Topological Classification of Multiple Saddle Connections

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Abstract

In this paper we deal with real analytic vector fields in an ambient space of dimension three. The existence of *connections of hyperbolic saddles* along the skeleton of the exceptional divisor is one of the major problems in order to get topological equivalences by means of a desingularization morphism. We give a complete topological classification of such multiple saddle connections under the assumption that the graph of connections has no cycles. For a given divisor and skeleton, the classifying space has a geometrical description, is finite and depends only on the distribution of the eigenvalues. We use in an essential way the control of the *inverse propagation* of a homeomorphism defined in a transversal disc of the one dimensional invariant variety to get a whole topological equivalence between two saddles.

1 Introduction

We give a complete topological classification for multiple saddle-connections along a normal crossings divisor, without cycles, in an ambient space of dimension three. We classify these dynamical systems, up to topological equivalence, by a finite set obtained from the eigenvalues in each stationary point. More precisely, we prove the following result

[*]Partially supported by the postdoctoral spanish program (EX-2003-0009).
[†]Partially supported by CNPq-Brasil
[‡]Partially supported by IMPA and spanish research project BFM 2001-2010.

Theorem 1 *Let M be an orientable real analytic variety of dimension three and $D = \bigcup_{i \in I} D_i$ a normal crossings divisor whose skeleton has no cycles. Let ξ, ξ' be two analytic vector fields on M of the type* multiple saddle-connections *along D. The following statements are equivalent:*

A. There is a topological equivalence $h : U \to U'$ between ξ and ξ' that respects the strata of D, defined in two neighborhoods U and U' of the compact part of the skeleton.

B. The eigenvalue functions λ_ξ and $\lambda_{\xi'}$ are equivalent.

A *normal crossings divisor* D is the union of a finite set of non-singular hypersurfaces D_i that intersect locally like the coordinate hyperplanes. It is typically obtained as the exceptional divisor of a procedure of reduction of singularities (see [10, 11, 12, 15, 22]). We say that the vector field ξ is of the type *multiple saddle-connections* along D if each component D_i is invariant, the stationary points of ξ are the triple points (corners) of D and they are of hyperbolic saddle type.

In dimension two, the vector fields whose reduction of singularities [22] is of hyperbolic type (without saddle-nodes) have been introduced in [7]. The multiple saddle-connections naturally appear for singular vector fields X at the origin of $\mathbb{R}^3$, having a desingularization (see [10]) without semi-hyperbolic points. For the topological study of such vector fields, we consider their behavior in the interior of the components of the exceptional divisor (see, for instance [6, 8]) and the situation along the intersections of two components (the *skeleton*). In some cases, we can "destroy" the saddle-connections in the interior of the divisors to get Morse-Smale conditions, by doing small perturbations of X not affecting the reduction of the singularities (see [9, 19]). Nevertheless, the saddle-connections along the skeleton are "rigid" in that sense, this forces us to do a study as in theorem 1.

Note that we can avoid the singularities of attractor type, since by "cutting" a small sphere around, any topological equivalence in the frontier of the sphere will be easily propagated to the interior. Thus, the acyclicity hypothesis refers not to the whole skeleton of the exceptional divisor, but just to the part corresponding to the connections between the saddle-type singularities. The case of a cycle of saddle-connections is not solved and the phenomena occurring for this situation is of a nature very different from the technics we use in this paper, in fact we should need to look at the transition functions along the cycle (see for instance [13, 14, 17, 21]).

The topological study of saddle-connections in dimension two has had an important contribution of J. Palis [18]. In the case of dimension three, the paper [5] considers a single saddle-connection, with a point of view not related with the reduction of singularities. The multiple saddle-connections for special types of skeletons have been classified in [1, 2].

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Let us give the structure of the paper. In sections 2 and 3 we introduce the combinatorial language and the definition of equivalent eigenvalues functions.

We use in an essential way the work by *basic zones*, where a basic zone is the closure of a connected component of the complement of the divisor. Around a corner point, we get eight basic zones and around an edge we have four ones. Next, we introduce the *path* of a corner point P in a basic zone Z. It is a path of edges in Z that depends only on the eigenvalues at the singular points (*eigenvalue function*). It starts with the Z-*mast* of P, the edge of the skeleton in Z contained in the invariant variety of dimension one. We give a weight α_P to the mast that is the quotient of the two eigenvalues of same sign. To pass from a β-weighted edge Γ to the next one Γ' through the corner point Q, we look at the flow-saturation of the curves $y = x^\beta$, traced in a transversal disc to Γ, by a *linear* vector field with the prescribed eigenvalues at Q. We get $y = x^{\beta'}$ in a transversal disc to Γ' and the procedure continues. It stops either at an open edge, or at a point R such that $\beta = \alpha_R$, in this case the path is reversible from R.

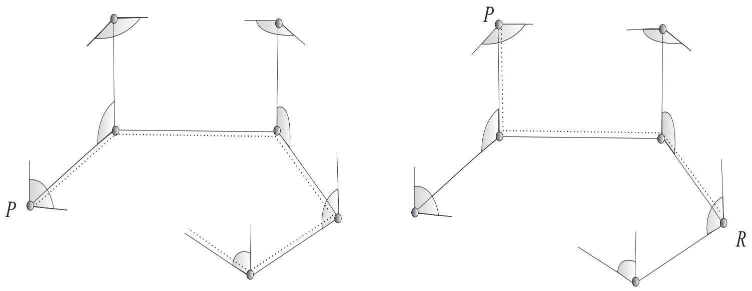

Figure 1

Two eigenvalues functions are *equivalent* if and only if they produce the same paths for any corner point and any basic zone. Since the set of possible collections of paths is finite, there are only finitely many equivalence classes.

A key technical part of the paper is the study of the admissible foliations introduced in [1, 3] and the transition of β-leaves through the corners, done in sections 5, 6 and 7. A homeomorphism $h : \mathbb{D}_{++} \to \mathbb{D}_{++}$ in the standard quadrant is said to *respect* α, α'*-admissible foliations* if and only if it lifts after doing the α, α'-weighted blowing-ups π_α, $\pi_{\alpha'}$ of polar coordinates. This kind of homeomorphisms are the only ones that extend to a whole topological equivalence around a pair of saddles in an inverse way: from the invariant variety of dimension one to the invariant variety of dimension two.

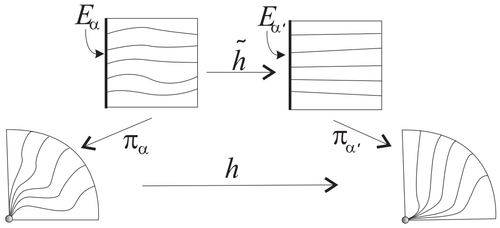

Figure 2

The β-leaves are the topologically horizontal leaves after the β-weighted blowing-up. We study in section 7 the transition of these leaves along a weighted path of edges. For this, we do a monoidal weighted blowing-up and we look to the corresponding lifted dynamical system. Having in mind these results (theorems 3 and 4), the first consequence (corollary 2) is that the path of a point P has a dynamical-topological characterization. The path is the intersection between the skeleton and the topological closure of the flow-saturation of a curve γ (*test-curve*) transversal to the two-dimensional variety of P. With this result we get immediately the part "A implies B" of theorem 1.

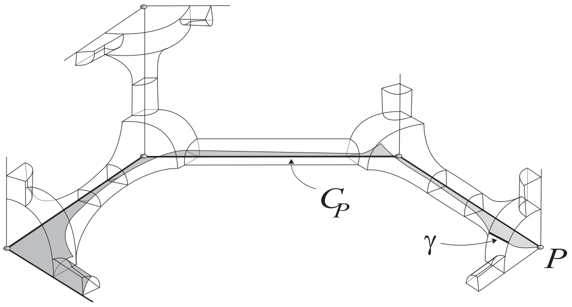

Figure 3

The proof of "B implies A" is the hardest part of the paper. We proceed by basic zones and by induction on the number of corner points.

First, we need to assure that the procedure glues at the intersections of a basic zone and a component of the divisor (*the walls*). In sections 8 and 9 we show how to codify a topological equivalence given "a priori" in a wall. We do it by means of *edge-seeds* and *point-seeds* satisfying a *glue-condition* along the wall determined by the flow. The edge-seeds are homeomorphisms defined between two transversal segments to ξ and ξ' contained in the wall, cutting the edge in an intermediary point. The point-seeds are homeomorphisms between two quart-circles around each *outer-point* of the wall (the corner points whose invariant variety of dimension two coincides with the wall).

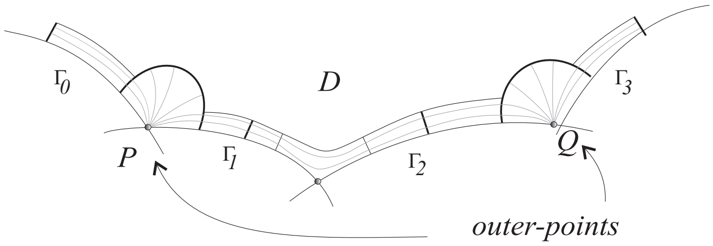

Figure 4

The assignation of point-seeds cannot be done independently for each wall. The first example is the case of a pair of single saddle-connections with two corner points P, Q having the same weights. The flow-saturation determines a necessary relation between the circles around P and around Q. This is a particular case of the structural identifications introduced in section 7. In section 8 we state this *global-compatibility condition* in a general way. It puts in relation the point-seeds in each pair of points that have the same path.

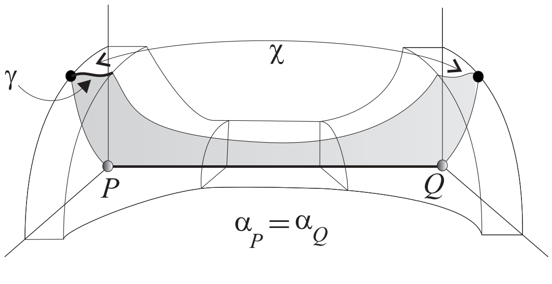

Figure 5

In theorem 5 of section 9 we prove the existence of such compatible data of seeds. This allows us to start our work by basic zones with an initial topological equivalence in each wall.

The point-seeds have an abstract version, that we call *abstract-seeds.* They are triples $\varsigma = (\beta, \beta', \sigma)$ with two weights $\beta, \beta' > 0$ and a homeomorphism $\sigma : E_\beta \to E_{\beta'}$ between the exceptional divisors of the weighted blowing-ups $\pi_\beta, \pi_{\beta'}$. Any homeomorphism that respects β, β'- admissible foliations induces an abstract-seed, by restriction of the lifted homeomorphism to the exceptional divisors. Theorem 2 is a key in order to start the construction of a topological equivalence. It states that we can find a homeomorphism respecting a finite list of abstract seeds, if the weights are ordered in such a way that $\beta_i < \beta_{i+1}$ and $\beta'_i < \beta'_{i+1}$.

Once we fix transversal discs and weights β, β' in an edge Γ, an abstract-seed can be attached to the weighted edge. The natural path of a corner point P, produces a pair of weights in each of the edges. In view of the structural identifications described in corollaries 3 and 4, a point-seed at P gives, by transition, abstract-seeds at each edge of the path. Any topological equivalence respecting the point-seed should also "respect" all the abstract-seeds along the path.

We enlarge the above construction by introducing in section 8 systems of abstract-seeds associated to double-weighted paths (exteriors systems of abstract-seeds). This will be necessary in our induction procedure by excision of a point. We divide the original basic zone in small pieces, and we work by induction in each one, but in a global coherent way. For this, we need to keep in mind the data of point-seeds in the exterior parts, when the path passes through the piece we are considering.

The construction that we do will be supported by a frame of cylinders and tubes that we introduce in section 4. In particular our equivalence neighborhoods, introduced in the section 11, will be a union of "chimneys" and tubes well connected, obtained by flow-saturation inside the frame of cylinders and tubes. We will also get refinement properties from the *windows* (transversal discs to the open edges) that we need in order to adjust the topological equivalences obtained by induction.

In section 12 we establish the induction procedure, by doing a statement (theorem 6) stronger than theorem 1. To prove how theorem 6 implies theorem 1, we need to give a starting homeomorphism at a window respecting the abstract-seeds induced by all the point-seeds in that window. This will be assured by corollary 5 that states that the abstract-seeds are ordered in a convenient way to apply theorem 2. We give also in this section 12 the proof of the induction step, given by a succession of refinements of the normalized neighborhoods.

Finally we concentrate the difficulty in the case of a single corner with an enlarged structure of abstract seeds, that come from the exterior part. This is done in section 13. We have two cases whether or not the starting window is in the mast. In the first case, the abstract-seeds in the mast represent all the abstract seeds and it is enough to describe the inverse propagation of the homeomorphism in the window to the interior of the chimney (proposition 16). In the second case, we extend the starting homeomorphism to the whole fence of the chimney, respecting the abstract seeds in the second window. In this way we get represented all the abstract seeds and we propagate, in a direct way, the homeomorphism in the fence to the chimney.

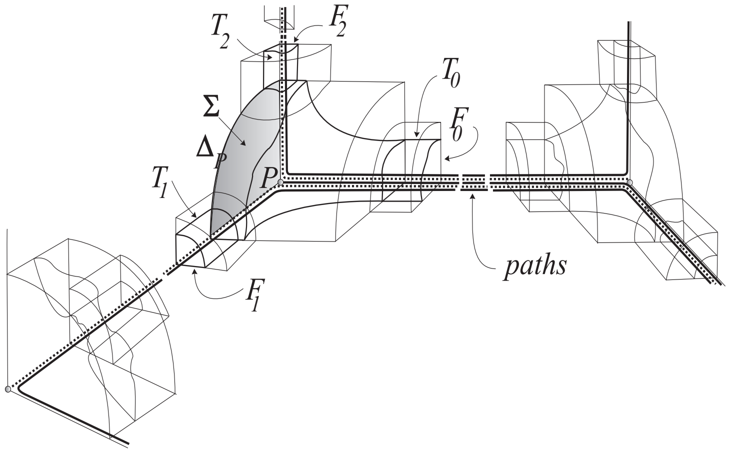

Figure 6

Acknowledgements: The first and third authors thank IMPA for its hospitality and the warm scientific ambient they found there during the preparation of this paper. We are grateful to J. Ribón for useful remarks.

2 Combinatorial Settings

The *ambient space* M is an orientable real analytic variety of dimension three. We fix a *normal crossings divisor* $D = \bigcup_{i\in I} D_i$ on M. We also fix a total ordering $\leq$ on the set of indices I. The *skeleton* $\mathcal{K}$ is defined by

$$\mathcal{K} = \bigcup_{i,j\in I,\, i\neq j} D_i \cap D_j.$$

The set S of *corner points* is the set of triple intersections of components of D. We assume $S \neq \emptyset$ and the

Acyclicity Hypothesis: The skeleton is simply connected.

The *edges* of the skeleton are the topological closures of the connected components of $\mathcal{K} - S$. There are two types: *open edges* homeomorphic to $[0, +\infty) \subset \mathbb{R}$, that have exactly one end-point $P \in \Gamma \cap S$, and *compact edges* homeomorphic to $[0, 1] \subset \mathbb{R}$, that have two end-points $P, Q \in \Gamma \cap S$. We denote by $\mathcal{J}_a$ the set of open edges, by $\mathcal{J}_c$ the set of compact edges and we put $\mathcal{J} = \mathcal{J}_a \cup \mathcal{J}_c$. We define $\mathbb{K} = \bigcup \{\Gamma;\ \Gamma \in \mathcal{J}_c\}$ to be the *compact part* of the skeleton.

Basic zones, walls Up to take M small enough around $\mathbb{K}$ we have the following *separating property* on the ambient pair (M, D):

> For any $\tilde{I} \subset I$, any $j \in I - \tilde{I}$ and any connected component U of $M - \bigcup_{i \in \tilde{I}} D_i$ then either $D_j \cap U = \emptyset$ or D_j divides U exactly in two connected components having $D_j \cap U$ as a common frontier.

A *basic zone* Z of (M, D) is the topological closure of a connected component of $M - D$. It is determined by the set $\mathcal{J}_Z = \{\Gamma \in \mathcal{J};\ \Gamma \subset Z\}$. Two edges having a common end point and contained in the same pair of divisors are called *opposite edges*. An *angle* $q = (\Gamma_1, P, \Gamma_2)$ is given by two edges Γ_1 and Γ_2 with a common end-point $P = \Gamma_1 \cap \Gamma_2$, that are not contained in the same pair of irreducible components of D. In particular the angle q determines a unique irreducible component D_q such that $\Gamma_j \subset D_q$ for $j = 1, 2$. The following remarks are straightforward:

- There are *eight* basic zones containing one corner point $P \in S$.
- There are *four* basic zones containing one edge $\Gamma \in \mathcal{J}$.
- There are *two* basic zones containing an angle $q = (\Gamma_1, P, \Gamma_2)$.

Let Z be a basic zone and $q = (\Gamma_1, P, \Gamma_2)$ an angle of Z. Then $W = Z \cap D_q$ is called a *wall* of Z. The frontier of Z is the union of the walls. A wall is a common wall exactly for two basic zones. We can order the set $\mathcal{J}_W = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_k\}$ of edges contained in W in such a way that $\Gamma_0, \Gamma_k \in \mathcal{J}_a$, $\Gamma_j \in \mathcal{J}_c$, for $0 < j < k$ and we have the angles $q_j = (\Gamma_{j-1}, P_j, \Gamma_j)$ where P_j are the corner points in the wall, for $j = 1, 2, \ldots, k$.

Respecting the strata A homeomorphism $h : M \to M$ *respects the strata of* D if $h(Z) = Z$ for any basic zone Z. This implies that $h(D_i) = D_i$ for any index $i \in I$, $h(\Gamma) = \Gamma$ for any edge Γ and $h(P) = P$ for any corner point P.

3 Eigenvalue Functions and Paths

Given a corner point $P \in S$, denote by $\mathcal{D}_P$ the set of non-ordered pairs $\{i,j\}$ such that $P \in D_i \cap D_j$. Then $\mathcal{D}_P$ has three elements $\{i,j\},\{j,k\},\{k,i\}$. An *eigenvalue function* is a function

$$\lambda : \bigcup_{P\in S} \{P\} \times \mathcal{D}_P \to \mathbb{R}-\{0\}; \quad (P,\{i,j\}) \mapsto \lambda^P_{ij},$$

such that the values $\lambda^P_{ij}, \lambda^P_{jk}, \lambda^P_{ki}$ have not the same sign. We define the triple $I^\lambda_P = (i,j,k)$ to be such that λ^P_{ij} has opposite sign to λ^P_{jk}, λ^P_{ki} and $i<j$ in the fixed ordering of the set of indices I. The λ-*weight* α^λ_P of P is

$$\alpha^\lambda_P = \lambda^P_{ik}/\lambda^P_{jk} \in \mathbb{R}_{>0}.$$

Given a basic zone Z and $P \in S_Z = S \cap Z$, there is one edge $\Gamma \in \mathcal{J}_Z$ having P as end-point such that $\Gamma \subset D_i \cap D_j$. It is called the (λ, Z)-*mast* of P, we say just Z-*mast* if there are no confusion with the chosen eigenvalue function. Note that a corner point P has exactly two (opposite) masts, each one is related with four of the eight basic zones around P.

