕ are tangent to the foliation associated to ω_1. Moreover, ω_1 is the pull-back by ϕ of a 1-form $\underline{\omega}_1$ on S. Recall that $\omega_2 = f_0\omega_0 + f_1\omega_1$ and that $df_1 \wedge d\omega_0 = 0$. Differentiating the identity

$$d\omega_2 = \omega_0 \wedge \omega_2 + \omega_1 \wedge \omega_2 = (f_1 - f_0)d\omega_0$$

it follows that $df_0 \wedge d\omega_0 = 0$. Consequently $\omega_2 = \phi^*\underline{\omega}_2$, where $\underline{\omega}_2$ is a meromorphic 1-form on S. At this point we can rewrite Ω as

$$\Omega = dz + \phi^*\underline{\omega}_0 + z\phi^*\underline{\omega}_1 + h \cdot \phi^*\underline{\omega}_2 \,,$$

where $h = \frac{z^2}{2} + \sum_{i=3}^{N} \frac{z^N}{N!} h_n$. The integrability of Ω implies that $dh \wedge \phi^*\underline{\omega}_2 = 0$, where d is the differential over M (i.e. $dz = 0$). This implies that each h_j belongs to $\phi^{-1}\mathcal{M}(S)$ and therefore $\omega_j = \phi^*\underline{\omega}_j$ for every j and some $\underline{\omega}_j$ on S. This proves that $l(\mathcal{G}) \leq N$. □

3. Proof of the main results

3.1. **Foliated Algebraic Reduction: the case $a(M) = 0$.** Recall first the classical

Lemma 3.1 (Lie). *Let $\mathcal{L}$ be a (finite dimensional) transitive Lie algebra of holomorphic vector fields defined on some neighborhood of $0 \in \mathbb{C}$. Then, after a change of local coordinate, we are in one of the following three cases:*

1. $\mathcal{L} = \mathbb{C} \cdot \partial_z$;
2. $\mathcal{L} = \mathbb{C} \cdot \partial_z + \mathbb{C} \cdot z\partial_z$;
3. $\mathcal{L} = \mathbb{C} \cdot \partial_z + \mathbb{C} \cdot z\partial_z + \mathbb{C} \cdot z^2\partial_z$.

In particular, $\mathcal{L}$ is a representation of a subalgebra of $sl(2, \mathbb{C})$.

Proof of Lemma 1.4 in the case $\mathbb{K} = \mathbb{C}$. Let G be a complex Lie group whose Lie algebra is isomorphic to $\mathcal{L}$. The subalgebra $\mathcal{L}'$ induces a (not necessarily closed) subgroup G' and the left cosets $g \cdot G'$ are the leaves of a regular codimension one holomorphic foliation $\mathcal{F}$ on G. By construction, the foliation $\mathcal{F}$ is invariant under the action of G on itself by left translations, and therefore by the corresponding infinitesimal action. We inherit a representation

$$\rho : \mathcal{L} \to \mathcal{X}(G, \mathcal{F})$$

of $\mathcal{L}$ to the Lie algebra of basic vector fields on G, i.e. of those vector fields on G preserving $\mathcal{F}$. On the other hand, given any transversal disc Δ to $\mathcal{F}$, the projection at any point $p \in \Delta$ of the tangent space $T_pG = T_p\mathcal{F} \times T_p\Delta$ onto the second factor induces representation

$$\pi : \mathcal{X}(G, \mathcal{F}) \to \mathcal{X}(\Delta)$$

to the Lie algebra of vector fields on Δ. Since G acts transitively on itself, the image $\pi \circ \rho(\mathcal{L})$ acts transitively on the disc. Applying Lemma 3.1 at a point $p \in \Delta$, we deduce that the composition

$$\pi \circ \rho : \mathcal{L} \to \mathcal{X}(\Delta)$$

factorises into a representation $\tilde{\rho} : sl(2, \mathbb{C}) \to \mathcal{X}(\Delta)$: there exists an homomorphism $\phi : \mathcal{L} \to sl(2, \mathbb{C})$ such that $\pi \circ \rho = \tilde{\rho} \circ \phi$. By construction, $\ker(\phi) \subset \mathcal{L}'$ and thus ϕ is not trivial. $\square$

We need a technical Lemma. Given a vector field X on a manifold M, we denote by L_X the Lie derivative on differential k-forms. Notice that, when X is a meromorphic vector field on a compact manifold M, then L_X is trivial on the 0-forms $\mathcal{M}(M)$

$$L_X f = 0, \quad \forall f \in \mathcal{M}(M)$$

if, and only if, the vector field X is actually tangent to the fibers of the algebraic reduction $\mathrm{red} : M \to \mathrm{red}(M)$ (see Theorem 2.21).

Lemma 3.2. *Let M be a compact manifold, ω be a meromorphic 1-form on M and X be a meromorphic vector field satisfying $\omega(X) = 1$ and $L_X\mathcal{M}(M) = 0$. Then, for all meromorphic vector field Y on M, we have*

$$L_X^{(i)}\omega(Y) = (-1)^i\omega(L_X^i Y)$$

where $L_X Y = [X, Y]$.

Proof. Since $\omega(X) = 1$, we have

$$L_X\omega = d(\omega(X)) + d\omega(X) = d\omega(X)\,.$$

Therefore, for any vector field Y, we have

$$L_X\omega(Y) = d\omega(X, Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y])\,.$$

By assumption, we have that $L_X(\omega(Y)) = L_X(\text{function}) = 0$ and $L_Y(\omega(X)) = L_Y(\text{constant}) = 0$. Thus we conclude that

$$L_X\omega(Y) = -\omega([X, Y])\,.$$

The proof immediately follows by induction on i. $\square$

Let M be a pseudo-parallelizable compact manifold with no non constant meromorphic function. Therefore, the Lie algebra $\mathcal{L}$ of meromorphic vector fields on M has dimension $n = \dim(M)$. If $\mathcal{F}$ is a foliation on M, then the Lie algebra $\mathcal{L}'$ of those vector fields tangent to $\mathcal{F}$ has dimension $n - 1$. Following Lemma 1.4, there exists a morphism $\phi : \mathcal{L} \to sl(2, \mathbb{C})$ such that $\ker(\phi) \subset \mathcal{L}' \subset \mathcal{L}$. Discussing on the codimension of $\ker(\phi)$, we construct a meromorphic vector field X satisfying $\omega(X) = 1$ (in particular $X \in \mathcal{L}\backslash\mathcal{L}'$) such that the Godbillon-Vey sequence $\omega_i := L_X^{(i)}\omega$ has length ≤ 2.

3.1.1. *First Case:* $\ker(\phi)$ *has codimension* 1. Therefore, $\mathcal{L}' = \ker(\phi)$. In particular, $\mathcal{L}'$ is an ideal of $\mathcal{L}$: $[\mathcal{L}, \mathcal{L}'] \subset \mathcal{L}'$. Let X be any meromorphic vector field satisfying $\omega(X) = 1$. For every $Y \in \mathcal{L}$, we can write

$$Y = c \cdot X + Y' \,,$$

where $c \in \mathbb{C}$ and $Y' \in \mathcal{L}'$. Thus

$$\omega([X, Y]) = \omega([X, Y']) = 0 \quad \forall Y \in \mathcal{L},$$

allowing us to conclude that $L_X\omega = 0$ (see Lemma 3.2). Finally, the Godbillon-Vey sequence given by $\omega_i := L_X^i\omega$ has length 0 ($\omega_1 = 0$) and $\omega_0 = \omega$ is closed.

3.1.2. *Second Case:* $\operatorname{codim}\ker(\phi) = 2$. One can choose a basis X_1, X_2 of $\mathcal{L}/\ker(\phi)$ such that $\omega(X_1) = 1$, X_2 is a basis for $\mathcal{L}'/\ker(\phi)$ and

$$\text{either } [X_1, X_2] = X_1, \quad \text{or } [X_1, X_2] = -X_2.$$

Indeed, after composing ϕ by an automorphism of $sl(2, \mathbb{C})$, the dimension two subalgebra $\phi(\mathcal{L})$ identifies with the Lie algebra generated by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\phi(\mathcal{L}')$ is the one dimensional subalgebra generated by A or B. Then, just choose X_1 and X_2 so that correspondingly $(\phi(X_1), \phi(X_2)) = (B, A)$ or (A, B) and normalize $\omega_0 := \frac{\omega}{\omega(X_1)}$ so that $\omega_0(X_1) = 1$. Therefore, $\omega_0(L_{X_1}^i X_2) = 0$ (i.e. $L_{X_1}^i X_2 \in \mathcal{L}'$) for $i = 1$ or 2. Finally, after writing every vector field $Y \in \mathcal{L}$ into the form

$$Y = c_1 \cdot X_1 + c_2 \cdot X_2 + Y' \,,$$

with $c_1, c_2 \in \mathbb{C}$ and $Y' \in \ker(\phi)$ and applying Lemma 3.2 as in Section 3.1.1, we conclude that the Godbillon-Vey sequence given by $\omega_i := L_{X_1}^i\omega_0$ has length 0 or 1: $\mathcal{F}$ is transversely affine.

