The compactified Picard scheme of the compactified Jacobian

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Abstract. Let C be an integral projective curve in any characteristic. Given an invertible sheaf $\mathcal{L}$ on C of degree 1, form the corresponding Abel map $A_{\mathcal{L}}: C \to \bar{J}$, which maps C into its compactified Jacobian, and form its pullback map $A_{\mathcal{L}}^*: \mathrm{Pic}^0_{\bar{J}} \to J$, which carries the connected component of 0 in the Picard scheme back to the Jacobian. If C has, at worst, double points, then $A_{\mathcal{L}}^*$ is known to be an isomorphism. We prove that $A_{\mathcal{L}}^*$ always extends to a map between the natural compactifications, $\mathrm{Pic}^-_{\bar{J}} \to \bar{J}$, and that the extended map is an isomorphism if C has, at worst, ordinary nodes and cusps.

1. Introduction

Let C be an integral projective curve of arithmetic genus g, defined over an algebraically closed field of any characteristic. Form its (generalized) Jacobian J, the connected component of the identity of the Picard scheme of C. If C is singular, then J is not projective. So for about forty years, numerous authors have studied a natural compactification of J: the (fine) moduli space $\bar{J}$ of torsion-free sheaves of rank 1 and degree 0 on C. It is called the *compactified Jacobian.*

Recently, the compactified Jacobian appeared in Laumon's preprint [**13**], where he identified, up to homeomorphism, affine Springer fibers with coverings of compactified Jacobians. For that identification, he used the autoduality of the compactified Jacobian, a property established in [**8**] and explained next.

From now on, assume C has, at worst, points of multiplicity 2 (or double points). For each invertible sheaf $\mathcal{L}$ of degree 1 on C, form the Abel map $A_{\mathcal{L}}: C \to \bar{J}$, given by $P \mapsto \mathcal{M}_P \otimes \mathcal{L}$ where $\mathcal{M}_P$ is the ideal sheaf of P; it is a closed embedding if C is not of genus 0. Form the pullback map

$$A_{\mathcal{L}}^*: \mathrm{Pic}^0_{\bar{J}} \to J,$$

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carrying the connected component of 0 in the Picard scheme back to the Jacobian. Then $A^*_{\mathcal{L}}$ is an isomorphism and is independent of $\mathcal{L}$; see [**8**, Thm. 2.1, p. 595].

Since the singularities are locally planar, $\bar{J}$ is integral by [**1**, (9), p. 8]. Hence, not only does $\mathrm{Pic}^0_{\bar{J}}$ exist, but also it admits a natural compactification: its closure $\mathrm{Pic}^-_{\bar{J}}$ in the compactified Picard scheme $\mathrm{Pic}^=_{\bar{J}}$, the (fine) moduli space of torsion-free sheaves of rank 1 on $\bar{J}$; see [**3**, Thm. (3.1), p. 28]. Does $A^*_{\mathcal{L}}$ extend to a map between the compactifications? If so, then is the extension an isomorphism?

These questions were posed to the authors by Sawon. As mentioned in his introduction to [**14**], his results on dual fibrations to fibrations by Abelian varieties, in the "nicest" cases, depend on "extending autoduality to the compactifications."

It is not true, for every map, that the pullback of a torsion-free sheaf is still torsion free. But, for $A_{\mathcal{L}}$, it is true! There are two basic reasons why: first, $A^*_{\mathcal{L}}$ is independent of $\mathcal{L}$; second, the maps $A_{\mathcal{L}}$ can be bundled up into a *smooth* map $C \times J \to \bar{J}$. Thus there exists an extended pullback map

$$A^*_{\mathcal{L}}\colon \mathrm{Pic}^-_{\bar{J}} \to \bar{J};$$

this existence statement is the content of Theorem (2.6) below. (The statement and its proof already appear on the web in the preliminary version of [**8**].)

Is the extended map $A^*_{\mathcal{L}}$ also an isomorphism? This question seems much harder.

From now on, assume that C has, at worst, ordinary nodes and cusps. Then the extended map $A^*_{\mathcal{L}}$ is, indeed, an isomorphism, according to Theorem (4.1), our main result. Here is a sketch of the proof.

First, recall the definition of the inverse $\beta\colon J \to \mathrm{Pic}^0_{\bar{J}}$ from [**8**, Prop. 2.2, p. 595], or rather [**8**, Rmk. 2.4, p. 597]. Let $\mathcal{I}$ be the universal sheaf on $C \times \bar{J}$, and $\mathcal{P}$ the determinant of cohomology of $\mathcal{I} \otimes \mathcal{L}^{\otimes g-1}$ with respect to the projection $C \times \bar{J} \to \bar{J}$. The sheaf $\mathcal{P}^{-1}$ has a canonical regular section, whose zero scheme we denote by Θ and call the *theta divisor* associated to $\mathcal{L}$. Let $p_1\colon \bar{J} \times J \to \bar{J}$ be the projection, and $\mu\colon \bar{J} \times J \to \bar{J}$ the multiplication map. Form the invertible sheaf

$$\mathcal{T} := \mathcal{O}_{\bar{J} \times J}(\mu^*\Theta - p_1^*\Theta).$$

Then $\mathcal{T}$ defines the map $\beta\colon J \to \mathrm{Pic}^0_{\bar{J}}$.

