Existence of global attractors for a class of nonlinear dissipative evolution equation

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Abstract

In this work we consider the Cauchy problem associated to dissipative perturbations of infinite dimensional Hamiltonian systems. we describe abstract conditions under which the problem is locally and globally well posed. Moreover we establish the existence of global attractor. Finally we present several applications of the theory.

1 Introduction.

In this paper we consider the Cauchy problem for dissipative perturbations of Hamiltonian systems. More precisely, we are interested in the properties of the solutions to problems of the form

$$\begin{cases} \partial_t u = -\mu A u + J\Phi'(u) + f \in X \\ u(0) = u_0 \in H, \end{cases} \tag{1}$$

where $\mu > 0$, H and X are reflexive Banach, spaces, $u : [0,T] \longrightarrow H$ for some $T > 0$, $A : \mathcal{D}(A) \subset X \longrightarrow X$ is a linear operator, $(-\mu Au)$ is the damping term, Φ' denotes the Gateaux (i.e. directional) derivative of the real valued functional Φ, called the Hamiltonian of the system[1], J is a skewsymmetric operator and f is a time independent external excitation. See [8]. A very large variety of problems arising in physics and engineering may be written as in (1), that is, as dissipative perturbations of conservative equations. Among these one finds nonlocal dispersive wave equations, the Ott-Sudan equation, the damped, modified Korteweg-de Vries equation, the damped Benjamin-Ono equation and dispersive nonlinear Schrödinger equations.

Our purpose here is to study the existence of global attractors associated to such problems. This work is organized as follows. In section 2 we discuss

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[1]The expression 'the hamiltonian' should be taken with a grain of salt. In applications one often encounters equations that may be written in different hamiltonian forms. See [1].

local and global well-posedness for (1). Section 3 deals with the existence of bounded absorbing sets and global attractors. Finally, in Section 4, we present applications, including some that, as far as we know, are new in the case of nonlocal equations.

We will use the following notations throughout this work. If X, Y, H... are Banach spaces, we denote their norms by $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_H$ and so on. $\mathcal{B}(Y,X)$denotes the set of all bounded operators from Y into X. In case $X=Y$ we write simply $\mathcal{B}(X)$. If Y is densely and continuously embedded in X we write $Y \hookrightarrow X$. If $\mathcal{O} \subset \mathcal{Y}$ is an open set, the symbols $F \in C(\mathcal{O}, X)$ means that F is a continuous function from $\mathcal{O}$ into X. Furthermore, $F'(y) \in \mathcal{B}(Y,X)$ represents the Gateaux (i.e. directional) derivative of F at $y \in \mathcal{O}$. Recall that if $X = \mathbb{R}$ then $F'(y) \in Y^*$. Finally we will use $\langle \cdot, \cdot \rangle$ to denote the duality pairing associated to any of the Banach spaces involved. No confusion will arise from this.

We say that (1) is locally well-posed (in time) if
(i) $\exists T > 0$ and $u \in C\big([0,T], H\big)$ such that $u(0) = u_0$ and the partial differential equation is satisfied with the time derivative computed with respect to norm of X;
(ii) The map $u_0 \mapsto u$ is continuous with respect to appropriate topologies (see Theorem 9 for a precise definition).

Note that our definition include the notion of persistence, that is, the solution remains in H. This is a non-trivial requirement (see [10], [6] and the references there in)
If the (i) and (ii) hold for all $T > 0$ we say that the problem is globally well-posed.

2 The Cauchy Problem.

We begin discussing local well-posedness for a slightly more general problem. Let H and X be as in the introduction, and consider

$$\begin{cases} \partial_t u = -\mu A u + F(u) + f \in X \\ u(0) = u_0 \in H. \end{cases} \tag{2}$$

We assume that
(A1) $H \hookrightarrow X$, (so that we have the dual inclusion $X^* \hookrightarrow H^*$ with X^* dense in H^*).
(A2) $F : H \longrightarrow X$ is continuous, $F(0) = 0$ and the following Lipchtz condition is satisfied

$$\|F(v) - F(w)\|_X \leq \gamma_o (\|v\|_H, \|w\|_H) \|v - w\|_H \tag{3}$$

for all v, $w \in H$, where $\gamma_o \colon I\!R_+^2 \to I\!R_+$ nondecreasing with respect to each of its arguments.
(A3) $(-A)$, generates a C^0 semigroup in X, so that $(-\mu A)$, $\mu > 0$, has the same propriety.
(A4) If $h \in H$ then $\exp(-\mu t A) h$ also belongs to H and the map $t \in [0, \infty) \longmapsto \exp(-\mu t A) h$ is continuous in the topology of H.

(A5) $\exp(-\mu tA) \in \mathcal{B}(X, H)$ for all $t > 0$, and satisfies the estimate

$$\|\exp(-\mu tA)\, x\|_H \leq g(t)\, \|x\|_X \tag{4}$$

for all $x \in X$ where $g \in L^1_{loc}([0,\infty))$.
(A6) The map $t \in (0,\infty) \longmapsto \exp(-\mu tA)\, x$ is continuous in the topology of H.
(A7) $f : [0,\infty) \longrightarrow X$ is continuous.

Example 1. *The above properties are satisfied with $A = \partial_x^2$, $X = H^s(\mathbb{R})$ and $H = H^{s+\lambda}(\mathbb{R})$, $s \in \mathbb{R}$, $\lambda \in [0,2)$. In this case (4) takes the form*

$$\left\|\exp\left(-\mu t\partial_x^2\right)\varphi\right\|_{s+\lambda} \leq K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^{\lambda}\right]^{\frac{1}{2}} \|\varphi\|_s\,, \tag{5}$$

for all $\varphi \in H^s(\mathbb{R})$. For a proof see [3] or[5].

Under assumptions (A1)-(A7) we have the following results. Since they are abstractions of some of the well-posedness results for various equations ocurring in the litereature, we present only a brief sketch of their proofs (See [10] and [5]).

Proposition 2. *Problem (2) is equivalent to the integral equation*

$$u(t) = \exp(-\mu tA)\, u_0 + \int_0^t \exp(-\mu(t-t')A)\,(F(u(t')) + f(t'))\, dt'. \tag{6}$$

More precisely, if $u \in C([0,T], H)$ is a solution of (2), then u satisfies (6). Conversely, if $u \in C([0,T], H)$ satisfiies (6) then $u(0) = u_0$, $u \in C^1([0,T], X)$ and satisfies the differential equation in (2).

Theorem 3. *Let $\mu > 0$ be fixed and assume that (A1)-(A7) are satisfied. Then problem (2) is locally well posed in the sense described in the introduction.*

Proof. We will show that we can choose $T > 0$ suffiently small so that the map

$$(\Psi v)(t) = \exp(-\mu tA)\, u_0 + \int_0^t \exp(-\mu(t-t')A)\,(F(v(t')) + f(t'))\, dt'. \tag{7}$$

is a contaction in the complete metric space $(\Xi(T), d)$ defined by

$$\begin{aligned} \Xi(T) &= \{v \in C([0,T], H)\, \|v(t) - \exp(-\mu tA)\, u_0\| \leq M\} \\ d(v,w) &= \sup_{[0,T]} \|v(t) - w(t)\|_H\,, \end{aligned} \tag{8}$$

where $M > 0$ a fixed constant. It is easy to see that if $v \in \Xi(T)$ then $\Phi v \in$

$C([0,T],H)$ for any $T>0$. Next note that

$$\begin{aligned}
&\left\|(\Psi v)(t)-\exp(-\mu tA)u_0\right\|_H \\
&\le \int_0^t \left\|\exp(-\mu(t-t')A)(F(v(t'))+f)\right\|_H dt' \\
&\le \int_0^t g(t-t')\left(\left\|F(v(t'))\right\|_X+\left\|f(t')\right\|_X\right)dt' \\
&\le \int_0^t g(t-t')\left[\gamma_o\left(\left\|v(t')\right\|_H,0\right)\left\|v(t')\right\|_H+\left\|f(t')\right\|_X\right]dt'.
\end{aligned} \tag{9}$$

$$\begin{aligned}
\left\|v(t')\right\|_X &\le \left\|v(t')-\exp(-\mu tA)\ u_0\right\|_X+\left\|u_0\right\|_X \\
&\le M+\left\|\exp(-\mu tA)\ u_0\right\|_X \\
&\le M+\left\|u_o\right\|_H,\ \text{ for all } t\in[0,T].
\end{aligned} \tag{10}$$

Therefore

$$\left\|(\Psi v)(t)-\exp(-\mu tA)u_0\right\|_H \le \ \Theta(M,u_0,f)\int_0^T g(t-t')\,dt', \quad \text{for all } t\in[0,T] \tag{11}$$

where

$$\Theta(M,u_0,f)=\left[\gamma_o\left(M+\left\|u_0\right\|_X,0\right)\left(M+\left\|u_0\right\|_X\right)+\sup_{[0,T]}\left\|f(t')\right\|_X\right]. \tag{12}$$

Since the right hand side of this inequality approaches zero as T tends to zero, we may choose $T>0$, sufficiently small, such that

$$\left\|(\Psi v)(t)-\exp(-\mu tA)u_0\right\|_H \le M \text{ for all } t\in[0,T] \text{ and } v\in\Xi(T). \tag{13}$$

A similar computation shows that we can choose $T>0$, small enough so that

$$d(\Psi v,\Psi w)\le \alpha d(v,w)\,,\text{ for all } v,w\in\Xi(T)\,, \tag{14}$$

with $\alpha\in(0,1)$. Banach's fixed point theorem implies existence and uniqueness in $\Xi(T)$. Uniqueness and continuous dependence in $C([0,T],H)$ follow from standard arguments. (See [2], [7]and [4] for example.) □

Following Kato ([8]), we now assume that Φ and J satisfy
(H1) $\Phi\in C(H,\mathbb{R})$, $\Phi(0)=0$ and $\Phi'\in C(H,H^*)$.
(H2) $J\in\mathcal{B}(H^*,X)\cap\mathcal{B}(X^*,H)$ and is skew-symmetric in the following sense

$$\langle Jx,h\rangle=-\langle x,Jh\rangle\,,\text{for all } x\in X^* \text{ and } h\in H^*.$$

Assume also that

(H3) There exists a continuous function $\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}$, nondecreasing with respect to each of its arguments, such that

$$\|\Phi'(v) - \Phi'(w)\|_{H^*} \leq \gamma(\|v\|_H, \|w\|_H) \|v - w\|_H, \tag{15}$$

for all $v, w \in H$.

It follows that (A3) is satisfied with $F(v) = J\Phi'(v)$ and Theorem 3 implies the local well posedness of (1) at once.