3.1.3. *Third Case:* $\operatorname{codim}\ker(\phi) = 3$. We construct a basis X_1, X_2, X_3 of $\mathcal{L}/\ker(\phi)$ such that $\omega(X_1) = 1$, X_2, X_3 is a basis for $\mathcal{L}'/\ker(\phi)$ and, after composing ϕ by an automorphism of $sl(2, \mathbb{C})$,

$$\phi(X_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \phi(X_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \phi(X_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$[X_1, [X_1, [X_1, X_3]]] \;=\; [X_1, [X_1, X_2]] \;=\; 0 \qquad \mod \ \ker(\phi)$$

and $\omega(L_{X_1}^3(Y)) = 0$ for all $Y \in \mathcal{L}$. Finally, the Godbillon-Vey sequence given by $\omega_i := L_{X_1}^i\omega$ has length 2: $\mathcal{F}$ is transversely projective.

Remark 3.3. In the proof of Theorem 1.1, case $a(M) = 0$, above, we do not need M to be pseudo-parallelizable, but only the existence of a meromorphic vector field X which is not identically tangent to $\mathcal{F}$. In this case, the foliation $\mathcal{F}$ is defined by the unique (and thus global) meromorphic 1-form ω satisfying $\omega(X) = 1$. Note that the Lie algebra $\mathcal{L}$ of meromorphic vector fields on M has dimension $n \leq \dim(M)$ and the subalgebra $\mathcal{L}'$ of those vector fields tangent to $\mathcal{F}$ has codimension one (because of X). Possibly, $\mathcal{L}'$ is trivial.

3.2. **Codimension one Lie subalgebras over $\mathbb{K}$.** Before proving Theorem 1.1 for arbitrary algebraic dimension $a(M)$, we need the following more complete statement for Lemma 1.4.

Lemma 3.4. *Let $\mathcal{L}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = 0$, having a codimension one Lie subalgebra $\mathcal{L}'$: $[\mathcal{L}', \mathcal{L}'] \subset \mathcal{L}'$. Then $\mathcal{L}$ also has a codimension ≤ 3 Lie-ideal $\mathcal{I}$, $[\mathcal{L}, \mathcal{I}] \subset \mathcal{I}$, which is contained in $\mathcal{L}'$ and we are in one of the following 3 cases:*

(1) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X$ *and* $\mathcal{L}' = \mathcal{I}$;
(2) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X + \mathbb{K} \cdot Y$ *with* $[X, Y] = X$ *and* $\mathcal{L}'/\mathcal{I} = \mathbb{K} \cdot Y$;
(3) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X + \mathbb{K} \cdot Y + \mathbb{K} \cdot Z$ *with* $[X, Y] = X$, $[X, Z] = 2Y$ *and* $[Y, Z] = Z$ *and* $\mathcal{L}'/\mathcal{I} = \mathbb{K} \cdot Y + \mathbb{K} \cdot Z$.

In other words, there exists a non trivial morphism $\phi : \mathcal{L} \to sl(2, \mathbb{K})$ whose kernel $\ker(\phi) = \mathcal{I}$ is contained in $\mathcal{L}'$.

Applying this Lemma to a finite dimensional transitive subalgebra $\mathcal{L} \subset \mathcal{X}(\mathbb{C}, 0)$ and to the subalgebra $\mathcal{L}'$ of those vector fields fixing 0, we retrieve a part of Lie's Lemma 3.1. In this sense, Lemma 3.4 may be considered as an algebraic version of Lie's Lemma.

Proof. We start with a general situation. Let $\mathbb{K}$ be a field (of arbitrary characteristic) and consider a (possibly infinite dimensional) Lie algebra $\mathcal{L}$ over $\mathbb{K}$ together with a Lie subalgebra $\mathcal{L}'$. Then, recursively define the vector subspaces $\mathcal{L} \supset \mathcal{L}' \supset \mathcal{L}'' \supset \cdots \supset \mathcal{L}^{(k)} \supset \cdots$ by

$$\mathcal{L}^{(k+1)} := \{W \in \mathcal{L}^{(k)} \ ; \ [\mathcal{L}, W] \subset \mathcal{L}^{(k)}\}.$$

We claim that the $\mathcal{L}^{(k)}$ are actually Lie subalgebra satisfying moreover

$$[\mathcal{L}^{(k+1)}, \mathcal{L}^{(l+1)}] \subset \mathcal{L}^{(k+l+1)} \quad \text{for all } k + l \geq -1.$$

We prove the formula by induction on $k + l$. Observe that the formula is already true for $k = -1$ or $l = -1$. By induction hypothesis, we may already assume that $[\mathcal{L}^{(k+1)}, \mathcal{L}^{(l+1)}] \subset \mathcal{L}^{(k+l)}$ since, for instance, $\mathcal{L}^{(k+1)} \subset \mathcal{L}^{(k)}$. By definition of $\mathcal{L}^{(k+l+1)}$, we just have to prove that $[\mathcal{L}, [\mathcal{L}^{(k+1)}, \mathcal{L}^{(l+1)}]]$ is still contained in $\mathcal{L}^{(k+l)}$. Following Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,$$

it suffices to show that the double brackets $[\mathcal{L}^{(l+1)},[\mathcal{L},\mathcal{L}^{(k+1)}]]$ and $[\mathcal{L}^{(k+1)},[\mathcal{L}^{(l+1)},\mathcal{L}]]$ are both contained in $\mathcal{L}^{(k+l)}$. But this immediately follows by induction hypothesis. For instance, $[\mathcal{L}^{(l+1)},[\mathcal{L},\mathcal{L}^{(k+1)}]] \subset [\mathcal{L}^{(l+1)},\mathcal{L}^{(k)}] \subset \mathcal{L}^{(k+l)}$. The claim is proved.

Now, assuming that $\mathcal{L}'$ has finite codimension, say n, *we claim* that

$$\dim_{\mathbb{K}} \mathcal{L}^{(k)}/\mathcal{L}^{(k+1)} \ \leq\ (\dim_{\mathbb{K}} \mathcal{L}/\mathcal{L}')^{k+1}$$

where these quotients are viewed as $\mathbb{K}$-vector spaces. Indeed, since $[\mathcal{L}',\mathcal{L}^{(k+1)}] \subset \mathcal{L}^{(k+1)}$, it follows that $\mathcal{L}^{(k+1)}$ actually coincides with the kernel of

$$\mathcal{L}^{(k)} \to \left(\mathcal{L}^{(k-1)}/\mathcal{L}^{(k)}\right)^n \ ;\ W \mapsto ([X_1,W],\ldots,[X_n,W])$$

where $(X_1,\ldots,X_n)$ denotes a basis for $\mathcal{L}/\mathcal{L}'$. Therefore,

$$\dim_{\mathbb{K}} \mathcal{L}^{(k)}/\mathcal{L}^{(k+1)} \ \leq\ \dim_{\mathbb{K}} \left(\mathcal{L}^{(k-1)}/\mathcal{L}^{(k)}\right)^n \ =\ n\cdot \dim_{\mathbb{K}} \mathcal{L}^{(k-1)}/\mathcal{L}^{(k)}$$

which recursively prove the claim. In particular, if $\mathcal{L}'$ has codimension 1 in $\mathcal{L}$, then there is a smallest $k_0 \in \mathbb{N}^* \cup \{\infty\}$ such that $\mathcal{L}^{(k_0)}$ is an ideal of $\mathcal{L}$ and

$$\dim_{\mathbb{K}}(\mathcal{L}^{(k)}/\mathcal{L}^{(k+1)}) = \begin{cases} 1 & \text{if} \quad k<k_0 \\ 0 & \text{if} \quad k \geq k_0 \end{cases}$$

Now, assume that $\mathcal{L}'$ is not an ideal (otherwise, we are in case (1)) and consider some $Y\in\mathcal{L}'$ generating $\mathcal{L}'/\mathcal{L}''$. We have

$$[X,Y] = aX + bY \mod \mathcal{L}'' \quad \text{for constants } a,b\in\mathbb{K},\ a\neq 0.$$

Replacing X and Y respectively by $X+\frac{b}{a}Y$ and $\frac{Y}{a}$, we may assume

$$[X,Y] = X \mod \mathcal{L}''.$$

Again, assume that $\mathcal{L}''$ is not an ideal (otherwise, we are in case (2)) and consider some $Z\in\mathcal{L}''$ generating $\mathcal{L}''/\mathcal{L}^{(3)}$. Maybe multiplying Z by a scalar, we can assume that $[X,Z]=Y$ modulo $\mathcal{L}''$ and we have

$$\begin{cases} [X,Y] &=& X+aZ \\ [X,Z] &=& Y+bZ \\ [Y,Z] &=& cZ \end{cases} \mod \mathcal{L}^{(3)},$$

for scalars $a,b,c\in\mathbb{K}$. Jacobi identity implies that $(c-1)Y=0$ modulo $\mathcal{L}''$, which shows that $c=1$. If $\mathrm{char}(\mathbb{K})\neq 2$, then we can assume $a=0$ after replacing X by $X+\frac{a}{2}Z$. We thus obtain

$$\begin{cases} [X,Y] &=& X \\ [X,Z] &=& Y+bZ \\ [Y,Z] &=& Z \end{cases} \mod \mathcal{L}^{(3)}.$$

If $\mathcal{L}^{(3)}$ is an ideal, then Jacobi identity applied in $\mathcal{L}/\mathcal{L}^{(3)}$ shows that $b = 0$ and we are in case (3).