We need only show that β extends to a map $\overline{\beta}\colon \bar{J} \to \mathrm{Pic}^-_{\bar{J}}$, or that $\mathcal{T}$ extends to a sheaf on $\bar{J} \times \bar{J}$ that is flat for $p_2\colon \bar{J} \times \bar{J} \to \bar{J}$ and whose fibers are torsion-free rank-1. In fact, as $\overline{\beta}$ is unique if it exists, we need only find a flat surjection $\zeta\colon W \to \bar{J}$ such that the pullback $\mathcal{T}_\zeta$ of $\mathcal{T}$ to $\bar{J} \times (\zeta^{-1}J)$ extends to a torsion-free $\mathcal{H}$ on $\bar{J} \times W/W$.

To define ζ, let C_0 be the smooth locus, and $P_1, \dots, P_m$ the singular points of C. Set $C_i := C_0 \cup \{P_i\}$ for $1 \leq i \leq m$. Set

$$H_m := C_m \times \cdots \times C_1 \subseteq C^{\times m} \text{ and } W := H_m \times J.$$

Define $\zeta\colon W \to \bar{J}$ as the natural map given on a pair, consisting of an m-tuple of points $(Q_m, \dots, Q_1)$ of C and an invertible sheaf $\mathcal{N}$ on C of degree 0, by

$$\zeta(Q_m, \dots, Q_1, \mathcal{N}) := \mathcal{M}_{Q_m} \otimes \cdots \otimes \mathcal{M}_{Q_1} \otimes \mathcal{N} \otimes \mathcal{L}^{\otimes m}$$

where $\mathcal{M}_{Q_i}$ is the ideal of Q_i in C for $i = 1, \dots, m$. Since ζ is a composition of base extensions of bigraded Abel maps, ζ is smooth by [**7**, Cor. 2.6, p. 5969]. It is also surjective, owing to our assumption on the singularities: since, at each local ring $\mathcal{O}_{C,P_i}$, the singularity degree δ is 1, any torsion-free rank-1 module is either free or isomorphic to the maximal ideal.

Next, we resolve the rational map $\mu \circ (1 \times \zeta)\colon \bar{J} \times W \to \bar{J}$ by using the *n-flag schemes* F_n of $\mathcal{I}/(C \times \bar{J})/\bar{J}$. The scheme F_n is the (fine) moduli space parameterizing n-chains of torsion rank-1 sheaves on C:

$$\mathcal{I}_n \subset \mathcal{I}_{n-1} \subset \cdots \subset \mathcal{I}_1 \subset \mathcal{I}_0$$

where $\mathcal{I}_0$ is of degree 0 and where each quotient $\mathcal{I}_{i-1}/\mathcal{I}_i$ is of length 1.

The scheme F_n comes equipped with two important maps. The first is the *multiplication map* $\gamma_n\colon F_n \times J \to \bar{J}$, which sends a pair consisting of a chain as above and an invertible sheaf $\mathcal{N}$ on C of degree 0 to the tensor product $\mathcal{I}_n \otimes \mathcal{N} \otimes \mathcal{L}^{\otimes n}$. The second is the *resolution map* $\widehat{\psi}^{(n)}\colon F_n \to \bar{J} \times C^{\times n}$, which sends a chain as above to the pair consisting of the sheaf $\mathcal{I}_0$ and the n-tuple $(Q_n, \ldots, Q_1) \in C^{\times n}$ such that $\mathcal{I}_{i-1}/\mathcal{I}_i$ is supported on Q_i for each i.

Set $\widetilde{F}_m := (\widehat{\psi}^{(m)})^{-1}(\bar{J} \times H_m)$. Then we have the following diagram, in which the right vertical map μ is only rational:

$$\begin{tikzcd} \widetilde{F}_m \times J \arrow[r, "\gamma_m"] \arrow[d, "\widehat{\psi}^{(m)} \times 1"'] & \bar{J} \\ \bar{J} \times H_m \times J \arrow[r, "1\times\zeta"] & \bar{J} \times \bar{J} \arrow[u, "\mu"] \end{tikzcd}$$

Note that $\zeta^{-1}(J) = C_0^{\times m} \times J$. It is not difficult to see that $\widehat{\psi}^{(m)}$ restricts to an isomorphism over $\bar{J} \times C_0^{\times m}$. Also, the composition $\mu \circ (1 \times \zeta) \circ (\widehat{\psi}^{(m)} \times 1)$ is defined on $(\widehat{\psi}^{(m)} \times 1)^{-1}(\bar{J} \times C_0^{\times m})$ and agrees with γ_m. Therefore, $\mathcal{T}_\zeta$ extends to the sheaf

$$\mathcal{H} := \mathcal{O}_{\bar{J}\times W}(-q_1^*\Theta) \otimes (\widehat{\psi}^{(m)} \times 1)_* \mathcal{O}_{\widetilde{F}_m \times J}(\gamma_m^*\Theta)$$

where $q_1\colon \bar{J} \times W \to \bar{J}$ is the projection.

The delicate part is now to prove that $\mathcal{H}$ is flat over W with torsion-free rank-1 fibers. It suffices to prove that $(\zeta \times 1)^*\mathcal{H}$ on $W \times W$ is flat over W with torsion-free rank-1 fibers. Now, there are base-change formulas for the determinant of cohomology and for the direct image. As the corresponding base-change maps, we use the horizontal maps in the following natural Cartesian square:

$$\begin{tikzcd} \widetilde{F}_{2m} \times J \arrow[r] \arrow[d, "\xi"'] & \widetilde{F}_m \times J \arrow[d, "\widehat{\psi}^{(m)} \times 1"] \\ W \times W \arrow[r, "\zeta\times 1"] & \bar{J} \times W \end{tikzcd}$$

where $\widetilde{F}_{2m} := (\widehat{\psi}^{(2m)})^{-1}(J \times H_m \times H_m)$ and ξ is, up to switching factors, $\widehat{\psi}^{(2m)} \times 1$. We now apply the technical, but essential, Lemma (3.3) to complete the proof.

