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On pairs of vectors achieving the maximal angle of a convex cone

Abstract. In this paper we explore the concept of antipodality relative to a closed convex cone $K \subset \mathbb{R}^d$. The problem under consideration is that of finding a pair of unit vectors in K achieving the maximal angle of the cone. We mention also a few words on the attainability of critical angles. By way of application of the general theory, we briefly discuss the problem of estimating the radius of pointedness of a cone.

Key words. Antipodal points - Antipodal faces - Convex cone - Pointedness - Maximal angle - Critical angle - Nonconvex optimization.

Dedicated to R.T. Rockafellar on his 70th Birthday

1. Introduction

An old problem of geometry is that of finding a pair of antipodal points in a compact set. Two points are antipodal (i.e., each is the antipode of the other) if they are diametrically opposite. Examples include endpoints of a line segment, or poles of a sphere. A less classical theory of antipodality, which applies to vertices of a convex polytope, has been developed by authors like Nguyen and Soltan [**?**]. In our note, we are interested in the concept of antipodality relative to a closed convex cone $K \subset \mathbb{R}^d$.

In what follows, the notation S_d refers to the unit sphere in $\mathbb{R}^d$, and

$$\operatorname{diam}(C) = \sup_{u,v\in C} \|u - v\|$$

stands for the diameter of a nonempty compact set $C \subset \mathbb{R}^d$. The distance between u and v is measured by means of the norm $\|\cdot\|$ associated to the standard Euclidean product

$$\langle u, v\rangle = u_1v_1 + \cdots + u_dv_d.$$

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For the sake of convenience, we introduce also the notation

$$\Xi(I\!R^d) = \{K \subset I\!R^d : \ K \text{ is a nonzero closed convex cone}\}.$$

Definition 1.1. One says that u and v are *antipodal points* of $K \in \Xi(I\!R^d)$ if

$$u, v \ \in K \cap S_d \quad \text{and} \ \ \|u - v\| = \operatorname{diam}(K \cap S_d). \tag{1}$$

What Definition 1.1 says is that, while remaining in the base $K \cap S_d$ of the cone K, the points u and v are as far away as possible from each other. The antipodality problem is somewhat related to the "farthest-point" problem, except that now we are looking for a pair of points maximizing a certain distance. By contrast, in the classical farthest-point problem one searches a point v in a compact set C that is as far away as possible from a given point u in the underlying space $I\!R^d$.

An equivalent definition of antipodality can be given in terms of

$$\theta_{max}(K) = \sup_{u,v \in K \cap S_d} \arccos \langle u, v \rangle, \tag{2}$$

the maximal angle of the cone K.

Definition 1.1. (bis) One says that u and v are *antipodal points* of $K \in \Xi(I\!R^d)$ if

$$u, v \ \in K \cap S_d \quad \text{and} \ \ \arccos\langle u, v \rangle = \theta_{max}(K). \tag{3}$$

According to this definition, antipodality is a matter of achieving the maximal angle of the cone. To see that (1) and (3) are equivalent, it is enough to recall the general identity

$$\|u - v\|^2 = 2(1 - \langle u, v \rangle) \qquad \forall u, v \in S_d.$$

Our attention will switch from one formulation to the other whenever the need arises. In fact, we also consider the variational problem (2) written in the minimization form

$$\cos \theta_{max}(K) = \inf_{u,v \in K \cap S_d} \langle u, v \rangle. \tag{4}$$

The purpose of this note is exploring in detail the concept of antipodality for a cone. Whether we use one formulation or the other, we always face a major difficulty: the optimization problem at hand is a nonconvex one. In order to compensate the lack of convexity we must investigate and exploit other structural properties.

2. Geometric and topological principles

In this section we review some basic principles concerning the concept of antipodality. We would like to have a better understanding of the nature of the set

$$\mathcal{A}(K) = \{(u, v) \in I\!R^d \times I\!R^d : \ u \text{ and } v \text{ are antipodal points of } K\}. \tag{5}$$

There are two trivial cases that can be removed from the discussion:

(i) *The case of a nonpointed cone.* That K is nonpointed simply means that $\theta_{max}(K) = \pi$. This is equivalent to saying that K contains a unit vector and its opposite. One can easily see that $\mathcal{A}(K) = \{(u, -u) : u \in S_d \cap \mathrm{lin}K\}$, with $\mathrm{lin}K = K \cap -K$ being the largest linear space included in K.

(ii) *The case of a ray.* A ray is a set of the form $K = \{te : t \in I\!R_+\}$, with $e \in I\!R^d$ being a unit vector. This time $\theta_{max}(K) = 0$, and $\mathcal{A}(K) = \{(e, e)\}$.

The smallest linear space containing the set K is denoted by $\mathrm{span}K$. Since K is a convex cone, one has

$$\mathrm{span}K = K - K.$$

The *relative interior* of K, which is usually denoted by $\mathrm{ri}K$, is defined as the interior which results when K is regarded as a subset of $\mathrm{span}K$. The next lemma tells us that the relative interior of the cone is not the right place for finding antipodal points.

Lemma 2.1. (Principle of the Relative Boundary) *Let $K \in \Xi(I\!R^d)$ be pointed but not a ray. If u and v are antipodal points of K, then u and v lie in the relative boundary of K.*

Proof. Suppose, for instance, that $u \in \mathrm{ri}K$. Since $u - v \in \mathrm{span}K$, the vector

$$u_\varepsilon = u + \varepsilon(u - v)$$

remains in K for all $\varepsilon > 0$ small enough. Notice that

$$\|u_\varepsilon\|^2 = 1 + 2\varepsilon(1 - \langle u, v\rangle) + \varepsilon^2\|u - v\|^2 \ \geq \ 1.$$

We now look at the inner product formed by v and the unit vector $\hat{u}_\varepsilon = \|u_\varepsilon\|^{-1}u_\varepsilon$. Recall that K is pointed and not a ray. This assumption ensures not only that $\hat{u}_\varepsilon \neq u$, but also

$$\langle \hat{u}_\varepsilon, v\rangle < \langle u, v\rangle. \tag{6}$$

To check the inequality (6), we evaluate at 0 the right-derivative of the function

$$\varepsilon \mapsto \varphi(\varepsilon) = \langle \hat{u}_\varepsilon, v\rangle = \frac{\langle u, v\rangle + \varepsilon(\langle u, v\rangle - 1)}{\|u_\varepsilon\|}.$$

A simple computation yields

$$\varphi'_+(0) = -[1 - \langle u, v\rangle^2] \ < 0,$$

proving that (6) holds for $\varepsilon > 0$ sufficiently small. Since the antipodality of the pair (u, v) is being contradicted, we conclude that u must be in $K \backslash \mathrm{ri}K$, the relative boundary of K. A similar argument applies to the component v. □

Corollary 2.2. *Let u and v be antipodal points of $K \in \Xi(\mathbb{R}^d)$. Let Q be a closed convex subcone of K containing u and v. If Q is pointed and not a ray, then Q contains u and v in its relative boundary.*

Proof. Observe that $u, v \in Q \cap S_d$ and $\arccos\langle u, v\rangle = \theta_{max}(Q) = \theta_{max}(K)$, so one just needs to apply Lemma 2.1 to the cone Q. □

The conclusion of Lemma 2.1 is very rough because the points u and v don't appear related to each other. According to common sense, u and v should be located on "opposite" sides of the relative boundary of K. The idea of opposition can be formalized in terms of the line

$$L_{u,v} = \{(1-t)u + tv : \ t \in \mathbb{R}\}$$

passing through the points u and v, and the segment

$$\mathrm{co}\{u, v\} = \{(1-t)u + tv : \ t \in [0, 1]\}$$

joining the points u and v.