Transitions of the weights Fix an eigenvalue function λ. The weight α^λ_Q is naturally associated to the mast of Q. More generally, we will associate weights to the edges and define, in a purely combinatorial way, transitions for the weights through the angles. Consider an edge Γ_1, an end-point P of Γ_1 and a *weight* $\beta_1 > 0$ attached to Γ_1. Assume that either Γ_1 is not a mast of P or $\beta_1 \neq \alpha^\lambda_P$. Let us define how the weight β_1 produces a weight $\beta_2 > 0$ in two opposite edges Γ_2 and Γ'_2 that form angles $q = (\Gamma_1, P, \Gamma_2)$ and $q' = (\Gamma_1, P, \Gamma'_2)$.

We use as a model the linear vector field $\sum_{i=1}^3 \lambda_i x_i \partial/\partial x_i$, having at the coordinate axes the prescribed eigenvalues given by λ. Next, we consider the flow-saturation A of a curve of the type $y = x^{\beta_1}, z = 1$, where $z=1$ is supposed to be transversal to Γ_1. The axis Γ_2 should be contained in the topological closure of A and cutting A by a transversal disc to Γ_2, we get a curve of the form $y' = x'^{\beta_2}$, where $\beta_2 > 0$ is the weight we will attach to Γ_2. Let us do it in a precise way as follows.

Given a corner point P with $I^\lambda_P = (i,j,k)$ and a weight $\beta > 0$ let us define $\varrho_+(\beta) = -\lambda^P_{jk}(\beta - \alpha^\lambda_P)/\lambda^P_{ij}$ and $\varrho_-(\beta) = -\lambda^P_{jk}(\alpha^\lambda_P - \beta)/(\lambda^P_{ij}\beta)$.

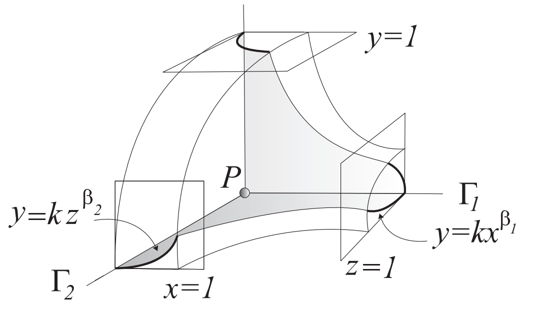

Figure 7

Definition 1 *Let* $q = (\Gamma_1, P, \Gamma_2)$ *be an angle and* $\beta_1 > 0$. *Assume that either* Γ_1 *is not a mast of* P *or* $\beta_1 \neq \alpha_P^\lambda$. *We say that* β_1 passes throughout q to get $\beta_2 > 0$ *and we denote* $\beta_1 \overset{\lambda,q}{\rightsquigarrow} \beta_2$ *iff one of the following situations holds:*

1. *If* $\Gamma_1 \subset D_i \cap D_j$, $\beta_1 > \alpha_P^\lambda$, *then* $\Gamma_2 \subset D_j \cap D_k$. *Moreover, if* $j > k$ *we put* $\beta_2 = \varrho_+(\beta_1)$ *and if* $j < k$ *we put* $\beta_2 = 1/\varrho_+(\beta_1)$.
2. *If* $\Gamma_1 \subset D_i \cap D_j$, $\beta_1 < \alpha_P^\lambda$, *then* $\Gamma_2 \subset D_i \cap D_k$. *Moreover, if* $i > k$ *we put* $\beta_2 = \varrho_-(\beta_1)$ *and if* $i < k$ *we put* $\beta_2 = 1/\varrho_-(\beta_1)$.
3. *If* $\Gamma_1 \subset D_k \cap D_j$, *then* $\Gamma_2 \subset D_i \cap D_j$. *Moreover, if* $j > k$ *we take* β_2 *such that* $\beta_1 = \varrho_-(\beta_2)$, *if* $j < k$ *we take* $\beta_1 = 1/\varrho_-(\beta_2)$.
4. *If* $\Gamma_1 \subset D_k \cap D_i$, *then* $\Gamma_2 \subset D_i \cap D_j$. *Moreover, if* $i > k$ *we take* β_2 *such that* $\beta_1 = \varrho_+(\beta_2)$, *if* $i < k$ *we take* $\beta_1 = 1/\varrho_+(\beta_2)$.

That is, if Γ_1 is a mast for P and $\beta_1 = \alpha_P^\lambda$, there is no transition of the weight. Otherwise, there is a transition in two angles $q = (\Gamma_1, P, \Gamma_2)$ and $q' = (\Gamma_1, P, \Gamma_2')$ where Γ_2 and Γ_2' are opposite edges (getting the same weight β_2). If Γ_1 is not a mast of P, we say that the transition is *direct* and if it is a mast (and $\beta_1 \neq \alpha_P^\lambda$) we have an *inverse transition*. Note the reversibility of the definition, in the sense that

$$\beta_1 \overset{\lambda,q}{\rightsquigarrow} \beta_2 \Longleftrightarrow \beta_2 \overset{\lambda,q^{-1}}{\rightsquigarrow} \beta_1.$$

where $q^{-1} = (\Gamma_2, P, \Gamma_1)$ if $q = (\Gamma_1, P, \Gamma_2)$.

Given the angle $q = (\Gamma_1, P, \Gamma_2))$ let us denote ϱ_q the function defined by

$$\beta_1 \overset{\lambda,q}{\rightsquigarrow} \beta_2 \Rightarrow \beta_2 = \varrho_q(\beta_1).$$

Then ϱ_q is one of the functions ϱ_+, ϱ_-, $1/\varrho_+$, $1/\varrho_-$, or one their inverse functions. We know that ϱ_q is monotonous decreasing or monotonous increasing and this property depends only on $\mathcal{I}_P = (i, j, k)$; that is, it depends only on the position of the mast of P, for the basic zones containing the angle q. Define the *sign coefficient* $\epsilon_q \in \{1, -1\}$ by the fact that $\epsilon_q \varrho_q$ is an increasing function. Then ϵ_q does not depend on the particular eigenvalue function, but just on the position of the masts.

Paths Fix a basic zone Z and an eigenvalue function λ. A (λ, Z)*-path* is any pair (C, β), where $C \subset \mathcal{J}_Z$ is a set of edges with connected support $|C|$ (the union of the edges) and $\beta : C \to \mathbb{R}_{>0}$ is a *weight-function* with the following properties. For any pair $\Gamma, \Gamma' \in C$ with $P = \Gamma \cap \Gamma'$, the weight $\beta(\Gamma)$ passes throughout the angle $q = (\Gamma, P, \Gamma')$ to get $\beta(\Gamma')$. Moreover (C, β) is *maximal* in the sense that there is no $(\tilde{C}, \tilde{\beta})$ with the above property such that $C \subset \tilde{C}$, $C \neq \tilde{C}$ and $\tilde{\beta}|_C = \beta$.

In view of the rules of transition of weights throughout the angles, given $\Gamma \in \mathcal{J}_Z$ and $\nu > 0$, there is exactly one (λ, Z)-path (C, β) such that $\Gamma \in C$ and $\beta(\Gamma) = \nu$. We denote it by $(C^{\lambda,Z}_{\nu,\Gamma}, \beta^{\lambda,Z}_{\nu,\Gamma})$. For any $P \in S_Z$ we also denote $(C^{\lambda,Z}_P, \beta^{\lambda,Z}_P)$ the (λ, Z)-path that contains the Z-mast Γ of P with weight α^{λ}_P.

Given a (λ, Z)-path (C, β), let us order (in one of the two possible ways) the set of edges $C = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_k\}$ in such a way that $q_j = (\Gamma_{j-1}, P_j, \Gamma_j)$ are the angles in the path, for $j = 1, 2, \ldots, k$. We have two possibilities:

1. *Open-path:* The two extremes Γ_0 and Γ_k are open edges and $k \geq 1$. Then (C, β) is not the path of a point.

2. *Point-path:* Either $k = 0$ or one of the extremes Γ_0 or Γ_k (or both) is a compact edge. Then, the support $|C|$ has one or two end-points and $(C, \beta) = (C^{\lambda,Z}_P, \beta^{\lambda,Z}_P)$ for any of the end-points P.

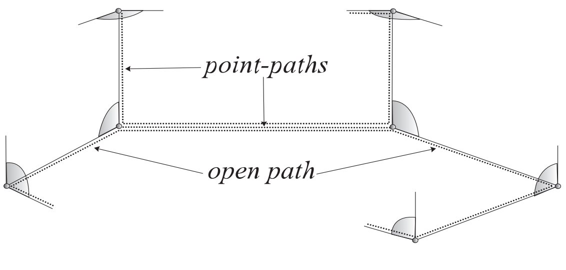

Figure 8

Denote by $\mathcal{P}_a^{\lambda,Z}$ the set of open paths and by $\mathcal{P}_c^{\lambda,Z}$ the set of point-paths. Put $\mathcal{P}_a^{\lambda} = \cup_Z \mathcal{P}_a^{\lambda,Z}$, $\mathcal{P}_c^{\lambda} = \cup_Z \mathcal{P}_c^{\lambda,Z}$ and $\mathcal{P}^{\lambda} = \mathcal{P}_a^{\lambda} \cup \mathcal{P}_c^{\lambda}$.

Remark 1 Consider two (λ, Z)-paths (C, β) and (C, β^*) having the same set of edges C. If they are point-paths, we have $(C, \beta) = (C, \beta^*)$ and hence the support $|C|$ determines the path. If we have two different open-paths, the weights $\beta(\Gamma)$ and $\beta^*(\Gamma)$ at all the edges $\Gamma \in C$ are different (the weight at an edge determines the path).

Equivalent eigenvalue functions Let $\mathcal{C}$ be the set of pairs $c = (P, Z)$ where Z is a basic zone and $P \in S_Z$. Put $P(c) = P$ and $Z(c) = Z$. We denote the (λ, Z)-path of P by $(C_c^{\lambda}, \beta_c^{\lambda})$ and we put C_c and β_c if there is no confusion about λ.

Definition 2 *Let λ, λ' be two eigenvalue functions. We say that λ and λ' are equivalent and denote it by $\lambda \equiv \lambda'$ iff $C_c^{\lambda} = C_c^{\lambda'}$ for any $c \in \mathcal{C}$.*

Remark 2 Since the set of possible supports of paths is finite, there is only finitely many possible equivalence classes for the eigenvalue functions λ.

Remark 3 If $\lambda \equiv \lambda'$, the sign coefficients are equal $\epsilon_q = \epsilon'_q$ for any angle q. This follows from the fact that the masts for λ and for λ' are the same ones.

Consider a pair of corner points $P \neq Q$ in a basic zone Z. Put $c = (P, Z)$ and $d = (Q, Z)$. We say that P and Q are (λ, Z)-*related* if and only if $(C_c, \beta_c) = (C_d, \beta_d)$; we also say that c, d are λ-*related*. A necessary and sufficient condition for this is that $C_c = C_d$, in view of the remark 1. We denote by $\mathcal{C}^{*\lambda}$ the quotient set of $\mathcal{C}$ that identifies λ-related pairs. Moreover, if $\lambda \equiv \lambda'$, then P and Q are (λ, Z)-related if and only if they are (λ', Z)-related; in particular $\mathcal{C}^{*\lambda} = \mathcal{C}^{*\lambda'}$ and we denote $\mathcal{C}^*$ if no confusion arises.

Open-paths assignations and weight-functions on the edges An eigenvalue function determines the point-paths. For induction reasons, we shall need to enlarge the point-paths by considering a finite set of additional open-paths. This will also generate a family of weights in each edge $\Gamma \in \mathcal{J}$.

Let $\mathcal{B}$ be a finite set whose elements are of the form $b = (j, Z)$, where $j \in \mathbb{Z}_{>0}$ and Z is a basic zone. We denote $j = j(b)$, and $Z = Z(b)$. An *open-path assignation* with *set of indices* $\mathcal{B}$ and relatively to λ is any injective map $\rho : \mathcal{B} \to \mathcal{P}_a^{\lambda}$ such that $\rho(b) \in \mathcal{P}_a^{\lambda, Z(b)}$ for any $b \in \mathcal{B}$. We denote also

$\rho(b) = (C_b^\lambda, \beta_b^\lambda) = (C_b, \beta_b)$ if there is no confusion about ρ or about λ. Let us remark that ρ extends in a natural way to an *injective* map

$$\tilde{\rho} : \mathcal{A}^\lambda \to \mathcal{P}^\lambda; \quad a \mapsto (C_a, \beta_a)$$

where $\mathcal{A}^\lambda = \mathcal{C}^{*\lambda} \cup \mathcal{B}$.

Definition 3 *Let $\lambda \equiv \lambda'$ be two equivalent eigenvalue functions. Consider two open-path assignations ρ, ρ' with same set of indices $\mathcal{B}$ relative to λ and λ' respectively. We say that ρ and ρ' are* equivalent *if the following holds:*

1. *For any $b \in \mathcal{B}$ we have $C_b^\lambda = C_b^{\lambda'}$.*
2. *For any pair $b_1, b_2 \in \mathcal{B}$ such that $C = C_{b_i} = C'_{b_i}$, $i = 1, 2$, and for any edge $\Gamma \in C$ then: $\beta_{b_1}(\Gamma) < \beta_{b_2}(\Gamma) \Leftrightarrow \beta'_{b_1}(\Gamma) < \beta'_{b_2}(\Gamma)$.*

We denote this situation by $(\lambda, \rho) \equiv (\lambda', \rho')$.

Remark 4 In order to verify the property 2) of the above definition it is enough to test just one edge $\Gamma \in C$. This follows from the fact that

$$\begin{array}{rcl}\beta_{b_1}(\Gamma) < \beta_{b_2}(\Gamma) & \Leftrightarrow & \epsilon_q \beta_{b_1}(\Gamma') < \epsilon_q \beta_{b_2}(\Gamma') \\ \beta'_{b_1}(\Gamma) < \beta'_{b_2}(\Gamma) & \Leftrightarrow & \epsilon_q \beta'_{b_1}(\Gamma') < \epsilon_q \beta'_{b_2}(\Gamma')\end{array}$$

for any angle $q = (\Gamma, P, \Gamma')$ in C.

Assume that $(\lambda, \rho) \equiv (\lambda', \rho')$. Take a basic zone Z and an edge $\Gamma \in \mathcal{J}_Z$. Put $\mathcal{A} = \mathcal{A}^\lambda = \mathcal{A}^{\lambda'}$ and consider the set $A_\Gamma^Z = \{a \in \mathcal{A};\ \Gamma \in C_a,\ Z = Z(a)\}$. We define the *weight functions* $\beta_\Gamma^Z, {\beta'}_\Gamma^Z : A_\Gamma^Z \to \mathbb{R}_{>0}$ by $\beta_\Gamma^Z(a) = \beta_a^\lambda(\Gamma)$ and ${\beta'}_\Gamma^Z(a) = \beta_a^{\lambda'}(\Gamma)$.

Proposition 1 *We have that $\beta_\Gamma^Z(a) < \beta_\Gamma^Z(b) \Leftrightarrow {\beta'}_\Gamma^Z(a) < {\beta'}_\Gamma^Z(b)$ and that $\beta_\Gamma^Z(a) = \beta_\Gamma^Z(b) \Leftrightarrow {\beta'}_\Gamma^Z(a) = {\beta'}_\Gamma^Z(b)$, for any $a, b \in A_\Gamma^Z$.*

Proof: Put $\beta_\Gamma = \beta_\Gamma^Z$, in order to simplify the notation. If $\beta_\Gamma(a) = \beta_\Gamma(b)$ we have that $a = b$ and hence ${\beta'}_\Gamma(a) = {\beta'}_\Gamma(b)$. Assume that $\beta_\Gamma(a) < \beta_\Gamma(b)$. Then we have either $C_a = C_b$ or $C_a \neq C_b$. In the first case $a, b \in \mathcal{B}$ and we get that ${\beta'}_\Gamma(a) < {\beta'}_\Gamma(b)$ by definition. Assume that $C_a \neq C_b$. In view of the increasing monotony of $\epsilon_q \varrho_q$ and $\epsilon_q \varrho'_q$, there are two possibilities

a) For any $\Gamma' \in C_a \cap C_b$ we have $(\beta_{\Gamma'}(a) - \beta_{\Gamma'}(b))\,(\beta'_{\Gamma'}(a) - \beta'_{\Gamma'}(b)) > 0$.

b) For any $\Gamma' \in C_a \cap C_b$ we have $(\beta_{\Gamma'}(a) - \beta_{\Gamma'}(b))\,(\beta'_{\Gamma'}(a) - \beta'_{\Gamma'}(b)) < 0$.

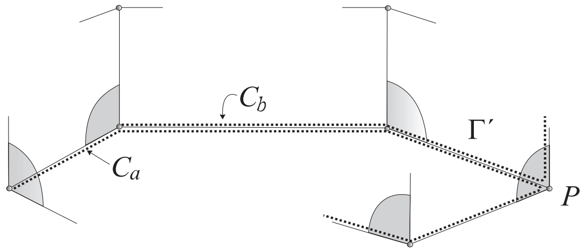

Figure 9

It is enough to show that property a) holds. Consider an end-edge $\Gamma' \in C_a \cap C_b$ where the paths are separated. Thus Γ' is the mast of a point P. Moreover, if we assume, without loss of generality, that $\beta_{\Gamma'}(a) > \beta_{\Gamma'}(b)$, then one of the following situations holds

$$1)\ \beta_a(\Gamma') < \alpha_P^\lambda < \beta_b(\Gamma');\ 2)\ \beta_a(\Gamma') = \alpha_P^\lambda < \beta_b(\Gamma');\ 3)\ \beta_a(\Gamma') < \alpha_P^\lambda = \beta_b(\Gamma').$$

The case 1) is equivalent to the fact that we have two different angles $q_1 = (\Gamma; P; \Gamma_1)$ and $q_2 = (\Gamma; P; \Gamma_2)$ in Z such that

$$\beta_a(\Gamma) \overset{\lambda;q_1}{\rightsquigarrow} \beta_a(\Gamma_1); \quad \beta_b(\Gamma) \overset{\lambda;q_2}{\rightsquigarrow} \beta_b(\Gamma_2).$$

where Γ_1 is the edge "over" Γ and Γ_2 is the edge "under" Γ. Since $C_a = C'_a$ and $C_b = C'_b$, we necessarily get the same edge transitions for ${\beta'}_a(\Gamma)$ and $\beta'_b(\Gamma)$ relatively to λ'. This implies that $\beta'_a(\Gamma) < \alpha_P^{\lambda'} < \beta'_b(\Gamma)$ as desired. We do the same arguments in the cases 2) and 3): one of the weights has no transition an the other one has a transition to the same edge for λ and λ'. $\square$

We shall refer to the property of the above proposition by saying that the functions β_Γ^Z and ${\beta'}_\Gamma^Z$ are *homologous*.