All our results are, in fact, proved not just for an individual curve C, but for a flat projective family of (geometrically) integral curves over an arbitrary base scheme. All schemes are implicitly assumed to be locally Noetherian.

In short, in Section 2, we prove that the autoduality map $A_{\mathcal{L}}^*$ extends if the curves have, at worst, double points. In Section 3, we study flag schemes. Finally, in Section 4, we prove that the extended map $A^*_{\mathcal{L}}$ is an isomorphism if the curves have, at worst, ordinary nodes and cusps.

2. Extension

(2.1) *The compactified Jacobian.* By a *flat projective family of integral curves* C/S, let us mean that C is a flat and projective S-scheme with geometrically integral fibers of dimension 1.

Given such a family C/S and given an integer n, recall from [**2**], [**3**], [**4**], and [**8**] that there exists a projective S-scheme $\bar{J}^n$, or $\bar{J}^n_{C/S}$, parameterizing the torsion-free rank-1 sheaves of degree n on the fibers of C/S. And there exists an open subscheme J^n, or $J^n_{C/S}$, parameterizing those sheaves that are invertible. Furthermore, forming these schemes commutes with changing the base S. For short, set $J := J^0$ and $\bar{J} := \bar{J}^0$. Customarily, J is called the (relative generalized) *Jacobian* of C/S, and $\bar{J}$ the *compactified Jacobian*.

More precisely, $\bar{J}^n$ represents the étale sheaf associated to the functor whose T-points are degree-n torsion-free rank-1 sheaves $\mathcal{I}$ on $C \times T/T$. Such an $\mathcal{I}$ is a T-flat coherent sheaf on $C \times T$ such that, for each point t of T, the fiber $\mathcal{I}(t)$ is torsion-free and of generic rank 1 on the fiber $C(t)$ and also

$$\chi(\mathcal{I}(t)) - \chi(\mathcal{O}_{C(t)}) = n.$$

(2.2) *Multiplication and translation.* Let C/S be a flat projective family of integral curves. Let m and n be arbitrary integers. Let $U^{n,m} \subseteq \bar{J}^n \times \bar{J}^m$ be the open subscheme that represents the étale sheaf associated to the subfunctor whose T-points are the pairs of torsion-free rank-1 sheaves $(\mathcal{I}, \mathcal{J})$ on $C \times T/T$ such that $\mathcal{I}$ is invertible where $\mathcal{J}$ is not. Define the *multiplication* map

$$\mu \colon U^{n,m} \to \bar{J}^{m+n} \text{ by } \mu(\mathcal{I}, \mathcal{J}) := \mathcal{I} \otimes \mathcal{J}.$$

For each invertible sheaf $\mathcal{M}$ of degree m on C/S, define the *translation* by $\mathcal{M}$

$$\mu_{\mathcal{M}} \colon \bar{J}^n \to \bar{J}^{m+n} \text{ by } \mu_{\mathcal{M}}(\mathcal{I}) := \mathcal{I} \otimes \mathcal{M}.$$

In other words, $\mathcal{M}$ defines a section $\sigma \colon S \to J^m$, and $\mu_{\mathcal{M}} := \mu \circ (1 \times \sigma)$; the composition makes sense because $U^{n,m} \supseteq \bar{J}^n \times J^m$.

(2.3) *The (bigraded) Abel map.* Let C/S be a flat projective family of integral curves. Let $\Delta \subset C \times C$ be the diagonal. Then its ideal defines a map $\iota \colon C \to \bar{J}^{-1}$.

Let m be an arbitrary integer. Let $W^m \subseteq C \times \bar{J}^{m+1}$ be the inverse image of $U^{-1,m+1}$ under $\iota \times 1$. Define the *bigraded Abel map* to be the composition

$$A := \mu \circ (\iota \times 1) \colon W^m \to \bar{J}^m.$$

Let $\mathcal{L}$ be an invertible sheaf of degree 1 on C/S, and $\sigma \colon S \to J^1$ be the corresponding section. Define the *Abel map* associated to $\mathcal{L}$ to be the composition

$$A_{\mathcal{L}} := A \circ (1 \times \sigma) \colon C \to \bar{J};$$

the composition makes sense because $C \times J^1 \subseteq W^0$.

(2.4) *Autoduality.* Let C/S be a flat projective family of integral curves. Assume that the curves (the geometric fibers of C/S) are locally planar. Then the projective S-scheme $\bar{J}^n$ is flat, and its geometric fibers are integral local complete intersections; see [**1**, (9), p. 8]. Hence, the Picard scheme $\mathrm{Pic}_{\bar{J}^n/S}$ exists, and is a disjoint union of quasi-projective S-schemes; see Théorème 3.1, p. 232-06, in [**10**], and Corollary (6.7)(ii), p. 96, in [**4**]. Also, by [**3**, Thm. (3.1), p. 28], there exists an S-scheme $\mathrm{Pic}^{=}_{\bar{J}^n/S}$, the *compactified Picard scheme*, that parameterizes torsion-free rank-1 sheaves on the fibers of $\bar{J}^n/S$; moreover, the connected components of $\mathrm{Pic}^{=}_{\bar{J}^n/S}$ are proper over S.