Proposition 2.3. (Tomographic Principle) *Let $K \in \Xi(\mathbb{R}^d)$ be pointed but not a ray. If u and v are antipodal points of K, then*

$$K \cap L_{u,v} = \mathrm{co}\{u, v\}. \tag{7}$$

Proof. If one looks more carefully at the proof of Lemma 2.1, one sees that the antipodal pair (u, v) satisfies the condition

$$K \cap \left[u + \mathbb{R}_+(u - v)\right] = \{u\}. \tag{8.a}$$

By exchanging the roles of u and v, one gets also

$$K \cap \left[v + \mathbb{R}_+(v - u)\right] = \{v\}. \tag{8.b}$$

One can easily prove that the tomographic condition (7) is equivalent to the system (8), so the details are omitted. □

Proposition 2.3 is sharper than Lemma 2.1, but we are still far from getting something satisfactory. Much finer localization results will be given in Section 4. We close this section with a theorem concerning the behavior of θ_{max} as function over the metric space $\Xi(\mathbb{R}^d)$. The metric considered in $\Xi(\mathbb{R}^d)$ is the standard one, to wit

$$\delta(K_1, K_2) = \sup_{\|z\| \le 1} |\mathrm{dist}[z, K_1] - \mathrm{dist}[z, K_2]|.$$

One usually refers to δ as the bounded Pompeiu-Hausdorff metric because it admits the equivalent formulation

$$\delta(K_1, K_2) = \text{haus}(K_1 \cap S_d, K_2 \cap S_d),$$

with

$$\text{haus}(C_1, C_2) = \max\{\sup_{z\in C_1} \text{dist}[z, C_2]\ ,\ \sup_{z\in C_2} \text{dist}[z, C_1]\}$$

denoting the Pompeiu-Hausdorff distance between the compact sets $C_1, C_2 \subset I\!R^d$. The next theorem concerns also the behavior of the antipodal map $\mathcal{A}$ introduced in (5). We recall first some classical semicontinuity concepts for set-valued maps.

Definition 2.4. Let $\Gamma : W \to Y$ be a set-valued map between two topological spaces. One says that Γ is *upper-semicontinuous* (respectively, *lower-semicontinuous*) if the set

$$\{w \in W :\ \Gamma(w) \subset O\} \qquad (\text{respectively}, \{w \in W :\ \Gamma(w) \cap O \neq \emptyset\})$$

is open, whenever $O \subset Y$ is open. Continuity of Γ means the combination of upper-semicontinuity and lower-semicontinuity.

Theorem 2.5. *Consider the space $\Xi(I\!R^d)$ equipped with the metric δ. Then,*

(a) *the function $\theta_{max} : \Xi(I\!R^d) \to [0, \pi]$ is continuous;*

(b) *for every $K \in \Xi(I\!R^d)$, the set $\mathcal{A}(K)$ is nonempty and compact;*

(c) *the antipodal map $\mathcal{A} : \Xi(I\!R^d) \to S_d \times S_d$ is upper-semicontinuous.*

Proof. A short and elegant proof of this result is based on Berge's maximum theorem [**?**]. As seen from (2), θ_{max} corresponds to the optimal-value function of a parametric optimization problem whose objective function

$$\arccos\langle\cdot,\cdot\rangle : S_d \times S_d \to [0, \pi]$$

is continuous, and whose feasible set

$$\Gamma(K) = \big[K \cap S_d\big] \times \big[K \cap S_d\big]$$

behaves continuously with respect to the parameter K. That $\Gamma : \Xi(I\!R^d) \to S_d \times S_d$ is a continuous set-valued map follows from standard rules of set-valued analysis (see, for instance, [**?**, **?**, **?**, **?**]). Berge's maximum theorem yields the continuity of the optimal-value function θ_{max}, as well as the upper-semicontinuity of the solution-set map

$$K \to \mathcal{A}(K) = \{(u, v) \in \Gamma(K) :\ \arccos\langle u, v\rangle = \theta_{max}(K)\}.$$

Part (b) is immediate. □

Berge's theorem doesn't guarantee the lower-semicontinuity of $\mathcal{A}$. More often than not, solution-set maps fail to be lower-semicontinuous. As shown in the next example, the antipodal map $\mathcal{A}$ is not an exception to this rule.

Example 2.6. It is not difficult to see that the sequence $\{K_n\}_{n\in I\!N}$ of elliptic cones

$$K_n = \{x \in I\!R^3 : [x_1^2 + 2x_2^2]^{1/2} \leq nx_3\}$$

converges in $\Xi(I\!R^3)$ toward the half-space $K = \{x \in I\!R^3 : x_3 \geq 3\}$. On the other hand, for each K_n, one has $\mathcal{A}(K_n) = \{(u_n, v_n), (v_n, u_n)\}$, with

$$u_n = [1+n^2]^{-1/2}\,(n,0,1) \;\to\; (1,0,0) \quad \text{and} \quad v_n = [1+n^2]^{-1/2}\,(-n,0,1) \;\to\; (-1,0,0)\,.$$

So, not every antipodal pair of K can be recovered as limit of elements taken from the successive $\mathcal{A}(K_n)$. This confirms that $\mathcal{A}$ is not lower-semicontinuous.

3. Antipodal faces

Facial analysis is specially useful when K is a polyhedral cone, that is to say, when K is representable as intersection of finitely many half-spaces. Whenever we speak about a face of a cone, it will be implicitly understood that the cone is polyhedral. Recall that a closed convex subcone F of $K \in \Xi(I\!R^d)$ is called a *face* of K if

$$x \in K\,,\; z - x \in K \;\text{ and }\; z \in F \quad \Longrightarrow \quad x \in F\,.$$

We use the notation $\mathcal{F}(K)$ to indicate the collection of all faces of K. The dimension of a face F is, by definition, the dimension of the linear space span$F$. The relative interior of a face F refers to the interior of F with respect to span$F$. We assume that the reader is acquainted with the basic arithmetic of faces as developed, for instance, in references [**?**, **?**].

Definition 3.1. Let $K \in \Xi(I\!R^d)$ be a polyhedral cone. Two faces $U, V \in \mathcal{F}(K)$ are said to be antipodal if there is a pair (u, v) of antipodal points of K such that $u \in ri(U)$ and $v \in ri(V)$.

Some comments on Definition 3.1 are in order. For each $z \in K$, there is exactly one face of K such that $z \in ri(K)$. This unique face, which we denote by $\Phi(z)$, corresponds to the smallest face of K containing z, i.e.,

$$\Phi(z) = \cap\{F \in \mathcal{F}(K) :\; F \ni z\}.$$

Antipodal points and antipodal faces are related through the function

$$\Phi_\otimes : K \times K \to \mathcal{F}(K) \times \mathcal{F}(K)$$
$$\Phi_\otimes(u, v) = (\Phi(u), \Phi(v)).$$

Recall the $\mathcal{A}(K)$ denotes the collection of all antipodal pairs of K. So,

$$\mathcal{B}(K) = \{(U, V) \in \Xi(I\!R^d) \times \Xi(I\!R^d) : \ U \text{ and } V \text{ are antipodal faces of } K\}$$

is the image of $\mathcal{A}(K)$ under $\Phi_\otimes$, that is to say,

$$\mathcal{B}(K) = \Phi_\otimes(\mathcal{A}(K)) = \{\Phi_\otimes(u, v) : \ (u, v) \in \mathcal{A}(K)\}.$$

It must be said, however, that $\Phi_\otimes$ is not a bijection between $\mathcal{A}(K)$ and $\mathcal{B}(K)$. The example given below shows that infinitely many pairs of antipodal points may produce just one pair of antipodal faces.

Example 3.2. The maximal angle of the positive orthant $I\!R^3_+$ is $\pi/2$. Thus,

$$\mathcal{A}(I\!R^3_+) = \{(u, v) \in I\!R^3_+ \times I\!R^3_+ : \ \|u\| = \|v\| = 1, \langle u, v\rangle = 0\}.$$

Consider, for instance, a pair $(u(\alpha), v) \in \mathcal{A}(I\!R^3_+)$ of the form $u(\alpha) = (\cos\alpha, \sin\alpha, 0),\ v = (0, 0, 1)$. Clearly, $V = \Phi(v) = I\!R_+(0, 0, 1)$, and $U = \Phi(u(\alpha)) = \{x \in I\!R^3_+ : x_3 = 0\}$ for any $\alpha \in]0, \pi/2[$.

Since antipodal points always exist, so do antipodal faces. Some elementary properties of antipodal faces are listed in the following proposition.