Finally, *we claim* that $\mathcal{L}$ has infinite dimension provided that $\mathcal{L}^{(3)}$ is not an ideal and $\mathrm{char}(\mathbb{K}) = 0$. In fact, starting from a generator W_3 of $\mathcal{L}^{(3)}/\mathcal{L}^{(4)}$, we recursively prove that $W_{k+1} := [Z, W_k] \in \mathcal{L}^{(k+1)}$ satisfies

$$\begin{cases} [X, W_{k+1}] = a_k W_k \quad \mod \quad \mathcal{L}^{(k)} \\ \quad [Y, W_k] = b_k W_k \quad \mod \quad \mathcal{L}^{(k+1)} \end{cases}$$

for non vanishing scalars $a_k, b_k \in \mathbb{K}$. We can assume $[X, W_3] = Z$ modulo $\mathcal{L}^{(3)}$. Jacobi identity

$$[X, [Y, W_k]] + [W_k, [X, Y]] + [Y, [W_k, X]] = 0$$

shows that $b_3 = 2$ for $k = 3$ and $a_{k-1}(b_k - b_{k-1} - 1) = 0$ for $k > 3$. On the other hand, Jacobi identity

$$[X, [Z, W_k]] + [W_k, [X, Z]] + [Z, [W_k, X]] = 0$$

shows that $a_3 = 2$ for $k = 3$ and $a_k = a_{k-1} + b_k$ for $k > 3$. It is now easy to conclude that $a_k, b_k > 0$ for all $k \geq 3$ which proves the last claim and the Lemma. $\square$

3.3. **Foliated Algebraic Reduction: the general case.** Let M be a pseudo-parallelizable compact manifold having algebraic dimension $a(M) < \dim M$ and let $\mathcal{F}$ be a foliation on M. We assume that $\mathcal{F}$ is generically transverse to the fibers given by the Algebraic Reduction Theorem, otherwise we conclude with Lemma 2.22 that we are actually in the second alternative of Theorem 1.1. The idea of the proof is to proceed as in section 3.1 along the fibers, but dealing only with objects (vector fields and functions) living on the ambient manifold M. Denote by $\mathcal{X}_0(M)$ the space of meromorphic vector fields that are tangent to the fibers. Recall that $\mathcal{L} := \mathcal{X}_0(M)$ is a Lie algebra of dimension $\dim(M) - a(M)$ over the field $\mathbb{K} := \mathcal{M}(M)$ of meromorphic functions on M. Consider $\mathcal{L}' \subset \mathcal{L}$ the Lie subalgebra of those vector fields that are tangent to the foliation $\mathcal{F}$. Clearly, $\mathcal{L}'$ has codimension 1 in $\mathcal{L}$. Applying Lemma 3.4 to this situation, we see that there is an ideal $\mathcal{I} \subset \mathcal{L}$ contained in $\mathcal{L}'$, and there is some $X \in \mathcal{L} \setminus \mathcal{L}'$ satisfying $L_X^3 Y \in \mathcal{I}$ for any $Y \in \mathcal{X}_0(M)$ (for instance, $L_X^3 X = L_X^3 Y = L_X^3 Z = 0$ modulo $\mathcal{I}$ in case (3) of Lemma 3.4). Let ω be the unique meromorphic 1-form defining the foliation $\mathcal{F}$ and satisfying $\omega(X) = 1$. Since $\mathcal{I} \subset \mathcal{L}'$, we deduce that $\omega(L_X^3 Y) = 0$ for any $Y \in \mathcal{X}_0(M)$.

Now, recall that $\mathcal{L} = \mathcal{X}_0(M)$ is a Lie ideal of the full Lie algebra $\mathcal{X}(M)$ of meromorphic vector fields on M: for any $Y \in \mathcal{X}(M)$, we have $L_X Y \in \mathcal{L}$ and therefore $\omega(L_X^4 Y) = 0$. Following Lemma 3.2, we deduce that $L_X^4\omega = 0$ and finally that the Godbillon-Vey sequence $(L_X^i\omega)_i$ has length ≤ 3.

In case $\mathcal{F}$ is not transversely projective ($(L_X^i\omega)_i$ has really length 3), then $\mathcal{F}$ is the pull-back of a foliation $\tilde{\mathcal{F}}$ on an algebraic surface S by a meromorphic map $\phi : M \dashrightarrow S$ following Theorem 2.26. Of course, ϕ factorizes through the algebraic reduction. But this implies that the fibers of the algebraic reduction are actually tangent to $\mathcal{F}$, contradicting our assumptions. We conclude that $\mathcal{F}$ is transversely projective, thus proving the Theorem 1.1.

3.4. **Some consequences.** In the special case where $a(M) = n - 1$, we directly obtain the

Proposition 3.5. *Let $\mathcal{F}$ be a foliation on a compact manifold M, defined by a global meromorphic 1-form ω. Assume that $a(M) = n-1$. Then*

(1) *either $\mathcal{F}$ is the pull-back of a foliation on the reduction red(M);*
(2) *or $\mathcal{F}$ is transversely affine.*

Proof. Assume that $\mathcal{F}$ is not tangent to the fibration of the algebraic reduction. Here $\mathcal{X}_0(M)$ is an algebra of dimension one over the field $\mathcal{M}(M)$ and is also an ideal of $\mathcal{X}(M)$. One defines a non trivial element $X \in \mathcal{X}_0(M)$ just by setting $\omega(X) = 1$. Then, for every $Y \in \mathcal{X}(M)$, we have that $ad_X^{(2)}Y = [X,[X,Y]] = 0$. In fact $[X,Y] = \lambda X$ and $X(\lambda) = 0$. Therefore $L_X^{(2)}\omega(Y) = \omega(ad_X^{(2)}Y) = 0$ implying that $L_X^{(2)}\omega \equiv 0$ and $l(\mathcal{F}) \leq 1$. □

In particular we have the

Corollary 3.6. *Let $\mathcal{F}$ be a foliation on a pseudo-parallelizable compact surface S. Assume that $a(M) < 2$. Then $\mathcal{F}$ is transversely affine.*

Proof. In the case $a(M) = 0$, we just notice that $\mathcal{X}(M)$ is a Lie algebra of dimension 2 over $\mathcal{M}(M) = \mathbb{C}$. In the case $a(M) = 1$ and $\mathcal{F}$ is tangent to the fibration of the algebraic reduction, then they actually coincide and $\mathcal{F}$ is transversely trivial, i.e. $\mathcal{F}$ has a meromorphic first integral. □

When the dimension of the ambient manifold is three, we resume

Corollary 3.7. *Let M be a 3-dimensional pseudo-parallelizable complex manifold and let $\mathcal{F}$ be a foliation of M. We have the following possibilities:*

1. $a(M) = 3$ *and $\mathcal{F}$ is bimeromorphically equivalent to an algebraic foliation of an algebraic manifold;*
2. $a(M) = 2$ *and either $\mathcal{F}$ is the meromorphic pull-back of a foliation on an algebraic surface, or $\mathcal{F}$ is transversely affine;*
3. $a(M) = 1$ *and $\mathcal{F}$ is transversely affine;*
4. $a(M) = 0$ *and $\mathcal{F}$ is transversely projective.*

Proof. Only the case $a(M) = 1$ does not follow from previous results. Let $f \in \mathcal{M}(M)$ be a non constant meromorphic function. We can suppose that $\mathcal{F}$ is generically transverse to the fibers of f, otherwise $\mathcal{F}$ is defined by df.