As is customary [**10**, p. 236-03], let $\mathrm{Pic}^0_{\bar{J}/S}$ denote the set-theoretic union of the connected components of the identity 0 in the fibers of $\mathrm{Pic}_{\bar{J}/S}$, and let $\mathrm{Pic}^{\tau}_{\bar{J}/S}$ denote the set of points of $\mathrm{Pic}_{\bar{J}/S}$ that have a multiple in $\mathrm{Pic}^0_{\bar{J}/S}$. The set $\mathrm{Pic}^{\tau}_{\bar{J}/S}$ is open; give it the induced scheme structure. Then, by general principles, forming $\mathrm{Pic}_{\bar{J}/S}$ and $\mathrm{Pic}^{\tau}_{\bar{J}/S}$ commutes with changing S.

Denote by $\mathrm{Pic}^{-}_{\bar{J}/S}$ the schematic closure of $\mathrm{Pic}^{\tau}_{\bar{J}/S}$ in $\mathrm{Pic}^{=}_{\bar{J}^n/S}$. Note that forming $\mathrm{Pic}^{-}_{\bar{J}/S}$ commutes with changing S via a flat map: it does so topologically because a flat map is open; whence, it does so schematically because a flat map carries associated points to associated points. If the fibers of C/S have, at worst, nodes and cups, then forming $\mathrm{Pic}^{-}_{\bar{J}/S}$ commutes with changing S via an arbitrary map, owing to Theorem (4.1), our main result, since forming $\bar{J}$ does so.

The Abel map $A_{\mathcal{L}}$ induces an S-map,

$$A^*_{\mathcal{L}}\colon \mathrm{Pic}_{\bar{J}/S} \to \textstyle\coprod_n J^n.$$

By [**8**, Thm. 2.1, p. 595], if the geometric fibers of C/S have, at worst, double points, then $\mathrm{Pic}^0_{\bar{J}/S} = \mathrm{Pic}^{\tau}_{\bar{J}/S}$. Furthermore, the Abel map induces an isomorphism,

$$A^*_{\mathcal{L}}\colon \mathrm{Pic}^0_{\bar{J}/S} \xrightarrow{\sim} J,$$

which is independent of the choice of the invertible sheaf $\mathcal{L}$ of degree 1 on C/S; in fact, the isomorphism exists whether or not any sheaf $\mathcal{L}$ does. Let us call this isomorphism the *autoduality isomoprhism.*

Proposition (2.5). *Let C/S be a flat projective family of integral curves, m and n integers, $\mathcal{M}$ an invertible sheaf of degree m on C/S. Suppose the curves have, at worst, double points. Then the translation map $\mu_{\mathcal{M}}$ induces an isomorphism*

$$\mu^*_{\mathcal{M}}\colon \mathrm{Pic}^0_{\bar{J}^{m+n}/S} \xrightarrow{\sim} \mathrm{Pic}^0_{\bar{J}^n/S},$$

which is independent of $\mathcal{M}$. If $m = 0$, then $\mu^*_{\mathcal{M}}$ is equal to the identity.

Proof. Note that $\mu_{\mathcal{O}_C} = 1_{\bar{J}^n}$. And, if $\mathcal{M}_1$ is also an invertible sheaf on C, then

$$\mu_{\mathcal{M}} \circ \mu_{\mathcal{M}_1} = \mu_{\mathcal{M}\otimes\mathcal{M}_1}.$$

So $\mu_{\mathcal{M}}$ is an isomorphism, whose inverse is $\mu_{\mathcal{M}^{-1}}$. Hence $\mu^*_{\mathcal{M}}$ is an isomorphism. Moreover, if $\mathcal{M}_1$ is of degree m too, then $\mathcal{M} \otimes \mathcal{M}_1^{-1}$ is of degree 0, and it suffices to prove that $\mu^*_{\mathcal{M}\otimes\mathcal{M}_1^{-1}} = 1$. Thus we may assume $m = 0$.

To prove that $\mu^*_{\mathcal{M}} = 1$, we may change the base via an étale covering, and so assume that the smooth locus of C/S admits a section σ. Set $\mathcal{L} := \mathcal{O}_C(\sigma(S))$.

Then $\mathcal{L}$ is an invertible sheaf on C. So,

$$\mu_{\mathcal{M}} = \mu_{\mathcal{L}^{\otimes n}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{L}^{\otimes -n}}.$$

Hence, since $\mathcal{L}$ is of degree 1 on C/S, we may assume $n = 0$.

Note that $\mu_{\mathcal{M}} \circ A_{\mathcal{L}} = A_{\mathcal{M}\otimes\mathcal{L}}$. Now, $A^*_{\mathcal{M}\otimes\mathcal{L}} = A^*_{\mathcal{L}}$ and $A^*_{\mathcal{L}}$ is an isomorphism; see (2.4). Thus $\mu^*_{\mathcal{M}} = 1$, and the proof is complete.

Theorem (2.6). *Let C/S be a flat projective family of integral curves with, at worst, double points. Then the autoduality isomorphism $\mathrm{Pic}^0_{\bar{J}/S} \xrightarrow{\sim} J$ extends uniquely to a map of compactifications $\mathrm{Pic}^-_{\bar{J}/S} \to \bar{J}$.*

Proof. Set $U := \mathrm{Pic}^0_{\bar{J}/S}$ and $\overline{U} := \mathrm{Pic}^-_{\bar{J}/S}$. Since J/S is smooth and admits a section (for example, the 0-section), by [**3**, Thm. (3.4)(iii), p. 40], $\bar{J} \times \overline{U} \,/\, \overline{U}$ carries a universal sheaf $\mathcal{P}$, which is determined up to tensor product with the pullback of an invertible sheaf on $\overline{U}$.