Proposition 3.3. *Suppose the polyhedral cone K is pointed but not a ray. Let (U, V) be a pair of antipodal faces of K. Then,*

(a) *$max\{dimU, dimV\} \leq dimK - 1$;*

(b) *neither U is contained in V, nor V is contained in U.*

Proof. Part (a) is a direct consequence of the Principle of the Relative Boundary. We prove the part (b) by negating its conclusion. Suppose, for instance, that $U \subset V$. Hence, the cone V is pointed and not a ray. Let (u, v) be a pair as in Definition 3.1. Since $U \subset V$, the points u and v are in V. By Corollary 2.2, both points are in the relative boundary of V. But, according to Definition 3.1, the point v is in the relative interior of V. We arrive in this way to a contradiction because relative boundaries and relative interiors do not intersect. □

Since a polyhedral cone K has a finite number of faces, the set $\mathcal{B}(K)$ is finite as well. In fact,

$$\text{card}\, \mathcal{B}(K) \leq \left[\text{card}\, \mathcal{F}(K)\right]^2.$$

The above cardinality bound can be sharpened in a substantial way because Proposition 3.3 reduces the number of potential candidates for membership in $\mathcal{B}(K)$.

4. Necessary conditions for antipodality: the general case

We start this section by writing a necessary condition for antipodality that applies to an arbitrary cone $K \in \Xi(I\!R^d)$. The notation

$$K^+ = \{y \in I\!R^d : \langle y, x\rangle \geq 0 \ \ \forall\, x \in K\}$$

refers to the positive dual cone of K.

Theorem 4.1. *If u and v are antipodal points of $K \in \Xi(I\!R^d)$, then*

$$u, v \in K \cap S_d\,, \quad v - \langle u, v\rangle u \in K^+\,, \quad u - \langle u, v\rangle v \in K^+. \tag{9}$$

Proof. We write the diameter maximization problem

$$\begin{cases} \text{maximize} \quad \|u - v\| \\ u, v \in K \cap S_d \end{cases}$$

in the equivalent form

$$\begin{cases} \text{minimize } \langle u, v\rangle \\ (u, v) \in K \times K, \ \|u\|^2 = \|v\|^2 = 1. \end{cases} \tag{10}$$

We then dualize with respect to the equality constraints in (10), that is to say, we introduce a Lagrangian function L of the type

$$L(u, v, \lambda_0, \lambda_1, \lambda_2) = \lambda_0 \,\langle u, v\rangle - \frac{\lambda_1}{2}\big(\|u\|^2 - 1\big) - \frac{\lambda_2}{2}\big(\|v\|^2 - 1\big).$$

The coefficients $\lambda_1, \lambda_2 \in I\!R$ are Lagrange multipliers associated to the equality constraints. The term $\lambda_0 \geq 0$ is a Fritz John multiplier. According to a general optimality theorem (cf.[**?**]), a necessary condition for (u, v) to solve (10) is that

$$-\nabla_{(u,v)} L(u, v, \lambda_0, \lambda_1, \lambda_2) \in N_{K\times K}(u, v) \tag{11}$$

for a suitable $(\lambda_0, \lambda_1, \lambda_2) \neq (0, 0, 0)$. The symbol $N_Q(z)$ stands for the normal cone to Q at z in the sense of convex analysis [**?**]. By writing the optimality condition (11) in the decomposed form

$$-\nabla_u L(u, v, \lambda_0, \lambda_1, \lambda_2) \in N_K(u), \quad -\nabla_v L(u, v, \lambda_0, \lambda_1, \lambda_2) \in N_K(v),$$

one gets

$$-(\lambda_0 v - \lambda_1 u) \in N_K(u), \quad -(\lambda_0 u - \lambda_2 v) \in N_K(v). \tag{12}$$

But K is a closed convex cone, so each part of (12) corresponds to a complementarity problem. More precisely, (12) can be written in the form

$$u \in K, \ \lambda_0 v - \lambda_1 u \in K^+, \ \langle u, \lambda_0 v - \lambda_1 u\rangle = 0,$$
$$v \in K, \ \lambda_0 u - \lambda_2 v \in K^+, \ \langle v, \lambda_0 u - \lambda_2 v\rangle = 0.$$

The choice $\lambda_0 = 0$ must be ruled out because it leads to a contradiction. So, there is no loss of generality in assuming that $\lambda_0 = 1$. This is a standard normalization procedure for getting rid of the Fritz John multiplier. One arrives then at the simplified system

$$u \in K,\ v - \lambda_1 u \in K^+,\ \langle u, v - \lambda_1 u\rangle = 0, \tag{13.a}$$

$$v \in K,\ u - \lambda_2 v \in K^+,\ \langle v, u - \lambda_2 v\rangle = 0, \tag{13.b}$$

getting in this way $\lambda_1 = \lambda_2 = \langle u, v\rangle$, and, of course, the announced result. □

Since the optimization problem (10) is not convex, the condition (9) is necessary for antipodality, but not sufficient. In fact, the condition (9) holds also when $u = v$, a choice corresponding rather to diameter minimization! The following definition is then important to capture the idea that a cone may admit a full spectrum of "critical" angles.

Definition 4.2. A *critical pair* for $K \in \Xi(I\!R^d)$ is any pair (u, v) satisfying (9). The set

$$\Sigma(K) = \{\arccos\langle u, v\rangle :\ (u, v) \text{ satisfies } (9)\}$$

is called the *angular spectrum* of K. Each element in this set is called a *critical angle* of K.

In agreement with standard optimization literature, one would probably replace the terminology critical pair by "stationary pair". The adjective "critical" is used sometimes with a slightly different meaning, but this should not be here a source of confusion.

In Section 8 we shall say a few words on the theory of angular spectra, but now we look back at Theorem 4.1 and write down some of its consequences.

Corollary 4.3. *If u and v are antipodal points of $K \in \Xi(I\!R^d)$, then*

$$u, v \in S_d\ , \quad u = \mathrm{Proj}_K[u - v + \langle u, v\rangle u]\ , \quad v = \mathrm{Proj}_K[v - u + \langle u, v\rangle v]. \tag{14}$$

Proof. In fact, (14) is just another way of writing the system (9). Let $K^- = -K^+$. Consider also the notation

$$\Psi_1(u, v) = -[v - \langle u, v\rangle u], \quad \Psi_2(u, v) = -[u - \langle u, v\rangle v].$$

As seen in the proof of Theorem 4.1, one has

$$u \in K,\ \Psi_1(u, v) \in K^-,\ \ \langle u, \Psi_1(u, v)\rangle = 0, \tag{15.a}$$

$$v \in K,\ \Psi_2(u, v) \in K^-,\ \ \langle v, \Psi_2(u, v)\rangle = 0. \tag{15.b}$$

By Moreau's orthogonal decomposition theorem [**?**, **?**], condition (15.a) is equivalent to saying that $u = \mathrm{Proj}_K[u + \Psi_1(u, v)]$. Similarly, (15.b) takes the form $u = \mathrm{Proj}_K[v + \Psi_2(u, v)]$. This confirms that the pair (u, v) solves the system (14). □

Remark: As pointed out by one of the referees, the proof of Corollary 4.3 could be somewhat simplified. Starting with the inclusion $-v + \langle u, v\rangle u \in N_K(u)$, obtained from the left relation in (12), we may exploit the fact that $b \in N_K(u)$ if and only if $\mathrm{Proj_K(u + b) = u}$. This immediately implies the respective relation in (14).