Let ω_0 define $\mathcal{F}$ and ω_1 be such that $d\omega_0 = \omega_0 \wedge \omega_1$. We can assume that $\omega_0 \wedge \omega_1 \neq 0$, otherwise ω_0 is closed. If $\omega_0 \wedge \omega_1 \wedge df = 0$, we conclude by lemma 2.25 that $\mathcal{F}$ is transversely affine. Thus suppose that $\omega_0 \wedge \omega_1 \wedge df \neq 0$ and let ω_2 be such that $d\omega_1 = \omega_0 \wedge \omega_2$. We can choose ω_2 without component in ω_0, i.e. $\omega_2 = f_1\omega_1 + g df$. Replacing ω_1 by $\tilde{\omega}_1 = \omega_1 - f_1\omega_0$ we have that

$$\begin{aligned} d\tilde{\omega}_1 &= d\omega_1 - f_1 d\omega_0 - df_1 \wedge \omega_0 \\ &= f_1\omega_0 \wedge \omega_1 + g\omega_0 \wedge df - f_1\omega_0 \wedge \omega_1 - df_1 \wedge \omega_0 \\ &= \omega_0 \wedge (g df + df_1) = h\omega_0 \wedge df\,, \end{aligned}$$

for some meromorphic function h satisfying $dh \wedge df = 0$ since $a(M) = 1$. After derivation, we deduce that $h d\omega_0 \wedge df = 0$ which implies that $h = 0$: $\tilde{\omega}_1$ is closed and $\mathcal{F}$ is transversely affine. □

The next proposition generalizes some of the results obtained by É. Ghys in [8] for the foliations on complex tori.

Proposition 3.8. *Let $\mathcal{F}$ be a foliation on a compact manifold M and assume that there exist $n = \dim(M)$ independent closed meromorphic 1-forms on M. Then we have the following alternative:*

- *either $\mathcal{F}$ is the pull-back of a foliation $\underline{\mathcal{F}}$ on red(M) via the algebraic reduction map $M \dashrightarrow$ red(M),*
- *or $\mathcal{F}$ is transversely euclidean, i.e. defined by a closed meromorphic 1-form.*

Proof. If the algebraic dimension $a(M)$ of M is n, the second alternative is trivially satisfied. Also, when $a(M) = 0$, any 1-form on M is closed since it is a linear combination of the given n closed ones with coefficients in $\mathcal{M}(M) = \mathbb{C}$; in particular, any 1-form defining $\mathcal{F}$ is closed.

Let $f_1, \ldots, f_q \in \mathcal{M}(M)$, $q = a(M)$, be such that $df_1 \wedge \ldots \wedge df_q \neq 0$. By our hypothesis we can find $p = n - q$ closed meromorphic 1-forms such that $\omega_1 \wedge \ldots \wedge \omega_p \wedge df_1 \wedge \ldots \wedge df_q \neq 0$. If ω is a 1-form defining $\mathcal{F}$ then we can write it as

$$\omega = \sum \lambda_i \omega_i + \sum \mu_j df_j \, ,$$

where the λ_i and the μ_j belong to $\mathcal{M}(M)$. If all the λ_i are zero then we are in the first case; if not we can suppose that $\lambda_1 = 1$.

Therefore

$$d\omega = \sum_{i=2}^{p} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \, ,$$

and the integrability condition writes:

$$\begin{aligned} 0 &= \omega_1 \wedge \left(\sum_{i \leq 2} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \right) \\ &+ \left(\sum_{i \leq 2} \lambda_i \omega_i + \sum_j \mu_j df_j \right) \wedge \left(\sum_{i \leq 2} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \right) . \end{aligned}$$

First suppose that $\dim(M) \geq 3$. Notice that the $d\lambda_i$ and $d\mu_j$ are in the $\mathcal{M}(M)$-vector space generated by the df_i. Since the meromorphic 3-forms $\omega_i \wedge \omega_j \wedge df_k$ together with $\omega_i \wedge df_j \wedge df_k$ are linearly independant over $\mathcal{M}(M)$, we deduce that the first term is zero: ω_1 does not occur when one developp the second term on the 3-forms above. Therefore

$$\sum_{i=2}^{q} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j = 0 \, ,$$

and consequently ω is closed.

When $\dim M = 2$, we just have to consider the case where $a(M) = 1$. Then, $\omega = \omega_1 + \lambda_1 df_1$ and $d\omega = d\lambda_1 \wedge df_1 = 0$ since $a(M) = 1$. $\square$

3.5. **Proof of Theorem 1.6.** In fact, we prove the more precise

Theorem 3.9. *Let $\mathcal{F}$ be a foliation admitting a finite Godbillon-Vey sequence $(\omega_0, \omega_1, \ldots, \omega_N)$ of length $N \geq 3$ with $\omega_N \neq 0$. Then*

- *either $\mathcal{F}$ is the pull-back by a meromorphic map $\Phi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation $\underline{\mathcal{F}}$ defined by*
$$dz + \underline{\omega}_0 + \underline{\omega}_1 z + \cdots + \underline{\omega}_N z^n$$
where $\underline{\omega}_k$ are meromorphic 1-forms on the curve C,
- *or $\mathcal{F}$ is transversely affine.*

In particular, we see that a purely transversely projective foliation cannot admits other finite Godbillon-Vey sequences than the projective triples.

Proof. Following Lemma 2.3, we have

$$\Omega = dz + \omega_0 + z\omega_1 + \left(\sum_{k=2}^{N} f_k \cdot z^k \right) \omega_N$$

for meromorphic functions $f_k \in \mathcal{M}(M)$, $f_N \equiv 1$. If $f_{N-1} = 0$, then integrability conditions imply that $d\omega_1 = 0$ (see Lemma 2.3) and $\mathcal{F}$ is transversely affine. Otherwise, after a change of Godbillon-Vey sequence of the form (7) (see Section 2.1), we may assume moreover $f_{N-1} = N$. Now, the change of coordinate $\tilde{z} = z + 1$ on Ω

$$\begin{aligned} \Omega &= d(\tilde{z} - 1) + \omega_0 + (\tilde{z} - 1)\omega_1 + \cdots + (\tilde{z} - 1)^N \omega_N \\ &= d\tilde{z} + \tilde{\omega}_0 + \tilde{z}\tilde{\omega}_1 + \cdots + \tilde{z}^N \tilde{\omega}_N \end{aligned}$$

provides a new sequence $(\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_N)$ of length N satisfying integrability conditions (3) (see Introduction). We take care that this is not a new Godbillon-Vey sequence for $\mathcal{F}$ (but for $\mathcal{F}_{\tilde{\omega}_0}$, whenever $\tilde{\omega}_0 \neq 0$). In fact, we have

$$\omega_0 = \tilde{\omega}_0 + \tilde{\omega}_1 + \tilde{\omega}_2 + \cdots + \tilde{\omega}_N.$$

We also note that $\tilde{\omega}_N = \omega_N$ and $\tilde{\omega}_{N-1} = 0$. Following Lemma 2.3, there exist meromorphic functions g_k satisfying

$$\tilde{\omega}_k = g_k \cdot \omega_N \quad \text{for} \quad k = 0, 2, \ldots, N-2 \tag{12}$$

and integrability conditions now write

$$d\tilde{\omega}_k = (k-1)\tilde{\omega}_1 \wedge \tilde{\omega}_k \quad \text{for} \quad k = 0, 2, \ldots, N-2 \tag{13}$$

and

$$d\omega_N = (N-1)\tilde{\omega}_1 \wedge \omega_N, \quad d\tilde{\omega}_1 = 0. \tag{14}$$

In particular, we see that ω_N is transversely affine and that

$$\omega_0 = \tilde{\omega}_1 + (g_0 + g_2 + \cdots + g_{N-2})\omega_N. \tag{15}$$

Following Lemma 3.10 below, there is a non constant meromorphic function $g \in \mathcal{M}(M)$ such that $dg \wedge \omega_N = 0$. It follows from Stein's Factorization Theorem that there exist:

- a meromorphic map $\phi : M \dashrightarrow C$ onto a smooth, compact, complex and connected curve C,
- a meromorphic function $\underline{g} : C \dashrightarrow \overline{\mathbb{C}}$,

such that $g = \underline{g} \circ \phi$ and the generic fibers $\phi^{-1}(c)$ are irreducible hypersurfaces of M. Let $\underline{\omega}$ be a non zero meromorphic 1-form on C. The 1-form $\omega := \phi^* \underline{\omega}$ on M is closed, non zero and $df \wedge \omega = 0$. Therefore, we can write $\tilde{\omega}_N = h_N \cdot \omega$ for a meromorphic function h and setting $h_k = h_N \cdot g_k$, we get

$$\tilde{\omega}_k = h_k \omega \quad \text{for} \quad k = 0, 2, \ldots, N-2, N. \tag{16}$$

From equations (13), we deduce that

$$\begin{cases} \text{either } h_k = 0, \\ \text{or } \left(\tilde{\omega}_1 - \frac{1}{k-1}\frac{dh_k}{h_k}\right) \wedge \omega = 0 \end{cases} \quad \text{for} \quad k = 0, 2, \ldots, N-2, N. \tag{17}$$