The extension $\eta\colon \overline{U} \to \bar{J}$ of the autoduality isomorphism is unique if it exists, because U is schematically dense in $\overline{U}$ and $\bar{J}$ is separated. Hence, by descent theory, it suffices to construct η after changing the base via an étale covering. So we may assume that the smooth locus of C/S admits a section σ. Set $\mathcal{L} := \mathcal{O}_C(\sigma(S))$. Then $\mathcal{L}$ is invertible of degree 1 on C/S. So the autoduality isomorphism is simply $A^*_{\mathcal{L}}$, and it suffices to prove that $(A_{\mathcal{L}} \times 1)^*\mathcal{P}$ is torsion-free rank-1 on $C \times \overline{U} \,/\, \overline{U}$.

Form the bigraded Abel map $A\colon C \times J^1 \to \bar{J}$. It is smooth by [**7**, Cor. 2.6, p. 5969]. Hence $(A \times 1)^*\mathcal{P}$ is torsion-free rank-1 on $C \times J^1 \times \overline{U} \,/\, \overline{U}$. It suffices to prove that $(A \times 1)^*\mathcal{P}$ is torsion-free rank-1 on $C \times J^1 \times \overline{U}/(J^1 \times \overline{U})$, since $(A_{\mathcal{L}} \times 1)^*\mathcal{P}$ is its fiber over the point of J^1 representing $\mathcal{L}$. Now, $(A \times 1)^*\mathcal{P}$ is flat over $J^1 \times \overline{U}$, by the local criterion, if its fiber is flat over the fiber $J^1(u)$ for each $u \in \overline{U}$.

Fix a $u \in \overline{U}$. Making a suitable faithfully flat base change S'/S, we may assume that the field $k(u)$ is equal to the field of the image of u in S. Set $\mathcal{I} := \mathcal{P}(u)$. It suffices to prove that $A(u)^*\mathcal{I}$ is a torsion-free rank-1 sheaf on $C(u) \times J^1(u)/J^1(u)$.

Let $\mathcal{M}$ be an invertible sheaf of degree 0 on C/S. Then the translation map $\mu_{\mathcal{M}}$ gives rise to the following commutative diagram:

$$\begin{array}{ccc} C \times J^1 & \xrightarrow{A} & \bar{J} \\ {\scriptstyle 1\times\mu_{\mathcal{M}}}\big\downarrow & & {\scriptstyle \mu_{\mathcal{M}}}\big\downarrow \\ C \times J^1 & \xrightarrow{A} & \bar{J} \end{array}$$

By Proposition (2.5), $\mu^*_{\mathcal{M}}$ is the identity on $\mathrm{Pic}^0_{\bar{J}/S}$, so on its closure $\overline{U}$ too. Thus $\mu_{\mathcal{M}}(u)^*\mathcal{I} = \mathcal{I}$. Now, the diagram is commutative; hence,

$$(1 \times \mu_{\mathcal{M}}(u))^* A(u)^*\mathcal{I} = A(u)^*\mathcal{I}. \tag{2.6.1}$$

Since $J^1(u)$ is integral, the lemma of generic flatness applies, and it implies that there is a dense open subset W of $J^1(u)$ over which $A(u)^*\mathcal{I}$ is flat. Now, by Part (ii)(a) of Lemma (5.12) on p. 85 of [**4**], it is an open condition on the base for a flat family of sheaves to be torsion-free rank-1 provided they are supported on a family whose geometric fibers are integral of the same dimension. Hence, since $A(u)^*\mathcal{I}$ is torsion-free rank-1 on $C(u) \times J^1(u)/k(u)$, after shrinking W, we may assume that the restriction of $A(u)^*\mathcal{I}$ to $C \times W \,/\, W$ is torsion-free rank-1. Fix an arbitrary point j_1 of W and one j_2 of $J^1(u)$.

Making a suitable faithfully flat base change S'/S, we may assume that each of j_1 and j_2 lies in the image of a section of J^1/S. These sections represent invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ of degree 1 on C/S; set $\mathcal{M} := \mathcal{M}_1 \otimes \mathcal{M}_2^{-1}$. Equation (2.6.1) implies that $A(u)^*\mathcal{I}$ is torsion-free rank-1 over $\mu_{\mathcal{M}}(u)^{-1}W$ as well. Now, j_2 is an arbitrary point of $J^1(u)$. Hence $A(u)^*\mathcal{I}$ is torsion-free rank-1 on $C(u) \times J^1(u)\big/J^1(u)$, and the proof is complete.

3. Flag schemes

Lemma (3.1). *Let X be a scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is invertible at each associated point of X, and is everywhere locally generated by two sections. Set $W := \mathbf{P}(\mathcal{F})$, and let $w\colon W \to X$ be the structure map. Then Serre's graded $\mathcal{O}_X$-algebra homomorphism α is an isomorphism:*

$$\alpha\colon \mathit{Sym}(\mathcal{F}) \xrightarrow{\sim} \bigoplus_{n\geq 0} w_*\mathcal{O}_W(n).$$

Proof. The question is local on X, and $\mathcal{F}$ is locally generated by two sections. So, setting $\mathcal{E} := \mathcal{O}_X^{\oplus 2}$, we may assume there is a short exact sequence of the form:

$$0 \to \mathcal{N} \xrightarrow{\varepsilon} \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \to 0.$$

This sequence induces a short exact sequence of graded $\mathit{Sym}(\mathcal{E})$-modules:

$$0 \to \mathcal{N} \otimes \mathit{Sym}(\mathcal{E})[-1] \to \mathit{Sym}(\mathcal{E}) \to \mathit{Sym}(\mathcal{F}) \to 0. \tag{3.1.1}$$

A priori, the sequence is only right exact. However, since ε is injective, every associated point of $\mathcal{N}$ is an associated point P of X; so we need only check for left exactness at such a P. By hypothesis, $\mathcal{F}$ is invertible at P; whence, $\mathcal{N}$ is too. Therefore, left exactness holds at P, so everywhere.