With the help of Corollary 4.3, one recovers again the Principle of the Relative Boundary. In fact, one gets the more general version:

Corollary 4.4. *Let (u, v) be a critical pair of $K \in \Xi(\mathbb{R}^d)$, with $|\langle u, v\rangle| \neq 1$. Then, u and v belong to the relative boundary of K.*

Proof. In view of (14), one has $u - v + \langle u, v\rangle u \notin K$ and $v - u + \langle u, v\rangle v \notin K$. So, the projections u and v must lie in the boundary of K. To see that u and v belong, in fact, to the relative boundary of K, we apply Moreau's orthogonal decomposition theorem again. This time the underlying space is not $\mathbb{R}^d$, but the linear space spanned by K. □

A similar argument leads us to a sort of "dual" boundary principle:

Corollary 4.5. *If (u, v) is a critical pair of $K \in \Xi(\mathbb{R}^d)$, then*

$$v - \langle u, v\rangle u \in bd(K^+) \quad and \quad v - \langle u, v\rangle u \in bd(K^+). \tag{16}$$

Proof. Consider again the notation used in the proof of Corollary 4.3. Moreau's orthogonal decomposition theorem tells us that (15.a) is equivalent to

$$\Psi_1(u, v) = \mathrm{Proj}_{K^-}[u + \Psi_1(u, v)].$$

Similarly, (15.b) amounts to saying that

$$\Psi_2(u, v) = \mathrm{Proj}_{K^-}[v + \Psi_2(u, v)].$$

Since u and v are nonzero vectors, it follows that $u + \Psi_1(u, v) \notin K^-$ and $v + \Psi_2(u, v) \notin K^-$. So, $\Psi_1(u, v)$ and $\Psi_2(u, v)$ must lie in the boundary of K^-. To complete the proof, it suffices to observe that $bd(K^+) = -bd(K^-)$. □

5. Necessary conditions for antipodality: the polyhedral case

We now look at the particular case of a polyhedral cone. Recall that a polyhedral cone can always be represented in the form

$$K = \mathrm{cone}\,\{g_1, \cdots, g_p\} = \left\{ \sum_{i=1}^{p} x_i g_i : \; x \in \mathbb{R}^p_+ \right\}. \tag{17}$$

Following a standard practice, we say that $\{g_1, \cdots, g_p\} \subset \mathbb{R}^d$ is a set of *generators* for K. There is no loss of generality in assuming that

$$\left\{ \begin{array}{l} \text{all generators have unit length,} \\ \text{no generator is a positive linear combination of the others.} \end{array} \right. \tag{18}$$

Some additional notation is needed to proceed further with the exposition. The $p \times p$ matrix M with components

$$M_{i,j} = \langle g_i, g_j \rangle \qquad \forall\, i, j \in \{1, \cdots, p\}$$

is called the *Gramian matrix* associated to the collection $\{g_1, \cdots, g_p\}$. Obviously, M is symmetric and positive semidefinite. The symbol x^t denotes the transpose of the column vector $x \in \mathbb{R}^p$.

Theorem 5.1. *Consider a polyhedral cone K as in (17)-(18). If u and v are antipodal points of K, then u and v are representable in the form*

$$u = \sum_{i=1}^{p} x_i g_i, \quad v = \sum_{i=1}^{p} y_i g_i, \tag{19}$$

with $(x, y) \in \mathbb{R}^p \times \mathbb{R}^p$ satisfying the following set of conditions:

$$x \in \mathbb{R}^p_+, \;\; y \in \mathbb{R}^p_+, \tag{20.a}$$
$$x^t M x = 1, \;\; y^t M y = 1, \tag{20.b}$$
$$My - (x^t M y) M x \in \mathbb{R}^p_+, \tag{20.c}$$
$$Mx - (x^t M y) M y \in \mathbb{R}^p_+, \tag{20.d}$$
$$x^t \big[My - (x^t M y) M x \big] = 0, \tag{20.e}$$
$$y^t \big[Mx - (x^t M y) M y \big] = 0. \tag{20.f}$$

Proof. The variational problem (10) takes now the form

$$\left\{ \begin{array}{l} \text{minimize } \; x^t M y \\ x \in \mathbb{R}^p_+, \; y \in \mathbb{R}^p_+, \; x^t M x = 1, \; y^t M y = 1. \end{array} \right. \tag{21}$$

The system (20) is obtained by writing down the standard Karush-Kuhn-Tucker optimality conditions for (21). Observe, incidentally, that (20.e) and (20.f) can be derived directly from (20.b). □

Since the generators $\{g_1, \cdots, g_p\}$ may not be linearly independent, the representation (19) is not necessarily unique. However, this is not a major problem. If the antipodal pair (u, v) admits the alternative representation

$$u = \sum_{i=1}^{p} \tilde{x}_i g_i, \quad v = \sum_{i=1}^{p} \tilde{y}_i g_i,$$

then the new $\tilde{x}_i$'s and $\tilde{y}_i$'s must also solve the system (20). In short, we don't need to bother in discussing which particular representation are we using.

Corollary 5.2. *Consider a polyhedral cone K as in (17)-(18). Suppose that K is pointed, but not a ray. Let $x \in \mathbb{R}^p_+$ and $y \in \mathbb{R}^p_+$ be such that*

$$u = \sum_{i=1}^{p} x_i g_i, \qquad v = \sum_{i=1}^{p} y_i g_i$$

form a pair of antipodal points of K. Then, for each $i \in \{1, \cdots, p\}$, at least one of the following three alternatives hold:

(a) $x_i = 0$,

(b) $y_i = 0$,

(c) $\langle g_i, u\rangle = \langle g_i, v\rangle = 0$.

Proof. Since x and y produce a pair of antipodal points of K, the system (20) is in force. In particular, for each $i \in \{1, \cdots, p\}$, one has

$$x_i\big[(My)_i - (x^t My)(Mx)_i\big] = 0,$$
$$y_i\big[(Mx)_i - (x^t My)(My)_i\big] = 0.$$

Taking into account (20.a), the above conditions are obtained by writing (20.e)-(20.f) in a componentwise manner. Fix now an arbitrary $i \in \{1, \cdots, p\}$. If neither (a) nor (b) hold, then

$$0 = (My)_i - (x^t My)(Mx)_i = \langle g_i, v\rangle - \langle u, v\rangle\langle g_i, u\rangle,$$
$$0 = (Mx)_i - (x^t My)(My)_i = \langle g_i, u\rangle - \langle u, v\rangle\langle g_i, v\rangle,$$

from where one gets $(1 - \langle u, v\rangle^2)\langle g_i, u\rangle = 0$ and $(1 - \langle u, v\rangle^2)\langle g_i, v\rangle = 0$. Since K is pointed and not a ray, one has $|\langle u, v\rangle| \neq 1$, and therefore $\langle g_i, u\rangle = \langle g_i, v\rangle = 0$. □

6. Do generators achieve the maximal angle?

It would be very convenient if the maximal angle of a polyhedral cone were achieved by at least one pair of generators. If that were the case, antipodality analysis of polyhedral cones would be a very simple business. Unfortunately, the example given below prevents us from being too optimistic.

Example 6.1. In the Euclidean space $\mathbb{R}^3$, consider the cone K generated by

$$g_1 = \frac{1}{2}\,(\sqrt{3}, 1, 0), \quad g_2 = \frac{1}{2}\,(-\sqrt{3}, 1, 0), \quad g_3 = \frac{1}{4}\,(0, -\sqrt{15}, 1).$$

As a matter of computation, one gets $\langle g_1, g_2\rangle = -3/4$ and $\langle g_1, g_3\rangle = \langle g_2, g_3\rangle = -\sqrt{15}/8$. Consider now the unit vector $z = g_1 + g_2 = (0, 1, 0) \in K$. Since

$$\langle z, g_3\rangle \;=\; -\sqrt{15}/4 \;< -3/4 \;< -\sqrt{15}/8,$$

$\theta_{max}(K)$ isn't achieved by a pair of generators.

In Example 6.1, the maximal angle of the cone is greater than $\pi/2$, and this is why we are getting into troubles. A much simpler situation is that of a cone which is not too wide open.

Proposition 6.2. *Consider a polyhedral cone K as in (17)-(18). Suppose that all pairs of generators form an acute angle, that is to say,*

$$\langle g_i, g_j\rangle \geq 0 \qquad \forall i \neq j. \tag{22}$$

If (g_1, g_2) is a pair forming the largest angle among the generators, then g_1 and g_2 are antipodal points of K.