Corollary 4. *Assume (A1), (A3)-(A-7) and (H1)-(H3) (so that (A2) is automatically satisfied). Then (1) is locally well posed in the sense described in the introduction.*

Remark 5. *Under appropriate condition it is possible to take the limit as $\mu \to 0^+$ and construct a solution of (1) with $\mu = 0$. Since this is not relevant for our proposes in this work we omit such results. (See Chapter 6 of [10] case of the generalized KdV equation is considered.)*

Next we discuss an abstract regularity result. To do this we introduce further assumptions. Let V be a real Banach space such that $V \hookrightarrow H$ and

(R1) $\exp(-\mu t A) \in \mathcal{B}(X, V)$ for all $t > 0$ and satisfies an estimate of the form

$$\|\exp(-\mu t A) x\|_V \leq \widetilde{g}(t) \|x\|_X \tag{16}$$

for all $x \in X$ where $\widetilde{g} \in L^1_{loc}([0, \infty))$.

(R2) The map $t \in (0, \infty) \longmapsto \exp(-\mu t A)$ is continuous for each fixed $x \in X$.

Theorem 6. *Assume that (A1)-(A7) and (R1), (R2) are satisfied. Then the solution constructed in Theorem 3 satisfies*

$$u \in C((0, T], V) \cap C^1([0, T], X). \tag{17}$$

Moreover, if F maps V into H and $A \in \mathcal{B}(V, H)$, then

$$\partial_t u \in C^1((0, T], H). \tag{18}$$

Proof. We already know that $u \in C^1([0, T], X)$, and it is easy to verify that, under the present assumptions, (6) implies $u \in C((0, T], V)$. An argument similar to the one employed in the proof that a solution of (6) is differentiable in the X topology, implies our last statement. $\square$

This result implies at once the following corollary.

Corollary 7. *Assume that the hypothesis of Corollary 4 are satisfied and assume further that $\Phi' \in C(V, X)$ and $J \in \mathcal{B}(X^*, H)$. Then the solution of (1) obtained in Corollary 4 satisfies*

$$u \in C([0, T], X) \cap C((0, T], V) \cap C^1([0, T], X) \text{ and } \partial_t u \in C^1((0, T], H). \tag{19}$$

Now we turn to global well posedness. Assume that
(G1) There exists a real, bilinear map $[\cdot,\cdot] : H \times X \longrightarrow \mathbb{R}$ that defines a real inner product in H and satisfies the estimate

$$\|h\|_X^2 \leq C\,[h,h]\,,\ \text{for all } h \in H, \tag{20}$$

and

$$[h, Ah] \geq 0 \text{ for all } h \in H \ \ (\ A \text{ is dissipative in } H). \tag{21}$$

(G2) Let Y be the completion of H with respect to $[\cdot,\cdot]$, we write $\|y\|_Y^2 = [y,y]$, $y \in H$, its easy to see that $H \subset Y \subset X$ continuously and densely.

(G3) $\langle h,\, y\rangle_{HH^*} = [y,\, h]$, $y \in Y$,$h \in H$.

(G4) $\Phi(u) \leq C_{11}\ \|u\|_H^2 + C_{21}\ \|u\|_Y^{p_1}\ \|u\|_H^{q_1}$, $0 \leq p_1$, $0 \leq q_1 < 2$ and $u \in H$.

(G5) $\Phi(u) \geq C_{12}\ \|u\|_H^2 - C_{22}\ \|u\|_Y^2 - C_{32}\ \|u\|_Y^{p_2}\ \|u\|_H^{q_2}$, $0 \leq p_2$, $0 \leq q_2 < 2$ and $u \in H$.

(G6) $[u, J\,\Phi'(u)] \leq C_{13}\ \|u\|_Y^2 + C_{23}\ \|u\|_Y^{p_3}\ \left\|A^{1/2}\,u\right\|_Y^{q_3}$, $0 \leq p_3 < (2-q_3)$, $0 \leq q_3 < 2$ and $u \in H$.

(G7) $-[\Phi'(u), Au] \leq -C_{14}\|u\|_V^2 + C_{24}\|u\|_H^{p_4}\|u\|_V^{q_4} + C_{34}\|u\|_H^{p_5}\|u\|_Y^{q_5}$, $0 \leq p_4 < (2-q_4)$, $0 \leq q_4 < 2$, $0 \leq p_5 < 2$, $0 \leq q_5$ and $u \in V$.

(G8) $[\Phi'(u), f] \leq C_{15}\|u\|_H^2 + C_{25}\|f\|_H^2 + C_{35}\|f\|_V^{p_6}\|u\|_Y^{q_6}\|u\|_H^{r_6}$ $\quad 0 \leq p_6$, $0 \leq q_6$, $0 \leq r_6 < 2$, $u \in H$ and $f \in V$.

(G9) There exists a constant $C_{16} > 0$ such that

$$\|J\,\phi\|_X \leq C_{16}\left\|A^{1/2}\phi\right\|_H, \quad \forall \phi \in H.$$

Theorem 8. *Suppose that the hypotheses of Corollary 7, conditions (G1)-(G8) are satisfied. Then if $u_o \in H$, there exists a unique $u \in C([0,+\infty),H) \cap C((0,+\infty),V)$ such that $u(0) = u_o$, $\partial_t u \in C([0,+\infty),X) \cap C((0,+\infty),H)$ and (2) is satisfied.*

Proof. From

$$\partial_t\,u = -\mu\,Au + J\Phi'(u) + f, \quad t > 0, \ \ u(t) \in V, \ \ \partial_t u \in H$$

$$\begin{aligned}\partial_t\Phi(u) &=< u_t, \Phi'(u) >_{HH^*}= \big[\Phi'(u), u_t\big]\\ &= \big[\Phi'(u), -\mu Au + J\Phi'(u) + f\big]\\ &= -\mu\big[\Phi'(u), Au\big] + \big[\Phi'(u), J\Phi'(u)\big] + \big[\Phi'(u), f\big]\\ &\leq \mu\Big[-C_{11}||u||_V^2 + C_{21}||u||_Y^{p_4}||u||_V^{q_4} + C_{31}||u||_H^{p5}||u||_Y^{q_5}\Big]+\\ &+\big[\Phi'(u), J\Phi'(u)\big] + [\Phi'(u), f]\end{aligned}$$

Since $J \in B(H^*, X) \cap B(X^*, H)$, $\Phi' \in C(V, X^*)$ and

$$< Jx, y >_{HH^*}= - < x, Jy >_{X^*X}, \quad \text{for all } x \in X^*, \text{ and } y \in H^*$$

then $\big[\Phi'(u), J\Phi'(u)\big] = 0$. Hence

$$\begin{aligned}\partial_t\Phi(u) \leq \quad &\Big[-C_{11}||u||_V^2 + C_{21}||u||_Y^{p_4}||u||_V^{q_4} + C_{31}||u||_H^{p_5}||u||_Y^{q_5}\Big]+\\ &C_{41}||u||_H||f||_H + C_{51}||f||_Y^{p_6}||u||_Y^{q_6}||u||_H^{r_6}.\end{aligned} \tag{22}$$

For $t > 0$ we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}||u||_Y^2 &= [u_t, u] = \big[-\mu Au + J\Phi'(u) + f, u\big]\\ &= -\mu[u, Au] + [u, J\Phi'(u)] + [f, u]\\ &= -\mu||A^{1/2}u||_Y^2 + C_{12}||u||_Y^2 + C_{22}||u||_Y^{p_3}||A^{1/2}u||_Y^{q_3} + ||f||_Y||u||_Y\end{aligned}$$

Since $q_3 < 2$ we obtain

$$\frac{d}{dt}||u||_Y^2 \leq C_{13}||u||_Y^2 + C_{23}(\mu) + ||f||_Y^2$$

Then

$$||u(t)||_Y \leq C(\mu, ||u_o||, ||f||_Y, t) \tag{23}$$

The a priori estimates follows from (22) and (23) □

Theorem 9. *Under the hypotheses of Theorem 8 and condition (G9), for any $T > 0$ the map*

$$\begin{aligned}\Lambda\colon H &\to C([0, T]; H)\\ u_o &\mapsto u\end{aligned}$$

is continuous.

Proof. Let u, v be solutions of the equation (2) with $u(0) = u_0$ and $v(0) = v_0$. Define $w = u - v$, thus w satisfies the equation

$$\partial_t w + \mu A w + J(\Phi'(u) - \Phi'(v)) = 0$$

Moreover

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|w\|_H^2 &= (w, w_t)_H \\
&= (w, -\mu A w + J(\Phi'(u) - \Phi'(v)))_H \\
&= -\left\|A^{1/2}w\right\|_H^2 + (w, J(\Phi'(u) - \Phi'(v)))_H \\
&\le -\left\|A^{1/2}w\right\|_H^2 - \langle Jw, \Phi'(u) - \Phi'(v)\rangle \\
&\le -\left\|A^{1/2}w\right\|_H^2 + \|\Phi'(u) - \Phi'(v)\|_{H^*}\|Jw\|_X \\
&\le -\left\|A^{1/2}w\right\|_H^2 + \gamma(\|u\|_H, \|v\|_H)\|Jw\|_X\|w\|_H \\
&\le -\left\|A^{1/2}w\right\|_H^2 + \gamma(\sup_{[0,t]}\|w\|_H, \sup_{[0,T]}\|v\|_H)\|Jw\|_X\|w\|_H \\
&\le -\left\|A^{1/2}w\right\|_H^2 + C_1(\mu, T, \|u_0\|_H, \|v_0\|_H, \|f\|_H)\|Jw\|_X\|w\|_H
\end{aligned}$$

Using condition (G9), it follows that

$$\frac{1}{2}\frac{d}{dt}\|w\|_H^2 \le C_3(\mu, T, \|u_0\|_H, \|v_0\|_H, \|f\|_H)\|w\|_H^2$$

the theorem follows from Gronwall's inequality. □

3 The Global Attractor.

We now introduce the following conditions
(G10) Exists a constant $C_{17} > 0$ such that

$$\left\|A^{1/2}\phi\right\|_Y^2 \ge C_{17}\|\phi\|_Y^2, \quad \forall\phi \in H$$

(G11) There are a constants $0 < \theta_o < 1$, $C_{18} > 0$ such that

$$\|\phi\|_H \le C_{18}\|\phi\|_Y^{\theta_0}\|\phi\|_V^{1-\theta_0}, \quad \forall\phi \in H$$

Proposition 10. *Assume that the hypotheses of Theorem (8) and condition (G10) are satisfied. Let the solution operator $\{S(t)\}_{t\ge 0}$ in H associated with the equation (2). If $f \in V$ there exist a constant $\rho_0 = \rho_0(\mu, ||f||_V)$ such that for every $R > 0$ there exist $T(R)$ such that*

$$||S(t)\,u_o||_Y \le \rho_0, \quad \forall u_o \in H, \quad ||u_o||_Y \le R, \quad \forall t \ge T(R). \tag{24}$$