Thus, for any $k, l = 0, 2, \ldots, N-2, N$ such that $h_k, h_l \neq 0$, we have

$$\left(\frac{1}{k-1}\frac{dh_k}{h_k} - \frac{1}{l-1}\frac{dh_l}{h_l}\right) \wedge \omega = 0 \tag{18}$$

and $\frac{h_k^{(l-1)}}{h_l^{(k-1)}}$ is a first integral for ω. Let $r = gcd\{k-1 \;;\; h_k \neq 0\}$: we have $\sum_{h_k \neq 0} n_k(k-1) = r$ for integers n_k. Set

$$h := \prod_{h_k \neq 0} h_k^{n_k}.$$

Therefore, summing equations (17) over l, we get

$$0 = \sum_{h_l \neq 0} n_l \left(\frac{l-1}{r}\frac{dh_k}{h_k} - \frac{k-1}{r}\frac{dh_l}{h_l}\right) \wedge \omega = \left(\frac{dh_k}{h_k} - \frac{k-1}{r}\frac{dh}{h}\right) \wedge \omega.$$

Thus, $\frac{h_k}{h^{\frac{k-1}{r}}}$ is a first integral for ω and we can write

$$\begin{cases}\text{either } h_k = 0, \\ \text{or } h_k = \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}}\end{cases} \quad \text{for} \quad k = 0, 2, \ldots, N-2, N \tag{19}$$

and meromorphic functions $\underline{h}_k : C \dashrightarrow \overline{\mathbb{C}}$. From equation (15), we deduce

$$\omega_0 = \tilde{\omega}_1 + \left(\sum_{k=0,2,\ldots,N} \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}} \right) \omega \tag{20}$$

(setting $\underline{h}_k = 0$ whenever $h_k = 0$). On the other hand, from (17) and (19) we get

$$\tilde{\omega}_1 \wedge \omega = \frac{1}{k-1} \frac{dh_k}{h_k} \wedge \omega = \frac{1}{r} \frac{dh}{h} \wedge \omega \tag{21}$$

and $\tilde{\omega}_1 = \frac{1}{r}\frac{dh}{h} + f\omega$ for a meromorphic function f. Since $\tilde{\omega}_1$ and ω are closed, we get after derivating equation (21) that $df \wedge \omega = 0$, i.e. we can write $f = \underline{f} \circ \phi$ for a meromorphic function $\underline{f} : C \dashrightarrow \overline{\mathbb{C}}$. Finally, we obtain

$$\omega_0 = \frac{1}{r} \frac{dh}{h} + \underline{f} \circ \phi \cdot \omega + \left(\sum_{k=0,2,\ldots,N} \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}} \right) \omega$$

and, setting $\Phi = (\phi, h) : M \dashrightarrow C \times \mathbb{C}P(1)$

$$rh\omega_0 = \Phi^* \left(dz + r(\underline{f} \cdot z + \sum_{k=0,2,\ldots,N} \underline{h}_k z^{\frac{k-1}{r}+1})\omega \right).$$

$\square$

Lemma 3.10. *Let $\mathcal{F}$ be a foliation admitting a finite Godbillon-Vey sequence $(\omega_0, \omega_1, \ldots, \omega_N)$ of length $N \geq 3$ with $\omega_N \neq 0$. Then*

- *either $\omega_N = f dg$ for meromorphic functions $f, g \in \mathcal{M}(M)$,*
- *or $\mathcal{F}$ is transversely affine.*

Proof. We start as in proof of Theorem 3.9, keeping the same notations. Substituting (12) into integrability conditions (13) yield

$$(dg_k + (N-k)g_k\tilde{\omega}_1) \wedge \tilde{\omega}_N = 0 \quad \text{for} \quad k = 0, 2, \ldots, N-2.$$

If there exist two distinct integers $k,l\in\{0,2,\dots,N-2\}$ such that $g_k,g_l\neq 0$, then we can deduce that

$$\left((N-k)\frac{dg_l}{g_l}-(N-l)\frac{dg_k}{g_k}\right)\wedge\tilde{\omega}_N=0\ ;$$

if moreover the left factor is not zero, then we can conclude that

$$dg\wedge\tilde{\omega}_N=0\quad\text{with}\quad g:=\frac{g_l^{(N-k)}}{g_k^{(N-l)}}\quad\text{non constant}$$

i.e. $\omega_N=fdg$ for some meromorphic function f. **Otherwise**, the discussion splits into many cases.

Case 1. Assume that $g_k=0$ for all $k\in\{0,2,\dots,N-2\}$. Then

$$\omega_0=\sum_{k=0}^{N}\tilde{\omega}_k=\tilde{\omega}_1+\tilde{\omega}_N$$

and, since $d\tilde{\omega}_1=0$, we have

$$d\omega_0=d\tilde{\omega}_N=(N-1)\tilde{\omega}_1\wedge\tilde{\omega}_N=(N-1)\tilde{\omega}_1\wedge\omega_0$$

and $\mathcal{F}$ is transversely affine.

Case 2. Assume that $g_k\neq 0$ for at least one $k\in\{0,2,\dots,N-2\}$ but

$$\frac{1}{N-l}\frac{dg_l}{g_l}=\frac{1}{N-k}\frac{dg_k}{g_k}$$

for all $k,l\in\{0,2,\dots,N-2\}$ such that $g_k,g_l\neq 0$: the closed 1-form

$$\beta=\tilde{\omega}_1+\frac{1}{N-k}\frac{dg_k}{g_k}$$

does not depend on k.

Subcase 2.1: $\beta=0$. Since

$$\omega_0=\tilde{\omega}_1+g\cdot\tilde{\omega}_N,\quad g=g_0+g_2+\cdots+g_{N-2}+1$$

we get that either $g=0$ and $\omega_0=\tilde{\omega}_1$ is closed, or $g\neq 0$ and we have

$$d(\frac{\omega_0}{g})=d\tilde{\omega}_N=(N-1)\tilde{\omega}_1\wedge\tilde{\omega}_N=(N-1)\tilde{\omega}_1\wedge\frac{\omega_0}{g};$$

in each case, we see that $\mathcal{F}$ is transversely affine.

Subcase 2.2: $\beta\neq 0$. Therefore, one can write $\tilde{\omega}_N=h\beta$ for some meromorphic function $h\neq 0$ and we have

$$d\tilde{\omega}_N=\frac{dh}{h}\wedge\tilde{\omega}_N.$$

Comparing with $d\tilde{\omega}_N=(N-1)\tilde{\omega}_1\wedge\tilde{\omega}_N$ and $\beta\wedge\tilde{\omega}_N=0$, we get

$$\left(\frac{dh}{h}-\frac{N-1}{N-k}\frac{dg_k}{g_k}\right)\wedge\tilde{\omega}_N=0.$$

Subsubcase 2.2.1: $\frac{N-1}{N-k}\frac{dg_k}{g_k} = \frac{dh}{h}$ for all $k \in \{0, 2, \ldots, N-2\}$ such that $g_k \neq 0$. Then

$$\omega_0 = \tilde{\omega}_1 + gh \cdot \beta, \quad g = g_0 + g_2 + \cdots + g_{N-2} + 1$$

with $dg \wedge dh = 0$. Since $\beta = \tilde{\omega}_1 + \frac{1}{N-1}\frac{dh}{h}$, we get

$$\omega_0 = (1+gh)\tilde{\omega}_1 + \frac{g}{N-1}dh.$$

Either $1+gh = 0$ and ω_0 is closed, or $1+gh \neq 0$ and $\frac{\omega_0}{1+gh}$ is closed; in each case, $\mathcal{F}$ is transversely affine.

Subsubcase 2.2.2: $\frac{N-1}{N-k}\frac{dg_k}{g_k} \neq \frac{dh}{h}$ for at least one k. Therefore, we can conclude that

$$dg \wedge \tilde{\omega}_N = 0 \quad \text{with} \quad g := \frac{h^{(N-k)}}{g_k^{(N-1)}} \quad \text{non constant}$$

i.e. $\omega_N = f dg$ for some meromorphic function f. $\square$

4. Examples

4.1. **Degree** 2 **foliations on** $\mathbb{C}P(n)$ **have length** $\leq$ 3**.** Here, we prove Theorem 1.7. In fact, given a degree 2 foliation $\mathcal{F}$ on $\mathbb{C}P(n)$, we prove that, after a convenient birational transformation

$$\Phi : \mathbb{C}P(n) \dashrightarrow \mathbb{C}P(n-1) \times \mathbb{C}P(1),$$

the tangency locus Δ between the foliation $\mathcal{F}' := \Phi_*\mathcal{F}$ and the projection $\pi : \mathbb{C}P(n-1) \times \mathbb{C}P(1) \to \mathbb{C}P(n-1)$ takes the following special form:

- either Δ is a vertical hypersurface, i.e. defined by $R \circ \pi = 0$ for a non constant rational function R on $\mathbb{C}P(n-1)$,
- or Δ is the union of a vertical hypersurface like above and the horizontal hyperplane at infinity $H_\infty := \mathbb{C}P(n-1) \times \{\infty\}$.