Proof. As usual, denote by M the Gramian matrix associated to the set of generators of the cone. Let $\alpha = \langle g_1, g_2\rangle$. We must show that $\alpha \leq \langle u, v\rangle \;\; \forall\, u, v \in K \cap S_d$. Pick up any pair

$$u = \sum_{i=1}^{p} x_i g_i, \quad v = \sum_{j=1}^{p} y_j g_j,$$

with $x, y \in I\!R^p_+$ such that $x^t M x = 1,\; y^t M y = 1$. A simple computation yields

$$\langle u, v\rangle = x^t M y = \sum_{i=1}^{p}\sum_{j=1}^{p} \langle g_i, g_j\rangle\, x_i y_j \geq \alpha\Big(\sum_{i=1}^{p} x_i\Big)\Big(\sum_{j=1}^{p} y_j\Big) = \alpha\, \|x\|_1\, \|y\|_1\,. \tag{23}$$

In the above line we are using the fact that g_1 and g_2 form the largest angle among the generators. Notice also that

$$1 = x^t M x \;\leq \Big(\sum_{i=1}^{p} x_i\Big)^2 = \|x\|_1^2\,,$$

$$1 = y^t M y \;\leq \Big(\sum_{j=1}^{p} y_j\Big)^2 = \|y\|_1^2\,,$$

from where one gets $\|x\|_1 \geq 1,\; \|y\|_1 \geq 1$. So, by using (23) and the fact that $\alpha \geq 0$, one arrives at the desired conclusion. □

Assumption (22) amounts to saying that the cone K is *acute* in the sense that

$$\langle u, v\rangle \geq 0 \qquad \forall u, v \in K. \tag{24}$$

Of course, acuteness of K can be defined equivalently in terms of the inequality $\theta_{max}(K) \leq \pi/2$. As shown below, $\pi/2$ is a sort of threshold value for the validity of Proposition 6.2.

Proposition 6.3. *For any $d \geq 3$ and $\theta \in]\pi/2, \pi[$, one can construct a polyhedral cone $K \subset I\!R^d$ such that*

(a) *the maximal angle of K equals θ;*

(b) *the maximal angle of K isn't achieved by a pair of generators.*

Proof. In the Euclidean space $I\!R^3$, consider the cone K_t generated by

$$g_1 = (\sqrt{1 - r_t^2}, r_t, 0), \quad g_2 = (-\sqrt{1 - r_t^2}, r_t, 0), \quad g_3 = (0, -t, \sqrt{1 - t^2}),$$

with $t \in]0, 1[$ and $r_t = 0.5\,\sqrt{2 - t}$. We are taking $d = 3$ just for simplicity. If $d > 3$, then it suffices to fill with zeroes the remaining components in each generator. A simple computation yields

$$\langle g_1, g_2 \rangle = 2r_t^2 - 1 = -\frac{t}{2}, \qquad \langle g_1, g_3 \rangle = \langle g_2, g_3 \rangle = -tr_t = -\frac{t}{2}\sqrt{2 - t}.$$

Consider now the unit vector $w = (0, 1, 0) \in K_t$. Observe that $\langle w, g_3 \rangle = -t$ is strictly smaller that $\langle g_1, g_2 \rangle$, and also strictly smaller that $\langle g_1, g_3 \rangle = \langle g_2, g_3 \rangle$. This proves that $\theta_{max}(K_t)$ isn't attained by a pair of generators. We now adjust the parameter $t \in]0, 1[$ so that $\theta_{max}(K_t) = \theta$. To prove that such a number t exists, we rely on a continuity argument. By Theorem 2.5, the function

$$t \in]0, 1[\mapsto \theta_{max}(K_t)$$

is continuous. On the other hand, one can easily check that

$$\lim_{t \to 0} \theta_{max}(K_t) = \pi/2, \qquad \lim_{t \to 1} \theta_{max}(K_t) = \pi.$$

So, the intermediate value $\theta \in]\pi/2, \pi[$ must be attained by some $t \in]0, 1[$. $\square$

We end this section by stating a necessary and sufficient condition for antipodality. The proof of Theorem 6.4 is quite involved and, unfortunately, it doesn't extend easily to the case of a cone generated by more than three vectors.

Theorem 6.4. *If $K \subset I\!R^d$ is a cone generated by three different vectors $\{g_1, g_2, g_3\} \subset S_d$, then the following two statements are equivalent:*

(a) *the pair (g_1, g_2) is critical for K and forms a maximal angle among generators;*

(b) *(g_1, g_2) is a pair of antipodal points of K.*

Proof. By Theorem 4.1, the implication $(b) \Rightarrow (a)$ is true even if K is generated by more than three vectors. To prove the reverse implication, consider the inner products

$$\alpha = \langle g_1, g_2 \rangle, \quad \beta = \langle g_1, g_3 \rangle, \quad \gamma = \langle g_2, g_3 \rangle$$

that can be formed among the generators. Observe that $\alpha, \beta, \gamma \in [-1, 1]$. We show next that there is no loss of generality in assuming that

$$-1 < \alpha \leq \beta \leq \gamma < 1, \tag{25.a}$$

$$\alpha < 0 \leq \gamma. \tag{25.b}$$

If $\alpha = -1$, then $\theta_{max}(K) = \arccos\langle g_1, g_2 \rangle = \pi$, so that (b) holds trivially. Clearly $1 \notin \{\alpha, \beta, \gamma\}$ because the vectors g_1, g_2, and g_3 are different. Thus, we may assume that $\alpha, \beta, \gamma \in]-1, 1[$. The second inequality in (25.a) holds because g_1 and g_2 form a maximal angle among generators. The third one may be assumed without loss of generality (otherwise, we interchange the roles of g_1 and g_2). This takes care of (25.a). For the left inequality of (25.b), we note that if $\alpha \geq 0$, then we are within the hypotheses of Proposition 6.2, in which case (b) holds even without assuming beforehand that (g_1, g_2) is a critical pair for K. We now look at the right inequality in (25.b). In the present setting, to say that (g_1, g_2) is a critical pair of K is the same as saying that

$$\gamma \geq \alpha\beta, \tag{26.a}$$

$$\beta \geq \alpha\gamma. \tag{26.b}$$

Due to (25.a) and (26.a), the coefficient γ cannot be negative. In short, (25.b) can also be assumed. Consider now an arbitrary pair (u, v) of antipodal points of K, and write

$$u = x_1 g_1 + x_2 g_2 + x_3 g_3, \quad v = y_1 g_1 + y_2 g_2 + y_3 g_3,$$

with $x, y \in I\!R^3_+$. We must prove that

$$\langle g_1, g_2 \rangle \leq \langle u, v \rangle. \tag{27}$$

We suppose that (u, v) isn't a pair of generators of K, because otherwise (27) holds trivially. By Lemma 2.1, u and v are in the boundary of K, so that $x_1x_2x_3 = 0$ and $y_1y_2y_3 = 0$. For the sake of the exposition, we split the remaining part of the proof in four cases:

- *Case* $x_3y_3 \neq 0$. By Corollary 5.2, one has $\langle g_3, u \rangle = \langle g_3, v \rangle = 0$, or equivalently,

$$\beta x_1 + \gamma x_2 + x_3 = 0,$$
$$\beta y_1 + \gamma y_2 + y_3 = 0.$$

Since $\gamma \geq 0$, one gets $x_1 \neq 0$ and $x_2 = 0$. This means that u lies in the relative interior of the cone $F_{\{1,3\}} = \text{cone}\{g_1, g_3\}$. Similarly, $y_1 \neq 0$ and $y_2 = 0$, so that also $v \in \text{ri } F_{\{1,3\}}$. We are then contradicting the statement of Corollary 2.2. In short, the case $x_3y_3 \neq 0$ cannot occur.

• *Case* $x_2y_2 \neq 0$. One follows the same line of argument as before. This time one works with the system

$$\alpha x_1 + x_2 + \gamma x_3 = 0,$$
$$\alpha y_1 + y_2 + \gamma y_3 = 0,$$

and concludes that u and v are both in the relative interior of $F_{\{1,2\}}$, the cone generated by g_1 and g_2. One arrives again at a contradiction.