Proof. We start with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,||u(t)||_Y^2 &= -\mu\,[u, Au] + [u, J\Phi'(u)] + [f, u] \\ &\leq -\mu\left\|A^{1/2}u\right\|_Y^2 + C_{13}\|u\|_Y^2 + C_{23}\|u\|_Y^{p_3}\left\|A^{1/2}u\right\|_Y^{q_3} + [f, u]\,.\end{aligned} \tag{25}$$

Since $q_3 < 3$ and for $\theta > 0$

$$-\theta\left\|A^{1/2}u\right\|_Y^2 + C_{23}\|u\|_Y^{p_3}\left\|A^{1/2}u\right\|_Y^{q_3} \leq C(\theta)\|u\|_Y^{\frac{2p_3}{2-q_3}}.$$

Then if $0 < \theta < \mu$, we obtain

$$\frac{1}{2}\frac{d}{dt}\,\|u(t)\|_Y^2 \leq -(\mu-\theta)\left\|A^{1/2}u\right\|_Y^2 - \theta\left\|A^{1/2}u\right\|_Y^2 + C_{13}\|u\|_Y^2 + C_{23}\|u\|_Y^{p_3}\left\|A^{1/2}u\right\|_Y^{q_3} + [f, u]$$

For α, β positive constant, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_Y^2 + \left(C_{17}(\mu-\theta) - C_{13} - \alpha - \beta\right)\|u\|_Y^2 \leq C(\theta, \alpha) + \frac{1}{\beta}\|f\|_Y^2$$

Choose μ, θ, α and β so that

$$C_{17}(\mu-\theta) - C_{13} - \alpha - \beta > 0.$$

This implies the existence of absorbing set in Y. □

Lemma 11. *Assume that the hypotheses of proposition (10) and condition (G11) are satisfied. Then exists absorbing set $\mathcal{B}$ in H.*

Proof. From

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\Phi(u(t)) + \mu\,\Phi(u(t)) &= \mu\big\{\Phi(u) - [\Phi'(u), Au]\big\} + [\Phi'(u), f] \\ &\leq \mu\Big\{C_{11}\|u\|_H^2 + C_{21}\|u\|_Y^{p_1}\|u\|_H^{q_1} - C_{14}\|u\|_V^2 + \\ &\quad C_{24}\|u\|_Y^{p_4}\|u\|_V^{q_4}C_{34}\|u\|_H^{p_5}\|u\|_Y^{q_5}\} + C_{15}\|u\|_H^2 + \\ &\quad C_{25}\|f\|_H^2 + C_{35}\|f\|_Y^{p_6}\|u\|_Y^{q_6}\|u\|_H^{r_6}.\end{aligned} \tag{26}$$

The lemma follows due to conditions (G4), (G11), (24) and the assumptions on q_1, q_4, p_5, q_5, q_6 and r_6 are < 2. □

Let $\{S(t)\}_{t\geq 0}$ denote the solution operator associated with (2) defined in H. It is a semigroup of continuous (nonlinear) operators in H, i.e. $\{S(t)\}_{t\geq 0}$ satisfies

$$\begin{cases} S(t+s) = S(t)\circ S(s), & \forall t,\, s \geq 0, \\ S(0) = I, & \end{cases} \tag{27}$$

and $S(t)$ is a continuous (nonlinear) operator from H into itself for any $t \geq 0$. We now introduce the following conditions:
(W1) $\Phi' \in C_w(H, H^*)$ (is weakly continuous in the sense that if $\{u_j\}_j \subset H$ and $u_j \rightharpoonup u$ in H, then $\Phi'(u_j) \rightharpoonup \Phi'(u)$ in H^*). [2]

Theorem 12. *Assume that the hypotheses of Theorem (8) and condition (W1). Then $S(t)$ is weakly continuous in H for each $t \geq 0$.*

Proof. Let ϕ, $\{\phi_n\}_n$, $n \in \mathbb{Z}^+$ belong to H and suppose that $\phi_n \rightharpoonup \phi$. Let $T > 0$ and $R > 0$ be such that $\|\phi\|_H \leq R$ and $\|\phi_n\|_H \leq R$ for all $n \in \mathbb{N}$.
Let the complete metric space $(\Xi(T, \psi), d)$ defined by

$$\begin{aligned} &\Xi(T, \psi) = \{v \in C([0, T], H) : \|v(t) - \exp(-\mu t A)\psi\|_H \leq M, \ \ \forall t \in [0, T]\} \\ &d(v, w) = \sup_{[0, T]} \|v(t) - w(t)\|_H \end{aligned} \tag{28}$$

where $M > 0$ is a fixed constant. Let the map

$$(\Psi(\psi, v))(t) = \exp(-\mu t A)\psi + \int_0^t \exp(-\mu(t - t')A) \ (J\Phi'(v(t')) + f) \ dt'.$$

for ψ such that $\|\psi\|_H \leq R$ and $v \in \Xi(T, \psi)$. We will show that we can choose $T > 0$ sufficiently small so that the map $\Psi(\psi, v)$ is a contraction in the complete metric space $(\Xi(T, \psi), d)$, where T is uniform in ψ with $\|\psi\|_H \leq R$, i. e. T is a constant depending only on R. Next note that

$$\begin{aligned} &\|(\Psi(\psi, v))(t) - \exp(-\mu t A)\psi\|_H \\ &\leq \int_0^t \left\|\exp(-\mu(t - t')A)\left[J\,\Phi'(v(t')) + f\right]\right\|_H dt' \\ &\leq \int_0^t g(t - t')\left(\|J\,\Phi'(v(t'))\|_X + \|f\|_X\right) dt' \\ &\leq \int_0^t g(t - t')\left\{\gamma\left(\|v(t')\|_H, 0\right)\|v(t')\|_H + \|f\|_X\right\} dt'. \end{aligned} \tag{29}$$

$$\begin{aligned} \|v(t')\|_X &\leq \|v(t')\|_H \\ &\leq \|v(t') - \exp(-\mu t A)\psi\|_H + \|\exp(-\mu t A)\psi\|_H \\ &\leq M + R, \end{aligned} \tag{30}$$

for all $t \in [0, T]$.
Therefore

$$\|(\Psi(\psi, v))(t) - \exp(-\mu t A)\psi\|_H \leq \Theta(M, R, f) \int_0^T g(s)\, ds, \text{for all } t \in [0, T] \tag{31}$$

[2] $\rightharpoonup$ denote weak convergence.

where

$$\Theta\left(M,\psi,f\right)=\left[\gamma\left(M+R,0\right)\left(M+R\right)+\|f\|_X\right]. \tag{32}$$

Since the right hand side of this inequality approaches zero as T tends to zero, we may choose $T_0>0$, sufficiently small, such that

$$\|(\Psi(\psi,\,v))\,(t)-\exp\left(-\mu tA\right)\psi\|_H\leq M \text{ for all } t\in[0,T_0] \text{ and } v\in\Xi\left(T_0,\,\psi\right). \tag{33}$$

Where $T_0=T_0\left(M,\,R,\,f\right)$ only depends on M, R and $\|f\|_X$.
Now we will show that we can choose $T_0>0$ uniformly in ψ, small enough so that

$$d\left(\Psi(\psi,\,v),\Psi(\psi,\,w)\right)\leq\alpha\,d\left(v,w\right), \text{ for all } v,w\in\Xi\left(T_0,\,\psi\right), \tag{34}$$

with $\alpha\in(0,1)$.
Next note that

$$\left(\Psi(\psi,v_2)\right)(t)-\left(\Psi(\psi,v_1)\right)(t)=\int_0^t\exp\left(-\mu(t-t')A\right)\,J\left[\Phi'\left(v_2(t')\right)-\Phi'\left(v_1(t')\right)\right]dt'. \tag{35}$$

Then

$$\|\left(\Psi(\psi,\,v_2)\right)(t)-\left(\Psi(\psi,\,v_1)\right)(t)\|_H\leq\gamma(M+R,\,M+R)\left(\int_0^{T_0}g(s)\,ds\right)d(v_2,\,v_1). \tag{36}$$

for all v_1, v_2 in $\Xi(\psi,\,T_0)$ and $\|\psi\|_H\leq R$.
Then exist $T_1\leq T_0$ such that, $T_1=T_1\left(M,\,R,\,f\right)$ only depends on M, R and $\|f\|_X$, moreover

$$\alpha=\gamma\left(M+R,\,M+R\right)\int_0^{T_1}g(s)\,ds<1.$$

Note that α depends only on M and R so that this condition is independent of ψ, with $\|\psi\|_H<R$. Then exits $u\in C\left([0,\,T_1],\,H\right)$ solution to equation (1) with $u(0)=\psi$ and $\|\psi\|_H\leq R$. By the uniqueness we obtain that

$$u=S(\cdot)\psi=\lim_{k\to+\infty}\Psi_k(\psi,\,v), \text{ in } \Xi\left(\psi,\,T_1\right).$$

uniformly in ψ with $\|\psi\|_H\leq R$, where

$$\Psi_k(\psi,\,v)=\begin{cases}v, & if\;\; k=0\\ \\ \Psi\left(\psi,\,\Psi_{k-1}(\psi,\,v)\right), & if\;\; k\geq1.\end{cases} \tag{37}$$

Since $J \in \mathcal{B}\big(H^*, X\big)$ and $\exp\left(-\mu\tau A\right) \in \mathcal{B}\big(X, H\big)$ for $\tau > 0$, then the condition (W1) implies the map

$$v(t') \mapsto \quad G(t,\, t') = \exp\left(-\mu(t-t')\, A\right)\, \left[J\Phi'\left(v(t')\right) + f\right]$$

is weakly continuous in H, for each $0 \le t' < t \le T_1$.
Given that $(t,\, t') \mapsto G(t, t')$ is continuous in H for $0 \le t' < t \le T_1$ and

$$t' \mapsto \chi_{[0,\, t]}\, \|G(t,\, t')\|_H \;\text{ is }\; L^1([0, T_1]).$$

for each $0 < t \le T_1$, it follows that

$$\int_0^t \exp\left(-\mu(t-t')A\right)\, \left(J\Phi'(v(t')) + f\right)\, dt' \tag{38}$$

is a Bochner integral. Hence, the weakly continuity the map

$$v \mapsto \int_0^t \exp\left(-\mu(t-t')\, A\right)\, \left[J\Phi'\left(v(t')\right) + f\right] dt'. \tag{39}$$

follows from the Dominated Convergence Theorem, for each $t \in [0,\, T_1]$.
We choose now $v_0(t) = \left(\Psi_0(\psi,\, v_0)\right)(t) = \exp\left(-\mu t A\right)\, \psi$, then the map

$$\psi \mapsto \quad \Psi_k(\psi,\, v_0). \tag{40}$$

is weakly continuous in H for each $k \in I\!\!N$.