4 Systems of cylinders and tubes

Let ξ be a real analytic vector field defined on M of the type *multiple saddle-connections* along D (see the introduction). It induces in an evident way an eigenvalue function $\lambda = \lambda_\xi$. In this section we introduce a frame of cylinders and tubes jointly with local coordinates that will support our constructions.

A *cylinder* (Λ_P, ω_P) for ξ at a corner point $P \in S$ is a closed neighborhood Λ_P of P with $\Lambda_P \cap S = \{P\}$, jointly with a coordinate system

$$\omega_P = (x, y, z) : \Lambda_P \to \mathbb{D} \times [-1, 1] \subset \mathbb{R}^3, \quad \mathbb{D} = \{(x, y);\ x^2 + y^2 \leq 1\},$$

such that $D_i \cap \Lambda_P = \{x = 0\}$, $D_j \cap \Lambda_P = \{y = 0\}$, $D_k \cap \Lambda_P = \{z = 0\}$, where $I_P^\lambda = (i, j, k)$. Moreover, we require the *transversality properties*

$$\begin{array}{rll} d(x^2 + y^2)(\xi) \neq 0 & \text{outside} & x = y = 0 \\ dz^2(\xi) \neq 0 & \text{outside} & z = 0. \end{array}$$

The *basis* $\Delta_P = \Lambda_P \cap D_k$ is isomorphic with $\mathbb{D}$, via the coordinates. We denote by $\delta\Delta_P$ the frontier of Δ_P, that is isomorphic to $\mathbb{S}^1$.

A *system of cylinders* $\Lambda = \{(\Lambda_P, \omega_P)\}_{P \in S}$ is a collection of cylinders with empty pairwise intersection.

Proposition 2 *Let ξ and ξ' be two vector fields of the type multiple saddle-connections along D. Assume that $\lambda_\xi \equiv \lambda_{\xi'}$. Then there is a common system of cylinders Λ for ξ and ξ'.*

Proof: By definition of normal crossings divisor, for any corner point P there is a coordinate neighborhood $(U, \tilde{\omega} = (x, y, z))$ with $U \cap D = \{xyz = 0\}$. Put $I_P^\lambda = I_P^{\lambda'} = (i, j, k)$. Up to re-ordering the coordinates, we have that

$$D_i \cap U = \{x = 0\},\ D_j \cap U = \{y = 0\},\ D_k \cap U = \{z = 0\}.$$

Taking U_P small enough we get the transversality conditions for ξ, since its linear part is diagonal with the given coordinates and the signs of the eigenvalues give $z = 0$ as the two dimensional invariant variety and $x = y = 0$ as the one dimensional invariant variety. Do the same thing with ξ' and take a common coordinate neighborhood U. Do it for all the points in such a way that the neighborhoods have empty pairwise intersection. Doing once more U small to get Λ_P, we have that $\tilde{\omega}(\Lambda_P)$ is of the cylindric form $\mathbb{D}_\epsilon \times [-\epsilon, \epsilon]$. Now, we get ω_P by multiplying the cylinder by the factor $1/\epsilon$. $\square$

Tubes Let us fix a system of cylinders Λ for ξ. Take a compact edge $\Gamma \in \mathcal{J}_c$ with end-points P, Q and suppose that $\Gamma \subset D_i \cap D_j$, with $i < j$. A (Λ, ξ)-*tube* for Γ is a pair (T_Γ, τ_Γ) where T_Γ is a compact set without common interior points with $\Lambda_P \cup \Lambda_Q$, and τ_Γ is an analytic isomorphism

$$\tau_\Gamma : T_\Gamma \to \mathbb{D} \times [-1, 1] \subset \mathbb{R}^3$$

that gives a topological equivalence between ξ and $\partial/\partial z$ (not necessarily preserving the orientation), such that $T_\Gamma \cap D_i = \{x = 0\}$, $T_\Gamma \cap D_j = \{y = 0\}$ and finally

$$T_\Gamma \cap \Lambda_P = (\tau_\Gamma)^{-1}(\mathbb{D} \times \{-1\}); \quad T_\Gamma \cap \Lambda_Q = (\tau_\Gamma)^{-1}(\mathbb{D} \times \{1\}).$$

For an open edge $\Gamma \in \mathcal{J}_a$ having end-point P, we define a (Λ, ξ)-*semi-tube* for Γ as a pair (T_Γ, τ_Γ) where T_Γ is a compact set, without common interior points with Λ_P and τ_Γ is an analytic isomorphism $\tau_\Gamma : T_\Gamma \to \mathbb{D} \times [0, 1]$ that gives a topological equivalence between ξ and $\partial/\partial z$, such that $T_\Gamma \cap D_i = \{x = 0\}$, $T_\Gamma \cap D_j = \{y = 0\}$ and $T_\Gamma \cap \Lambda_P = (\tau_\Gamma)^{-1}(\mathbb{D} \times \{1\})$.

We define the *central disc* Ω_Γ by $\Omega_\Gamma = (\tau_\Gamma)^{-1}(\mathbb{D} \times \{0\}) = \{z = 0\}$, both for tubes and semi-tubes.

Let Γ be a compact edge with end-points P, Q. The tube T_Γ can be considered as the union of two semi-tubes T_Γ^P and T_Γ^Q in an evident way. In the case that Γ is an open edge with end-point P, we also denote $T_\Gamma = T_\Gamma^P$.

Lemma 1 *Given a system of cylinders Λ for ξ, there is a system of tubes $T = \{T_\Gamma\}_{\Gamma \in \mathcal{J}}$ for ξ adapted to Λ.*

Proof: Take a small enough transversal section Ω_Γ at a point of Γ not in the cylinders, such that it is isomorphic to a disc. Now we use the flow to get the tube by flow-saturation and also to get the chart. □

The following result has a standard proof and we do not do it:

Proposition 3 *For a system (Λ, T) of tubes and cylinders for ξ, the set*

$$V_{\Lambda,T} = \bigcup_{P \in S} \Lambda_p \cup \bigcup_{\Gamma \in \mathcal{J}} T_\Gamma$$

is a compact neighborhood of the compact part $\mathbb{K}$ of the skeleton. Moreover, the sets $V_{\Lambda,T}$ form a fundamental system of neighborhoods of $\mathbb{K}$.

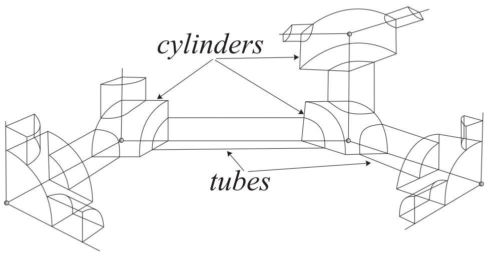

Figure 10

Given a subset $U \subset M$ and $A \subset U$ we denote by $\mathrm{Sat}_U^\xi(A)$ the saturation of A by the flow of $\xi|_U$, that is, the union of the ξ-trajectories in U of the points of A. In particular, if $A = \{Q\}$ is a single point, then $\mathrm{Sat}_U^\xi(\{Q\})$ is the ξ-trajectory of Q in U. The particular properties of the tubes and cylinders, jointly with the acyclicity hypothesis, allow us to state the following result

Proposition 4 *Let (Λ, T) be a system of tubes and cylinders for ξ. Then for any point $Q \in V_{\Lambda,T}$ the intersection of $\mathrm{Sat}_{V_{\Lambda,T}}^\xi(\{Q\})$ with each semi-tube or cylinder has at most one connected component.*

Corollary 1 *For any union U of semi-tubes and cylinders and any subset $A \subset U$, we have that $\mathrm{Sat}_U^\xi(A) = U \cap \mathrm{Sat}_{V_{\Lambda,T}}^\xi(A)$.*

Normalized quadrants Consider a basic zone Z. Given a cylinder Λ_P, there is a sign-change isomorphism ϵ given by $(x, y, z) \mapsto (\pm x, \pm y, \pm z)$, such that $\omega_P^Z = \epsilon \circ \omega_P$ sends $\Lambda_P \cap Z = \Lambda_{P,Z}$ to the first octant $x, y, z \geq 0$. Analogously, given a tube T_Γ, there is a sign-change $\tilde{\epsilon}$ such that

$$\tau_\Gamma^Z = \tilde{\epsilon} \circ \tau_\Gamma : T_{\Gamma,Z} = T_\Gamma \cap Z \to \mathbb{D}_{++} \times [-1, 1],$$

where $\mathbb{D}_{++} = \mathbb{D} \cap \{x, y \geq 0\}$. Note that if $\Omega_{\Gamma,Z} = \Omega_\Gamma \cap Z$, this isomorphism restricts to $\tau_\Gamma^Z : \Omega_{\Gamma,Z} \to \mathbb{D}_{++}$.

5 Admissible foliations

Given $\alpha > 0$, the *α-weighted blowing-up* $\pi_\alpha : \mathbb{I}_{s,\theta}^2 = [0,1] \times [0,1] \to \mathbb{D}_{++}$ is the map defined by $\pi_\alpha(\theta, s) = (s \cos \pi\theta/2, s^\alpha \sin \pi\theta/2)$ if $\alpha \geq 1$ and by $\pi_\alpha(\theta, s) = (s^{\frac{1}{\alpha}} \cos \frac{\pi}{2}\theta, s \sin \frac{\pi}{2}\theta)$ if $0 < \alpha < 1$. We say that $E_\alpha = \pi_\alpha^{-1}(0) = \{s = 0\}$ is the *exceptional divisor* of π_α.

Definition 4 *A singular foliation $\mathcal{F}$ on $\mathbb{D}_{++}$ is* α-admissible *if it is of the form $\mathcal{F} = (\pi_\alpha)_*(\phi_*\{dv = 0\})$ for a homeomorphism $\phi : \mathbb{I}_{u,v}^2 \to \mathbb{I}_{s,\theta}^2$ that respects the square sides. We denote $\mathcal{F} = \mathcal{F}_{\alpha,\phi}$.*

The *α-standard* foliation $\mathcal{F}_\alpha$ defined by $\omega_\alpha = xdy - \alpha y dx$ is α-admissible. Its leaves are given by the equations $y = Kx^\alpha$, $K > 0$.

We need to enlarge the definition of the admissible foliations to more general domains than $\mathbb{D}_{++}$, that we call *abstract-window quadrants*. They are compact subsets $F \subset \mathbb{D}_{++}$ that are homeomorphic to $\mathbb{D}_{++}$ by a homeomorphism $h : \mathbb{D}_{++} \to F \subset \mathbb{D}_{++}$ that respects the origin and sends axes into axes. Given an abstract-window quadrant $F \subset \mathbb{D}_{++}$ a foliation $\mathcal{F}$ over F is called *α-admissible* if and only if there is a smaller abstract-window quadrant $\tilde{F} \subset F$ such that $\mathcal{F}|_{\tilde{F}}$ extends to an α-admissible foliation of $\mathbb{D}_{++}$.

The next results in this section are essentially contained in [1]:

Proposition 5 *Consider two abstract-window quadrants F and F' and two positive weights $\alpha, \alpha' > 0$. The following statements are equivalent for any homeomorphism $h : F \to F'$ that respects the axes:*

1. *There is an α-admissible foliation $\mathcal{F}$ over F such that the image $h_*\mathcal{F}$ is α'-admissible.*
2. *The image $h_*\mathcal{F}$ is α'-admissible, for any α-admissible foliation $\mathcal{F}$ over F.*
3. *The homeomorphism h lifts to $\widetilde{h}_{\alpha,\alpha'} : \pi_\alpha^{-1}(F) \to \pi_{\alpha'}^{-1}(F')$.*

Proof: 1) $\Rightarrow$ 3). Let us define the lifting $\widetilde{h}_{\alpha,\alpha'}$ on the exceptional divisors E_α and $E_{\alpha'}$ as follows. Given $(0, v) \in E_\alpha$, let L_v be the leaf of $\mathcal{F}$ such that $(0, v)$ is in the adherence of $\pi_\alpha^{-1}(L_v)$. We put $\widetilde{h}_{\alpha,\alpha'}(0, v) = (0, v')$ the only point of $E_{\alpha'}$ in the adherence of $h(L_v)$. The continuity of $\widetilde{h}_{\alpha,\alpha'}$ follows straightforward and the reversibility of the construction shows that it is a homeomorphism.

3) $\Rightarrow$ 2). If $\mathcal{F} = \mathcal{F}_{\alpha,\phi}$, then $h_*(\mathcal{F}) = \mathcal{F}_{\alpha',\phi'}$, where $\phi' = \widetilde{h}_{\alpha,\alpha'} \circ \phi$. □

We denote by $\mathrm{Adm}_{F,F'}(\alpha_i, \alpha'_i)$ the set of homeomorphisms satisfying the above equivalent properties. In the case $F = F' = \mathbb{D}_{++}$, we write just $\mathrm{Adm}(\alpha_i, \alpha'_i)$.

Proposition 6 *Let $\phi : \mathbb{D}_{++} \to \mathbb{D}_{++}$ be an analytic monomorphism that respects the axes and $F \subset \mathbb{D}_{++}$ an abstract-window quadrant. Then $\phi(F) \subset \mathbb{D}_{++}$ is also an abstract-window quadrant. Moreover, a foliation $\mathcal{F}$ over F is α-admissible if and only if the image $\phi_*(\mathcal{F})$ is α-admissible over $\phi(F)$.*

Proof: The first part is evident. In view of the previous proposition 5, it is enough tho show that the image of the standard foliation $\mathcal{F}_\alpha$ is α-admissible. Since $\mathcal{F}_\alpha$ is given by the differential form $xdy - y\alpha dx = 0$, the image is also given by a differential form with same linear part and hence it induces an α-admissible foliation. □

An *abstract-seed* ς of weights α, $\alpha' > 0$ is a triple $\varsigma = (\alpha, \alpha', \sigma)$ where $\sigma : [0,1] \to [0,1]$ is a homeomorphism that respects $0, 1$. Analogously we define *abstract-degenerate seeds* of weight 0, respectively ∞ as pairs $\varsigma = (0, \sigma)$, respectively $\varsigma = (\infty, \sigma)$. Next theorem is one of the keys for our arguments:

Theorem 2 *Let* $\{\varsigma_i = (\alpha_i, {\alpha'}_i, \sigma_i)\}_{i=1}^n$ *be a finite set of abstract seeds, such that* $\alpha_i < \alpha_{i+1}$ *and* ${\alpha'}_i < {\alpha'}_{i+1}$ *for* $i = 1, \ldots, n-1$. *Consider two degenerate abstract-seeds* $\varsigma_0 = (0, \sigma_0)$ *and* $\varsigma_\infty = (\infty, \sigma_\infty)$. *Let* F *and* F' *be two abstract-window quadrants of axes ending in* $(a, 0), (0, b)$ *and* $(a', 0), (0, b')$ *respectively. Assume that* $\sigma_0(b) = b'$ *and* $\sigma_\infty(a) = a'$. *Then, there exists* $h \in Adm_{F,F'}(\alpha_i, {\alpha'}_i)$, *for* $i = 1, \ldots, n$, *such that*

1. $h\,|_{\{0\}\times[0,b]} = \sigma_0|_{[0,b]}$, $h\,|_{[0,a]\times\{0\}} = \sigma_\infty|_{[0,a]}$.
2. $\widetilde{h}_{\alpha_i {\alpha'}_i}\,|_{E_{\alpha_i}} = \sigma_i$, $i = 1, ..., n$.

We identify $[0,1]$ *with* $\{0\} \times [0,1]$, $[0,1] \times \{0\}$ *and* E_α *in a standard way.*

Proof: (See also [1]). Take a small enough $\epsilon > 0$ such that $\mathbb{D}^{2\epsilon}_{++} \subset F, F'$, where 2ϵ is the radius of the quadrant and let us work with the quadrants $\mathbb{D}^{\epsilon}_{++} \subset F, F'$. In fact, up to the isomorphism $\Phi : \mathbb{D}^{\epsilon}_{++} \to \mathbb{D}_{++}$, $\Phi(x) = x/\epsilon$, we shall work on $\mathbb{D}_{++}$. Take two lists of positive weights $\{\beta_j\}_{j=1}^{3n}$ and $\{{\beta'}_j\}_{j=1}^{3n}$ such that $\beta_j < \beta_{j+1}$, ${\beta'}_j < {\beta'}_{j+1}$, for $j = 1, 2, \ldots, 3n-1$ and

$$\beta_{3i-1} = \alpha_i; \quad {\beta'}_{3i-1} = \alpha'_i; \quad i = 1, 2, \ldots, n.$$

Let us construct two lists of points $P_j = (x_j, y_j)$ and $P'_j = (x'_j, y'_j)$ in $\mathbb{S}^1_{++} = \mathbb{S}^1 \cap \{x, y > 0\}$, for $j = 1, 2, \ldots, 3n$, satisfying that $x_j < x_{j+1}$, $x'_j < x'_{j+1}$ for $j = 1, 2, \ldots, 3n-1$ and that

$$\pi^{-1}_{\alpha_i}(P_j) = (1, v_j);\ \pi^{-1}_{\alpha'_i}(P'_j) = (1, \sigma_i(v_j)); \quad j = 3i-2, 3i-1, 3i,$$

for $i = 1, 2, \ldots, n$. To do it we can proceed inductively on $i \in 1, 2, \ldots, n$.