One can easily deduce from this geometric picture that $\mathcal{F}'$ is actually defined by a unique rational integrable 1-form

$$\Omega = dz + \sum_{k=0}^{N} \omega_k z^k$$

where ω_k are rational 1-forms on $\mathbb{C}P(n-1)$ and z is the $\mathbb{C}P(1)$-variable. A Godbillon-Vey sequence of length $\leq N$ is therefore provided by $(L_X^{(k)}\Omega)_k$ where $X = \partial_z$ is the vertical vector field. We will also prove that $N \leq 3$ in our case. In the first case of the alternative above, we have $N \leq 2$: Δ is vertical, $\mathcal{F}'$ is a Riccati foliation with respect to π and is in particular transversely projective. In the second case, $N = 2 + m$ where m is the multiplicity of contact between $\mathcal{F}'$ and the projection

π along the hyperplane at infinity H_∞. Actually, it is better to view Δ as a positive divisor, defined in charts by the holomorphic function $\omega(X)$ where X is a non vanishing holomorphic vector field tangent to the fibration given by π and ω a holomorphic 1-form defining $\mathcal{F}'$ with codimension ≥ 2 zero set. Then, m is the weight of Δ along H_∞.

Let $\mathcal{F}$ be a degree 2 foliation on $\mathbb{C}P(n)$. In order to construct Φ and reach the geometrical picture above, the rought idea is to find a rational pencil on $\mathbb{C}P(n)$ such that the tangency locus Δ between the foliation and the pencil intersects each rational fiber once. In fact, we choose any singular point p of the foliation $\mathcal{F}$ and consider the pencil of lines passing through p. Of course, the number of tangencies between a line and $\mathcal{F}$, counted with multiplicities, is 2, the degree of $\mathcal{F}$; but looking at the pencil passing through p, we expect that the tangency occuring at the singular point disappear after blowing up the point p. Let us compute.

A foliation $\mathcal{F}$ of degree ≤ 2 on $\mathbb{C}P(n)$ is given in an affine chart $\mathbb{C}^n \subset \mathbb{C}P(n)$ by a polynomial 1-form with codimension ≥ 2 zero set having the special form

$$\Omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$$

where ω_i is homogeneous of degree i and ω_3 is radial (see [5]): we have $\omega_3(\mathcal{R}) = 0$, where $\mathcal{R} := x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$ is the radial vector field. Saying that $\mathcal{F}$ is not of degree less than 2 just means that, if ever $\omega_3 = 0$, then ω_2 is not radial. Let us assume $p = 0$ be singular for $\mathcal{F}$, i.e. $\omega_0 = 0$. The tangency locus between $\mathcal{F}$ and the pencil of lines passing through 0 is given by $\mathrm{tang}(\mathcal{F}, \mathcal{R}) = \Omega(\mathcal{R}) = 0$. If $\Omega(\mathcal{R})$ is the zero polynomial, then this means that $\mathcal{F}$ is actually radial; we avoid this by choosing another singular point p. Therefore, $\mathrm{tang}(\mathcal{F}, \mathcal{R})$ is a cubic hypersurface which is singular at p. After blowing-up the origin, the foliation lifts-up in the chart

$$\pi : (t_1, \ldots, t_{n-1}, z) \mapsto (zt_1, \ldots, zt_{n-1}, z) = (x_1, \cdots, x_{n-1}, x_n)$$

just by lifting-up the 1-form Ω which now takes the special form

$$\pi^*\Omega = z\left((f_0(t) + zf_1(t))dz + z\tilde{\omega}_1 + z^2\tilde{\omega}_2 + z^3\tilde{\omega}_3\right)$$

where f_0 and f_1 are polynomial functions of $t = (t_1, \ldots, t_{n-1})$ and $\tilde{\omega}_i$ are polynomial 1-forms depending only on t. We observe that $\mathrm{tang}(\mathcal{F}, \mathcal{R})$ is now defined by $\{z(f_0(t) + zf_1(t)) = 0\}$, has possibly some vertical components given by common factors of f_0 and f_1 and has exactly 2 non vertical components defined by $z = 0$ and $z = -f_0/f_1$ (the two tangencies between any line of the pencil with $\mathcal{F}$). Also, as expected, the first section $z = 0$ is irrelevant since it disappears after division of

$\pi^*\Omega$: the tangency locus between the lifted foliation $\tilde{\mathcal{F}}$ and the lifted pencil (the vertical line bundle $\{t = \text{constant}\}$) actually reduces to $\{f_0(t) + zf_1(t) = 0\}$ in the chart above. We now discuss on this set.

If $f_0 \equiv 0$, then $\frac{\Omega}{z^2 f_1(t)}$ is Riccati with wingular set over $\{f_1(t) = 0\}$: $\mathcal{F}$ has length ≤ 2. Recall that we have supposed $\mathcal{F}$ non radial and thus f_0 and f_1 cannot vanish identically simultaneously.

If $f_0 \not\equiv 0$, then the non vertical component of $\text{tang}(\mathcal{F}, \mathcal{R})$ is the section $z = s(t)$, $s(t) := -\frac{f_0(t)}{f_1(t)}$. If $f_1 \equiv 0$, then this section is the hyperplane at infinity $\{z = \infty\}$: $\frac{\Omega}{f_0(t)}$ is already in the expected geometrical normal form and has length ≤ 3. If $f_1 \not\equiv 0$, it suffices to push it towards infinity by a meromorphic change of coordinate of the form $\tilde{z} := \frac{z}{z - s(t)}$; after this birational transformation, we are in the previous case $y_1 \equiv 0$ and we have done. Precisely, the foliation is defined by

$$d\tilde{z} - \tilde{z}\frac{\tilde{\omega}_1}{f_0} + \tilde{z}^2(\frac{2\tilde{\omega}_1}{f_0} - \frac{df_0}{f_0 f_1} + \frac{df_1}{f_1^2} - \frac{\tilde{\omega}_2}{f_1}) - \tilde{z}^3(\frac{\tilde{\omega}_1}{f_0} - \frac{\tilde{\omega}_2}{f_1} + \frac{f_0\tilde{\omega}_3}{f_1^2}).$$

In order to finish the proof of Theorem 1.7, we note that a generic degree 2 foliation of $\mathbb{C}P(2)$ has length 3, i.e. is not transversely projective. Actually, this is a well known fact. For instance, it immediately follows from Corollary 2.15 and the fact that a generic degree $d \geq 2$ foliation on $\mathbb{C}P(2)$ has no invariant algebraic curve. An explicit example is given in Section 4.2.

Remark 4.1. If $\mathcal{F}$ is a foliation of $\mathbb{C}P(2)$ given by a 1-form of the type $\omega = \omega_\nu + \omega_{\nu+1} + f_{\nu+1}(xdy - ydx)$ then, for generic ω as above, $\text{Tang}(\mathcal{F}, \mathcal{E})$ is a rational curve and an argument similar to the one used above implies that $\mathcal{F}$ also satisfies $l(\mathcal{F}) \leq 3$.

4.2. **The examples of Jouanolou.** In [11], Jouanolou exhibited the first examples of holomorphic foliations of the projective plane without algebraic invariant curves. His examples, one for each degree greater than or equal to 2, are the foliations of $\mathbb{C}P(2)$ induced by the homogeneous 1-forms in $\mathbb{C}^3$

$$\Omega_d = \det \begin{pmatrix} dx & dy & dz \\ x & y & z \\ y^d & z^d & x^d \end{pmatrix} .$$

The automorphism group of the foliation $\mathcal{J}_d$, induced by Ω_d, is isomorphic to a semi-direct product of $\mathbb{Z}/(d^2+d+1)\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$ and is generated by the transformations $\psi_d(x:y:z) = (\delta^{d^2}x : \delta^d y : \delta z)$ and $\rho(x:y:z) = (y:z:x)$, where δ is a primitive $(d^2+d+1)^{\text{th}}$ root of the unity.

In [14] it is observed that the foliations $\mathcal{J}_d$ can be presented in a different way. If $\mathcal{F}_d$ is the degree 2 foliation of $\mathbb{C}P(2)$ induced by the 1-form

$$\omega_d = \det \begin{pmatrix} dx & dy & dz \\ x & y & z \\ x(-x+dy) & y(-y+dz) & z(-z+dx) \end{pmatrix} ,$$

and $\phi_d : \mathbb{C}P(2) \dashrightarrow \mathbb{C}P(2)$ is the rational map(of degree d^2+d+1) given by

$$\phi_d(x:y:z) = (y^{d+1} \cdot z : z^{d+1} \cdot x : x^{d+1} \cdot y)$$

then the foliation $\mathcal{J}_d$ is the pull-back of the foliation $\mathcal{F}_d$ under ϕ_d, i.e., $\mathcal{J}_d = \phi_d^* \mathcal{F}_d$. Conversely we can say that $\mathcal{F}_d$ is birationally equivalent to the quotient of $\mathcal{J}_d$ by the group generated by ψ_d.