On the other hand,

$$\langle g, S(t)\phi_n\rangle - \langle g, S(t)\phi\rangle = \langle g, S(t)\phi_n\rangle - \langle g, \left(\Psi_k(\phi_n,\, v_0^n)\right)(t)\rangle + \tag{41}$$
$$\langle g, \left(\Psi_k(\phi_n,\, v_0^n)\right)(t)\rangle - \langle g, \left(\Psi_k(\phi,\, v_0)\right)\rangle + \langle g, \left(\Psi_k(\phi,\, v_0)\right)(t)\rangle - \langle g, S(t)\phi\rangle.$$

Where $\langle\cdot,\cdot\rangle$ denote the duality pairing associated to $(H^*,\, H)$, $g \in H^*$ and $v_0^n(t) = \exp\left(-\mu t A\right)\phi_n$.
Notice that $\Psi_k(\phi_n,\, v_0^n) \to S(t)\phi_n$ and $\Psi_k(\phi,\, v_0) \to S(t)\phi$, strongly in H and uniformly in n, as $k \to +\infty$. Then for $\epsilon > 0$ exist $k_o \in I\!\!N$ such that

$$|\langle g, S(t)\phi_n\rangle - \langle g, \left(\Psi_{k_0}(\phi_n,\, v_0^n)\right)(t)\rangle| < \frac{\epsilon}{2} \tag{42}$$

$$\text{and} \quad |\langle g, S(t)\phi\rangle - \langle g, \left(\Psi_{k_0}(\phi,\, v_0)\right)(t)\rangle| < \frac{\epsilon}{2}, \tag{43}$$

for all $n \in I\!\!N$.
Note also that $\Psi_{k_0}(\phi_n,\, v_0^n) \rightharpoonup \Psi_{k_0}(\phi,\, v_0)$, weakly in H, as $n \to +\infty$, then

$$\limsup_{n\to+\infty} |\langle g, S(t)\phi_n\rangle - \langle g, S(t)\phi\rangle| \le \epsilon \;\text{ for all }\; \epsilon > 0. \tag{44}$$

Then $S(t)\phi_n \rightharpoonup S(t)\phi$ weakly in H, for each $t \in [0,\, T_1]$.

If $m\,T_1 \le t < (m+1)\,T_1$ for some $m \in I\!N$. Since

$$S(t)\phi = S(T_1) \circ S(T_1) \circ \cdots \circ S(T_1) \circ S(r)\,\phi$$

where $r = t - m\,T_1$, the proof is complete. □

We now introduce the following conditions:

(AS1) $\Phi = \Phi_0 + \Phi_1$ such that $\Phi_0 \colon H \to I\!R^+$ is continuous, is bounded on bounded subsets of H and if $\{u_j\}_j$ is bounded in H, $\{t_j\} \subset I\!R^+$ with $t_j \to +\infty$, such that $S(t_j)\,u_j \rightharpoonup w$ weakly in H, $S(t_j)\,u_j \to w$ strongly in Y, and

$$\limsup_{j\to+\infty} \Phi_0\left(S(t_j)u_j\right) \le \Phi_0(w),$$

then

$$S(t_j)u_j \to w \quad \text{strongly in } H.$$

(AS2) $\Phi_1 \colon H \to I\!R$ is *asymptotical weakly continuous* in the sense that if $\{u_j\}_j$ is bounded in H, $\{t_j\} \subset I\!R^+$, $t_j \to +\infty$, $S(t_j)\,u_j \rightharpoonup w$ weakly in H, then

$$\Phi_1\left(S(t_j)u_j\right) \to \Phi_1(w).$$

(AS3) $K \colon \bigcup_{t>0} S(t)H \to I\!R$,

$$K(u) \;=\; [u, J\Phi'(u)] + [f, u]$$

is *asymptotically weakly continuous* in the sense that if $\{u_j\}_{j\in N}$ is bounded in H, $\{t_j\}_{j\in N} \subset I\!R^+$, $t_j \to +\infty$, and $S(t_j)\,u_j \rightharpoonup w$ weakly in H, then

$$\lim_{j\to\infty} \int_0^t e^{-\mu(t-s)}\,K(S(s+t_j)u_j)\,ds \;=\; \int_0^t e^{-\mu(t-s)}\,K(S(s)w)\,ds, \quad \forall t > 0$$

where it is assumed that $s \mapsto K\left(S(s)u\right)$ belongs to $L^1(0,\,t)$, for each $t > 0$.
(AS4) $L \colon \bigcup_{t>0} S(t)H \to I\!R$,

$$L(u) \;=\; \mu\left([u, Au] - ||u||_Y^2\right).$$

is *asymptotically weakly lower semi-continuous* in the sense that if $\{u_j\}_{j\in N}$ is bounded in H, $\{t_j\}_{j\in N} \subset I\!R^+$, $t_j \to +\infty$, and $S(t_j)\,u_j \rightharpoonup w$ weakly in H, then

$$\int_0^t e^{-\mu(t-s)}\,L(S(s)w)\,ds \le \liminf_{j\to\infty} \int_0^t e^{-\mu(t-s)}\,L(S(s+t_j)u_j)\,ds\,, \quad \forall t > 0$$

where it is assumed that $s \mapsto L\left(S(s)u\right)$ belongs to $L^1(0,\,t)$, for each $t > 0$.
(AS5) $G \colon \bigcup_{t>0} S(t)H \to I\!R$,

$$G(u) \;=\; \big[\Phi'(u), f\big]$$

is asymptotically weakly continuous in the same sense of K.
(AS6) $M\colon \bigcup_{t>0} S(t)H \to I\!R$,

$$M(u) \;=\; \mu\left([\Phi'(u), Au] - \Phi(u)\right).$$

is asymptotically weakly lower semi-continuous in the same sense of L.
For a set $B \subset H$, we define its ω-limit set by $\omega(B) \;=\; \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)\,B}$. It is easy to prove the following well known characterization of an ω-limit set:

$$w \in \omega(B) \Longleftrightarrow \exists \{w_j\}_{j\in N} \subset B,\ \exists \{t_j\}_{j\in N} \subset I\!R^+ \text{ such that } t_j \to +\infty \text{ and } S(t_j)\,w_j \to w \text{ in } H.$$

A related concept is that of asymptotic compactness. One says that $\{S(t)\}_{t\geq 0}$ is *asymptotically compact* in H if the following condition holds:

$$\begin{gathered}\text{If } \{u_j\}_{j\in N} \subset H \text{ is bounded and } \{t_j\}_{j\in N} \subset I\!R^+,\ t_j \to +\infty \\ \text{then } \{S(t_j)\,u_j\}_{j\in N} \text{ is precompact in } H.\end{gathered} \tag{45}$$

We can now proceed as in I. Moise, R. Rosa, and X. Wang.

Lemma 13. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup of continuous (nonlinear) operators in H such that $u(t) = S(t)u_o$ is the global solution to the equation (2), assume that hypotheses of Theorem (12), the conditions (AS3) and (AS4) are satisfied. It follows that if $\{u_n\}_{n\in N}$ is bounded in H and $\{S(t)\,u_n\}_{n\in N} \subset I\!R^+$, $t_n \to +\infty$, then $S(t_{n'})u_{n'} \to w$ strongly in Y for some $w \in H$ and some subsequence $\{n'\}$.*

Proof. For $t > 0$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}||u||_Y^2 \quad &= [u_t, u] = \big[-\mu A u + J\Phi'(u) + f, u\big] \\ &= -\mu[u, Au] + [u, J\Phi'(u)] + [f, u].\end{aligned} \tag{46}$$

Then

$$\frac{1}{2}\frac{d}{dt}||u||_Y^2 + \mu\,||u||_Y^2 + L(u) \;=\; K(u). \tag{47}$$

Let then $\{u_n\}_{n\in N} \subset H$ be bounded and let $\{t_n\}_{n\in N} \subset I\!R^+$, $t_n \to +\infty$. Since $\{S(t_n)u_n\}_{n\in N}$ is bounded due to the existence of a bounded absorbing set $\mathcal{B}$ and H is reflexive, it follows that

$$S(t_{n'})\,u_{n'} \rightharpoonup w \;\text{ weakly in } H. \tag{48}$$

for some $w \in H$. Similarly, $\{S(t_{n'} - T)\,u_{n'}\}$ has a weakly convergent subsequence for each $T > 0$,

$$\{S(t_{n'} - T)\,u_{n'}\} \rightharpoonup w_T \;\text{ weakly in } H, \quad \forall T > 0 \tag{49}$$

with $w_T \in H$. Note then by the weak continuity of $S(T)$ that

$$w \;=\; S(T)\, w_T, \quad \forall T > 0 \tag{50}$$

Now, since the trajectories of $\{S(t)\}_{t\geq 0}$ are continuous, integrating we obtain the energy equation (47) from 0 to T with $u_o \;=\; S(t_{n'} \;-\; T)\, u_{n'}$ that

$$\begin{aligned} &||S(T)\, u_o||_Y^2 \;+\; 2\int_0^T e^{-2\mu(T-\tau)}\, L(S(\tau)\, u_o)\, d\tau = \\ &||u_o||_Y^2\, e^{-2\mu\, T} \;+\; 2\int_0^T e^{-2\mu(T-\tau)}\, K(S(\tau)\, u_o)\, d\tau, \end{aligned} \tag{51}$$

for all $T > 0$, for all $t_{n'} > T$. From (24) and the assumptions on K, we can take the lim sup in (51) to find

$$\begin{aligned} &\limsup_{n'\to+\infty} ||S(t_{n'})\, u_{n'}||_Y^2 + 2\int_0^T e^{-2\mu(T-\tau)}\, L(S(\tau)\, w_T)\, d\tau \leq \\ &\rho_o^2\, e^{-2\mu\, T} \;+\; 2\int_0^T e^{-2\mu(T-\tau)}\, K(S(\tau)\, w_T)\, d\tau, \end{aligned} \tag{52}$$

for all $T > 0$.
Combining the energy equation, now with $u_o \;=\; w_T$, and (50) we obtain

$$\begin{aligned} &||S(T) w_T||_Y^2 \;+\; 2\int_0^T e^{-2\mu(T-\tau)}\, L(S(\tau)\, w_T)\, d\tau = \\ &||w_T||_Y^2\, e^{-2\mu T} \;+\; 2\int_0^T e^{-2\mu(T-\tau)}\, K(S(\tau)\, w_T)\, d\tau. \end{aligned} \tag{53}$$

Subtract (53) from (52) to find

$$\limsup_{n'\to+\infty} ||S(t_{n'})\, u_{n'}||_Y^2 \leq ||w||_Y^2 \;+\; \rho_o^2\, e^{-2\mu\, T}, \quad \forall T > 0 \tag{54}$$