• *Case* $x_1y_1 \neq 0$. This case is more interesting to deal with. What is happening now is that

$$x_1 + \alpha x_2 + \beta x_3 = 0, \tag{28.a}$$
$$y_1 + \alpha y_2 + \beta y_3 = 0, \tag{28.b}$$

so that two configurations are possible: either $(x_3, y_2) = (0, 0)$ or $(x_2, y_3) = (0, 0)$. It suffices to explore the first option, because the situation is symmetric with respect to u and v. Plugging this choice into (28), one gets

$$x_1 + \alpha x_2 = 0, \tag{29.a}$$
$$y_1 + \beta y_3 = 0. \tag{29.b}$$

Next we add the equations

$$x_1^2 + 2\alpha x_1 x_2 + x_2^2 = 1, \tag{29.c}$$
$$y_1^2 + 2\beta y_1 y_3 + y_3^2 = 1, \tag{29.d}$$

which are derived from the normalization conditions $\|u\|^2 = 1$ and $\|v\|^2 = 1$. The system (29) is solvable if and only if the coefficients α and β are negative, in which case one gets

$$x_1 = -\alpha/\sqrt{1-\alpha^2}, \quad x_2 = 1/\sqrt{1-\alpha^2}, \quad y_1 = -\beta/\sqrt{1-\beta^2}, \quad y_3 = 1/\sqrt{1-\beta^2}.$$

One ends up with

$$\langle u, v\rangle = \frac{\gamma - \alpha\beta}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}} \geq 0,$$

proving in this way the inequality (27).

• *Case* $x_1y_1 = x_2y_2 = x_3y_3 = 0$. By a symmetry argument, there is no loss of generality in assuming that x has exactly one null component, and y has exactly two null components. Three cases must be considered:

(a) $x = (1, 0, 0), y = (0, y_2, y_3)$. This means that $u = g_1$ and $v \in \mathrm{ri}(F_{\{2,3\}})$. One has

$$\langle u, v\rangle = \langle g_1, y_2 g_2 + y_3 g_3\rangle = \alpha y_2 + \beta y_3.$$

In addition, the normalization condition $\|v\|^2 = 1$ yields

$$y_2^2 + 2\gamma y_2 y_3 + y_3^2 = 1. \tag{30}$$

Since $\gamma \in [0,1[$, the equation (30) implies that $y_3 \in [0,1]$, and

$$y_2 = -\gamma y_3 \pm \sqrt{1-(1-\gamma^2)y_3^2}.$$

Hence,

$$\langle u,v\rangle = (\beta - \alpha\gamma)y_3 \pm \alpha\sqrt{1-(1-\gamma^2)y_3^2}.$$

We now look at the problem of minimizing the function

$$t \in [0,1] \mapsto \varphi_1(t) = (\beta - \alpha\gamma)t \pm \alpha\sqrt{1-(1-\gamma^2)t^2}.$$

If t^* minimizes this function, one has $\langle u,v\rangle \geq \varphi_1(t^*)$. If $t^*=0$, then $(y_2,y_3)=(1,0)$, that is to say, $v=g_2$. Similarly, if $t^*=1$, then $(y_2,y_3)=(0,1)$, so that $v=g_3$. Since we are assuming that (u,v) isn't a pair of generators, t^* must be in $]0,1[$, in which case $\varphi_1'(t^*)=0$. One can derive an explicit expression for t^*, but we will consider a function more general that φ_1, which will be useful in the following two items. Take $\theta,\sigma,\eta \in I\!R$, with $\theta \geq 0$ and $\eta > 0$, and define

$$\psi(t) = \theta t \pm \sigma\sqrt{1-\eta t^2}.$$

It is easy to check that the derivative of ψ vanishes at $t^* = \theta\,[\theta^2\eta + \sigma^2\eta^2]^{-1/2}$. This value is obtained by solving the equation $\psi'(t)=0$, that is to say,

$$\theta = \frac{\pm\sigma\eta t}{\sqrt{1-\eta t^2}}.$$

Regardless of the sign of σ, one always get

$$\psi(t^*) = \sqrt{\sigma^2 + \eta^{-1}\theta^2} \;\geq 0.$$

Note that $\varphi_1(t)=\psi(t)$ with $\theta=\beta-\alpha\gamma$, $\eta = 1-\gamma^2$, $\sigma=\alpha$, so that η and θ have the correct signs by (25.a) and (26.b). We conclude that $\langle u,v\rangle \geq \varphi_1(t^*) = \psi(t^*) \geq 0$, so that (27) holds.

(b) $x=(0,1,0)$, $y=(y_1,0,y_3)$. One has $\langle u,v\rangle = \alpha y_1 + \gamma y_3$, with

$$y_1 = -\beta y_3 \pm \sqrt{1-(1-\beta^2)y_3^2}.$$

We get again $\langle u,v\rangle \geq \psi(t^*) \geq 0$, with t^* being a minimum of ψ over $]0,1[$. This time, of course, the function ψ is defined in terms $\theta = \gamma - \alpha\beta$, $\sigma=\alpha$, and $\eta = 1-\beta^2$.

(c) $x=(0,0,1)$, $y=(y_1,y_2,0)$. One has $\langle u,v\rangle = \beta y_1 + \gamma y_2$, with

$$y_1 = -\alpha y_2 \pm \sqrt{1-(1-\alpha^2)y_2^2}.$$

Everything is the same as before, but now $\theta=\gamma-\alpha\beta$, $\sigma=\beta$, and $\eta=1-\alpha^2$. □

7. When a pair of generators is critical?

The pairs $(g_1, g_1), \cdots, (g_p, g_p)$ are all critical, but, of course, they provide no relevant information on the angular structure of the cone. The interesting question is

$$\begin{cases} \text{given two different indices, say } i, j \in \{1, \cdots, p\}, \text{ is it possible} \\ \text{to check easily whether or not the pair } (g_i, g_j) \text{ is critical ?} \end{cases}$$

The answer is yes. Below we state a simple test for checking criticalness of a given pair of generators.

Proposition 7.1. *Let $K \in \Xi(I\!\!R^d)$ be a polyhedral cone as in (17)-(18). For given indices $i, j \in \{1, \cdots, p\}$, criticalness of the pair (g_i, g_j) is equivalent to the combination of*

$$\min_{1 \le k \le p} \{\langle g_k, g_j \rangle - \langle g_i, g_j \rangle \langle g_k, g_i \rangle\} = 0 \tag{31.a}$$

and

$$\min_{1 \le k \le p} \{\langle g_k, g_i \rangle - \langle g_i, g_j \rangle \langle g_k, g_j \rangle\} = 0. \tag{31.b}$$

Proof. If K admits the representation (17)-(18), then its positive dual cone is given by

$$K^+ = \{z \in I\!\!R^d : \langle g_k, z \rangle \ge 0 \ \ \forall k = 1, \cdots, p\}.$$

Hence,

$$z \in K^+ \iff \min_{1 \le k \le p} \langle g_k, z \rangle \ge 0. \tag{32}$$

If z is in the boundary of K^+, then the right-hand side of (32) occurs as an equality. In view of Corollary 4.5, it suffices to work out the particular cases

$$z = g_j - \langle g_i, g_j \rangle g_i \quad \text{and} \quad z = g_i - \langle g_i, g_j \rangle g_j,$$

producing the conditions (31.a) and (31.b), respectively. □

Corollary 7.2. *Let $K \in \Xi(I\!\!R^d)$ be a polyhedral cone as in (17)-(18). Suppose that the number of generators of K is at least three, and that*

$$\forall\, i \ne j \ \ \exists\, k \notin \{i, j\} \ \text{ s.t. } \ \langle g_k, g_j \rangle - \langle g_i, g_j \rangle \langle g_k, g_i \rangle < 0 \ \text{ or } \ \langle g_k, g_i \rangle - \langle g_i, g_j \rangle \langle g_k, g_j \rangle < 0. \tag{33}$$

Then, two different generators cannot form a critical pair.

Proof. Given $i \ne j$, we pick up any $k \notin \{i, j\}$ as in (33). We are either violating condition (31.a) or (31.b), so the pair (g_i, g_j) cannot be critical. □

Remark: Assumption (33) holds, for instance, if the generators of K are pairwise obtuse, that is to say, if $\langle g_i, g_j \rangle < 0 \ \ \forall\, i \ne j$.

We end this section by examining the case of a very interesting polyhedral cone, namely, the cone Φ_d that defines the Schur ordering in the Euclidean space $I\!R^d$.