Now, consider the β_j-standard leaves $L_j = \{y = c_j x^{\beta_j}\} \subset \mathbb{D}_{++}$ such that $P_j = L_j \cap \mathbb{S}^1_{++}$. The leaves have empty pairwise intersection since $x_j < x_{j+1}$ and $\beta_j < \beta_{j+1}$. Define also β'_j-standard leaves L'_j by the fact that $P'_j = L'_j \cap \mathbb{S}^1_{++}$. Define the regions

$$R_i = \{(x, y) \in \mathbb{D}_{++};\ c_{3i} x^{\beta_{3i}} \leq y \leq c_{3i-2} x^{\beta_{3i-2}}\};\ i = 1, 2, \ldots, n.$$

Note that any (different from the axes) leaf L of the standard foliation $\mathcal{F}_{\alpha_i}$ has the same germ at the origin as $L \cap R_i$.

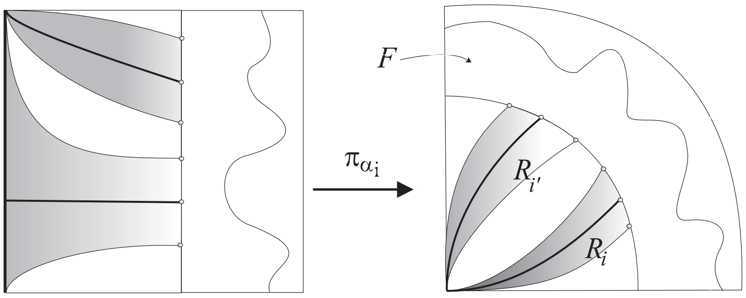

Figure 11

We also get regions R'_i as above, constructed from the L'_j, with pairwise intersection only at the origin.

Let us construct homeomorphisms $h_i : R_i \to R'_i$ respecting the pairs of weights and the seeds. We give h_i by their lifting $\tilde{h}_i : \pi_{\alpha_i}^{-1}(R_i) \to \pi_{\alpha'_i}^{-1}(R'_i)$, such that $\pi_{\alpha'_i} \circ \tilde{h}_i = h_i \circ \pi_{\alpha_i}$. Consider

$$g_i^+ : \pi_{\alpha_i}^{-1}(L_{3i-2}) \to \pi_{\alpha'_i}^{-1}(L'_{3i-2}); \quad g_i^- : \pi_{\alpha_i}^{-1}(L_{3i}) \to \pi_{\alpha'_i}^{-1}(L'_{3i})$$

defined by $(u,v) \mapsto (u', \sigma_i(v))$. Then $\tilde{h}_i(u,v)$ is the intersection of the horizontal line through $(0, \sigma_i(v))$ and the segment joining $g_i^+(u, v^+)$ and $g_i^-(u, v^-)$, where (u, v^+) and (u, v^-) are the respective intersections of the vertical line through (u,v) with $\pi_{\alpha_i}^{-1}(L_{3i-2})$ and $\pi_{\alpha_i}^{-1}(L_{3i})$.

Now, we consider the regions $R_i^* = \Phi^{-1}(\bar{R}_i)$ and $R_i^{\prime *} = \Phi^{-1}(\bar{R}'_i)$, as well as the homeomorphisms $h_i^* : R_i^* \to R_i^{\prime *}$ given by the lifting of h_i by Φ. Since the regions R_i^*, respectively $R_i^{\prime *}$, intersect pairwise only at the origin, we can define a homeomorphism $h : F \to F'$ that extends each of the h_i^* and induces on the axes the degenerate seeds. By construction h is the desired homeomorphism. $\square$

Remark 5 Given a homeomorphism h as in the theorem, we say that h *respects the abstract-seeds* ς_i and the *degenerate abstract-seeds* ς_0, ς_∞.

Weighted germs of leaves For certain purposes, we do not use a whole α-admissible foliation, but just the leaves in an isolated way. Thus, given a curve $L \subset \mathbb{D}_{++}$, we say that L is an *α-leaf* if it defines at the origin the germ of a leaf of an α-admissible foliation $\mathcal{F}$ over an abstract-window quadrant $F \subset \mathbb{D}_{++}$ and it does not coincide locally with one of the axes. In this case, we know that $\pi_\alpha^{-1}(L)$ accumulates in a single point $(0,\theta) \in E_\alpha$, with $0 < \theta < 1$. We call $(0,\theta)$ the *limit point* of L.

6 Getting equivalent eigenvalue functions

In this section, we prove the part "A) implies B)" of the theorem 1, by giving a dynamical-topological characterization of the point-paths associated to the eigenvalue functions $\lambda = \lambda_\xi$ and $\lambda' = \lambda_{\xi'}$. This result depends on theorems 3 and 4, that we state here but we prove in the next section.

Transitions of germs of admissible leaves Consider a system of tubes and cylinders (Λ, T) associated to ξ. Let Z be a basic zone and $P \in S_Z$. A *test-curve* for P, Z is a continuous injective map

$$\gamma : [0,1] \to \Lambda_P \cap Z = \Lambda_{P,Z}$$

such that $\gamma(0) \in \Delta_{P,Z} \setminus \mathcal{K}$, $\gamma((0,1]) \subset \Lambda_{P,Z} \setminus D$ and for $t_1 \neq t_2$ the points $\gamma(t_1)$ and $\gamma(t_2)$ are in different $\Lambda_{P,Z}$-orbits of ξ (and hence in different $V_{\Lambda,T}$-orbits, in view of corollary 1). We denote by $|\gamma|$ the image of γ.

Recall that for any $\Gamma \in \mathcal{J}_Z$ we have an analytic isomorphism between quadrants $\tau_\Gamma^Z : \Omega_{\Gamma,Z} \to \mathbb{D}_{++}$.

Theorem 3 *Let γ be a test-curve for P, Z and Γ the Z-mast of P. Then the set $\tau_\Gamma^Z(\mathrm{Sat}^\xi_{V_{\Lambda,T}}(|\gamma|) \cap \Omega_{\Gamma,Z})$ is an α_P^λ-leaf.*

Theorem 4 *Consider $\Gamma \in \mathcal{J}_Z$ and $\beta > 0$. Let L be a β-leaf and let us put $\mathcal{L} = (\tau_\Gamma^Z)^{-1}(L)$. Given an edge $\Gamma' \in \mathcal{J}_Z$, then $\Gamma' \in C_{\beta,\Gamma}^{\lambda,Z}$ if and only if $\Gamma' \cap V_{\Lambda,T}$ is contained in the topological adherence of $\mathrm{Sat}^\xi_{V_{\Lambda,T}}(\mathcal{L})$.*

The next corollary is straightforward from theorems 3 and 4:

Corollary 2 *Let γ be a test-curve for P, Z. Put $c = (P, Z) \in \mathcal{C}$. Given an edge $\Gamma \in \mathcal{J}_Z$, then $\Gamma \in C_c$ if and only if $\Gamma \cap V_{\Lambda,T}$ is contained in the topological adherence of $\mathrm{Sat}^\xi_{V_{\Lambda,T}}(|\gamma|)$.*

Proof of "A implies B" Assume that we have a topological equivalence $h : V \to V'$ between ξ and ξ', for two neighborhoods V, V' of $\mathbb{K}$, with the properties stated in the main theorem. By restricting the neighborhood V and in view of the proposition 3, we can assume that $V = V_{\Lambda,T}$ and that $V_{\Lambda',T'} \subset V' \subset V_{\Lambda'',T''}$, where $V_{\Lambda,T}$ corresponds to a system of tubes and cylinders for ξ and $V_{\Lambda',T'}$, $V_{\Lambda'',T''}$ are associated to two respective systems of tubes and cylinders for ξ'.

Consider $c = (P, Z) \in \mathcal{C}$. Recall that $h(V \cap Z) \subset Z$ and $h(P) = P$, since h respects the strata of D. Moreover, the fact that h gives a topological equivalence at P implies that the (λ, Z)-mast Γ of P is also the (λ', Z)-mast of P and $h(\Delta_P) \subset \Delta''_P$. Let γ be a test-curve for c, relative to (Λ, T) and ξ. Taking $\gamma(0)$ close enough to P and γ "short enough", we get $h(|\gamma|) \subset \Lambda'_P$. Now $h \circ \gamma$ is a test-curve for c, relative to Λ', T' and ξ'. To see this, we have only to verify that the ξ'-orbits in $V_{\Lambda',T'}$ cut $|h \circ \gamma|$ in at most one point, but this is a consequence of the fact that h is a topological equivalence and $V_{\Lambda',T'} \subset V'$.

Denote by A, A' and A'' the respective topological closures of $\mathrm{Sat}^{\xi}_{V_{\Lambda,T}}(|\gamma|)$, $\mathrm{Sat}^{\xi'}_{V_{\Lambda',T'}}(|h \circ \gamma|)$ and $\mathrm{Sat}^{\xi'}_{V_{\Lambda'',T''}}(|h \circ \gamma|)$. Since h is a topological equivalence we have that $A' \subset h(A) \subset A''$. Consider and edge $\Gamma \in \mathcal{J}_Z$. Note that $h(V \cap \Gamma) = V' \cap \Gamma$. Assume that $\Gamma \in C^{\lambda}_c$, then $\Gamma \cap V \subset A$ by the corollary 2 and hence $V' \cap \Gamma \subset h(A)$. Then $\Gamma \cap V_{\Lambda'',T''} \subset A''$ and we have that $\Gamma \in C^{\lambda'}_c$. Conversely, assume that $\Gamma \in C^{\lambda'}_c$, then $\Gamma \cap V_{\Lambda',T'} \subset A'$ and reversing the preceding argument we get $\Gamma \in C^{\lambda}_c$. Finally, we have $C^{\lambda}_c = C^{\lambda'}_c$ as desired.

7 Monoidal weighted blowing-ups and structural identifications

In this section we prove theorems 3 and 4 by using *weighted monoidal blowing-ups* with center the mast of a corner point P. We also introduce the *structural identifications* that identify, in a flow-compatible way, each pair of exceptional divisors E_{β_1}, E_{β_2} of the transversal discs to two edges Γ_1 and Γ_2 in a weighted path. Moreover, we give another structural identification that identifies the border of the basis $\delta\Delta_P$ at a point P with any of the exceptional divisors along the path of P.

Fix a system of tubes and cylinders (Λ, T) for ξ. Consider a basic zone Z. Put $\lambda = \lambda_\xi$. Theorem 4 is a consequence of the following proposition

Proposition 7 *Let $q = (\Gamma_1, P, \Gamma_2)$ be an angle in Z and $\beta_1 > 0$. Consider any β_1-leaf L_1 and denote by A the topological closure of $\mathrm{Sat}^{\xi}_{V_{\Lambda,T}}(\mathcal{L}_1)$, where $\mathcal{L}_1 = (\tau^Z_{\Gamma_1})^{-1}(L_1)$. The following statements are equivalent:*

1. *There is $\beta_2 > 0$ with $\beta_1 \overset{\lambda;q}{\rightsquigarrow} \beta_2$.*

2. *$(\Gamma_1 \cup \Gamma_2) \cap \Lambda_{P,Z} \subset A$.*

In this case, $L_2 = \tau^Z_{\Gamma_2}(\mathrm{Sat}^{\xi}_{V_{\Lambda,T}}(\mathcal{L}_1) \cap \Omega_{\Gamma_2,Z}) \subset \mathbb{D}_{++}$ *is a* β_2*-leaf.*

The correspondence $L_1 \mapsto L_2$ defined in the above proposition produces a *structural identification* as follows

Corollary 3 *Let* $q = (\Gamma_1, P, \Gamma_2)$ *be an angle in* Z *and assume that* $\beta_1 \overset{\lambda,q}{\rightsquigarrow} \beta_2$. *There is a homeomorphism* $\chi^{\xi,Z}_{(\beta_1,q,\beta_2)} : E_{\beta_1} \to E_{\beta_2}$, $t_1 \mapsto t_2$, *such that for any* β_1*-leaf* L_1 *with limit point* t_1*, the corresponding* β_2*-leaf* L_2 *has limit point* t_2.

Also we shall get the following corollary of theorem 3:

Corollary 4 *Let* $P \in S_Z$ *and assume that* Γ *is the* Z*-mast of* P. *There is a homeomorphism* $\chi^{\xi,Z}_{P,\Gamma} : \delta\Delta_{P,Z} \to E_{\alpha^{\lambda}_P}$, $Q \mapsto t$, *such that if* γ *is a test-curve for* P, Z *with* $\gamma(0) = Q$*, then* t *is the limit point of* $L = \tau^Z_{\Gamma}(\mathrm{Sat}^{\xi}_{V_{\Lambda,T}}(|\gamma|) \cap \Omega_{\Gamma,Z})$.

Weighted monoidal blowing-up Let us consider a weight $\alpha > 0$. We define the *standard* α*-weighted monoidal blowing-up* Π_α with center $x = y = 0$ to be the map

$$\begin{array}{rcl} \Pi_\alpha : \mathbb{I}^3_{s,\theta,z} = [0,1] \times [0,1] \times [0,1] & \longrightarrow & \mathbb{D}_{++} \times [0,1] \subset \mathbb{R}^3_{x,y,z} \\ (s, \theta, z) & \mapsto & (s \cos \pi\theta/2, s^\alpha \sin \pi\theta/2, z) \end{array}$$

if $\alpha \geq 1$ and $\Pi_\alpha(\theta, s, z) = (s^{\frac{1}{\alpha}} \cos \frac{\pi}{2}\theta, s \sin \frac{\pi}{2}\theta, z)$ if $0 < \alpha < 1$. We denote by $\mathbb{E}_\alpha = \Pi^{-1}_\alpha(\{x = y = 0\}) = \{s = 0\}$ the *exceptional divisor* of Π_α.

Proof of theorem 3 Denote ϑ the vector field in $\mathbb{D}_{++} \times [0,1]$ that corresponds to ξ by $\omega^Z_P : \Lambda_{P,Z} \to \mathbb{D}_{++} \times [0,1]$. Put $\zeta = \omega^Z_P \circ \gamma$. In order to prove theorem 3 is enough to show that

$$L = \mathrm{Sat}^{\vartheta}_{\mathbb{D}_{++}\times[0,1]}(|\zeta|) \cap (\mathbb{D}_{++} \times \{1\}) \subset \mathbb{D}_{++}$$

is an α^λ_P-leaf. In fact the passage from L to a leaf in $\Omega_{\Gamma,Z}$ is done just by analytic functions and then we apply proposition 6. Let us denote $\alpha = \alpha^\lambda_P$ and $\alpha' = \alpha^{\lambda'}_P$ to simplify the notation.

Lemma 2 *There is a* $\mathcal{C}^1$*-vector field* $\tilde{\vartheta}$ *defined in* $\mathbb{I}^3_{s,\theta,z}$ *that is* Π_α*-related with* ϑ *and satisfies the following properties:*

a) The restriction of $\tilde{\vartheta}$ *to the exceptional divisor* $\mathbb{E}_\alpha = \{s = 0\}$ *is of the form* $z\eta$*, where* η *is a non-singular vector field transversal to the levels of the function* z*. More precisely, under the identification* $\mathbb{E}_\alpha = \mathbb{I}^2_{\theta,z}$*, the vector field* η *is conjugated to* $\pm\partial/\partial z$.

b) The restriction of $\tilde{\vartheta}$ to $z = 0$ is of the form $s\kappa$, where κ is a non-singular vector field transversal to the levels of the function s. More precisely, under the identification $\Pi_\alpha^{-1}(0,0,0) = \mathbb{I}^2_{s,\theta}$, the vector field κ is conjugated to $\pm\partial/\partial s$.

c) At any point $(0,\theta,0)$ the vector field $\tilde{\vartheta}$ is locally tangent to a non-singular codimension-one foliation $\mathcal{F}$, transversal to $s = z = 0$.

Proof: We get a),b) by straightforward computations. To get c) is enough to apply the theorems of reduction to the central variety: see [16, 20] for instance (see also [1, 2, 4] and the proof of next lemma 3). $\square$

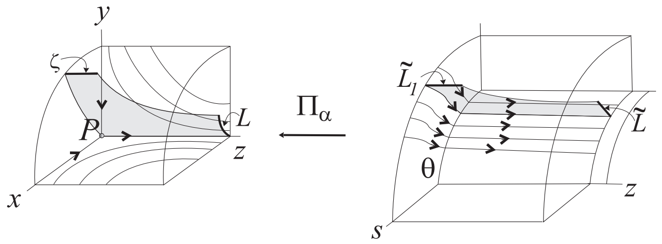

Figure 12

The description given in lemma 2 is the key for the proof of the theorem 3. Consider $\tilde{L}_1 = \Pi_\alpha^{-1}(|\zeta|)$ and denote

$$\tilde{L} = \mathrm{Sat}^{\tilde{\vartheta}}(\tilde{L}_1) \cap (\{z = 1\}).$$

Note that $\tilde{L} = \Pi_\alpha^{-1}(L)$ and that the restriction of Π_α to $z = 1$ is naturally identified with π_α. Hence, we have to show that $\tilde{L}$ accumulates at exactly one point $(0,\theta_0,1)$, for $0 < \theta_0 < 1$. Consider the accumulation point $(s,\theta_1,0)$ of $\tilde{L}_1$ at $z = 0$. By lemma 2-b), the trajectory of $(s,\theta_1,0)$ by the flow of κ cuts $s = z = 0$ in a point $(0,\theta',0)$, $0 < \theta' < 1$. This is the only point of the saturation of $\tilde{L}_1$ in $s = z = 0$, in view lemma 2-c). Now, by lemma 2-a) the flow of η gives the only point $(0,\theta_0,1)$ of accumulation of $\tilde{L}$ at $s = 0$.

Note that $\{s = 1, z = 0\}$ is isomorphic to $\delta\Delta_{P,Z}$. If we particularize the above construction to the points in $\{s = 1, z = 0\}$, we get a homeomorphism

$$(1,\theta_1,0) \mapsto (0,\theta',0) \mapsto (0,\theta_0,1)$$

not depending on the particular choice of the test-curve. It gives the structural identification of corollary 4.

Proof of proposition 7 Assume first that Γ_2 is the Z-mast of P. Then the statements 1 and 2 always hold. The fact that L_2 is a β_2-leaf follows by reversing the arguments in the case that Γ_1 is the Z-mast. Assume that Γ_1 is the Z-mast of P and that property 1 of proposition 7 holds. We shall proceed by a double monoidal blowing-up that we explain next.