By letting $T \to +\infty$ we see that

$$\limsup_{n'\to+\infty} ||S(t_{n'})\, u_{n'}||_Y^2 \leq ||w||_Y^2 \tag{55}$$

which, together with the weak convergence (48), implies, since Y is a Hilbert space, the strong convergence

$$S(t_{n'})\, u_{n'} \to w, \text{ strongly in } Y \tag{56}$$

□

Theorem 14. *Suppose that the assumptions of lemma (13) and conditions (AS1), (AS2), (AS5) and (AS6) are satisfied. Then, the solution operator $\{S(t)\}_{t\geq 0}$, in H associated with equation (2) possesses a connected global attractor in H, i.e., a compact (in H), connected, invariant set which attracts all the orbits (in the H-metric) of the system, uniformly on bounded sets of initial conditions.*

Proof. For $t > 0$.

$$\begin{aligned} \partial_t \Phi(u) &= < u_t, \Phi'(u) >_{HH^*} = \big[\Phi'(u), u_t\big] \\ &= \big[\Phi'(u), -\mu A u + J\Phi'(u) + f\big] \\ &= -\mu\big[\Phi'(u), Au\big] + \big[\Phi'(u), J\Phi'(u)\big] + \big[\Phi'(u), f\big] \end{aligned} \tag{57}$$

Then

$$\partial_t \Phi(u) \,+\, \mu\, \Phi(u) \,+\, M(u) \,=\, G(u). \tag{58}$$

Let $\{u_n\}_{n\in N} \subset H$ be bounded and let $\{t_n\}_{n\in N} \subset I\!R^+$, $t_n \to +\infty$. We need to show that $\{S(t_n)u_n\}_{n\in N}$ is precompact in H. Since $\{S(t_n)u_n\}_{n\in N}$ is bounded due to the existence of a bounded absorbing set $\mathcal{B}$ and the space H is reflexive, it follows that

$$S(t_{n'})\, u_{n'} \rightharpoonup w \;\text{ weakly in } H. \tag{59}$$

for some $w \in H$. Similarly, $\{S(t_{n'} \,-\, T)\, u_{n'}\}$ has a weakly convergent subsequence for each $T > 0$,

$$\{S(t_{n'} \,-\, T)\, u_{n'}\} \rightharpoonup w_T \;\text{ weakly in } H, \quad \forall T > 0 \tag{60}$$

with $w_T \in H$. Note then by the weakly continuity of $S(T)$ that

$$w \,=\, S(T)\, w_T, \quad \forall T > 0 \tag{61}$$

Now, we repeat the arguments lemma above for Φ, we obtain by integration the energy equation (58) from 0 to T with $u_o \,=\, S(t_{n'} \,-\, T)\, u_n$ that

$$\begin{aligned} &\Phi\,(S(T)\, u_o) \,+\, 2\int_0^T e^{-2\mu(T-\tau)}\, M(S(\tau)\, u_o)\, d\tau = \\ &\Phi\,(u_o)\; e^{-2\mu\, T} \,+\, 2\int_0^T e^{-2\mu(T-\tau)}\, G(S(\tau)\, u_o)\, d\tau \end{aligned} \tag{62}$$

for all $T > 0$, for all $t_{n'} > T$.
Similarly for $u_o \,=\, w_T$:

$$\begin{aligned} &\Phi\,(S(T) w_T) \,+\, 2\int_0^T e^{-2\mu(T-\tau)}\, M(S(\tau)\, w_T)\, d\tau \,= \\ &\Phi\,(w_T)\; e^{-2\mu T} \,+\, 2\int_0^T e^{-2\mu(T-\tau)}\, G(S(\tau)\, w_T)\, d\tau. \end{aligned} \tag{63}$$

By subtracting (63) from (62) and the assumptions on Φ_1, G and M, taking the limit as n' goes to infinity, and using (60) and (61), we obtain

$$\limsup_{n'\to+\infty}\, \Phi_o\,(S(t_{n'})\, u_{n'}) \le \Phi_o\,(w) \,+\, C\, e^{-2\mu\, T}, \quad \forall T > 0 \tag{64}$$

where C is a constant independent of T. By letting T go to infinity in (64) we find that

$$\limsup_{n'\to+\infty}\, \Phi_o\,(S(t_{n'})\, u_{n'}) \le \Phi_o\,(w) \tag{65}$$

This, together with the strongly convergence in Y (56) and assumption (AS1), implies $S(t_{n'})\,u_{n'}$ converges strongly to w in H. This proves the asymptotic compactness property of the solution operator and, hence, the existence of the global attractor. □

4 Applications.

In the sequel we will use the following notations and definition

$J^s = (1-\Delta)^{s/2}$ and $D^s = (-\Delta)^{s/2}$ denote the Bessel and Riesz potentials of order $-s$, respectively.

$H^s(\mathbb{R}) = J^{-s}\,L^2(\mathbb{R})$ whose norm will be denoted by $||f||_s = \int\,(J^s f)^2\,dx$ and scalar product by $(f,\,g) = \int\, J^s f\, J^s g\,dx$ where the integral is over $\mathbb{R}^n$. In particular, $H^0 = L^2\,(\mathbb{R}^n)$.

We denote $\mathcal{F}\,f = \hat{f}$ and $\mathcal{F}^{-1}\,g = \check{g}$ denote Fourier transform and Fourier inverse transform of f, respectively. We will use these symbols in all contexts were the transforms are defined.

4.1 A class of fifth order model evolution equations.

This application is concerned with the initial-value problem for fifth-order evolution equation of the form

$$\begin{aligned} &u_t\,+\,\mu\,u_{xxx}\,+\,\alpha\,u_{xxxxx}\,+\,\beta\,uu_{xxx}\,+\,\delta\,u_x u_{xx}\,+\,q\,(u^2)_x\,+\,r\,(u^3)_x = \\ &u_t\,+\,\left[\,\mu\,u_{xx}\,+\,\alpha\,u_{xxxx}\,+\,\beta\,uu_{xx}\,+\,\gamma\,u_x^2\,+\,q\,u^2\,+\,r\,u^3\,\right]_x\,=\,0, \end{aligned} \tag{66}$$

where $\delta = \beta + 2\gamma$. Here, μ, α, β, δ, γ, q and r are constants, $u = u(x,t)$ is a real-valued function (see [9]). In this work we study the existence of the global attractor in the space phase $H^s(\mathbb{R})$ of a damped, forced fifth order evolution equation of the form

$$\begin{cases} u_t + \left[\,\mu\,u_{xx}\,+\,\alpha\,u_{xxxx}\,+\,\beta\,uu_{xx}\,+\,\frac{\beta}{2}\,u_x^2\,+\,q\,u^2\,+\,r\,u^3\,\right]_x + \\ \epsilon\left(D^\lambda u + u\right) = f. \end{cases} \tag{67}$$

where $D = \left(-\partial_x^2\right)^{1/2}$ and real $\lambda > 0$. First we write the equation as the perturbation of a Hamiltonian

$$\partial_t u = -\epsilon A u + J\Phi_1'\,(u) + f. \tag{68}$$

where

$$A = D^\lambda + \frac{\mu}{\epsilon}\,\partial_x^3 + \frac{\alpha}{\epsilon}\,\partial_x^5, \quad J = -\partial_x,$$

$$\Phi_1(u) = \int_{-\infty}^{+\infty} \left\{ -\frac{\beta}{2}\, u\, u_x + \frac{q}{3}\, u^3 + \frac{r}{4}\, u^4 + \frac{\epsilon}{2}\, u^2 \right\} dx$$

and

$$\Phi_1'(u) = \frac{\beta}{2}\, u_x^2 + \beta\, u\, u_{xx} + q\, u^2 + r\, u^4 + \epsilon\, u.$$

Set $H = H^s(I\!R)$, $X = H^{s-5}(I\!R)$ for $s \geq 2$ and $\lambda = 4$. Then the properties (A1), (A3)-(A7) and (H1) -(H3) are satisfied, so that problem (67) is locally well possed in $H^s(I\!R)$ for all $s \geq 2$, i.e. exist $T > 0$ and a unique

$$u \in C\big([0,T], H^s(I\!R)\big) \cap C\big((0,T], H^\infty(I\!R)\big) \text{ such that } \partial_t u \in C\big([0,T], H^{s-5}\big)$$

and satisfies (67). Moreover, if $V = H^{s+1}(I\!R)$ the conditions (R1), (R2) are satisfied and $\Phi_1' \in C\,(V,\, X)$, then exist $T_1 > 0$ such that the solution of (67) satisfies

$$u \in C\big((0,T_1], H^{s+1}(I\!R)\big) \cap C^1\left([0,\, T_1],\, H^{s-5}(I\!R)\right) \text{ and } \partial_t u \in C\big((0,T_1], H^s(I\!R)\big), \tag{69}$$

for all $s \geq 2$.

Now we turn to global well posedness in $H^2(I\!R)$, we write the equation in Hamiltonian form

$$u_t = -\epsilon\, Bu + J\, \Phi_2'(u) + f \tag{70}$$

where

$$B = D^\lambda + 1, \quad \lambda > 0, \quad J = -\partial_x$$

and

$$\Phi_2(u) = \int_{-\infty}^{+\infty} \left\{ \frac{\alpha}{2}\, u_{xx}^2 - \frac{\mu}{2}\, u_x^2 - \frac{\beta}{2}\, u\, u^2 + \frac{q}{3}\, u^3 + \frac{r}{4}\, u^4 \right\} dx.$$

with

$$\Phi_2'(u) = \mu\, u_{xx} + \alpha\, u_{xxxx} + \beta\, u u_{xx} + \frac{\beta}{2}\, u_x^2 + q\, u^2 + r\, u^3.$$

We use two real reflexive Banach spaces $X = H^{-3}(I\!R)$, $H = H^2(I\!R)$, $Y = L^2(I\!R)$ and $V = H^{2+s}(I\!R)$, for $0 < s < 2$. Defining $[f,\, g] = \int_{-\infty}^{+\infty} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi$, for all $f,\, g \in S(I\!R)$, then are satisfied the conditions (G1), (G2) and (G3). Moreover

$$\Phi_2(u) \leq \left(\frac{|\alpha| + |\mu|}{2}\right) \|u\|_2^2 + C\, \left(\frac{\beta}{2} \|u\|_0^{7/4}\, \|u\|_2^{5/4} + |q|\, \|u\|_0^{11/4}\, \|u\|_2^{1/4} + |r|\, \|u\|_0^{7/2}\, \|u\|_2^{1/2}\right), \tag{71}$$

and for $\alpha > 0$

$$\Phi_2(u) \geq \frac{\alpha}{2} \|u\|_2^2 + C \left\{ -\frac{1}{4} (|\mu| + \mu) \|u\|_0 \|u\|_2 - \frac{1}{4} (|\alpha - \mu| + \alpha - \mu) \|u\|_0^2 - \frac{|\beta|}{2} \|u\|_0^{7/4} \|u\|_2^{5/4} - \frac{|q|}{3} \|u\|_0^{11/4} \|u\|_2^{1/4} - \frac{|r|}{4} \|u\|_0^{7/2} \|u\|_2^{1/2} \right\}. \tag{72}$$

Then Φ_2 satisfies (G4) and (G5). Integrate by parts to obtain

$$\int_{-\infty}^{+\infty} u\, \partial_x \Phi_2(u)\, dx = 0.$$

Then Φ_2 satisfies the condition (G6).
The argument to show that Φ_2 satisfies (G7) and (G8) is similar. Since

$$\|\partial_x \phi\|_{-3} \leq C \left\| \left(1 + D^4\right)^{1/2} \phi \right\|_2,$$

(G9) is satisfied. Moreover, conditions (G10) and (G11) are automatically satisfied. Using the Theorems (8) and (9) exits unique solution u the problem (70) such that

$$\begin{aligned} u \in C\left([0, +\infty), H^2(I\!R)\right) \cap C\left((0, +\infty), H^{2+s}(I\!R)\right) \quad \text{and} \\ \partial_t u \in C\left([0, +\infty), H^{-3}(I\!R)\right) \cap C\left((0, +\infty), H^2(I\!R)\right). \end{aligned} \tag{73}$$

for $0 < s < 2$. In particular, $S(t)\phi$ satisfies the following a-priori estimates.