From the results of the previous section it follows that $\mathcal{F}_d$ has length at most 3. Pulling-back a Godbillon-Vey sequence by ϕ_d we obtain that the length of $\mathcal{J}_d$ is also bounded by 3 and since it does not admit invariant algebraic curves its length is precisely 3. We have therefore proved the

Corollary 4.2. *The foliations $\mathcal{J}_d$, for every $d \geq 2$, have length 3.*

4.3. **A new component of the space of foliations on $\mathbb{C}P(3)$.** We start by considering the transversely projective foliation on $\mathbb{C}P(2)$ given in the affine chart $\{(x,y)\} = \mathbb{C}^2 \subset \mathbb{C}P(2)$ by the 1-form

$$\omega = xdy - ydx + P_2dx + Q_2dy + R_2(xdy - ydx)\,.$$

where P_2, Q_2, R_2 are generic homogeneous polynomials of degree 2. This is a degree 2 foliation of $\mathbb{C}P(2)$ transverse to the Hopf fibration $x/y = \mathit{const}$ outside three distinct lines. Let us consider the homogenization Ω_3 of ω in the coordinates (x,y,z) of $\mathbb{C}^3$:

$$\Omega_3 = z^2(xdy - ydx) + z(P_2dx + Q_2dy) + R_2(xdy - ydx) - R_3dz\,,$$

where $R_3(x,y) = xP_2 + yQ_2$. The genericity condition on P_2, Q_2, R_2 implies that $d\Omega_3$ has only one zero on $\mathbb{C}^3$ which is isolated and located at the origin. Of course, Ω_3 defines a transversely projective foliation of $\mathbb{C}^3 \subset \mathbb{C}P(3)$. We will twist this foliation by a polynomial automorphism of $\mathbb{C}^3$. More precisely, if $\sigma(x,y,z) = (x,y,z+x^2)$ then

$$\Omega := \sigma^*\Omega_3 = \Omega_3 + \Omega_4 + \Omega_5 \quad \text{with}$$

$$\begin{cases} \Omega_3 &= z^2(xdy - ydx) + z(P_2dx + Q_2dy) + R_2(xdy - ydx) - R_3dz \\ \Omega_4 &= 2zx^2(xdy - ydx) + x^2(P_2dx + Q_2dy) - 2xR_3dx \\ \Omega_5 &= x^4(xdy - ydx) \end{cases}$$

The 1-form Ω defines a degree 4 foliation on $\mathbb{C}P(3)$ which is transverse to the Hopf fibration(induced by the Euler vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$) outside the union of the four hyperplanes $\Omega_4(E) = x^2R_3(x,y) = 0$. If P_2, Q_2, R_2 are generic, then these four hyperplanes are distinct.

Let $\mathcal{F}'$ be a foliation of degree 4 close to $\mathcal{F}_\Omega$: $\mathcal{F}'$ is given in the affine chart $\mathbb{C}^3$ by a polynomial 1-form

$$\Omega' = \Omega'_0 + \Omega'_1 + \Omega'_2 + \Omega'_3 + \Omega'_4 + \Omega'_5\,,$$

where the Ω'_k are homogeneous of degree k and $\Omega'_5(E) \equiv 0$.

After normalization, we can suppose that the coefficients of Ω' are close to those of Ω. Since $d\Omega_3$ has an isolated singularity at 0, there exists (see [2]) a point $0'$ where the 2-jet of Ω' is zero, and the Euler vector field centered at $0'$ is in the kernel of the 3-jet. Therefore, after translating $0'$ to 0, we can suppose that $\mathcal{F}'$ is given by

$$\Omega' = \Omega'_3 + \Omega'_4 + \Omega'_5\,.$$

We verify that Ω' is transversely projective (with poles contained in $\Omega'_4(E)$). In fact, since $\mathcal{F}$ is not transversely affine, the same holds for $\mathcal{F}'$. Therefore every element $\mathcal{F}'$ of the component of $\mathcal{F}(3,4)$ containing $\mathcal{F}$ is actually transversely projective.

4.4. Transversely projective foliations that are not pull-back.

Example 4.3 (Example 8.6 of [9])**.** Let Γ be discrete torsion free subgroup of $PSL(2,\mathbb{R})^n$ such that the quotient $PSL(2,\mathbb{R})^n/\Gamma$ is compact. For $n \geq 2$, there exists examples such that the projection $\pi(\Gamma)$ on the first factor is a dense subgroup of $PSL(2,\mathbb{R})$ (see [1]). The action of Γ on $\mathbb{H}^n$, the n product of the upper half-plane, is free, cocompact and preserves the regular foliation induced by the projection on the first factor. In this way, we obtain a regular transversely projective foliation $\mathcal{F}$ on a n-dimensional compact complex manifold M such that every leaf is dense and the generic leaf is biholomorphic to $\mathbb{H}^{n-1}$. Observe that $\mathcal{F}$ is not the pull-back of a foliation on a lower dimensional manifold, otherwise there would exist compact subvarieties in $\mathbb{H}^{n-1}$.

Example 4.4 (Hilbert Modular Foliations)**.** Let K be a totally real number field of degree $n \geq 2$ over the rational numbers $\mathbb{Q}$ and let $\mathcal{O}_K$ be the ring of integers of K. The group $\Gamma = PSL(2,\mathcal{O}_K)$ is dense in $PSL(2,\mathbb{R})$, but considering the n embeddings $i \circ \sigma : K \hookrightarrow \mathbb{R}$ given by the action $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we get an embedding $\Gamma \hookrightarrow PSL(2,\mathbb{R})^n$ as a discrete subgroup of the product. The quotient of $\mathbb{H}^n$, the n-product of the upper-half plane $\mathbb{H}$, by Γ is a quasiprojective variety V which can be singular due to torsion elements of Γ. One can compactify and desingularize V and obtain a projective manifold M. The n fibrations on $\mathbb{H}^n$ given by the projections on each of the factors induce n foliations on M which are regular and pair-wise transversal outside the invariant hypersurfaces coming from the compactification and desingularization of V. By construction, they are transversely projective and all leaves apart from the invariant hypersurface above are dense in M. In [21] and [15], some basic properties of these foliations are described.

When $K = \mathbb{Q}(\sqrt{5})$, the resulting variety is birationally equivalent to the projective plane. In [15] explicit equations for the foliations associated to the two projections $\mathbb{H}^2 \to \mathbb{H}$, denoted by $\mathcal{F}_2$ and $\mathcal{F}_3$, are determined. We give below an explicit projective triple for them. The corresponding suspensions $\mathcal{H}_2$ and $\mathcal{H}_3$ defined by

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

can be seen as singular foliations on $\mathbb{C}P(2) \times \mathbb{C}P(1)$ or equivalently on $\mathbb{C}P(3)$. Although the leaves of $\mathcal{F}_2$ are dense, we note that the same is not true for $\mathcal{H}_2$ since the monodromy lie in $PSL(2,\mathbb{R})$.

A projective triple for $\mathcal{F}_2$

$$
\begin{aligned}
\omega_0 &= \frac{(80\,y-60\,x\,y-80\,x^2)dx+(36\,x^2-y-32\,x)dy}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}\\
\omega_1 &= \frac{16}{3}\,\frac{24\,y+3\,y^2+62\,x\,y-24\,x^2-108\,x^2\,y-368\,x^3+432\,x^4}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dx\\
&- \frac{4}{15}\,\frac{150\,y+3\,y^2+192\,x-172\,x\,y-872\,x^2+720\,x^3}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dy\\
\omega_2 &= \frac{32}{45}\,\frac{144\,y+84\,y^2+3\,y^3+852\,x\,y-48\,x\,y^2-144\,x^2-1472\,x^2\,y-36\,x^2\,y^2-4416\,x^3+336\,x^3\,y+4928\,x^4}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2)}dx\\
&- \frac{32}{225}\,\frac{441\,y+288\,x-372\,x\,y+12\,x\,y^2-2292\,x^2-56\,x^2\,y+1472\,x^3-108\,x^3\,y+720\,x^4}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dy
\end{aligned}
$$

A projective triple for $\mathcal{F}_3$

$$
\begin{aligned}
\omega_0 &= \frac{(-\frac{5}{4}\,y^2+20\,x\,y-60\,x^3)dx+(-y+\frac{3}{4}\,x\,y+x^2)dy}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}\\
\omega_1 &= 4\,\frac{-80\,x^2\,y+3\,y^2-208\,x^3+288\,x^4+48\,x\,y}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dx\\
&- \frac{2}{5}\,\frac{40\,y+y^2-168\,x^2+192\,x^3-50\,x\,y}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dy\\
\omega_2 &= \frac{32}{5}\,\frac{-9\,y^2+80\,x\,y+8\,x\,y^2-16\,x^2\,y-368\,x^3-48\,x^3\,y+320\,x^4}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dx\\
&+ \frac{32}{25}\,\frac{-36\,y+63\,x\,y+164\,x^2-28\,x^2\,y-304\,x^3+144\,x^4}{-720\,x^3\,y+1728\,x^5+80\,x\,y^2-y^3+640\,x^2\,y-1600\,x^4-64\,y^2}dy
\end{aligned}
$$

Theorem 4.5. *The explicit suspensions $\mathcal{H}_2$ and $\mathcal{H}_3$ above are not the meromorphic pull-back of a foliation on a surface.*

Proof. Suppose that there exists a foliation $\underline{\mathcal{H}}_2$ on a surface S and a meromorphic map $\Phi : \mathbb{C}P(2) \times \mathbb{C}P(1) \dashrightarrow S$ such that $\Phi^* \underline{\mathcal{H}}_2 = \mathcal{H}_2$.