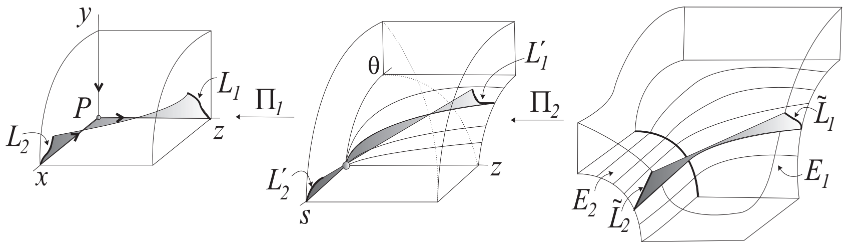

Figure 13

We take as above $\omega_P^Z : \Lambda_{P,Z} \to \mathbb{D}_{++} \times [0,1]$ and we denote by ϑ the vector field in $\mathbb{D}_{++} \times [0,1]$ that corresponds to ξ. Moreover, by doing an analytic isomorphism respecting the flow along the corresponding tube, we can suppose that $L_1 \subset \mathbb{D}_{++} \times \{1\}$. Note that either $\beta_1 > \alpha$ or $\beta_1 < \alpha$. Let us work in the case $\beta_1 > \alpha$, the other one is done in a similar way. Denote by A the topological adherence of the saturation of L_1 by the flow of ϑ. We have to show that $\{x = y = 0\} \cup \{y = z = 0\} \subset A$ and that $L_2 = A \cap \{x^2 + y^2 = 1\}$ is a β_2-leaf at the point $(1, 0, 0)$ (note that this is done up to the flow-isomorphism in the tube corresponding to the edge Γ_2).

Let us first do the monoidal weighted blowing-up of weight β_1 with center $x = y = 0$. That is, we consider $\Pi_1 = \Pi_{\beta_1} : \mathbb{I}^3_{s,\theta,z} \to (\mathbb{D}_{++} \times [0,1])_{x,y,z}$. Then $L_1' = \Pi_1^{-1}(L_1)$ is contained in $z = 1$ and accumulates at a single point $Q_1 = (0, \theta_1, 1)$, $0 < \theta_1 < 1$, in the exceptional divisor $\{s = 0\}$. Let us consider the subset

$$U = [0,1] \times \mathbb{D}_{++} \subset \mathbb{I}^3_{s,\theta,z}.$$

Then, the β_2-weighted blowing-up $\Pi'_{\beta_2} : \mathbb{I}^3_{s,t,\rho} \to U = ([0,1] \times \mathbb{D}_{++})_{s,\theta,z}$ with center $\theta = z = 0$ can be extended by chart-gluing to a morphism

$$\Pi_2 : H \to \mathbb{I}^3_{s,\theta,z}.$$

Note that $H = H_1 \cup H_2$ is the union of the two charts $H_1 = \mathbb{I}^3_{s,t,\rho}$ and $H_2 = \mathbb{I}^3_{s,\theta,z} \setminus \{\theta = z = 0\}$. Consider the composition $\Pi = \Pi_1 \circ \Pi_2 : H \to \mathbb{D}_{++} \times [0,1]$.

Let us denote $\mathbb{E}_1 = \Pi^{-1}\{x = y = 0\}$ and $\mathbb{E}_2 = \Pi_2^{-1}\{\theta = z = 0\}$. Note that $\mathbb{E}_2$ is contained in the chart H_1. Next lemma provides the essential arguments to get proposition 7 as well as corollary 3.

Lemma 3 *There is a $\mathcal{C}^1$ vector field $\tilde{\vartheta}$ on H that is Π-related with ϑ. It satisfies the following properties*

1. *The surfaces $\theta = 0$, $\theta = 1$ in H_2 and $\rho = 0$ and $\rho = 1$ in H_1 are invariant surfaces.*
2. *The surface $\mathbb{E}_1$ is invariant for $\tilde{\vartheta}$. In the chart H_2 the restriction $\tilde{\vartheta}|_{\mathbb{E}_1 \cap H_2}$ is non-singular and transversal to the levels of the function z. In the chart H_1, the restriction $\tilde{\vartheta}|_{\mathbb{E}_1 \cap H_1}$ is of the form $t\eta$, where η is a non-singular vector field transversal to the levels of the function t.*
3. *The surface $\mathbb{E}_2 \subset H_1$ is invariant for $\tilde{\vartheta}$. The restriction $\tilde{\vartheta}|_{\mathbb{E}_2}$ is of the form $s\kappa$, where κ is a non-singular vector field transversal to the levels of the function s.*
4. *There is a non-singular codimension one foliation $\mathcal{G}$ defined locally in each point $(s = 0, t = 0, \rho)$ of $\mathbb{E}_1 \cap \mathbb{E}_2$ (and compatible with $\rho = 0$ and $\rho = 1$) that is transversal to $\mathbb{E}_1 \cap \mathbb{E}_2$, such that $\tilde{\vartheta}$ is tangent to $\mathcal{G}$ and induces a two dimensional saddle in each leaf.*

Proof: (See also [3] for more details). Up to multiply it by a constant, the vector field ϑ is written in the space $\mathbb{D}_{++} \times [0, 1]$ as follows

$$\vartheta = A(x, y, z)x\frac{\partial}{\partial x} + B(x, y, z)y\frac{\partial}{\partial y} + C(x, y, z)z\frac{\partial}{\partial z}$$

where A, B and C are analytic functions such that $A(0) = -1$, $B(0) = -\alpha$ and $C(0) = \nu$. Let us show how ϑ lifts to the space $\mathbb{I}^3_{s,\theta,z}$ by the blowing-up Π_1. First, we have that $z\partial/\partial z = z\partial/\partial z$ and

$$\begin{aligned} x\frac{\partial}{\partial x} &= f(\theta)\left\{(\pi/2)\cot(\pi\theta/2)s\frac{\partial}{\partial s} - \beta_1\frac{\partial}{\partial \theta}\right\} \\ y\frac{\partial}{\partial y} &= f(\theta)\left\{(\pi/2)\tan(\pi\theta/2)s\frac{\partial}{\partial s} + \frac{\partial}{\partial \theta}\right\}, \end{aligned}$$

where

$$f(\theta) = \frac{2}{\pi}\frac{\sin(\pi\theta/2)\cos(\pi\theta/2)}{\cos^2(\pi\theta/2) + \beta_1\sin^2(\pi\theta/2)}.$$

Then ϑ lifts to a vector field ϑ_1 of the form

$$\vartheta_1 = A_1(s,\theta,z)s\frac{\partial}{\partial s} + B_1(s,\theta,z)\theta\frac{\partial}{\partial \theta} + C_1(s,\theta,z)z\frac{\partial}{\partial z}$$

where A_1, B_1 and C_1 are $\mathcal{C}^1$ functions. (Note that in order to get B_1, we have done the division $f(\theta)/\theta$). The invariance of $\theta = 1$ and $\theta = 0$ by ϑ_1 are evident from the corresponding one for ϑ. Also $\mathbb{E}_1 = \{s = 0\}$ is invariant from the expression of ϑ_1. Moreover, we have that $A_1(0) = -1$, $B_1(0) = \beta_1 - \alpha$ and $C_1(0) = \nu$. More precisely $C_1(0,\theta,z) \neq 0$ if $z > 0$, since ϑ is transversal to the function z. Hence the restriction $\tilde{\vartheta}|_{\mathbb{E}_1\cap H_2} = \vartheta_1|_{\mathbb{E}_1\setminus\{z=0\}}$ is non-singular and transversal to the levels of z. Now, let us do the blowing-up Π_2 of weight $\beta_2 = (\beta_1-\alpha)/\nu$. Assume that $\beta_2 > 1$, otherwise the computations are similar but with the symmetric expression of the blowing-up. We repeat the above computations with the vector field ϑ_1 (the analyticity of the coefficients of ϑ is not really necessary) and we get

$$\tilde{\vartheta} = \tilde{A}(s,t,\rho)s\frac{\partial}{\partial s} + \tilde{B}(s,t,\rho)t\frac{\partial}{\partial t} + \tilde{C}(s,t,\rho)\rho\frac{\partial}{\partial \rho}$$

at the origin of the chart H_1. The surface $\mathbb{E}_2$={t=0} is invariant and

$$\tilde{A}(s,0,\rho) = \tilde{a}(s,\rho); \quad \tilde{C}(s,0,\rho) = s\tilde{c}(s,\rho)$$

where $\tilde{a}(0,\rho) \neq 0$ for any ρ. This gives property 3. In order to see the remaining parts of property 2, note that $\mathbb{E}_1 \cap H_1 = \{s = 0\}$. We have that $\tilde{B}(0,t,\rho) = b'(t,\rho)$ and $\tilde{C}(0,t,\rho) = tc'(t,\rho)$, where $b'(0,\rho) \neq 0$. This ends the proof of the statement 2. Finally, in the line $s = t = 0$ we get normally hyperbolic singularities and then we can apply the results in [16, 20]. This proves property 4. $\square$

Let us show how lemma 3 gives the proof of proposition 7 and corollary 3. Consider $\tilde{A} = \Pi^{-1}(A \setminus \{xyz = 0\})$. It is clear that $\tilde{A}$ is the $\tilde{\vartheta}$-flow saturated of $\tilde{L}_1 = \Pi^{-1}(L_1)$. Let us look at the adherence of $\tilde{A}$ in H. Recall that $\tilde{L}_1$ accumulates at the only point $Q_1 = (0,\theta_1,1) \in (H_2)_{s,\theta,z}$. If we follow the $\tilde{\vartheta}$-trajectory ℓ_1 of Q_1 in $\mathbb{E}_1$, we get a single point $Q_0 \in \mathbb{E}_1 \cap \mathbb{E}_2$ of the form

$$Q_0 = (0,0,\rho_0) \in (H_1)_{s,t,\rho}, \quad 0 < \rho_0 < 1,$$

in view of the flow description in properties 2 and 3 of lemma 3. The trajectory ℓ_0 of Q_0 by κ cuts $\mathbb{E}_2 \cap \{s = 1\}$ in a point $Q_2 = (1,0,\rho_2)$, with

$0 < \rho_2 < 1$. By property 4 of lemma 3, we see that the topological adherence of $\tilde{A}$ is precisely $\tilde{A} \cup \ell_1 \cup \ell_2$. Now, since $\Pi(\ell_1) = \{x = y = 0\}$ and $\Pi(\ell_2) = \{y = z = 0\}$, we get the statement 2 of proposition 7. Now the intersection $\tilde{L}_2$ of $\tilde{A}$ with $s = 1$ projects over L_2. Since the restriction of Π to $s = 1$ coincides with the blowing-up of weight β_2, we get that L_2 is a β_2-leaf. Finally, the homeomorphism $\theta_1 \mapsto \rho_2$ defined by

$$Q_1 = (0, \theta_1, 1) \mapsto Q_0 = (0, 0, \rho_0) \mapsto Q_2 = (1, 0, \rho_2)$$

does not depend on the particular leaf L_1 and gives the structural identification stated in the corollary 3. The implication "2 implies 1" follows also by reversing the preceding arguments.

8 Seeds, compatibility conditions

Here, we start the proof of the part "B) implies A)" of theorem 1. Hence we consider two vector fields ξ and ξ' of the type multiple saddle-connections along D, with equivalent eigenvalue functions $\lambda = \lambda_\xi$ and $\lambda' = \lambda_{\xi'}$. Also we consider a common system of cylinders Λ for ξ and ξ' and we fix a system of tubes $T = \{T_\Gamma\}_{\Gamma \in \mathcal{J}}$ for ξ and a system of tubes $T' = \{T'_\Gamma\}_{\Gamma \in \mathcal{J}}$ for ξ', both adapted to Λ.

Let W be a wall. Recall that W is exactly a wall for two basic zones Z_1 and Z_2 and that there is a component D_k of D such that $W \subset D_k$. A corner point $P \in S_W = S \cap W$ is an *outer-point* for W if and only if $\Delta_P \subset D_k$. This is equivalent to say that no one of the masts of P is contained in W. Let S_W^{out} be the set of outer-points of W. For $P \in S_W^{out}$, we denote

$$\begin{array}{ccccccccc} \Delta_{P,W} & = & \Delta_{P,Z_1} & = & \Delta_{P,Z_2} & = & \Delta_P \cap Z_1 & = & \Delta_P \cap Z_2 \\ \delta\Delta_{P,W} & = & \delta\Delta_{P,Z_1} & = & \delta\Delta_{P,Z_2} & = & \delta\Delta_P \cap Z_1 & = & \delta\Delta_P \cap Z_2 \end{array}$$

Given an edge $\Gamma \in \mathcal{J}_W$ adjacent to P, we denote $P_{\Gamma,W} = \delta\Delta_{P,W} \cap \Gamma$.

A *point-seed* at $P \in S_W^{out}$ is any homeomorphism $\sigma_{P,W} : \delta\Delta_{P,W} \to \delta\Delta_{P,W}$ such that $\sigma_{P,W}(P_{\Gamma,W}) = P_{\Gamma,W}$ for any of the two edges Γ adjacent to P.

Consider $\Gamma \in \mathcal{J}_W$. We have two closed segments $\Omega_{\Gamma,W} = \Omega_\Gamma \cap W$ and $\Omega'_{\Gamma,W} = \Omega'_\Gamma \cap W$; denote $R_\Gamma = \Gamma \cap \Omega_\Gamma$ and $R'_\Gamma = \Gamma \cap \Omega'_\Gamma$. An *edge-seed* at Γ is any homeomorphism $\sigma_{\Gamma,W} : \Omega_{\Gamma,W} \to \Omega'_{\Gamma,W}$ such that $\sigma_{\Gamma,W}(R_\Gamma) = R'_\Gamma$.

A *system of point-seeds* $\mathcal{PS}$ is the data of a point-seed $\sigma_{P,W}$ for each wall W and each outer-point $P \in S_W^{out}$. Analogously, a *system of edge-seeds* $\mathcal{ES}$

is the data of an edge-seed $\sigma_{\Gamma,W}$ for each wall W and each edge $\Gamma \in \mathcal{J}_W$. A complete system of seeds is the union $\mathcal{S} = \mathcal{PS} \cup \mathcal{ES}$.

Definition 5 (Glue Condition) *A complete system of seeds $\mathcal{S}$ satisfies the* glue condition *if the following holds for any wall W. There are "gluing-segments" $\nabla_{P,\tilde{\Gamma}}, \nabla'_{P,\tilde{\Gamma}} \subset \delta\Delta_{P,W}$, for any edge $\tilde{\Gamma} \in \mathcal{J}_W$ adjacent to an outer point $P \in S_W^{out}$ and "gluing-segments" $\nabla_\Gamma \subset \Omega_{\Gamma,W}$, $\nabla'_\Gamma \subset \Omega'_{\Gamma,W}$, for any edge $\Gamma \in \mathcal{J}_W$, such that the following conditions are satisfied:*

1. *$P_{\tilde{\Gamma}} \in \nabla_{P,\tilde{\Gamma}} \cap \nabla'_{P,\tilde{\Gamma}}$ and $R_\Gamma \in \nabla_\Gamma$, $R'_\Gamma \in \nabla'_\Gamma$.*

2. *Each seed "respects" the corresponding gluing-segments, in the sense that $\sigma_{P,W}(\nabla_{P,\tilde{\Gamma}}) = \nabla'_{P,\tilde{\Gamma}}$ and $\sigma_{\Gamma,W}(\nabla_\Gamma) = \nabla'_\Gamma$.*

3. *The restrictions of the seeds to the gluing-segments are compatible with the flows of ξ and ξ' along the wall W. That is if ∇, ∇' and ∇_1, ∇'_1 are two associated pairs of gluing-segments, with corresponding seeds σ and σ_1, we have that*

$$\sigma(Q) \in Sat_W^{\xi'}(\sigma_1(Q_1)) \Leftrightarrow Q \in Sat_W^{\xi}(Q_1)$$

for any pair of points $Q \in \nabla$ and $Q_1 \in \nabla_1$.

Not that the glue-condition could have been stated just for a system of point-seeds, by considering only gluing segments of the type $\nabla_{P,\tilde{\Gamma}}$, $\nabla'_{P,\tilde{\Gamma}}$.

Lemma 4 *A system of point-seeds $\mathcal{PS}$ that satisfies the glue condition can be completed with a system of edge seeds $\mathcal{ES}$, in such a way that the complete system of seeds $\mathcal{S}$ satisfies the glue condition.*

Proof: In each wall W having at least one outer point, the point-seeds determine by the flows the edge-seeds in the segments $\Omega_{\Gamma,W}$, this is done in a coherent way since $\mathcal{PS}$ satisfies the glue condition. Given a wall W without outer points, select an arbitrary edge-seed and the other ones will be determined by the flows. $\square$

Transition of Seeds Consider a system of point-seeds $\mathcal{PS} = \{\sigma_{P,W}\}$, an element $c = (P, Z) \in \mathcal{C}$ and an edge $\Gamma \in C_c$. We construct an *abstract-seed*

$$\varsigma_{c,\Gamma} = (\beta_c(\Gamma), \beta'_c(\Gamma), \sigma_{c,\Gamma})$$

as follows. Put $\beta = \beta_c(\Gamma)$ and $\beta' = \beta'_c(\Gamma)$ in order to simplify notation. Let Γ_0 be the Z-mast of P and $q_j = (\Gamma_{j-1}, P_j, \Gamma_j)$, $j = 1, 2, \ldots, k$, the consecutive angles in the set of edges C_c to reach $\Gamma = \Gamma_k$ from Γ_0. Using the corollary 3, we extend the definition of the structural identification $\chi^{\xi,Z}_{P,\Gamma_0} = \chi_{c,\Gamma_0}$ given in the corollary 4 to a *structural identification*

$$\chi_{c,\Gamma} : \delta\Delta_{P,Z} \to E_\beta; \quad \chi_{c,\Gamma} = \chi_{(\beta_{k-1},q_k,\beta)} \circ \cdots \circ \chi_{(\alpha^\lambda_P,q_0,\beta_2)} \circ \chi_{c,\Gamma_0}.$$

Now, let W be the wall of Z such that P is an outer-point for W. We define the homeomorphism $\sigma_{c,\Gamma} : E_\beta \to E_{\beta'}$ by

$$\sigma_{c,\Gamma} = \chi'_{c,\Gamma} \circ \sigma_{P,W} \circ (\chi_{c,\Gamma})^{-1}.$$

We call $\mathcal{A}(\mathcal{PS}) = \{\varsigma_{c,\Gamma}\}$ the *system of abstract-seeds associated to* $\mathcal{PS}$.

Remark 6 Note that the system of abstract-seeds in a path (C_c, β_c) satisfies to the following *transition rule* for two edges $\Gamma_1, \Gamma_2 \in C_c$

$$\chi'_{\beta'_1,\Gamma_1,\Gamma_2} \circ \sigma_{c,\Gamma_1} = \sigma_{c,\Gamma_2} \circ \chi_{\beta_1,\Gamma_1,\Gamma_2}$$

where $\chi_{\beta_1,\Gamma_1,\Gamma_2}$ is the structural identification obtained by composition of the identifications of the corollary 3 for sequence of angles joining Γ_1 and Γ_2 with initial weight $\beta_1 = \beta_c(\Gamma_1)$. We do a similar definition for χ'.