Lemma 15. *The solution operator $\{S(t)\}_{t \geq 0}$ in $H^2(I\!R)$ associated with the equation (67) satisfies*

$$\|S(t)\phi\|_0 \leq \|\phi\|_0 \, e^{-\epsilon t} + \frac{\|f\|_0}{\epsilon} \left(1 - e^{-\epsilon t}\right) \tag{74}$$

for all $t \geq 0$, $\phi \in H^2(I\!R)$ and $f \in L^2(I\!R)$. Moreover

$$\begin{aligned} \frac{d}{dt} \|S(t)\phi\|_2^2 + \frac{\epsilon}{2} \|S(t)\phi\|_4^2 \quad &\leq \quad C(\beta, q, r, \epsilon) \left(\|S(t)\phi\|_0^{30/7} + \|S(t)\phi\|_0^{22/3} \right) \\ &\quad + \frac{8}{\epsilon} \|f\|_2^2. \end{aligned} \tag{75}$$

for all $t > 0$, $\phi \in H^2(I\!R)$ and $f \in H^2(I\!R)$.

Proof. Let $S(t)\phi = u$. Multiply equation (67) by $Bu = u + \partial_x^4 u$ and integrate to obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right) u_t \, dx \quad &= \quad -\epsilon \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right)^2 dx - \int_{-\infty}^{+\infty} \partial_x^4 u \, \partial_x \Phi_2'(u) \, dx \\ &\quad - \int_{-\infty}^{+\infty} u \, \partial_x \Phi_2'(u) \, dx + \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right) f \, dx \end{aligned} \tag{76}$$

Integrating by parts we get

$$\frac{d}{dt}\,\|u\|_0^2 + \epsilon \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right)^2\,dx = -\epsilon \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right)^2\,dx - 2\int_{-\infty}^{+\infty} \partial_x^4 u\,\partial_x \Phi_2'(u)\,dx$$
$$+2\int u\,f + 2\int_{-\infty}^{+\infty} \partial_x^2 u\,\partial_x^2 f\,dx \qquad (77)$$

Another integration by parts in (77) gives

$$\int_{-\infty}^{+\infty} \partial_x^4 u\,\partial_x \Phi_2'(u)\,dx = \beta \int_{-\infty}^{+\infty} u\,\partial_x^2 u\,\partial_x^4 u\,dx + \frac{\beta}{2}\int_{-\infty}^{+\infty} \left(\partial_x u\right)^2\,\partial_x^4 u\,dx$$
$$+\,q\int_{-\infty}^{+\infty} u^2\,\partial_x^4 u\,dx + r\int_{-\infty}^{+\infty} u^3\,\partial_x^4 u\,dx \qquad (78)$$

Integrating by parts each of the terms on the right of (78), we obtain

$$\int_{-\infty}^{+\infty} u\,\partial_x^2 u\,\partial_x^4 u\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \left(\partial_x^2 u\right)^3\,dx - \int_{-\infty}^{+\infty} u\,\left(\partial_x^3 u\right)^2\,dx \qquad (79)$$

$$\int_{-\infty}^{+\infty} \left(\partial_x u\right)^2\,\partial_x^4 u\,dx = \int_{-\infty}^{+\infty} \left(\partial_x^2 u\right)^3\,dx \qquad (80)$$

$$\int_{-\infty}^{+\infty} u^2\,\partial_x^4 u\,dx = -2\int_{-\infty}^{+\infty} \left(\partial_x u\right)^2\,\partial_x^2 u\,dx - 2\int_{-\infty}^{+\infty} u\,\left(\partial_x^2 u\right)^2\,dx \qquad (81)$$

$$\int_{-\infty}^{+\infty} u^3\,\partial_x^4 u\,dx = 6\int u\,\left(\partial_x u\right)^2\,\partial_x^2 u + 3\int_{-\infty}^{+\infty} u^2\,\left(\partial_x^2 u\right)^2\,dx \qquad (82)$$

Combining interpolation and Gagliardo-Nirenberg inequality in (79 - 82) obtain

$$\left|\int_{-\infty}^{+\infty} \left(\partial_x^2 u\right)^3 dx\right| \leq C_1 \|u\|_0^{11/8} \|u\|_4^{13/8} \tag{83}$$

$$\left|\int_{-\infty}^{+\infty} u \left(\partial_x^3 u\right)^2 dx\right| \leq C_2 \|u\|_0^{11/8} \|u\|_4^{13/8} \tag{84}$$

$$\left|\int_{-\infty}^{+\infty} \left(\partial_x u\right)^2 \partial_x^2 u \, dx\right| \leq C_3 \|u\|_0^{15/8} \|u\|_4^{9/8} \tag{85}$$

$$\left|\int_{-\infty}^{+\infty} u \left(\partial_x^2 u\right)^2 dx\right| \leq C_4 \|u\|_0^{15/8} \|u\|_4^{9/8} \tag{86}$$

$$\left|\int_{-\infty}^{+\infty} u \left(\partial_x u\right)^2 \partial_x^2 u \, dx\right| \leq C_5 \|u\|_0^{22/8} \|u\|_4^{10/8} \tag{87}$$

$$\left|\int_{-\infty}^{+\infty} u^2 \left(\partial_x^2 u\right)^2 dx\right| \leq C_6 \|u\|_0^{11/4} \|u\|_4^{5/4}. \tag{88}$$

Using inequalities (83 - 88), (78) and Young's inequality give us

$$\begin{aligned}\frac{d}{dt}\|u\|_2^2 + \epsilon \int_{-\infty}^{+\infty} \left(u + \partial_x^4 u\right)^2 dx \quad &\leq \quad C\left(\beta,\, q,\, r,\, \epsilon\right) \left(\|u\|_0^{30/7} + \|u\|_0^{22/3}\right) \\ &+ \frac{\epsilon}{2}\|u\|_0^2 + \epsilon\left\|\partial_x^2 u\right\|_o^2 + \frac{8}{\epsilon}\|f\|_0^2 + \frac{4}{\epsilon}\left\|\partial_x^2 f\right\|_0^2\end{aligned} \tag{89}$$

and the proof of (75) follows.

Multiply the equation (67) by u and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 = \epsilon \int_{-\infty}^{+\infty} u\left(-\epsilon\, Bu + J\Phi_2'(u) + f\right) dx \tag{90}$$

Integrating by parts

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \epsilon \int_{-\infty}^{+\infty} \left(u^2 + \left(\partial_x^2 u\right)^2\right) dx = \int_{-\infty}^{+\infty} u\, f\, dx. \tag{91}$$

Then

$$\frac{d}{dt}\|u\|_0 + \epsilon\|u\|_0 \leq \|f\|_0. \tag{92}$$

This implies (74). □

Now we turn to existence the global attractor in $H^2(\mathbb{R})$.

Lemma 16. *The map $\Phi_2'\colon H^2(\mathbb{R}) \to H^{-2}(\mathbb{R})$ satisfies the condition (H3) in $H^2(\mathbb{R})$.*

Proof. Let u, v in $H^2(\mathbb{R})$ and $\psi \in H^2(\mathbb{R})$, observe that

$$\left|\int_{-\infty}^{+\infty} (u_{xx} - v_{xx})\ \psi\, dx\right| = \left|\int_{-\infty}^{+\infty} (u - v)\ \psi_{xx}\, dx\right| \le \|u - v\|_2\, \|\psi_{xx}\|_{-2}, \tag{93}$$

$$\left|\int_{-\infty}^{+\infty} (u_{xxxx} - v_{xxxx})\ \psi\, dx\right| = \left|\int_{-\infty}^{+\infty} (u - v)\ \psi_{xxxx}\, dx\right| \le \|u - v\|_2\, \|\psi_{xxxx}\|_{-2}, \tag{94}$$

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} (uu_{xx} - vv_{xx})\ \psi\, dx\right| &= \left|\int (u - v)\ u_{xx}\,\psi + \int (u_{xx} - v_{xx})\ v\,\psi\right| \\ &\le \|u - v\|_0\, \|\psi\|_{L^\infty}\, \|u_{xx}\|_0 + \\ &\quad \|u_{xx} - v_{xx}\|_0\, \|\psi\|_{L^\infty}\, \|v\|_0,\end{aligned} \tag{95}$$

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \left(u_x^2 - v_x^2\right)\ \psi\, dx\right| &= \left|\int_{-\infty}^{+\infty} (u_x - v_x)\ (u_x + v_x)\ \psi\, dx\right| \\ &\le \|u_x - v_x\|_0\, \|\psi\|_{L^\infty}\, \left(\|u_x\|_0 + \|v_x\|_0\right),\end{aligned} \tag{96}$$

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \left(u^2 - v^2\right)\ \psi\, dx\right| &= \left|\int_{-\infty}^{+\infty} (u - v)\ (u + v)\ \psi\, dx\right| \\ &\le \|u - v\|_0\, \|\psi\|_{L^\infty}\, \left(\|u\|_0 + \|v\|_0\right),\end{aligned} \tag{97}$$

and

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \left(u^3 - v^3\right)\ \psi\, dx\right| &= \left|\int_{-\infty}^{+\infty} (u - v)\ \left(u^2 + uv + v^2\right)\ \psi\, dx\right| \\ &\le \|u - v\|_0 \Big(\|u\|_{L^\infty}^2 + \|u\|_{L^\infty}\, \|v\|_{L^\infty} + \\ &\quad \|v\|_{L^\infty}^2\Big)\ \|\psi\|_0.\end{aligned} \tag{98}$$

From the estimates (93)-(98), obtain

$$\|\Phi_2'(u) - \Phi_2'(v)\|_{-2} \le \gamma\left(\|u\|_2,\, \|v\|_2\right)\, \|u - v\|_2.$$

where γ is nondecreasing with respect to each of this arguments. This completes the proof.