Set $V := \mathbf{P}(\mathcal{E})$, and let $v\colon V \to X$ be the structure map. Then φ induces a closed embedding $\iota\colon W \hookrightarrow V$ such that $w = v \circ \iota$. Moreover, applying the exact functor "tilde" to (3.1.1), we obtain the following short exact sequence on V:

$$0 \to v^*\mathcal{N} \otimes \mathcal{O}_V(-1) \to \mathcal{O}_V \to \mathcal{O}_W \to 0,$$

in which $\mathcal{O}_V \to \mathcal{O}_W$ is the comorphism of ι.

For convenience, given a coherent $\mathcal{O}_X$-module $\mathcal{G}$ and integer i, set

$$R(\mathcal{G}, i) := R^1 v_*(v^*\mathcal{G} \otimes \mathcal{O}_V(i)),$$

and let $b(\mathcal{G}, i)$ denote the following natural map:

$$b(\mathcal{G}, i)\colon \mathcal{G} \otimes \mathit{Sym}_i(\mathcal{E}) \longrightarrow v_*(v^*\mathcal{G} \otimes \mathcal{O}_V(i)).$$

For every $i \geq 0$, consider the natural commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \otimes \mathit{Sym}_{i-1}(\mathcal{E}) \arrow[r] \arrow[d, "b(\mathcal{N}, i-1)"'] & \mathit{Sym}_i(\mathcal{E}) \arrow[r] \arrow[d, "b(\mathcal{O}_V, i)"'] & \mathit{Sym}_i(\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & v_*(v^*\mathcal{N} \otimes \mathcal{O}_X(i-1)) \arrow[r] & v_*\mathcal{O}_V(i) \arrow[r] & w_*\mathcal{O}_W(i) \arrow[r] & R(\mathcal{N}, i-1).
\end{tikzcd}$$

Since $\mathcal{E}$ is free, $b(\mathcal{O}_V, i)$ is an isomorphism by Serre's computation. Therefore, to prove the lemma, it is enough to prove that $R(\mathcal{N},\, i-1) = 0$ and that $b(\mathcal{N},\, i-1)$ is an isomorphism.

Fix $i \geq -1$. Given a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0, \tag{3.1.2}$$

consider the following induced diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}' \otimes Sym_i(\mathcal{E}) \arrow[r] \arrow[d, "b(\mathcal{G}',i)"'] & \mathcal{G} \otimes Sym_i(\mathcal{E}) \arrow[r] \arrow[d, "b(\mathcal{G},i)"'] & \mathcal{G}'' \otimes Sym_i(\mathcal{E}) \arrow[r] \arrow[d, "b(\mathcal{G}'',i)"'] & 0 \\
0 \arrow[r] & v_*(v^*\mathcal{G}' \otimes \mathcal{O}_V(i)) \arrow[r] & v_*(v^*\mathcal{G} \otimes \mathcal{O}_V(i)) \arrow[r] & v_*(v^*\mathcal{G}'' \otimes \mathcal{O}_V(i)) \arrow[r] & R(\mathcal{G}', i).
\end{tikzcd}$$

Since $\mathcal{E}$ is free, the upper sequence is exact. Since v is flat, (3.1.2) pulls back to a short exact sequence on V; whence, the lower sequence is exact.

If $R(\mathcal{G}', i) = 0$ and if $b(\mathcal{G}', i)$ and $b(\mathcal{G}'', i)$ are isomorphisms, then $b(\mathcal{G}, i)$ is one too. If, in addition, $R(\mathcal{G}'', i) = 0$, then also $R(\mathcal{G}, i) = 0$.

On the other hand, if $b(\mathcal{G}, i)$ and $b(\mathcal{G}'', i)$ are isomorphisms, then $b(\mathcal{G}', i)$ is one too, and $R(\mathcal{G}', i) \subseteq R(\mathcal{G}, i)$. If, in addition, $R(\mathcal{G}, i) = 0$, then also $R(\mathcal{G}', i) = 0$.

Since $\mathcal{N} \subset \mathcal{O}_X^{\oplus 2}$, there is a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{N} \to \mathcal{J} \to 0$$

where $\mathcal{I} \subseteq \mathcal{O}_X$ and $\mathcal{J} \subseteq \mathcal{O}_X$. It is, therefore, enough to prove that $R(\mathcal{G}, i) = 0$ and that $b(\mathcal{G}, i)$ is an isomorphism when $\mathcal{G} \subseteq \mathcal{O}_X$.

Finally, suppose $\mathcal{G} \subseteq \mathcal{O}_X$, and let $Y \subseteq X$ be the subscheme defined by $\mathcal{G}$. By Serre's computation, $b(\mathcal{O}_X, i)$ and $b(\mathcal{O}_Y, i)$ are isomorphisms. Also, $R(\mathcal{O}_X, i) = 0$. Hence, $R(\mathcal{G}, i) = 0$, and $b(\mathcal{G}, i)$ is an isomorphism. The proof is now complete.