Definition 6 *We say that a system of point-seeds $\mathcal{PS}$ satisfies the* global compatibility condition *if for any λ-related $c, d \in \mathcal{C}$ we have that $\sigma_{c,\Gamma} = \sigma_{d,\Gamma}$, and in fact $\varsigma_{c,\Gamma} = \varsigma_{d,\Gamma}$.*

Remark 7 The above definition means that the assignation of an abstract seed to an edge from an element $c = (P, Z)$ depends only on the set of edges C_c of the paths (C_c, β_c), (C_c, β'_c). That is, this assignation is in fact done from the quotient set $\mathcal{C}^*$.

If we have a system of edge-seeds $\mathcal{ES} = \{\sigma_{\Gamma,W}\}$, we can associate a degenerate abstract seed $\varsigma_{\Gamma,W}$ to each wall W and each $\Gamma \in \mathcal{J}_W$ as follows. Assume that $\Gamma \subset D_i \cap D_j$, with $i < j$. Then either $W \subset D_i$ or $W \subset D_j$. If $W \subset D_i$ we say that the weight of $\varsigma_{\Gamma,W}$ is 0 and we put $\varsigma_{\Gamma,W} = (0, \sigma_{\Gamma,W})$, where in

$$\sigma_{\Gamma,W} : \Omega_{\Gamma,W} \to \Omega'_{\Gamma,W}$$

we identify $\Omega_{\Gamma,W}$ and $\Omega'_{\Gamma,W}$ with $[0,1]$, via the maps τ_Γ and τ'_Γ. Note that we have the same notation for the edge-seed and the homeomorphism in the abstract-seed, this will not cause confusion. In the case that $W \subset D_j$, we do the same construction but with weight ∞ and we put $\varsigma_{\Gamma,W} = (\infty, \sigma_{\Gamma,W})$.

Then, we obtain a system $\mathcal{A}(\mathcal{S}) = \{\varsigma_{c,\Gamma}\} \cup \{\varsigma_{\Gamma,W}\}$ of abstract-seeds associated to a complete system of seeds $\mathcal{S}$.

Exterior Systems of abstract-Seeds Assume that we have two open-paths assignations ρ and ρ' such that $(\lambda,\rho) \equiv (\lambda',\rho')$. By an *exterior system of abstract-seeds* $\mathcal{EXT} = \{\varsigma_{b,\Gamma}\}$ we mean a family of abstract-seeds $\varsigma_{b,\Gamma} = (\beta_b(\Gamma), {\beta'}_b(\Gamma), \sigma_{b,\Gamma})$ for each $b \in \mathcal{B}$ and each edge $\Gamma \in C_b$, that satisfies to the following *transition rule* for any $\Gamma_1, \Gamma_2 \in C_b$

$$\chi'_{\beta'_1,\Gamma_1,\Gamma_2} \circ \sigma_{b,\Gamma_1} = \sigma_{b,\Gamma_2} \circ \chi_{\beta_1,\Gamma_1,\Gamma_2},$$

where $\beta_1 = \beta_b(\Gamma_1)$ and $\beta'_1 = \beta'_b(\Gamma_1)$ (see remark 6).

Consider a basic zone Z and an edge $\Gamma \in \mathcal{J}_Z$. Denote by $\mathcal{W}_{\Gamma,Z}$ the image of the double-weight map (recall the definition of A^Z_Γ in section 3)

$$\begin{array}{rcl}(\beta_{\Gamma,Z}, \beta'_{\Gamma,Z}) : A^Z_\Gamma & \to & \mathbb{R}^2_{>0} \\ a & \mapsto & (\beta_a(\Gamma), \beta'_a(\Gamma))\end{array}$$

We say that $\mathcal{W}_{\Gamma,Z}$ is the set of *pairs of weights associated to* Γ, Z. The following result is a direct consequence of the definitions

Proposition 8 *Assume that* $(\lambda,\rho) \equiv (\lambda',\rho')$. *Consider a system of point-seeds* $\mathcal{PS}$ *that satisfies the global compatibility condition and a system* $\mathcal{EXT}$ *of exterior abstract-seeds associated to* ρ, ρ'. *Then there is a bijection between* $\mathcal{W}_{\Gamma,Z}$ *and* $\{\varsigma_{a,\Gamma}\}_{a\in A^Z_\Gamma}$ *such that to each pair of weights* (β,β') *corresponds exactly one abstract-seed of the form* (β,β',σ).

By the above proposition 8 and noting that the weight functions are homologous (proposition 1), we deduce that the abstract-seeds at an edge Γ in a basic zone Z can be expressed following the conditions of the theorem 2:

Corollary 5 *In the situation of the above proposition, the abstract-seeds* $\{\varsigma_{a,\Gamma}\}_{a\in A^Z_\Gamma}$ *associated to* Γ, Z *can be ordered in a finite list*

$$(\beta_1,\beta'_1,\sigma_1), (\beta_2,\beta'_2,\sigma_2), \ldots, (\beta_k,\beta'_k,\sigma_k)$$

such that $\beta_1 < \beta_2 < \cdots < \beta_k$ *and* $\beta'_1 < \beta'_2 < \cdots < \beta'_k$.

9 Existence of compatible data of seeds

We will construct the topological equivalence by basic zones. Hence we need to assure us that they "glue" at the walls. To do this, we start from topological equivalences defined at each wall and we try to extend them to each basic zone. Unfortunately, not all the systems of topological equivalences defined in the walls will be compatible with an extension to the basic zones. To get this property we need to assure the existence of a system of seeds satisfying both the glue-condition and the global compatibility condition. This section is devoted to prove the existence of such systems of seeds.

Theorem 5 *There exists a complete system of seeds $\mathcal{S}$ that satisfies both the glue condition and the global compatibility condition.*

In view of lemma 4, it is enough to find a system of point seeds $\mathcal{PS}$ that satisfies both the glue-condition and the global compatibility condition.

Lemma 5 *Let W be a wall, $P \in S_W^{out}$ and $\sigma : \delta\Delta_{P,W} \to \delta\Delta_{P,W}$ a point-seed. For each W-outer point Q we can chose a point-seed $\sigma_{Q,W}$ satisfying the glue conditions and such that $\sigma_{P,W} = \sigma$.*

Proof: Once we fix σ, the flows of ξ, ξ' induce in the wall a "part" of the new seeds in each one of the outer points Q adjacent to P. Chose $\sigma_{Q,W}$ in order to respect these "parts" and continue with the next adjacent outer points in a correlative way. □

Let us introduce now the *graph of the walls* $(\mathbb{W}, \mathcal{E})$. This will be made from λ or λ' and we will get the same graph since $\lambda \equiv \lambda'$. The set of *vertices* $\mathbb{W}$ is the set of walls. And *edge* $e = (P_1, Z, P_2) \in \mathcal{E}$ between two walls W_1 and W_2 of a basic zone Z is a triple where P_i is an outer point of W_i, for $i = 1, 2$ and we have the equality of paths $C_{(P_1,Z)} = C_{(P_2,Z)}$.

Proposition 9 *The graph $(\mathbb{W}, \mathcal{E})$ is simple without cycles.*

Proof: Let us first prove that it is a simple graph. Assume that we have two different edges $e = (P_1, Z, P_2)$ and $e' = (P_1', Z', P_2')$ between two walls W_1 and W_2. Consider C_1 the topological path (union of skeleton-edges) joining P_1 and P_2 in Z. Then C_1 is unique and $C_1 = |C_{(P_1,Z)}| = |C_{(P_2,Z)}|$. Let C_2 be the minimum topological path of skeleton-edges joining P_2 and P_2' in W_2. Let C_3 be the topological path joining P_2' and P_1' in Z. Finally let C_4 be the topological path joining P_1' and P_1 in W_1. Then $C_1 \cup C_2 \cup C_3 \cup C_4$ contains a cycle in the skeleton, contradiction. In the same way a cycle in the graph $(\mathbb{W}, \mathcal{E})$ will produce a cycle in the skeleton. □

Let us fix a *level* function $\mathcal{L} : \mathbb{W} \to \mathbb{Z}_{\geq 0}$ for $(\mathbb{W}, \mathcal{E})$, that is, if we put $\mathbb{W}_n = \mathcal{L}^{-1}(n)$, we have the following two properties:

1. For any $W \in \mathbb{W}_n$, $n \geq 0$, there are no $W' \in \mathbb{W}_n$ connected with W.
2. For any $W \in \mathbb{W}_n$, $n \geq 1$, there is exactly one $W' \in \cup_{k<n}\mathbb{W}_k$ that is connected with W. Moreover $W' \in \mathbb{W}_{n-1}$.

The following statement gives theorem 5 by taking $n \geq 0$ big enough:

> Fix $n \geq 0$. Then there is a system $\mathcal{PS}_n$ of point-seeds $\sigma_{P,W}$ for any $P \in S_W^{out}$ and $W \in \mathbb{W}_{\leq n}$ satisfying the glue condition and the global compatibility condition.

Let us prove this statement by induction. If $n = 0$, we put one point-seed arbitrarily in one outer-point of each $W \in \mathbb{W}_0$ and we complete to the other outer-points by lemma 5. The glue-condition is satisfied and, since there are no wedges $e = (P_1, Z, P_2)$ between two walls of the level 0, then the global compatibility condition is empty. Assume the statement proved for $n-1$. We keep the point-seeds $\sigma_{P,W}$ for $W \in \mathbb{W}_{\leq n-1}$. Consider a wall $W_1 \in \mathbb{W}_n$. Then, there is exactly one $W_2 \in \mathbb{W}_{n-1}$ connected with W_1. Call $e = (P_1, Z, P_2)$ the connecting edge. In view of the rules of transition of seeds in remark 6, the point-seed σ_{P_2,W_2} induces a point-seed at P_1, W_1 that we call σ_{P_1,W_1}. By lemma 5 we complete the point-seeds in W_1. The glue-condition and the global compatibility condition are satisfied by construction.

10 Excision of a corner point

We shall use an induction process on the number of corner points in D. In this section we describe the effect of "eliminate one corner point" from the skeleton. We assume that $(\lambda, \rho) \equiv (\lambda', \rho')$, that we have a system of seeds $\mathcal{S}$ satisfying the glue-condition and the global compatibility condition and an exterior system of abstract-seeds $\mathcal{EXT}$.

Let us consider a corner point P. Recall that we have a system Λ of cylinders and two systems T and T' of tubes around the compact part $\mathbb{K}$ of the skeleton $\mathcal{K}$. Let us denote by $\mathcal{J}_P^a$ and $\mathcal{J}_P^c$ the set of open edges, respectively the set of compact edges having P as an end point. Put $\mathcal{J}_P = \mathcal{J}_P^a \cup \mathcal{J}_P^c$. Let us consider the compact neighborhood of P defined by

$$V_{\Lambda,T}^P = \Lambda_P \cup \bigcup_{\Gamma \in \mathcal{J}_P^a} T_\Gamma \cup \bigcup_{\Gamma \in \mathcal{J}_P^c} T_\Gamma^P$$

(Recall that T_Γ^P is the semi-tube of T_Γ corresponding to the end P.) The following statement is a direct consequence of the acyclicity hypothesis:

> There is a one-to-one identification between the set $\mathcal{J}_P^c$ of compact edges at P and the set of connected components of $V_{\Lambda,T} \setminus V_{\Lambda,T}^P$.

Given an edge $\Gamma \in \mathcal{J}_P^c$ the corresponding connected component $V_{\Lambda,T}^\Gamma$ of $V_{\Lambda,T} \setminus V_{\Lambda,T}^P$ is determined by the property that $\Gamma \cap V_{\Lambda,T}^\Gamma \neq \emptyset$. Note that the common frontier between $V_{\Lambda,T}^\Gamma$ and $V_{\Lambda,T}^P$ is Ω_Γ and that $V_{\Lambda,T}^\Gamma \cup \Omega_\Gamma$ is a compact subset of the ambient space M that is union of cylinders and semi-tubes.

Next proposition has a straightforward proof and we do not do it:

Proposition 10 *There are open sets M_P and M_Γ, for $\Gamma \in \mathcal{J}_P^c$ such that the following hold:*

1. $V_{\Lambda,T}^P \cup V_{\Lambda,T'}^P \subset M_P$.
2. *For any $\Gamma \in \mathcal{J}_P^c$, we have $V_{\Lambda,T}^\Gamma \cup V_{\Lambda,T'}^\Gamma \subset M_\Gamma$.*
3. *For any $\Gamma_1, \Gamma_2 \in \mathcal{J}_P^c$ with $\Gamma_1 \neq \Gamma_2$ we have $M_{\Gamma_1} \cap M_{\Gamma_2} = \emptyset$.*

Moreover the components of $D \cap M_P$ and $D \cap M_\Gamma$ correspond one-to-one to the components of $D \cap V_{\Lambda,T}^P$ and $D \cap V_{\Lambda,T}^\Gamma$.

Let us describe how we get a system of seeds and an exterior system of abstract-seeds in M_{Γ_0}, for a fixed $\Gamma_0 \in \mathcal{J}_P^c$. We have a new ambient space $\tilde{M} = M_{\Gamma_0}$ and a divisor $\tilde{D} = D \cap \tilde{M}$. The set of edges $\tilde{\mathcal{J}}$ corresponds to the edges $\Gamma \in \mathcal{J}$ such that $\Gamma \cap \tilde{M} \neq \emptyset$. Moreover

$$\tilde{\mathcal{J}}_c = \{\Gamma \in \mathcal{J}_c; \quad \Gamma \cap \tilde{M} \neq \emptyset \text{ and } \Gamma \neq \Gamma_0\}.$$

That is, Γ_0 is a new open edge for $(\tilde{M}, \tilde{D})$. The set of corners is $\tilde{S} = S \cap \tilde{M}$ and we get a system of cylinders $\tilde{\Lambda}$ in an evident way. To obtain the system of tubes $\tilde{T}$ and $\tilde{T}'$, we just substitute the tubes T_{Γ_0} and T'_{Γ_0} by the semi-tubes complementary to $T_{\Gamma_0}^P$ and ${T'}_{\Gamma_0}^P$.

The system of point-seeds $\mathcal{PS} = \{\sigma_{Q,W}\}$ induces in $(\tilde{M}, \tilde{D})$ a system of point-seeds $\widetilde{\mathcal{PS}}$ given by the $\sigma_{Q,W}$ such that $Q \in \tilde{S}$. Note that the walls in $(\tilde{M}, \tilde{D})$ that contain $Q \in \tilde{S}$ are in bijection with the walls in (M, D) that contain Q. Analogously, the system of edge-seeds $\mathcal{ES} = \{\sigma_{Q,\Gamma}\}$ induces a system of edge-seeds $\widetilde{\mathcal{ES}}$ in $(\tilde{M}, \tilde{D})$. The glue-condition is satisfied, by an evident restriction, for the new system of seeds $\tilde{\mathcal{S}} = \widetilde{\mathcal{PS}} \cup \widetilde{\mathcal{ES}}$.

The new eigenvalue functions $\tilde{\lambda}$, $\tilde{\lambda}'$ in $\tilde{M}, \tilde{D}$ obtained by restriction are equivalent, since locally the rules of construction of a path are the same ones. Let us describe how we get new open-paths assignations $\tilde{\rho}$ and $\tilde{\rho}'$ and a new system $\widetilde{\mathcal{EXT}}$ of exterior abstract- seeds. The set $\widetilde{\mathcal{B}}$ is the union of two sets $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}' \cup \widetilde{\mathcal{B}}''$. The elements of $\widetilde{\mathcal{B}}'$ are the $b \in \mathcal{B}$ such that C_b has at least one edge in $\tilde{\mathcal{J}}$. The weighted open-paths (C_b, β_b) and (C_b, β'_b) give by restriction weighted open paths $\tilde{\rho}(b)$ and $\tilde{\rho}'(b)$. Note that the connectedness of the union of $\widetilde{C}_b$ is guaranteed since C_b do not pass twice through the point P.

The set $\widetilde{\mathcal{B}}''$ is the set of elements $b'' = [c = (Q, Z)] \in \mathcal{C}^*$, where C_c has at least one edge in $\tilde{\mathcal{J}}$ and no one of the ends of $|C_c|$ are in $\tilde{M}$. We get the open-paths $\tilde{\rho}(b'')$ and $\tilde{\rho}'(b'')$ by evident restriction of (C_c, β_c) and (C_c, β'_c). By construction, we have that $(\tilde{\lambda}, \tilde{\rho}) \equiv (\tilde{\lambda}', \tilde{\rho}')$.

The new system of seeds $\widetilde{\mathcal{S}}$ satisfies the glue-condition and the global compatibility condition. Define the exterior system of abstract-seeds $\widetilde{\mathcal{EXT}}$ by keeping the abstract-seeds coming from $\mathcal{S}$ and $\mathcal{EXT}$, in such a way that we recover the same weights and abstract-seeds as above in each edge $\Gamma \in \tilde{\mathcal{J}}$.

In an analogous way, we define a system of seeds, open-paths assignations and an exterior system of abstract-seeds, for the ambient data M_P, D_P.

11 Normalized neighborhoods

In this section we construct *normalized neighborhoods* V associated to a system of tubes and cylinders Λ, T and to the vector field ξ. Moreover, given a system of seeds $\mathcal{S}$ that satisfies the glue-condition, we shall get pairs of normalized neighborhoods V, V' for ξ and ξ' *compatible with the system of seeds* $\mathcal{S}$. We also have a *refinement process* from the windows, that is important in order to "adjust" the neighborhoods where we get the topological equivalence in the statement "A" of theorem 1.

Let $P \in S$ be a corner point. Consider the compact neighborhood $V^P_{\Lambda,T}$ of P. If we denote also $T_\Gamma = T^P_\Gamma$ for the open edges $\Gamma \in \mathcal{J}^a_P$, then

$$V^P_{\Lambda,T} = \Lambda_P \cup \bigcup_{\Gamma \in \mathcal{J}_P} T^P_\Gamma .$$

For any edge Γ, a Γ-*window* is any subset $F \subset \Omega_\Gamma$ homeomorphic to $\mathbb{D}$ such that for any basic zone Z with $\Gamma \subset Z$ the set $\tau^Z_\Gamma(F) \subset \mathbb{D}_{++}$ is an abstract-window quadrant. The (F,P)-*semitube* $T_{F,P}$ is by definition

$$T_{F,P} = \mathrm{Sat}^{\xi}_{T^P_\Gamma}(F) \subset T^P_\Gamma$$

(There is no confusion with the "frame semitubes" T^P_Γ).