Let $U \subset \mathbb{C}P(2) \times \mathbb{C}P(1)$ be the Zariski open subset where Φ is holomorphic and $U_0 = U \cap (\mathbb{C}P(2) \times \{0\})$. After blowing-up S, one can assume $\Phi(U_0)$ having codimension ≤ 1. The generic rank of Φ restricted to $U_0 = U \cap (\mathbb{C}P(2) \times \{0\})$ is 2, otherwise we are in one of the following contradicting situations

(1) The closure of $\Phi(U_0)$ is a proper submanifold of S non-invariant by $\underline{\mathcal{H}}_2$. In particular $\mathcal{F}_2$ is the pull-back of a foliation of a foliation on a curve and is transversely euclidean; contradiction.
(2) The closure of $\Phi(M_0)$ is a proper submanifold of S invariant by $\underline{\mathcal{H}}_2$ (and not contained in the singular set of $\underline{\mathcal{H}}_2$). Reasoning in local coordinates at the neighborhood of a generic point $p \in \Phi(U_0)$, we see that $\mathbb{C}P(2) \times \{0\}$ is invariant by $\mathcal{H}_2$ obtaining a contradiction.

We conclude therefore that $\Phi|_{U_0}$ is dominant and $\underline{\mathcal{H}}_2 = \Phi_* \mathcal{F}_2$ has dense leaves (in fact all but finitely many). Therefore, the same density property holds for the pull-back $\mathcal{H}_2 = \Phi^* \underline{\mathcal{H}}_2$ providing a contradiction: the Riccati foliation $\mathcal{H}_2$ has no dense leaf since its monodromy is contained in $PSL(2, \mathbb{R})$. This proves the Theorem. □

5. Integrable 1-forms in Positive Characteristic

Due to the algebraic nature of many of the arguments used through this paper it is natural to ask if it would be possible carry on a similar study for integrable 1-forms on varieties defined over fields of positive characteristic.

The surprising fact, at least for us, is that over fields of positive characteristic every 1-form admits a Godbillon-Vey sequence of length one. In the case of 1-forms on the projective plane this is already implicitly proved in [17].

Our argument is based on the following

Lemma 5.1. *Let M be a m-dimensional smooth projective variety defined over an arbitrary field. If ω is an integrable rational 1-form then there exists $m-1$ rationally independent vector fields $X_1, \ldots, X_{m-1}$ such that*

(1) $[X_i, X_j] = 0$ *for every* $i, j \in 1, \ldots, m-1$*;*
(2) $\omega(X_i) = 0$ *for every* $i \in 1, \ldots, m-1$.

Proof. Let $f_1, \dots, f_{m-1} \in k(M)$ be rational functions such that

$$\omega \wedge df_1 \wedge \cdots \wedge df_{m-1} \neq 0\,.$$

If $\omega_m = \omega$ and $\omega_i = df_i$, for $i = 1 \dots m-1$ then $\{\omega_i\}_{i=1}^m$ form a basis of the $k(M)$-vector space of rational 1-forms over M.

Let $\{X_i\}_{i=1}^m$ be a basis of the space of rational vector fields on M dual to $\{\omega_i\}_{i=1}^m$, i.e., $\omega_i(X_j) = \delta_{ij}$. It is clear that $\omega(X_i) = 0$ for every $i = 1 \dots m-1$. We claim that $[X_i, X_j] = 0$ for every $i, j = 1 \dots m-1$. It is sufficient to show that

$$\omega_k([X_i, X_j]) = 0 \text{ for every } k = 1 \dots m \tag{22}$$

For $k = m$ the integrability of ω implies that (22) holds. For $k < m$ we have that

$$\begin{aligned} \omega_k([X_i, X_j]) &= X_i(\omega_k(X_j)) - X_j(\omega_k(X_i)) + d\omega_k(X_i, X_j) = \\ &= X_i(\delta_{kj}) - X_j(\delta_{ki}) + d^2 f_k(X_i, X_j) = 0\,. \end{aligned}$$

This shows that (22) holds for every $k = 1 \dots m$ and concludes the proof of the lemma. □

Theorem 5.2. *Let M be a smooth projective variety defined over a field K of characteristic $p > 0$ and ω be a rational 1-form. If ω is integrable then ω admits an "integrating factor", i.e., there exist a rational function $F \in K(M)$ such that $F\omega$ is closed. Equivalently we have that*

$$d\omega = \omega \wedge \frac{dF}{F}\,.$$

Proof. Let m be the dimension of M and $X_1, \dots, X_{m-1}$ be the rational vector fields given by lemma 5.1. We will distinguish two cases:

(1) for every $i = 1 \dots m-1$ we have that $\omega(X_i^p) = 0$
(2) there exists $i \in \{1, \dots, m-1\}$ such that $\omega(X_i^p) \neq 0$

Let $\mathcal{F}$ be the unique saturated subsheaf of the tangent sheaf of M which coincides with the kernel of ω over the generic point of M. The integrability of ω implies that $\mathcal{F}$ is involutive. If we are in the case (1) then we have also that $\mathcal{F}$ is p-closed. From [16, propositions 1.7 and 1.9, p. 55–56] it follows that $\omega = g df$ where $g, f \in k(M)$.

In case (2) we can suppose that $\omega(X_1^p) \neq 0$. If $F = \omega(X_1^p)^{-1}$ then

$$d(F\omega) = F\omega \wedge L_{X_1^p}(F\omega)\,.$$

To conclude we have just to prove that $L_{X_1^p}(F\omega) = 0$. In fact since $F\omega(X_1^p) = 1$ it follows that

$$L_{X_1^p}(F\omega) = i_{X_1^p} d(F\omega)\,.$$

Moreover for every $i = 1 \dots m-1$ we have that $[X_1^p, X_i] = 0$, since X_1 commutes with X_i, and therefore

$$i_{X_1^p} d(F\omega)(X_i) = F\omega([X_1^p, X_i]) - X_1^p(F\omega(X_i)) + X_i(F\omega(X_1^p)) = 0\,.$$

This is sufficient to show that $L_{X_1^p}(F\omega) = 0$ concluding the proof of the Theorem. $\square$

As a corollary we obtain a codimension one version of the main result of [17].

Corollary 5.3. *Let ω be a polynomial integrable 1-form on $\mathbb{A}_k^n$, where k is a field of positive characteristic. If $d\omega \neq 0$ then there exists an irreducible algebraic hypersurface H such that $i^*\omega = 0$, where $i : H \to \mathbb{A}_k^n$ denotes the inclusion.*

Proof. Of course ω can be interpreted as rational 1-form over $\mathbb{P}_k^n$ which is regular over $\mathbb{A}_k^n$. From Theorem 5.2 there exists a rational function $F \in k(x_1, \dots, x_n)$ such that

$$d\omega = \omega \wedge \frac{dF}{F}\,.$$

Since $d\omega \neq= 0$ we have that $dF \neq 0$, i.e., F is not a p-th power. In particular the polar set of dF/F is not empty. It is an easy exercise to show that every irreducible component H of the polar set of dF/F satisfies $i^*\omega = 0$, where $i : H \to \mathbb{A}_k^n$ denotes the inclusion $\square$

In fact the same proof as above yields the stronger

Corollary 5.4. *Let ω be a regular integrable 1-form over a smooth quasiprojective algebraic variety M defined over k, a field of positive characteristic. Suppose that $H^0(M, \mathcal{O}_M^*) = k^*$. If $d\omega \neq 0$ then there exists an irreducible algebraic hypersurface H such that $i^*\omega = 0$, where $i : H \to M$ denotes the inclusion.*

Observe that the result above can be applied to projective varieties since there exists such varieties with global regular 1-forms which are not closed, see [16].

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