(3.2) *Flag schemes.* Let $f\colon X \to T$ be a map of (locally Noetherian) schemes, and $\mathcal{I}$ a coherent sheaf on X. As is conventional, for each T-scheme U, set $X_U := X \times U$, and let $\mathcal{I}_U$ denote the pullback of $\mathcal{I}$ to X_U. Fix $m \geq 0$. By a *m-flag of $\mathcal{I}_U/X_U/U$*, let us mean a chain of coherent sheaves,

$$\mathcal{I}_m \subset \mathcal{I}_{m-1} \subset \cdots \subset \mathcal{I}_1 \subset \mathcal{I}_0 := \mathcal{I}_U,$$

such that, for $1 \leq i \leq m$, the ith quotient $\mathcal{I}_{i-1}/\mathcal{I}_i$ is U-flat of relative length 1. Denote the set of all these m-flags by $\mathbf{F}_m(U)$.

Since the quotients are flat, for each U-scheme V, the m-flag pulls back to a m-flag of $\mathcal{I}_V/X_V/V$. So, as U varies, the $\mathbf{F}_m(U)$ form a contravariant functor $\mathbf{F}_m$.

Clearly, $\mathbf{F}_0$ is representable by T. Suppose $\mathbf{F}_{m-1}$ is representable by a T-scheme F_{m-1}, and consider the universal $(m-1)$-flag:

$$\mathcal{K}_{m-1} \subset \mathcal{K}_{m-2} \subset \cdots \subset \mathcal{K}_1 \subset \mathcal{K}_0 := \mathcal{I}_{F_{m-1}}.$$

Then, clearly, $\mathbf{F}_m$ is representable by the Quot scheme

$$F_m := \mathrm{Quot}^1_{(\mathcal{K}_{m-1}/X \times F_{m-1}/F_{m-1})};$$

furthermore, the universal m-flag on $X \times F_m$ is the chain

$$\mathcal{J}_m \subset \mathcal{J}_{m-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 := \mathcal{I}_{F_m}$$

where $\mathcal{J}_i$ is the pullback of $\mathcal{K}_i$ for $0 \le i < m$ and where $\mathcal{J}_m$ is the universal subsheaf of $\mathcal{J}_{m-1}$. Call F_m the *m-flag scheme of* $\mathcal{I}/X/T$.

According to [**12**, Prop. (2.2), p. 109], we have $F_m = \mathbf{P}(\mathcal{K}_{m-1})$; furthermore, if $(\phi_m, \tau_m)\colon F_m \to X \times F_{m-1}$ denotes the structure map, then, on $X \times F_m$, we have

$$\mathcal{J}_{m-1}/\mathcal{J}_m = (\phi_m, 1)_* \mathcal{O}_{F_m}(1).$$

And there are natural maps,

$$\psi^{(m,n)}\colon F_m \to X \times F_{m-1} \to X \times (X \times F_{m-2}) \to \cdots \to X^{\times n} \times F_{m-n}.$$

Set $\tau_{j,i} := \tau_{i+1} \cdots \tau_j$; so $\tau_{j,i}\colon F_j \to F_i$. Set $\psi^{(m)} := \psi^{(m,m)}$; so $\psi^{(m)}\colon F_m \to X^{\times m}$. Set $\psi_i := \phi_i \circ \tau_{m,i}$; so $\psi_i\colon F_m \to X$. Then $\psi_m = \phi_m$ and $\psi^{(m)} = (\psi_m, \dots, \psi_1)$. Let $q_i\colon X \times F_i \to F_i$ denote the projection.

Lemma (3.3). *Under the conditions of* (3.2)*, assume that* $f\colon X \to T$ *is a flat projective family of integral curves that are locally planar, and assume that* $\mathcal{I}$ *is invertible. Let* p_i *and* $p_{i,j}$ *be the projections of* $X^{\times m}$ *onto the indicated factors, and let* p *be its structure map. Let* $\mathcal{X}$ *be the ideal of the diagonal* Δ *of* $X \times X$*. Set*

$$\mathcal{H} := \psi^{(m)}_* \big(\mathcal{D}_{q_m}(\mathcal{J}_m)^{-1}\big) \text{ and } \mathcal{F} := \mathcal{D}_f(\mathcal{I})^{-1}$$

where $\mathcal{D}_{q_m}$ *and* $\mathcal{D}_f$ *mean determinant of cohomology. Let* $X_1, \dots, X_m \subset X$ *be open subschemes such that* $X_i \cap X_j \cap X_k$ *is* T*-smooth for all distinct* i, j, k*. Set* $H := X_m \times \cdots \times X_1$*. Then*

$$\mathcal{H} \mid H = \left(\Big(\bigotimes_{i<j} p_{i,j}^* \mathcal{X} \Big) \otimes \Big(\bigotimes_i p_i^* \mathcal{I} \Big) \otimes p^* \mathcal{F} \right) \Big| \, H.$$

Proof. The additivity of the determinant of cohomology yields

$$\mathcal{D}_{q_m}(\mathcal{J}_m)^{-1} = \Big(\bigotimes_{i=1}^{m} \mathcal{D}_{q_m}(\mathcal{J}_{i-1}/\mathcal{J}_i) \Big) \otimes \mathcal{D}_{q_m}(\mathcal{I}_{F_m})^{-1}.$$

Set $\mathcal{M}_i := \tau_{m,i}^* \mathcal{O}_{F_i}(1)$. Then $\mathcal{J}_{i-1}/\mathcal{J}_i = (\phi_i, 1)_* \mathcal{M}_i$. So $\mathcal{D}_{q_m}(\mathcal{J}_{i-1}/\mathcal{J}_i) = \mathcal{M}_i$. Now, forming the determinant commutes with changing the base; so

$$\mathcal{D}_{q_m}(\mathcal{I}_{F_m})^{-1} = \big(p\, \psi^{(m)}\big)^* \mathcal{D}_f(\mathcal{I})^{-1} = \big(\psi^{(m)}\big)^* p^* \mathcal{F}.$$

Since $\mathcal{F}$ is invertible, we may apply the projection formula to $p^*\mathcal{F}$ and $\psi^{(m)}_*$. We therefore have to prove that

$$\psi^{(m)}_* \Big(\bigotimes_{i=1}^{m} \mathcal{M}_i \Big) \Big| \, H = \left(\Big(\bigotimes_{i<j} p_{i,j}^* \mathcal{X} \Big) \otimes \Big(\bigotimes_i p_i^* \mathcal{I} \Big) \right) \Big| \, H. \tag{3.3.1}$$

Since $F_1 = \mathbf{P}(\mathcal{I})$ and $\mathcal{I}$ is invertible, $F_1 = X$ and $\mathcal{M}_1 = \mathcal{I}$ and $\psi^{(m)} = 1$. So (3.3.1) holds when $m = 1$. We proceed by induction on m.