Let Γ_1, Γ_2 be the masts of P and consider a Γ_i-window F_i, for $i = 1, 2$. Put $\Lambda_{F_i,P} = T_{F_i,P} \cap \Lambda_P$, that is homeomorphic to $\mathbb{D}$. The *chimney* $\Upsilon_{F_i,P}$ associated to F_i is by definition the topological adherence of $\mathrm{Sat}^{\xi}_{\Lambda_P}(\Lambda_{F_i,P})$. In fact we have that

$$\Upsilon_{F_i,P} = \Delta_P \cup \mathrm{Sat}^{\xi}_{\Lambda_P}(\Lambda_{F_i,P}).$$

For a basic zone Z, the image $\omega^Z_P(\Lambda_{F_i,P} \cap Z) \subset \mathbb{D}_{++}$ is an abstract-window quadrant. We call $\Lambda_{F_i,P}$ the *top* of the chimney and $\Sigma_{F_i,P} = \Upsilon_{F_i,P} \cap \{x^2+y^2 = 1\}$ the *fence* of $\Upsilon_{F_i,P}$. Note that $\delta\Delta_P \subset \Sigma_{F_i,P}$ and that $\Sigma_{F_i,P}$ is homeomorphic to $\mathbb{S}^1 \times [0,1]$ in such a way that $\delta\Delta_P \simeq \mathbb{S}^1 \times \{0\}$. The *border* $\delta\Sigma_{F_i,P}$ is the closed curve corresponding to $\mathbb{S}^1 \times \{1\}$. Moreover, after sign-change, each basic zone Z gives $\Sigma_{F_i,P} \cap Z \simeq \mathbb{S}^1_{++} \times [0,1]$.

Let $\Gamma \in \mathcal{J}_P$ be an edge different from Γ_1, Γ_2, that is (Γ_1, P, Γ) is an angle. Consider a Γ-window F. We define as above the intersection $\Lambda_{F,P} = T_{F,P} \cap \Lambda_P$ and we denote by $\delta\Lambda_{F,P}$ the corresponding border. We say that $T_{F,P}$ is *well connected with* the chimney $\Upsilon_{F_1,P}$ if for any (of the two ones) basic zone Z containing the angle (Γ_1, P, Γ) we have that $\Lambda_{F,P} \cap Z \subset \Sigma_{F_1,P}$.

Consider a window assignation to the edges $\mathbb{F} : \Gamma \mapsto F_\Gamma = \mathbb{F}(\Gamma)$. We say that $\mathbb{F}$ is *normalized* if $T_{F,P}$ is well connected with the chimney $\Upsilon_{F_1,P}$ for any angle (Γ, P, Γ_1) as above. The neighborhood of $\mathbb{K}$ defined by

$$V_{\mathbb{F}} = \bigcup_{P,F} \Upsilon_{F,P} \cup \bigcup_{P,F} T_{F,P}$$

is called the *normalized neighborhood associated* to $\mathbb{F}$. We say that another normalized neighborhood $V_{\mathbb{F}^*}$ *refines* $V_{\mathbb{F}}$ if $V_{\mathbb{F}^*} \subset V_{\mathbb{F}}$ (this is equivalent to say that $\mathbb{F}^*(\Gamma) \subset \mathbb{F}(\Gamma)$ for any edge Γ).

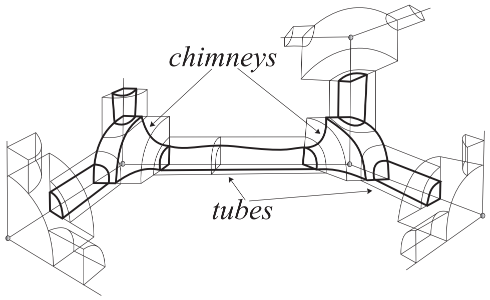

Figure 14

Proposition 11 *There is at least one normalized neighborhood $V_{\mathbb{F}}$ of $\mathbb{K}$. Moreover, let $V_{\mathbb{F}}$ be a normalized neighborhood , consider an edge $\Gamma \in \mathcal{J}$ and let $F^* \subset \mathbb{F}(\Gamma)$ be a Γ-window. Then, there is a normalized neighborhood $V_{\mathbb{F}^*}$ that refines $V_{\mathbb{F}}$ and such that $F^* = \mathbb{F}^*(\Gamma)$.*

Proof: (See also [3] for more details). We do the proof by induction on the number n of corner points. If $n = 1$ it is not difficult in view of the properties of the flow-saturation along saddles and tubes. The passage from $n-1$ to n is done by excision of a corner point containing the desired window. $\square$

Compatibility with the seeds Consider two Γ-windows $F \subset \Omega_\Gamma$ and $F' \subset \Omega'_\Gamma$, corresponding to the same edge Γ. We say that the pair (F, F') is *$\mathcal{S}$-compatible* if for any wall W that contains Γ, the segments $F \cap W$ and $F' \cap W$ are contained in the corresponding pair of gluing-segments and

$$\sigma_{\Gamma,W}(F \cap W) = F' \cap W.$$

In particular, the edge-seed $\sigma_{\Gamma,W}$ induces by restriction a homeomorphism $\sigma_{\Gamma,W} : F \cap W \to F' \cap W$. A pair of normalized neighborhoods $V_{\mathbb{F}}$, ${V'}_{\mathbb{F}'}$ is *$\mathcal{S}$-compatible* if each pair (F, F') of associated windows is $\mathcal{S}$-compatible. In the next result we show that the refinements can be done respecting the $\mathcal{S}$-compatibility.

Proposition 12 *There is at least one $\mathcal{S}$-compatible pair of normalized neighborhoods $V_{\mathbb{F}}$, ${V'}_{\mathbb{F}'}$. Moreover, let $F^* \subset \mathbb{F}(\Gamma)$ and $F'^* \subset \mathbb{F}'(\Gamma)$ be a pair of $\mathcal{S}$-compatible Γ-windows. There is a $\mathcal{S}$-compatible pair $V_{\mathbb{F}^*}$, ${V'}_{\mathbb{F}'^*}$ of normalized neighborhoods that refines $V_{\mathbb{F}}$, ${V'}_{\mathbb{F}'}$ and $F^* = \mathbb{F}^*(\Gamma)$, $F'^* = \mathbb{F}'^*(\Gamma)$.*

Proof: (See also [3]). We do an induction following the same ideas of the proof of proposition 11 to get the pair of normalized neighborhoods. $\square$

Consider a homeomorphism $H : V_{\mathbb{F}} \to {V'}_{\mathbb{F}'}$ between two normalized neighborhoods. We say that H *respects the structure of the normalized neighborhoods* if it respects the divisors, the basic zones and also the chimneys and the tubes. Assume that, in addition, H gives a topological equivalence between ξ and ξ' and the pair of normalized neighborhoods is $\mathcal{S}$-compatible, then we say that H *respects the edge-seeds $\mathcal{ES}$* if H induces $\sigma_{\Gamma,W} : F \cap W \to F' \cap W$, for any Γ, W.

Corollary 6 *Let $H : V_{\mathbb{F}} \to V'_{\mathbb{F}'}$ be a topological equivalence between ξ and ξ' defined in a $\mathcal{S}$-compatible pair of normalized neighborhoods. Assume that H respects the structure of the normalized neighborhoods and the edge-seeds. Let $F^* \subset \mathbb{F}(\Gamma)$ be a Γ-window and $F'^* = H(F^*)$. Then there is a $\mathcal{S}$-compatible pair of normalized neighborhoods $V_{\mathbb{F}^*}$, $V'_{\mathbb{F}'^*}$ that refines $V_{\mathbb{F}}$, $V'_{\mathbb{F}'}$, such that $F^* = \mathbb{F}^*(\Gamma)$, $F'^* = \mathbb{F}'^*(\Gamma)$ and $H(V_{\mathbb{F}^*}) = V'_{\mathbb{F}'^*}$.*

12 The Induction Statement

In this section we introduce a statement, called the "Induction Statement", stronger than "B implies A" in theorem 1, that we shall prove by induction on the number of corner points. Here we shall do the proof of the induction step and in the next section the case of a single corner point.

Theorem 6 (Induction Statement) *Assume that $(\lambda, \rho) \equiv (\lambda', \rho')$. Consider a system of cylinders and tubes Λ, T and T' for ξ and ξ', a system of seeds $\mathcal{S}$ that satisfies the glue condition and the global compatibility condition and a system of exterior abstract-seeds $\mathcal{EXT}$. Let $\Gamma \in \mathcal{J}_a$ be an open edge and $h : F \to F'$ a homeomorphism between two Γ-windows of a pair V, V' of $\mathcal{S}$-compatible normalized neighborhoods. Assume that h respects the abstract-seeds at Γ. Then, there is a refinement V^*, V'^* of V,V' and a homeomorphism $H : V^* \to V'^*$ such that:*

1. *H respects the structure of the normalized neighborhoods and induces a topological equivalence between ξ and ξ'.*

2. *The restriction of H to the Γ-windows F^* and F'^* of V^* and V'^* is the same one as the restriction of h to those windows.*

3. *For each $\Gamma_1 \in \mathcal{J}_a$, the induced homeomorphism $h_1 : F_1^* \to F'^*_1$ between the Γ_1-windows of V^* and V'^* respects the abstract-seeds at Γ_1.*

Let us show how this theorem 6 gives the part "B implies A" of the main theorem 1. First of all, we put $\mathcal{B} = \emptyset$, that is we do no assignation of open-paths. In view of propositions 2 and 1, we can construct systems Λ, T and T' of tubes and cylinders for ξ and ξ'. By the theorem 5 there is a system of seeds $\mathcal{S}$ that satisfies the glue-condition and the global compatibility condition. We can construct a $\mathcal{S}$-compatible pair of normalized neighborhoods V,V' in view of proposition 12. Choose an open edge $\Gamma \in \mathcal{J}_a$ and the pair F, F' of

Γ-windows. Recall that by the corollary 5, the weights and the abstract-seeds at Γ in a basic zone Z can be listed as

$$(0, \sigma_0), (\beta_1, \beta'_1, \sigma_1), (\beta_2, \beta'_2, \sigma_2), \ldots, (\beta_k, \beta'_k, \sigma_k), (\infty, \sigma_\infty)$$

where $\beta_i < \beta_{i+1}$ and $\beta'_i < \beta'_{i+1}$, for $i = 1, 2, \ldots, k-1$. This allows us to use theorem 2, working in each of the four basic zones around Γ, to get a homeomorphism $h : F \to F'$ that respects the abstract-seeds at Γ. Now the "Induction Statement" applies and we get the desired topological equivalence H in a neighborhood of the compact part $\mathbb{K}$ of the skeleton.

Proof of the induction step We start here the proof of theorem 6, by doing the induction step "$(n-1)$ implies n", where n stands for the number of corner points $P \in S$. In the next section we end the paper with a proof for the case $n = 1$.

Let P be the end-point of the open edge Γ corresponding to the windows F, F'. We shall do an excision of this point, then we will apply the induction hypothesis and successive refinements of the situation. By section 10, we get a new ambient space M_P around P and new ambient spaces $M_{\Gamma'}$ for each compact edge $\Gamma' \in \mathcal{J}_P^c$ having P as one of the end-points. In each of these spaces the induction applies, since the number of corner points is strictly lower than n. Let us denote by V_P, V'_P the corresponding normalized neighborhoods for M_P obtained from V and V' by excision. Also we denote by $V_{\Gamma'}$, $V'_{\Gamma'}$ the corresponding normalized neighborhoods for $M_{\Gamma'}$.

The induction statement, applied to M_P, gives a refinement V_P^1, ${V'}_P^1$ of V_P, V'_P and a homeomorphism

$$H_P^1 : V_P^1 \to {V'}_P^1$$

satisfying the properties 1, 2 and 3 in theorem 6. Let $F_{\Gamma'}$, $F'_{\Gamma'}$ be the pair of windows of V, V' corresponding to Γ'. They are also windows of $V_{\Gamma'}$, $V'_{\Gamma'}$. The Γ'-windows $F_{\Gamma'}^1, {F'}_{\Gamma'}^1$ of V_P^1, ${V'}_P^1$ are respectively contained in $F_{\Gamma'}, F'_{\Gamma'}$ and $\mathcal{S}$-compatibles. By proposition 12, we can refine each pair of normalized neighborhoods $V_{\Gamma'}$, $V'_{\Gamma'}$ in order to get that $F_{\Gamma'}^1, {F'}_{\Gamma'}^1$ are the Γ'-windows of the refined normalized neighborhoods $V_{\Gamma'}^1$, ${V'}_{\Gamma'}^1$. This allows to glue together V_P^1 and the $V_{\Gamma'}^1$, respectively ${V'}_P^1$ and the ${V'}_{\Gamma'}^1$ to get new normalized neighborhoods V^1 and ${V'}^1$.

Now H_P^1 induces $h_{\Gamma'}^1 : F_{\Gamma'}^1 \to {F'}_{\Gamma'}^1$ and we apply induction to the pair $V_{\Gamma'}^1$, ${V'}_{\Gamma'}^1$ of normalized neighborhoods. We get a refinement $V_{\Gamma'}^2$, ${V'}_{\Gamma'}^2$ of $V_{\Gamma'}^1$,

${V'}^1_{\Gamma'}$ and a homeomorphism $H^2_{\Gamma'} : V^2_{\Gamma'} \to {V'}^2_{\Gamma'}$ that satisfies the properties 1, 2 and 3 in theorem 6. By corollary 6, we do “small enough” refinements V^2_P, ${V'}^2_P$ of V^1_P, ${V'}^1_P$ that respects H^1_P in order to get that the new Γ'-windows of V^2_P, ${V'}^2_P$ are contained in the corresponding Γ'-windows of $V^2_{\Gamma'}$, ${V'}^2_{\Gamma'}$ (note that we cannot “adjust” immediately the windows, because we are working simultaneously with all the edges Γ'). We end with a new refinement $V^3_{\Gamma'}$, ${V'}^3_{\Gamma'}$ of $V^2_{\Gamma'}$, ${V'}^2_{\Gamma'}$ respecting $H^2_{\Gamma'}$, that allows us to adjust the windows and to recover a pair normalized neighborhoods V^*, ${V'}^*$ and a homeomorphism H with the properties 1,2 and 3 of theorem 6.

13 The case of a single corner

In this section we prove theorem 6 for the case of a single corner P. This gives the “starting part” of the induction process and completes the proof of theorem 6. As a consequence, the proof of theorem 1 is finished.

Assumptions and notations We assume that S has a single element P, hence we have six (open) edges and eight basic zones. We also suppose that $(\lambda, \rho) \equiv (\lambda', \rho')$ and we have a system of cylinders and tubes Λ, T,T' for ξ, ξ' a system of seeds $\mathcal{S}$ with the glue condition (the global compatibility condition is evident) and an exterior system $\mathcal{EXT}$ of abstract-seeds. We start with a $\mathcal{S}$-compatible pair $V = V_{\mathbb{F}}$ and $V' = V'_{\mathbb{F}'}$ of normalized neighborhoods. In this situation we have two chimneys Υ_1, Υ_2 for $V_{\mathbb{F}}$ and Υ'_1, Υ'_2 for $V'_{\mathbb{F}}$.

Consider a basic zone Z. Put $V_Z = V \cap Z$ and $V'_Z = V' \cap Z$. We have three edges in Z, that we denote Γ_0, Γ_1 and Γ_2, where Γ_0 is the mast of P. The corresponding windows are called F_0, F_1 and F_2. We deal just with one chimney Υ having fence Σ and top Λ_{F_0}. Denote $\Upsilon_Z = \Upsilon \cap Z$, $\Sigma_Z = \Sigma \cap Z$, $\Delta_Z = \Delta_P \cap Z$, and $T_i \subset T_{\Gamma_i}$, $i = 1, 2, 3$, the tubes associated to the windows. The corresponding objects for ξ' will be F'_0, F'_1, F'_2, Υ',T'_0, T'_1 and T'_2.

Topological equivalences and transition of seeds Here we introduce the key result that identifies the transition of abstract-seeds (or between point-seeds and abstract-seeds) defined in the remark 6 of section 8 and the transition induced by a topological equivalence.

Proposition 13 *Let $H : V_Z \to V'_Z$ be a topological equivalence between ξ and ξ' that respects the structure of the normalized neighborhoods.*

a) Let $\varsigma_{P,W} = (\alpha, \alpha', g)$ be the abstract-seed in the mast Γ_0 associated to the point-seed $\sigma_{P,W}$ at P. Then the restriction $h_0 = H|_{F_0} : F_0 \to F_0'$ respects $\varsigma_{P,W}$ if and only if $H|_{\delta\Delta_Z} = \sigma_{P,W}$.

b) Consider the angle (Γ_0, P, Γ_i), $i \in \{1,2\}$, and let ς_0, ς_i be two abstract-seeds (degenerate or not) related by the transition rule throughout this angle. Then h_0 respects ς_0 if and only if $h_i = H|_{F_i}$ respects ς_i.

Proof: Recall the relationship between the point-seed $\sigma_{P,W} : \delta\Delta_Z \to \delta\Delta_Z$ and the abstract-seed $\varsigma_{P,W} = (\alpha, \alpha', g)$, where $g : E_\alpha \to E_{\alpha'}$. We have that

$$\sigma_{P,W} = {\chi'}^{-1} \circ g \circ \chi,$$

where $\chi : \delta\Delta_Z \to E_\alpha$ and $\chi' : \delta\Delta_Z \to E_{\alpha'}$ are the structural identifications defined in corollary 4. The fact that h_0 respects ς means that $g = \tilde{h}_0|_{E_\alpha}$, where $\tilde{h}_0$ is the lifting of h_0 by the weighted blowing-ups (after doing the chart identifications). Let us then describe the map

$$\sigma_{P,W} = {\chi'}^{-1} \circ \tilde{h}_0|_{E_\alpha} \circ \chi$$

in view of the constructions of χ, $\tilde{h}_0$ and χ'. Consider a point $Q \in \delta\Delta_Z$ and a test-curve $\gamma \subset \Sigma_Z$ such that $\gamma(0) = Q$. Put $L = |\gamma|$. The set $L_0 = \mathrm{Sat}^{\xi}(L) \cap F_0$ is an α-leaf of F_0 (after doing the chart identification). Denote $L_0' = h_0(L_0)$ and $L' = \mathrm{Sat}^{\xi'}(L_0') \cap \Sigma_Z'$. Then $Q' = \sigma_{P,W}(Q)$ is the only point in the intersection between $\delta\Delta_Z'$ and the adherence of L'. Note also that L' is the support of a test-curve for Q'. Since H is a topological equivalence, the fact that $H(L_0) = L_0'$ implies that $H(L) = L'$. By continuity, we get $H(Q) = Q'$. This proves the "sufficient" condition of a). The "necessary" condition is done by a reversing argument.