□

The proof the weakly continuity of $\{S(t)\}_{t\geq 0}$ is almost identical to the one given by I. Moise, R. Rosa, and X. Wang (see lemma 4.3.3, [11]). It is a consequence of Theorem (12).

Observe that

$$K(u) = \int_{-\infty}^{+\infty} u\, \left(-\partial_x\, \Phi_2'(u)\right)\, dx + \int_{-\infty}^{+\infty} f\, u\, dx. \tag{99}$$

Integrate by parts, to obtain

$$K(u) = \int_{-\infty}^{+\infty} f\, u\, dx. \tag{100}$$

Moreover

$$L(u) = \epsilon \left(\int_{-\infty}^{+\infty} u\, \left(D^4 + 1\right) u\, dx - \int_{-\infty}^{+\infty} u^2\, dx \right). \tag{101}$$

Integrate by parts, to get

$$L(u) = \epsilon \int_{-\infty}^{+\infty} u_{xx}^2\, dx. \tag{102}$$

Moreover

$$G(u) = \int_{-\infty}^{+\infty} \Phi_2'(u)\, f\, dx, \tag{103}$$

and

$$M(u) = \epsilon \left(\int_{-\infty}^{+\infty} \Phi_2'(u)\, \left(u + \partial_x^4 u\right)\, dx - \Phi_2(u) \right). \tag{104}$$

By (74) and (75), for $\phi \in H^2(I\!R)$, $t' \mapsto S(t')\phi$ belong to $L^2\left([0,\, t];\, H^4(I\!R)\right)$, for each $t > 0$.

Let $\{\phi_j\}_{j\in I\!N} \subset H^2(I\!R)$ bounded and $\{t_j\}_{j\in I\!N}$, $t_j \to +\infty$, let $j_0 \in I\!N$ such that $t + t_j > T(M)$ for all $j > j_0$, where $T(M) = \dfrac{1}{\epsilon}\, \ln\left(\dfrac{\epsilon\, M}{\|f\|_0}\right)$ and $\|\phi_j\|_2 \leq M$, for all $j \in I\!N$, of (74) follows

$$\|S(t' + t_j)\phi_j\|_0 \leq \frac{2\, \|f\|_0}{\epsilon}. \tag{105}$$

for $0 \leq t' \leq t$ and for all $j > j_0$.

Using (105) in (75) and integrate to obtain

$$\|S(t + t_j)\phi_j\|_2^2 + \frac{\epsilon}{2} \int_0^t \|S(t' + t_j)\phi_j\|_4^2\, dt' \leq C_1\left(\beta,\, q,\, r,\, \epsilon,\, \|f\|_2\right)\, t. \tag{106}$$

for all $j > j_0$ and each $t > 0$. Then $\{S(\cdot + t_j)\ \phi_j\}_{j\in I\!N}$ is bounded in $L^2\left([0,\,t];\ H^4(I\!R)\right)$, for each $t > 0$. Of the observations above, the conditions (AS1) - (AS6) for K, L, G and M are satisfied, then using the Theorem (14), obtain

Theorem 17. *The solution operator $\{S(t)\}_{t\geq 0}$ in $H^2(I\!R)$ associated with equation (67) possesses a connected global attractor in $H^2(I\!R)$, i.e., a compact (in $H^2(I\!R)$), connected, invariant set which attracts all the orbits (in the $H^2(I\!R)$ - metric) of the system, uniformly on bounded sets of initial conditions.*

4.2 On the Benney-Lin and Kawahara Equations.

In this application consider of the initial value problem associated to the Benney-Lin and Kawahara equation,

$$\begin{cases} u_t + u\,u_x + u_{xxx} + \beta\,(u_{xx} + u_{xxxx}) + \eta\,u_{xxxxx} = 0 \\ \\ u(x,\,0) = \phi(x). \end{cases} \tag{107}$$

where $\beta > 0$, $\eta \in I\!R$ (see [12]). Consider the damped, forced Benney-Lin and Kawahara equation.

$$\begin{cases} u_t + u\,u_x + u_{xxx} + \beta\,(u_{xx} + u_{xxxx}) + \eta\,u_{xxxxx} + \mu\,B\,u = f \\ \\ u(x,\,0) = \phi(x). \end{cases} \tag{108}$$

where

$$B\,u = u \ \text{ or } \ B\,u = u + \left(-\partial_x^2\right)^{\lambda/2} u, \ \ \lambda > 0, \tag{109}$$

$\mu > 0$ and f is a time independent, external excitation.

Using Hamiltonian of the KdV equation,

$$\Phi_2(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{1}{3}\,u^3 - (u_x)^2\right]\,dx,$$

(108) can be written as

$$u_t = -\mu\,Au + J\Phi_2'(u) + f, \tag{110}$$

where $Au = \frac{\beta}{\mu}\left(\partial_x^2\,u + \partial_x^4\,u\right) + \frac{\eta}{\mu}\partial_x^5\,u + B\,u$ and $J = -\partial_x$.

For the case $B\,u = u$, note that $\mathrm{e}^{-\mu\,t\,A}\phi = \mathrm{e}^{-\mu\,t}\,E_{\eta,\,\beta}(t)\,\phi$, where $\{E_{\eta,\,\beta}(t)\}_{t\geq 0}$ is C^0-semigroup associated to linear equation (2.1) in [12]. Set $H = H^s(I\!R)$, $X = H^{s-5}(I\!R)$ for $s \geq 0$. It follows from Proposition 2.1., in [12], that properties (A1), (A3)-(A7) and (H1)-(H3) are satisfied, then the problem (110) is locally well possed in $H^s(I\!R)$ for all $s \geq 0$, i.e. exist $T > 0$ and unique

$$u \in C\left([0,\,T],\,H^s(I\!R)\right) \cap C\left((0,\,T],\,H^\infty(I\!R)\right) \ \text{ such that } \ \partial_t u \in C\left([0,\,T],\,H^{s-5}(I\!R)\right)$$

and satisfies (110). Note that for the case $B\,u = u + \left(-\partial_x^2\right)^{\lambda/2}\, u$, we obtain the same results which can be proved analogously to the preceding case, with $\lambda < 4$ (see [13]).

Now we turn to global well posedness in $H^2(I\!R)$, we write the equation as the perturbation of a Hamiltonian

$$u_t = -\mu\, A\, u + J\, \Psi'(u) + f \tag{111}$$

where $Au = \frac{\beta}{\mu}\,(u_{xx} + u_{xxxx}) + B\,u,\ B\,u$ as in (109), $\mu > 0,\ J = -\partial_x$ and

$$\Psi(u) = \int_{-\infty}^{+\infty} \left(\frac{\eta}{2}\, u_{xx}^2 - \frac{1}{2}\, u_x^2 + \frac{1}{6}\, u^3\right) dx, \quad \eta > 0$$

with

$$\Psi'(u) = \eta\, u_{xxxx} + u_{xx} + \frac{1}{2}\, u^2.$$

We use to real Banach spaces reflexive $X = H^{-3}(I\!R)$, $H = H^2(I\!R)$, $Y = L^2(I\!R)$ and $V = H^{2+s}(I\!R)$, for $0 < s < 4$. Defining $[f,\, g] = \int_{-\infty}^{+\infty} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi$, for all $f\;\; g \in S(I\!R)$, then are satisfied the conditions (G1), (G2) and (G3). Moreover

$$\Psi(u) \leq \frac{\eta}{2}\, \|u\|_2^2 + C_1\, \|u\|_0^{11/4}\, \|u\|_2^{1/4}, \quad \forall u \in H^2(I\!R).$$

and

$$\Psi(u) \geq \frac{\eta}{2}\, \|u\|_2^2 - \frac{\eta}{2}\, \|u\|_0^2 - C_2\, \|u\|_0\, \|u\|_2 - C_3\, \|u\|_0^{11/4}\, \|u\|_2^{1/4}, \quad \forall u \in H^2(I\!R).$$

The Ψ satisfies (G4) and (G5). Integrate by parts obtain

$$\int_{-\infty}^{+\infty} u\, J\, \Psi'(u)\, dx = 0$$

Then Ψ satisfies the condition (G6).

As a consequence of Proposition 2.1 part 3 in ([12]), $\left\{e^{-\mu\, t\, A}\right\}_{t\geq 0}$ satisfies

$$\left\|e^{-\mu\, t\, A}\right\|_{\mathcal{B}(H^s)} \leq e^{(\mu - \frac{\beta}{4})\, t},$$

for the weak dissipation $Bu = u$. Then $\left\{e^{-\mu\, t\, A}\right\}_{t\geq 0}$ is contraction semigroup in $H^s(I\!R)$ $\forall s \in I\!R$, if $\mu > \dfrac{\beta}{4}$. The case $Bu = u + \left(-\partial_x^2\right)^{\lambda/2}\, u$ is analogy (see [13]). Hence we obtain

$$\left(1 - \frac{\beta}{4\,\mu}\right)^{-1/2} \left\|B^{1/2}\, u\right\|_0 \geq \|u\|_0,$$

and (G9) is satisfied.
The argument to show that Ψ satisfies (G7) and (G8) is similar, and condition (G10) and (G11) are automatically satisfied.
Using the Theorems (8) and (9) we obtain a unique solution u the problem (108) such that

$$\begin{aligned} u \in C\left([0,\,+\infty),\, H^2(I\!R)\right) \cap C\left((0,\,+\infty),\, H^{2+s}(I\!R)\right) \quad \text{and} \\ \partial_t u \in C\left([0,\,+\infty),\, H^{-3}(I\!R)\right) \cap C\left((0,\,+\infty),\, H^2(I\!R)\right). \end{aligned} \tag{112}$$

for $0 < s < 4$. In particular, $S(t)u_0$ satisfies estimates a-priori analogy as in Lemma (15). Moreover Ψ satisfies the condition (H3) in $H^2(I\!R)$, the proof is similar to that of Lemma (16).

The proof the weak continuity de $\{S(t)\}_{t\geq 0}$ is almost identical to that given by I. Moise, R. Rosa, and X. Wang (see lemma 4.3.3, [11]). It is a consequence of the Theorem (12).