Example 7.3. Take $d \geq 4$. The Schur ordering in $I\!R^d$ is the partial-order relation $\succ$ defined by

$$\varrho \succ \vartheta \quad \Longleftrightarrow \quad \sum_{i=1}^{r} \varrho_i \geq \sum_{i=1}^{r} \vartheta_i \quad \forall\, r \in \{1, \cdots, d-1\} \quad \text{and} \quad \sum_{i=1}^{d} \varrho_i = \sum_{i=1}^{d} \vartheta_i.$$

The cone $\Phi_d = \{\varrho \in I\!R^d : \ \varrho \succ 0\}$ of Schur-positive vectors of $I\!R^d$ is clearly a polyhedral one. In fact, one has

$$\Phi_d = \text{cone}\{g_1, \cdots, g_{d-1}\},$$

where $g_i = (e_i - e_{i+1})/\sqrt{2}$ for $i = 1, \cdots, d-1$. As usual, $\{e_1, \cdots, e_d\}$ denotes the canonical basis of $I\!R^d$. Observe that

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ -1/2 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

so it is not difficult to see that $\{g_1, \cdots, g_{d-1}\}$ satisfies the assumption (33). In conclusion, a critical pair of Φ_d cannot be formed with two different generators. Of course, this rules out the possibility of achieving the maximal angle of Φ_d by restricting our attention to the set of generators. In other words, higher dimensional faces of Φ_d must be brought into consideration.

8. Rudiments of the theory of critical angles

In this section we are concerned not just with the maximal angle of a cone, but rather with the full collection of critical angles. As pointed out in Definition 4.2, the set of all critical angles of K is called the angular spectrum of K.

Besides the minimal angle 0 and the maximal angle $\theta_{max}(K)$, the angular spectrum of K may contain also intermediate critical angles. The minimal angle 0 is a parasitic object because it says absolutely nothing on the structure of the cone. By contrast, intermediate critical values may provide a valuable information on the shape of the cone. The next example serves to illustrate this point.

Example 8.1. Consider the V-shape cantilever $K = \{x \in I\!R^3 : \ x_3 \geq |x_2|\}$. Since K contains the bottom line $I\!R(1,0,0)$, one has $\theta_{max}(K) = \pi$. One can easily check that

$$u = (0, \sqrt{2}/2, \sqrt{2}/2) \quad \text{and} \quad v = (0, -\sqrt{2}/2, \sqrt{2}/2)$$

form a critical pair for K. Since u and v are orthogonal, they produce $\pi/2$ as critical angle. This is precisely the angle formed by both (two-dimensional) faces of the cantilever.

How many elements can be found in the angular spectrum of a given cone? Is the angular spectrum of a cone always discrete, or, on the contrary, is it possible to find a whole interval of critical angles? To answer this type of question we need to open a big parenthesis and say a few words on the theory of cone-constrained eigenvalue problems as developed in references [**?**, **?**]. In what follows, M_n denotes the space of real matrices of size $n \times n$.

Definition 8.2. The spectrum of $A \in M_n$ relative to the cone $Q \in \Xi(I\!R^n)$, denoted by $\sigma(A, Q)$, is the set of all $\lambda \in I\!R$ for which the linear complementarity problem

$$z \in Q, \quad Az - \lambda z \in Q^+, \quad \langle z, Az - \lambda z \rangle = 0 \tag{34}$$

admits a nonzero solution $z \in I\!R^n$. The term λ is called an eigenvalue of A relative to Q, and $z \neq 0$ is referred to as an eigenvector of A relative to Q.

As shown in [**?**], an arbitrary matrix A admits only a finite number of eigenvalues relatively to a polyhedral cone. In fact, it is possible to obtain a uniform bound for the cardinality of $\sigma(A, Q)$.

Lemma 8.3. (Seeger-Torki [**?**]) *If $Q \in \Xi(I\!R^n)$ is a polyhedral cone, then*

$$\sup_{A \in M_n} card\,[\sigma(A, Q)] \;\leq\; \sum_{k=1}^{dimQ} k\, f_Q(k), \tag{35}$$

where $f_Q(k)$ stands for the number of k-dimensional faces of Q.

Corollary 8.4. (Seeger-Torki [**?**]) *Let $Q \in \Xi(I\!R^n)$ be a polyhedral cone. One has:*

(a) *if Q is generated by a collection of q vectors, then*

$$\sup_{A \in M_n} card\,[\sigma(A, Q)] \;\leq\; \sum_{k=1}^{min\{q,n\}} k\, C_k^q \;\leq\; q\, 2^{q-1}, \tag{36}$$

(b) *if Q is the intersection of r half-spaces, then*

$$\sup_{A \in M_n} card\,[\sigma(A, Q)] \;\leq\; \sum_{k=0}^{min\{r,n\}} (n-k)\, C_k^r \;\leq\; (2n-r)\, 2^{r-1}. \tag{37}$$

The bounds given by Corollary 8.4 are easy to evaluate because they depend only on the integers q, r, and n. The notation C_k^m is standard and refers to the binomial coefficient

$$C_k^m = \frac{m!}{k!\,(m-k)!}\;.$$

We now close the parenthesis on cone-constrained eigenvalues and come back to the theory of critical angles.

Theorem 8.5. *A polyhedral cone admits only a finite number of critical angles.*

Proof. Let $K \in \Xi(R^d)$ be the polyhedral cone under consideration. Take any θ in $\Sigma(K)$. By definition of the angular spectrum, $\theta = \arccos\langle u, v\rangle$ for some critical pair (u, v) of K. As seen in the proof of Theorem 4.1, we can express the criticalness of (u, v) by means of the system (13). Since both Lagrange multipliers in (13) are the same, we arrive in this way to the cone-constrained eigenvalue problem (34) with particular data

$$z = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A = I_d \oslash I_d = \begin{bmatrix} 0 & I_d \\ I_d & 0 \end{bmatrix}, \quad Q = K \times K.$$

The notation I_d refers to the identity matrix of size $d \times d$. The common Lagrange multiplier $\lambda = \langle u, v\rangle$ is then an eigenvalue of A relative to Q. Since λ can take only a finite number of values (cf. Lemma 8.3), so does the variable $\theta = \arccos \lambda$. In short, the angular spectrum of K is discrete. □

The proof of Theorem 8.5 yields a lot of extra information concerning the angular spectrum of a cone. We shall not exploit the full power of this proof, but we mention at least a few remarkable facts. First of all, notice that the inclusion

$$\Sigma(K) \subset \{\arccos \lambda : \ \lambda \in \sigma(I_d \oslash I_d, K \times K)\} \tag{38}$$

applies for an arbitrary cone $K \in \Xi(I\!R^d)$, be it polyhedral or not! As far as the reverse inclusion is concerned, one needs to distinguish between two mutually exclusive cases:

(i) if $K \cap K^+$ equals $\{0\}$ or contains a pair of orthogonal unit vectors, then

$$\Sigma(K) = \{\arccos \lambda : \ \lambda \in \sigma(I_d \oslash I_d, K \times K)\}, \tag{39}$$

(ii) if $K \cap K^+$ is different from $\{0\}$ and doesn't contain a pair of orthogonal unit vectors, then

$$\Sigma(K) = \{\arccos \lambda : \ \lambda \in \sigma(I_d \oslash I_d, K \times K)\} \backslash \{\pi/2\}. \tag{40}$$

Formulas (39) and (40) are obtained by working out the proof of Theorem 8.5 in the backward order. Withdrawing the spurious value $\pi/2$ in the right-hand side of (40) is necessary. To understand this phenomenon, observe that a pair of the form $(u, 0)$, with $u \neq 0$, can be an eigenvector of $I_d \oslash I_d$ relative to $K \times K$, but it cannot be critical for K because its second component is zero.

In view of (38), any upper bound for the cardinality of $\sigma(I_d \oslash I_d, K \times K)$ serves as upper bound for the cardinality of $\Sigma(K)$. A direct application of Lemma 8.3 yields

$$\mathrm{card}[\Sigma(K)] \ \leq \sum_{k=1}^{\dim(K \times K)} k \, f_{K \times K}(k). \tag{41}$$

If K is representable in terms of p generators, then $K \times K$ can be expressed in terms of $q = 2p$ generators, getting in this way

$$\mathrm{card}[\Sigma(K)] \;\leq\; p\, 2^{2p}. \tag{42}$$

On the other hand, if K is an intersection of s half-spaces, then $K \times K$ is representable as intersection of $r = 2s$ half-spaces, getting in this way

$$\mathrm{card}[\Sigma(K)] \;\leq\; (2d - s)\, 2^{2s}. \tag{43}$$

The general bounds of Seeger-Torki are uniform and don't take into account the specific structure of the matrix A. In our particular setting, however, the matrix A has a very special form. So, it is possible to sharpen the estimates (41)-(43), but we shall not indulge in this matter.