The statement b) in the case that ς_0 and ς_i are degenerate abstract-seeds is a direct consequence of the fact that the system of seeds satisfy the glue condition and the pair of normalized neighborhoods is $\mathcal{S}$-compatible. Assume then that $\varsigma_0 = (\beta_0, \beta_0', \sigma_0)$ and $\varsigma_i = (\beta_i, \beta_i', \sigma_i)$ are non-degenerate abstract-seeds that are related through the angle (Γ_0, P, Γ_i). Our argument is now very close to the precedent one. Recall the structural identifications

$$\chi : E_{\beta_0} \to E_{\beta_i}; \quad \chi' : E_{\beta_0'} \to E_{\beta_i'}$$

introduced in the corollary 4. We have that $\sigma_i = \chi' \circ \sigma_0 \circ \chi^{-1}$. Assume that h_0 respects ς_0. Hence h_0 lifts to $\tilde{h}_0 : E_{\beta_0} \to E_{\beta_0'}$ after the (β_0, β_0')-weighted blowing-ups and $\sigma_0 = \tilde{h}_0|_{E_{\beta_0}}$. This implies that $\sigma_i = \chi' \circ \tilde{h}_0|_{E_{\beta_0}} \circ \chi^{-1}$. Let us consider

$$\tilde{h}_i^* : \pi_{\beta_i}^{-1}(F_i) \setminus E_{\beta_i} \to \pi_{\beta_i'}^{-1}(F_i') \setminus E_{\beta_i'}$$

the lifting of $H|_{F_i}$ by the β_i, β_i'-weighted blowing-ups to the complementary of the divisors E_{β_i} and $E_{\beta_i'}$. Now, it is enough to prove that $\tilde{h}_i^*$ can be extended continuously by σ_i, to get a lifting $\tilde{h}_i$ of $h_i = H|_{F_i}$. Put $Q' = \sigma_i(Q)$. Let us show that $\tilde{h}_i$ is continuous at Q.

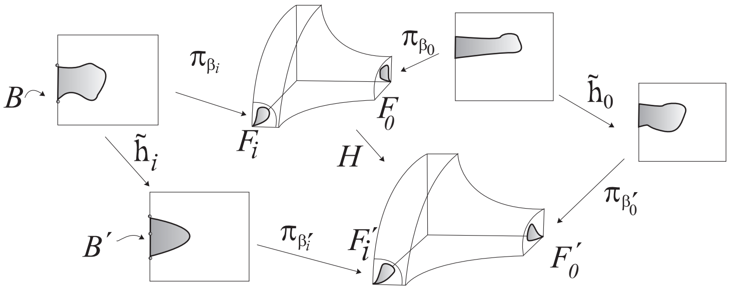

Figure 15

Consider a fundamental system of closed neighborhoods B' of Q' in $\pi_{\beta_i'}^{-1}(F_i')$ whose frontier is a Jordan curve of the form

$$\delta B' = [Q'_-, Q'_+] \cup A'$$

where $[Q'_-, Q'_+]$ is a segment in $E_{\beta_i'}$ that contains Q' and A' is an arc joining Q'_-, Q'_+ with $A' \setminus \{Q'_-, Q'_+\} \subset \pi_{\beta_i'}^{-1}(F_i') \setminus E_{\beta_i'}$. If we project the germ of A' at Q'_- by $\pi_{\beta_i'}$ we obtain a β_i'-leaf of F_i', we get the same result for the germ at Q'_+. Using a similar argument as in a), the description of $\sigma_i = \chi' \circ \tilde{h}_0|_{E_{\beta_0}} \circ \chi^{-1}$ gives that the topological adherence of $(\tilde{h}_i^*)^{-1}(A' - E_{\beta_i'})$ is also a closed arc A that joins the points $Q_- = \sigma_i^{-1}(Q'_-)$ and $Q_+ = \sigma_i^{-1}(Q'_+)$. This implies that $\tilde{h}_i^{-1}(B') = B$, where B has a similar structure as B' and hence is a neighborhood of Q. Then $\tilde{h}_i$ is continuous. □

Push-Functions and Reduction to Basic Zones We will define a system of *push-functions* in each normalized neighborhood, that are a particular case of "Liapunov type" functions (see [8]) and we shall use them in the construction of the topological equivalence H. Actually, we construct H respecting the levels of the push-functions. This property, jointly with the fact that H must respect the point-seeds and the edge-seeds, provides uniqueness for the restriction of H to the walls. This allow us to work in each basic zone independently, and to glue-together the obtained topological equivalences.

Recall that we have isomorphisms $\tau_\Gamma : T_\Gamma \to \mathbb{D} \times [0,1]$. We define the *push-function* $\epsilon_\Gamma : T_\Gamma \to [0,1]$ to be the corresponding projection of τ_Γ. Note that ξ is transversal to the levels of ϵ_Γ. It induces by restriction a push-function on the (F,P)-tube $T_{F,P}$, where F is the Γ-window of V.

Consider the cylinder Λ_P and the chimney Υ corresponding to one of the masts of P. A function $\epsilon_\Upsilon : \Upsilon \to [-1,1]$ is a push-function if it satisfies the following properties:

1. There is an open neighborhood U of P in Υ such that $\epsilon_\Upsilon|_U$ is given by the function $x^2 + y^2 - z^2$, via the coordinate chart ω_P.

2. The level -1 coincides with the top $\Lambda_{F,P}$ of the chimney. The level $+1$ is the fence $\Sigma_{F,P}$.

3. The levels $\epsilon_\Upsilon^{-1}(t)$, for $-1 < t < 1$, are connected surfaces, transversal to the vector field ξ (except for the point P) and each one divides Υ in two connected components, one of them containing the top and the other one containing the fence. Moreover, the levels (except at the point P) are transversal to the components of the divisor and they have only one connected component in each wall.

Proposition 14 *There are push-functions ϵ_{Υ_1} and ϵ_{Υ_2} for each of the chimneys Υ_1 and Υ_2. Moreover, they can be chosen such that $\epsilon_{\Upsilon_1}|_{\Delta_P} = \epsilon_{\Upsilon_2}|_{\Delta_P}$.*

Proof: It is enough to do a standard construction of partitions of the unity, in order to adjusts the levels of the push function to the top and the fence of the chimney. □

We say that ϵ_{Υ_1} and ϵ_{Υ_1} are *compatible* if $\epsilon_{\Upsilon_1}|_{\Delta_P} = \epsilon_{\Upsilon_2}|_{\Delta_P}$.

Corollary 7 *Let Z be a basic zone and W a wall of Z. Consider two homeomorphisms $H_1, H_2 : V_Z \to V'_Z$ that induce topological equivalences between ξ and ξ', respect the structure of the normalized neighborhoods, the push functions and the abstract-seeds. Then $H_1|_{W\cap V_Z} = H_2|_{W\cap V'_Z}$.*

Proof: Using the trajectories of the vector fields, each point of W is determined by a point in the segments of definition of the seeds and a level of the push-function. □

From now on we take compatible systems of push-functions for ξ and ξ'. Hence we have well defined push-functions in each wall W.

Reduction to the octant case The next proposition will be enough to achieve the starting case $n = 1$ of the induction for the theorem 6.

Proposition 15 (Octant case) *Consider a basic zone Z. Let Γ be an edge in Z and $h : F \to F'$ a homeomorphism between the Γ-windows of V, V' that respects the abstract-seeds at Γ. There are respective refinements V^*, V'^* of V, V' and a topological equivalence $H_Z : V_Z^* \to {V'}_Z^*$ between ξ and ξ' such that:*

1. *For any basic zone $Z_1 \neq Z$, we have that $V_{Z_1}^* = V_{Z_1}$ and $V_{Z_1}'^* = V_{Z_1}'$.*
2. *The structure of the normalized neighborhoods and the push-functions are respected by H_Z and the restriction of H_Z to F_Z^*, ${F'}_Z^*$ coincides with the restriction of h, where F^*, F'^* are the Γ-windows of V^*, V'^*.*
3. *For each $\bar{\Gamma} \in \mathcal{J}_Z$, the induced homeomorphism $\bar{h} : \bar{F}_Z^* \to \bar{F'}_Z^*$ respects the abstract-seeds at $\bar{\Gamma}$ related with the basic zone Z, where $\bar{F}^*$, $\bar{F'}^*$ are the $\bar{\Gamma}$-windows of V^*,V'^*.*

Before proving proposition 15, let us show how we use it to prove theorem 6 in the case of a single corner point. First, we fix push-functions as in proposition 14. We consider the four basic zones around Γ and we apply successively proposition 15. We glue the topological equivalences obtained around Γ to get a homeomorphism defined in a half of the normalized neighborhoods. Now, let $\bar{\Gamma}$ be the edge that is opposite to Γ and consider the $\bar{\Gamma}$-windows $\bar{F}$, $\bar{F'}$. Take a homeomorphism $\bar{h} : \bar{F} \to \bar{F'}$ obtained from theorem 2 and repeat the argument, the obtained topological equivalence glue with the precedent one in view of corollary 7.

The rest of this section is devoted to the proof of proposition 15.

Starting from the Mast In this paragraph we prove proposition 15 in the case that the edge $\Gamma = \Gamma_0$ is the Z-mast of P. We start with a homeomorphism $h : F_{0,Z} \to F_{0,Z}'$ that respects the abstract-seeds at Γ_0 related with the basic zone Z. By using the flows and the push-functions, we extend h in a unique way to a homeomorphism

$$H_{T_0} : T_{0,Z} = T_0 \cap Z \to T_0' \cap Z = T_{0,Z}'$$

that respects the levels of the push-functions in the tubes and gives a topological equivalence between ξ and ξ'. In particular H_{T_0} gives by restriction a homeomorphism h_{F_0} between the Z-tops $\Lambda_{F_0,Z}$ and $\Lambda'_{F'_0,Z}$. Using as above the push functions, we can extend h_{F_0} in a unique way to a homeomorphism

$$H_{\Upsilon-\Delta} : \Upsilon_Z \setminus \Delta_Z \to \Upsilon'_Z \setminus \Delta_Z$$

that respects the levels of the push functions and gives a topological equivalence between ξ and ξ'. In order to prove that $H_{\Upsilon-\Delta}$ extends to a homeomorphism between Υ_Z, Υ'_Z is enough to show that we can extend it to the border $\delta\Delta_Z$, the extension to the basis Δ_Z will be done in view of the push-functions and the structure of the vector fields in Δ_Z.

Proposition 16 *The seed $\sigma_{P,W} : \delta\Delta_Z \to \delta\Delta_Z$ extends $H_{\Upsilon-\Delta}$ by continuity.*

Proof: Let us recall, as in the proof of proposition 13 that

$$\sigma_{P,W} = {\chi'}^{-1} \circ \tilde{h}|_{E_\alpha} \circ \chi,$$

since h respects the abstract-seed corresponding to $\sigma_{P,W}$. We do now an argument very similar to that of the proof proposition 13-b), where the fence has the role of the total space of the blowing-up. First, we get a description of $Q' = \sigma_{P,W}(Q)$ as in the proof of proposition 13-a). In order to prove that $H_{\Upsilon-\Delta}$ can be extended continuously by $\sigma_{P,W}$, we will construct a special fundamental system of neighborhoods B' of Q'. Consider two points Q'_+ and Q'_- in $\delta\Delta_Z$ close to Q' and each one at one "side" of Q'. Let $A' \subset \Sigma'_Z$ be an arc having Q'_+ and Q'_- as end-points, that gives a Jordan curve by completing it with the segment Q'_+, Q'_-. Note that A' gives a germ of test-curve at Q'_+ and Q'_-. Denote by C' the interior component of this Jordan curve, it is a neighborhood of Q' in Σ'. Choose a level $\epsilon' = t$ of the push-function, for $0 \ll t < 1$ and consider the set

$$B' = \mathrm{Sat}^{\xi'}(C') \cap \{t \leq \epsilon' \leq 1\}.$$

It is a closed neighborhood of Q' and the neighborhoods of this form are a fundamental system of neighborhoods of Q' in the chimney Υ'. By construction $H^{-1}_{\Upsilon-\Delta}(B' \setminus \Delta)$ is of the form $B \setminus \Delta$, where B has the same structure as B'. Moreover, we get the same level $\epsilon = t$ and end-points $Q_+ = \sigma^{-1}_{P,W}(Q'_+)$, $Q_- = \sigma^{-1}_{P,W}(Q'_-)$. Hence we get a neighborhood of Q and we obtain the desired continuous extension. $\square$

Let us denote $H_\Upsilon : \Upsilon_Z \cup T_{0,Z} \to \Upsilon' \cup T'_{0,Z}$ the extension of $H_{\Upsilon-\Delta}$ and H_{T_0}. Let us do the refinements V^* of V and V'^* of V' in order to extend H_Υ to V^*_Z, ${V'}^*_Z$. Actually we shall only make small the tubes T_1,T_2 and T'_1,T'_2. More precisely the new tubes T^*_1,T^*_2 and ${T'}^*_1$,${T'}^*_2$ will satisfy that

$$T^*_i \cap Z_1 = T_i \cap Z_1; \quad {T'}^*_i \cap Z_1 = {T'}_i \cap Z_1; \quad 1 = 1, 2,$$

for any basic zone $Z_1 \neq Z$. This allows to get the condition 1 in proposition 15. Take the notation $\Lambda_{i,Z} = \Lambda_P \cap T_i \cap Z$, $i = 1, 2$ (we adopt analogous notations for the tubes T'_i, T^*_i, ${T'}^*_i$). In order to get T^*_i, ${T'}^*_i$ it is enough to determine $\Lambda^*_{i,Z} \subset \Lambda_{i,Z}$ in such a way that

1. $\Lambda^*_{i,Z} \cap W = \Lambda_{i,Z} \cap W$, for any wall $W \supset \Gamma_i$.
2. If we put ${\Lambda'}^*_{i,Z} = H_\Upsilon(\Lambda^*_{i,Z})$, then ${\Lambda'}^*_{i,Z} \subset {\Lambda'}_{i,Z}$.

In fact the property ${\Lambda'}^*_{i,Z} \cap W = {\Lambda'}_{i,Z} \cap W$ will now be a consequence of the fact that H_Υ respects the point-seed $\sigma_{P,W}$ and the edge-seed $\sigma_{\Gamma_i,W}$. Let us show how to choose $\Lambda^*_{i,Z} \subset \Lambda_{i,Z}$ with the properties 1 and 2 above. To do this, we consider the set ${\Lambda'}_{i,Z} \cap H_\Upsilon(\Lambda_{i,Z})$. It contains the segments ${\Lambda'}_{i,Z} \cap W$ for the walls $W \supset \Gamma_i$. Now, by elementary topological arguments, we can consider a subset homeomorphic to a quadrant ${\Lambda'}^*_{i,Z} \subset {\Lambda'}_{i,Z} \cap H_\Upsilon(\Lambda_{i,Z})$ and such that ${\Lambda'}_{i,Z} \cap W = {\Lambda'}^*_{i,Z} \cap W$. Now, take $\Lambda^*_{i,Z} = {H_\Upsilon}^{-1}({\Lambda'}^*_{i,Z})$.

The extension of H_Υ to T^*_1 and T^*_2 is now straightforward, by using the push-functions and the flows. We have then a topological equivalence

$$H_Z : V^*_Z \to {V'}^*_Z$$

that extends h and respects the push-functions. In order to verify the conditions of proposition 15 it is now enough to show that H_Z respects the abstract-seeds at Γ_i for $i = 1, 2$, related with Z. But this is a consequence of proposition 13 and the fact that all the abstract-seeds are represented in the mast Γ_0.

Starting from the fence We prove here proposition 15 in the case that the edge $\Gamma = \Gamma_1$ is not the Z-mast of P. We start with a homeomorphism $h = h_1 : F_{1,Z} \to F'_{1,Z}$ that respects the abstract-seeds at Γ_1 related with the basic zone Z. By using the flows and the definition of the push-functions in the tubes as in the preceding paragraph, we extend h in a unique way to a homeomorphism $H_{T_1}: T_{1,Z} \to T'_{1,Z}$ that respects the levels of the push functions and gives a topological equivalence between ξ and ξ'. In particular H_{T_1} gives by restriction a homeomorphism

$$h_{F_1}: \Lambda_{F_1,Z} \to \Lambda'_{F_1,Z}.$$

Where $\Lambda_{F_1,Z} = \Sigma_Z \cap T_{1,Z}$ and $\Lambda'_{F_1,Z} = \Sigma'_Z \cap T'_{1,Z}$. Now, we apply theorem 2 to get a homeomorphism $h_2 : F_{2,Z} \to F'_{2,Z}$ that respects the abstract-seeds at Γ_2. As above, we extend h_2 to H_{T_2} and it gives by restriction $h_{F_2} : \Lambda_{F_2,Z} \to \Lambda'_{F_2,Z}$. Note that the facts that h_1 and h_2 respects the degenerate abstract seeds, that $\mathcal{S}$ satisfies the glue condition and V, V' is a pair of $\mathcal{S}$-compatible normalized neighborhoods implies that

$$h_{F_i}|_{\delta\Delta_Z \cap \Lambda_{F_i,Z}} = \sigma_{P,W}|_{\delta\Delta_Z \cap \Lambda_{F_i,Z}}; \quad i = 1, 2.$$

We can extend h_{F_1} and h_{F_2} to a homeomorphism $h_\Sigma : \Sigma_Z \to \Sigma'_Z$ with the condition that $h_\Sigma|_{\delta\Delta_Z} = \sigma_{P,W}$. By using the flows and the push functions, we get in a direct way $h_\Upsilon : \Upsilon_Z \to \Upsilon'_Z$ that extends h_Σ. Finally, we extend from the top to the tube T_0 as before and we get a topological equivalence $H_Z : V_Z \to V'_Z$ between ξ and ξ' that respects the structure of the normalized neighborhoods, the push-functions, the abstract-seeds at Γ_1 and Γ_2 (related with Z) and such that

$$H_Z|_{\delta\Delta_Z} = \sigma_{P,W}$$

where W is the Z-wall that contains Δ_Z. By proposition 13 we deduce that H_Z respects also the abstract-seeds at Γ_0, since given an abstract-seed ς_0 at Γ_0 it is either related with an abstract-seed at Γ_1, or at Γ_2 or it corresponds to the point-seed $\sigma_{P,W}$.

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