Observe that

$$K(u) = \int_{-\infty}^{+\infty} u\,\left(J\,\Psi'(u)\right)\,dx + \int_{-\infty}^{+\infty} f\,u\,dx. \tag{113}$$

Integrate by parts, obtain

$$K(u) = \int_{-\infty}^{+\infty} f\,u\,dx. \tag{114}$$

Moreover

$$L(u) = \mu\left(\int_{-\infty}^{+\infty} [u\,Bu]\,dx - \int_{-\infty}^{+\infty} u^2\,dx\right). \tag{115}$$

where

$$\mu\,[u,\,Bu] = \mu \int_{-\infty}^{+\infty} u\,Bu\,dx = \beta\,\|u_{xx}\|_0^2 - \beta\,\|u_x\|_0^2 + \mu\,\|u\|_0^2.$$

Moreover

$$G(u) = \int_{-\infty}^{+\infty} \Psi'(u)\,f\,dx. \tag{116}$$

and

$$M(u) = \mu\left(\int_{-\infty}^{+\infty} \Psi'(u)\,B\,u\,dx - \Psi(u)\right). \tag{117}$$

As in the problem in (4.1), the conditions (AS1) - (AS6) for K, L, G and M are satisfied. Then using the Theorem (14), obtain

Theorem 18. *The solution operator $\{S(t)\}_{t\geq 0}$ in $H^2(I\!R)$ associated with the equation (108) possesses a connected global attractor in $H^2(I\!R)$, i.e., a compact (in $H^2(I\!R)$), connected, invariant set which attracts all the orbits (in the $H^2(I\!R)$ - metric) of the system, uniformly on bounded sets of initial conditions.*

4.3 Generalizations the Korteweg-de Vries Burgers.

This application is concerned with the initial value problem (see [14])

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{j=1}^{n} \frac{\partial}{\partial x_j} \left[f(u) + \alpha \mathbf{L}\, u \right] \,+\, \epsilon \mathbf{B}\, u = g(x), \;\; x \,\in\, I\!R^n, \; t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{118}$$

with periodic and non-periodic boundary conditions.
In problem (118), $u = u(x,t)$ is a real valued function, f and g are scalar functions, α and $\epsilon > 0$ are constants. We assume that $\mathbf{L}$ stands for a differential or pseudo-differential operator

$$\mathbf{L} : D(\mathbf{L}) \subset L^2(\Pi_n) \to L^2(\Pi_n),$$

where $\Pi_n = [0, 2\pi] \times [0, 2\pi] \times \cdots \times [0, 2\pi]$ n-times and

$$(\mathbf{L}\phi)(x) = \sum_{k \in Z\!\!\!Z^n} e^{i\,k\cdot\,x}\, l(k) \hat{\phi}(k), \;\; \forall \phi \in C^{\infty}_{\pi}(I\!R^n). \tag{119}$$

where $l \in L^{\infty}_{loc}(Z\!\!\!Z^n, I\!R)$ and $C^{\infty}_{\pi} = \{\psi \in C^{\infty}(I\!R^n) \; : \psi \text{ is } 2\pi-\text{periodic in x } \}$. It follow that $\mathbf{L}$ is self-adjoint. $\mathbf{B}$ stands for a differential or pseudo-differential operator with non negative symbol $b(k)$ where

$$(\mathbf{B}\phi)(x) = \sum_{k \in Z\!\!\!Z^n} e^{i\,k\cdot\,x}\, b(k) \hat{\phi}(k), \;\; \forall \phi \in C^{\infty}_{\pi}(I\!R^n), \tag{120}$$

$b \in L^{\infty}_{loc}(Z\!\!\!Z^n, I\!R)$. Thus $\mathbf{B}$ is self-adjoint and non negative. Consider Hilbert space scale associated self-adjoint operator $\mathbf{L}$, i. e. if $s \geq t \geq 0$,

$$H^s_{\pi}(\mathbf{L}) \subset H^t_{\pi}(\mathbf{L}) \subset H^0_{\pi}(\mathbf{L}) \subset H^{-t}_{\pi}(\mathbf{L}) \subset H^{-s}_{\pi}(\mathbf{L})$$

for $s \geq 0$, $H^s_{\pi}(\mathbf{L}) = (1 + |\mathbf{L}|)^{-\frac{s}{2}} L^2(\Pi)$ with scalar product $\langle u, v \rangle_{H^s_{\pi}(\mathbf{L})} = \langle (1 + |\mathbf{L}|)^{\frac{s}{2}} u, (1 + |\mathbf{L}|)^{\frac{s}{2}} v \rangle_{L^2(\Pi)}$
Similar by in the case of Sobolev spaces, if $s \geq t \geq 0$,

$$H^s(\mathbf{L}) \subset H^t(\mathbf{L}) \subset H^0(\mathbf{L}) \subset H^{-t}(\mathbf{L}) \subset H^{-s}(\mathbf{L})$$

for $s \geq 0$, $H^s(\mathbf{L}) = (1 + |\mathbf{L}|)^{-\frac{s}{2}} L^2(I\!R^n)$ with scalar product $\langle u, v \rangle_{H^s(\mathbf{L})} = \langle (1 + |\mathbf{L}|)^{\frac{s}{2}} u, (1 + |\mathbf{L}|)^{\frac{s}{2}} v \rangle_{L^2(I\!R^n)}$

Consider energy integral

$$\Phi(u) \,=\, \int_{I\!R^n} \left[\frac{\alpha}{2} u\,\mathbf{L}u \,-\, G(u)\right]\, dx.$$

where $G(s) = \int\limits_0^s f(\xi)\, d\xi$. Here we write the equation as the perturbation of a Hamiltonian

$$\partial_t\, u = -\epsilon\, B\, u + J\, \Phi'(u) \,+\, g. \tag{121}$$

where $J\phi = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}\, \phi$, for all $\phi \in S(I\!R^n)$.

We consider only the following class studied originally Saut ([14])

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{j=1}^{n} \frac{\partial}{\partial x_j}\, [f(u) + \alpha \mathbf{L}\, u] \,+\, \epsilon\, \left(u + (-1)^k \Delta^k\, u\right) = g(x),\; x \,\in\, I\!R^n,\, t > 0 \\ u(x, 0) = u_0(x) \end{cases} \tag{122}$$

where $\mathbf{L} = \mathbf{H} + \mathbf{K}$, $\mathbf{K}$ is operator defined for

$$\mathbf{K}\, u(x) = (2\pi)^{-n/2} \int\limits_{I\!R^n} r(x,\, \xi)\, \hat{u}(\xi)\, \mathrm{e}^{x\cdot\xi}\, d\xi \tag{123}$$

with $r \in C^\infty\, (I\!R^n \times I\!R)$, such that

$$\left|D_\xi^\alpha\, D_x^\beta\, r(x,\, \xi)\right| \le C_{\alpha,\,\beta}\, \left(1 + |\xi|\right)^{-1-\rho\, |\alpha| + \delta\, |\beta|}\,,\;\; (x,\, \xi) \in I\!R^n \times I\!R^n, \tag{124}$$

for $0 \le \delta < \rho \le 1$ (r belongs to the class $S^{-1}_{\rho,\,\beta}\, (I\!R^n)$ of Hörmander [15]) .
The operator $\mathbf{H}$ is defined for

$$\mathbf{H}\, u(x) = (2\pi)^{-n/2} \int\limits_{I\!R^n} p(\xi)\, \hat{u}(\xi)\, \mathrm{e}^{x\cdot\xi}\, d\xi, \tag{125}$$

where p satisfies the following conditions:

(i) $p \in L^\infty_{loc}\, (I\!R^n,\, I\!R)$,

(ii) $p(\xi) = p(-\xi)$ a. e.,

(iii) Exists λ, μ, $0 \le \lambda \le \mu$ and constants R, C_1, $C_2 > 0$, such that

$$C_1\, |\xi|^\lambda \le p(\xi) \le C_2\, |\xi|^\mu \;\text{ for }\; |\xi| \ge R.$$

(iv) $p(\xi) > 0$, $\forall \xi \in I\!R^n$.

We consider $\mathbf{H}\colon D(\mathbf{H}) \subset L^2(I\!R^n) \to L^2(I\!R^n)$. This operator not bounded. It follows the condition (i) - (iv) that $\mathbf{H}$ is densely defined, symmetric, self-adjoint. These properties are satisfied by $\mathbf{H}^{1/2}$, which is well defined due to condition (iv).

We define the Hilbert space $\mathcal{H} = D(\mathbf{H}^{1/2})$, with inner product

$$\langle u, v\rangle_{\mathcal{H}} = \int\limits_{I\!R^n} u\, v\, dx + \int\limits_{I\!R^n} \mathbf{H}^{1/2} u\, \mathbf{H}^{1/2} v\, dx.$$

It is easy to see that

$$H^{\mu/2}(I\!R^n) \subset \mathcal{H} \subset H^{\lambda/2}(I\!R^n),$$

with the inclusion continuous and dense, suppose that $\lambda > n+2$ and $g \in H^\infty(I\!R^n)$ (see [14]).

Remark 19. *The above hypotheses are verified by many physical examples: $p(\xi) = \xi^2$, the Korteweg-de Vries equation; $p(\xi) = |\xi|$, the Benjamin-Ono equation; $p(\xi) = (1+|\xi|)^{1/2}$ the Smith equation; $p(\xi) = \xi^2\,(\ln|\xi|+c)$, $c \in I\!R$ the Pritchard equation and $p(\xi) = \xi^2\,(K_0(k\,|\xi|)+c)$, $c \neq 0$, $k > 0$ where K_0 is Modified Bessel function of order 0, the Leibovitch-Benjamin-Bona-Mahony equation.*

Suppose that f is a polynomial, $f(u) = \sum\limits_{i=1}^{d} a_i\,u$, such that $d < \dfrac{2\lambda}{n} + 1$, $a_i \in I\!R$ are constants.

We consider the problem (122) on the phase space $\mathcal{H}$, under these assumptions the properties (A1), (A3)-(A7) and (H1)-(H3) are satisfied, then the problem (122) is locally well possed in $\mathcal{H}$, i.e. exist $T > 0$ and unique

$$u \in C\left([0,\,T],\,\mathcal{H}\right) \cap C\left((0,\,T],\,H^\infty(I\!R^n)\right).$$

and satisfies (122) with $u_0 \in \mathcal{H}$.

Now we turn to global well posedness in $\mathcal{H}$. Verification of the conditions (G1) - (G11) is straightforward (see [14]). For example

$$\Phi(u) \leq \frac{1}{2}\,\|u\|_{\mathcal{H}}^2 + \sum_{k=0}^{d-1} c_k\,\|u\|_0^{q_k(n,\,\lambda,\,d)}\,\|u\|_{\mathcal{H}}^{\frac{n(d-k-1)}{\lambda}}.$$

$q_k(n,\,\lambda,\,d) > 0$ and $\frac{n(d-k-1)}{\lambda} < 2$ if $d < \dfrac{2\lambda}{n} + 1$.

The proof the weak continuity of $\{S(t)\}_{t\geq 0}$ the solution operator in $\mathcal{H}$ associated with the equation (122) is consequence of the Theorem (12).

As the Theorem 1 in ([14]), the conditions (AS1) - (AS6) for Φ, K, L, G and M are satisfied. Using the Theorem (14), obtain

Theorem 20. *The solution operator $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}$ associated with the equation (122) possesses a connected global attractor in $\mathcal{H}$, i.e., a compact (in $\mathcal{H}$), connected, invariant set which attracts all the orbits (in the $\mathcal{H}$ - metric) of the system, uniformly on bounded sets of initial conditions.*

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