Below we work out in detail the case of a cone represented by p vectors. The next theorem serves for identifying the elements of the angular spectrum. First, some words on notation. For nonempty subsets $I, J \subset \{1, \cdots, p\}$, denote by $M_{I,J}$ the principal matrix of M which is obtained by deleting the i-th row and the j-th column of M, whenever $i \notin I$ and $j \notin J$. The symbol $|I|$ stands for the cardinality of I. So, $M_{I,J}$ is a rectangular matrix of size $|I| \times |J|$. Because the generators $\{g_1, \cdots, g_p\}$ are not necessarily linearly independent, it is helpful to write

$$I \in \mathcal{N}(g_1 \ \cdots \ g_p) \iff \begin{cases} I \subset \{1, \cdots, p\} \text{ is nonempty and the set} \\ \{g_i : i \in I\} \text{ is linearly independent.} \end{cases}$$

Notice that $M_{I,I}$ is the Gramian matrix associated to the sub-collection $\{g_i : i \in I\}$. So, for $I \in \mathcal{N}(g_1 \ \cdots \ g_p)$, the matrix $M_{I,I}$ is nonsingular. Everything is now in place to state:

Theorem 8.6. *Let $K \in \Xi(R^d)$ be a polyhedral cone as in (17)-(18), and M be the Gramian matrix associated to the generators of K. Let $\theta \notin \{0, \pi\}$ be a critical angle of K, and write $\lambda = \cos\theta$. Under these assumptions, there are sets $I, J \in \mathcal{N}(g_1 \ \cdots \ g_p)$, with $I \neq J$, and vectors $\xi \in int(\mathbb{R}_+^{|I|})$ and $\eta \in int(\mathbb{R}_+^{|J|})$ such that*

$$\begin{bmatrix} 0 & M_{I,I}^{-1} M_{I,J} \\ M_{J,J}^{-1} M_{J,I} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \lambda \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{44}$$

$$\sum_{j \in J} M_{kj} \eta_j - \lambda \sum_{i \in I} M_{ki} \xi_i \geq 0 \quad \forall\, k \notin I, \tag{45.a}$$

$$\sum_{i \in I} M_{li} \xi_i - \lambda \sum_{j \in J} M_{lj} \eta_j \geq 0 \quad \forall\, l \notin J. \tag{45.b}$$

$$\Big\| \sum_{i \in I} \xi_i g_i \Big\| = 1\,, \qquad \Big\| \sum_{j \in J} \eta_j g_j \Big\| = 1. \tag{46}$$

Proof. Suppose that θ derives from the critical pair (u, v). As done in (19), one can express u in terms of the full collection $\{g_1, \cdots, g_p\}$ of generators. This time, however, we need to follow a much subtler strategy. According to the cone version of Caratheodory's Theorem (cf. Section 1.6 in [?]), the nonzero vector u is representable in the form

$$u = \sum_{i \in I} \xi_i g_i \quad \text{with } \; I \in \mathcal{N}(g_1 \; \cdots \; g_p) \; \text{ and } \; \xi_i > 0 \;\; \forall\, i \in I.$$

Similarly, the nonzero vector v admits the representation

$$v = \sum_{j \in J} \eta_j g_j \quad \text{with } \; J \in \mathcal{N}(g_1 \; \cdots \; g_p) \; \text{ and } \; \eta_j > 0 \;\; \forall\, j \in J.$$

Since u and v have unit length, we take $\xi \in \operatorname{int}(I\!R_+^{|I|})$ and $\eta \in \operatorname{int}(I\!R_+^{|J|})$ as in (46). Criticalness of (u, v) leads to the system

$$\Big\langle g_k, \sum_{j \in J} \eta_j g_j - \lambda \sum_{i \in I} \xi_i g_i \Big\rangle \geq 0 \qquad \forall\, k = 1, \cdots, p, \tag{47.a}$$

$$\Big\langle g_l, \sum_{i \in I} \xi_i g_i - \lambda \sum_{j \in J} \eta_j g_j \Big\rangle \geq 0 \qquad \forall\, l = 1, \cdots, p. \tag{47.b}$$

Observe that

$$0 = \langle u, v - \lambda u \rangle = \Big\langle \sum_{k \in I} \xi_k g_k, \sum_{j \in J} \eta_j g_j - \lambda \sum_{i \in I} \xi_i g_i \Big\rangle = \sum_{k \in I} \xi_k \Big\langle g_k, \sum_{j \in J} \eta_j g_j - \lambda \sum_{i \in I} \xi_i g_i \Big\rangle.$$

Hence, equality in (47.a) occurs for each $k \in I$. Similarly, equality in (47.b) occurs for each $l \in J$. In short, the system (47) decomposes into (45) and

$$\sum_{j \in J} M_{kj} \eta_j - \lambda \sum_{i \in I} M_{ki} \xi_i = 0 \quad \forall i \in I, \tag{48.a}$$

$$\sum_{i \in I} M_{li} \xi_i - \lambda \sum_{j \in J} M_{lj} \eta_j = 0 \quad \forall j \in J. \tag{48.b}$$

Writing (48) in matrix form

$$M_{I,J} \eta = \lambda \; M_{I,I} \xi, \tag{49.a}$$

$$M_{J,I} \xi = \lambda \; M_{J,J} \eta, \tag{49.b}$$

one arrives finally at (44). In order to complete the proof we need to show that $I \neq J$. Observe that (44) is an eigenvalue problem with eigenvector (ξ, η) of size $|I| + |J|$. This large eigenvalue problem can be decoupled in two problems of smaller sizes. Indeed, by working directly with (49), one gets

$$M_{I,I}^{-1} M_{I,J} M_{J,J}^{-1} M_{J,I} \; \xi = \lambda^2 \xi, \tag{50.a}$$

as well as

$$M_{J,J}^{-1}M_{J,I}M_{I,I}^{-1}M_{I,J}\,\eta = \lambda^2\eta. \tag{50.b}$$

The case $I = J$ must be ruled out because (50.a) would reduce to $\xi = \lambda^2\xi$, implying that $\lambda^2 = 1$, and contradicting the fact that $\theta \notin \{0, \pi\}$. □

9. By way of application

As measure for the degree of pointedness of a cone $K \in \Xi(\mathbb{R}^d)$, reference [?] suggests considering the number

$$\rho(K) = \inf_{Q\in\mathcal{M}(\mathbb{R}^d)} \delta(Q, K),$$

where $\mathcal{M}(\mathbb{R}^d)$ denotes the set of nonpointed cones in $\Xi(\mathbb{R}^d)$. Since $\mathcal{M}(\mathbb{R}^d)$ is compact in the metric space $(\Xi(\mathbb{R}^d), \delta)$, the above infimum is actually attained. An important question is then

$$\begin{cases}\text{given a pointed cone } K \in \Xi(\mathbb{R}^d), \text{ how to constuct a} \\ \text{nonpointed cone } Q \in \Xi(\mathbb{R}^d) \text{ at minimal distance from } K\ ?\end{cases}$$

Of course, one would like also to evaluate such a minimal distance. This issue is, in fact, quite involved. It would be too space consuming to explain all the details, so we state below without proof the main result of our work [?]. As the reader will notice, the concept of antipodality plays here a prominent role.

Theorem 9.1. *Suppose the maximal angle of $K \in \Xi(\mathbb{R}^d)$ doesn't exceed $2\pi/3$. Then,*

$$\rho(K) = cos\left(\frac{\theta_{max}(K)}{2}\right).$$

Suppose, in addition, that K is not a ray, and admits (u, v) as pair of antipodal points. Then, $K + \mathbb{R}(u - v)$ is a member of $\mathcal{M}(\mathbb{R}^d)$ lying at minimal distance from K.

The recipe for constructing Q is clear enough. If we are able to find a pair (u, v) of antipodal points of K, then we are done. It suffices to set $Q = K + \mathbb{R}(u - v)$. Attention, however, with the angular specification. If the angle of K is bigger that $2\pi/3$, then we are in trouble.

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