Suppose $m \ge 2$. Set $\mathcal{N}_i := \tau_{m-1,\,i}^* \mathcal{O}_{F_i}(1)$ for $i = 1, \dots, m-1$. Let u_i and $u_{i,j}$ be the projections of $X^{\times (m-1)}$ onto the indicated factors. Set

$$\mathcal{G} := \Big(\bigotimes_{i<j} u_{i,j}^* \mathcal{X} \Big) \otimes \Big(\bigotimes_i u_i^* \mathcal{I} \Big) \text{ and } G := X_{m-1} \times \cdots \times X_1.$$

Then the induction hypothesis yields

$$\psi_*^{(m-1)}\Big(\bigotimes_{i=1}^{m-1}\mathcal{N}_i\Big)\,\Big|\,G=\mathcal{G}\big|G. \tag{3.3.2}$$

Since $\mathcal{M}_i=\tau_m^*\mathcal{N}_i$ for $i=1,\dots,m-1$, the projection formula yields

$$(\phi_m,\tau_m)_*\Big(\bigotimes_{i=1}^{m}\mathcal{M}_i\Big)=q_{m-1}^*\Big(\bigotimes_{i=1}^{m-1}\mathcal{N}_i\Big)\otimes(\phi_m,\tau_m)_*\mathcal{O}_{F_m}(1). \tag{3.3.3}$$

For $i=0,\dots,m-1$, set

$$\mathcal{G}_i:=\Big(\bigotimes_{j=1}^{i}p_{1,m-j+1}^*\mathcal{X}\Big)\otimes p_1^*\mathcal{I}.$$

For $i>0$, set $\widetilde{\mathcal{G}}_{i-1}:=(u_{m-i},1)^*\mathcal{G}_{i-1}$ and $H_i:=(u_{m-i},1)^{-1}H$. Now, $\mathcal{I}$ is invertible. Since $u_{m-i}(H_i)\subseteq X_m\cap X_i$ and since $X_m\cap X_i\cap X_j$ is T-smooth for each $j<i$, the pullback $\widetilde{\mathcal{G}}_{i-1}$ is invertible.

Set $\widetilde{G}:=\big(\psi^{(m-1)}\big)^{-1}G\subset F_{m-1}$ and $G_1:=X_m\times\widetilde{G}$. For $i=1,\dots,m-1$, set $\gamma_i:=1\times\psi_{m-1,i}$; so $\gamma_i\colon X\times F_{m-1}\to X\times X$. Let us prove that

$$\mathcal{K}_i\big|G_1=\bigg(\Big(\bigotimes_{j=1}^{i}\gamma_j^*\mathcal{X}\Big)\otimes\mathcal{I}_{F_{m-1}}\bigg)\,\bigg|\,G_1 \tag{3.3.4}$$

by induction on $i\geq 0$. First off, $\mathcal{K}_0=\mathcal{I}_{F_{m-1}}$. So suppose (3.3.4) holds for $i-1$. Let $\Gamma_i\subset X\times F_{m-1}$ be the image of F_{m-1} under $(\psi_{m-1,i},1)$. Then $\Gamma_i=\gamma_i^{-1}\Delta$, where $\Delta\subset X\times X$ is the diagonal. Since Δ/X is flat, forming its ideal $\mathcal{X}$ commutes with changing the base. Hence $\gamma_i^*\mathcal{X}$ is the ideal of Γ_i.

Set $\mathcal{O}_{\Gamma_i}(1):=(\psi_{m-1,i},1)_*\mathcal{N}_i$. Then $\mathcal{K}_i$ is the kernel of the composition

$$\mathcal{K}_{i-1}\twoheadrightarrow\mathcal{O}_{\Gamma_i}\otimes\mathcal{K}_{i-1}\twoheadrightarrow\mathcal{O}_{\Gamma_i}(1). \tag{3.3.5}$$

The kernel of the first surjection is the ideal-module product $(\gamma_i^*\mathcal{X})\cdot\mathcal{K}_{i-1}$. Hence $(\gamma_i^*\mathcal{X})\cdot\mathcal{K}_{i-1}\subset\mathcal{K}_i$. Now, $\Gamma_i\cap G_1\subseteq(X_m\cap X_i)\times\widetilde{G}$. Also, $\mathcal{K}_{i-1}\big|(X\times\widetilde{G})$ is the structure sheaf, whence invertible, off the union of the $X_j\times\widetilde{G}$ for $j=1,\dots,i-1$. Since, by hypothesis, $X_m\cap X_i\cap X_j$ is T-smooth for $j=1,\dots,i-1$, the restriction $\mathcal{K}_{i-1}\big|(\Gamma_i\cap G_1)$ is invertible. Thus the second surjection in (3.3.5) is an isomorphism on G_1, and hence $(\gamma_i^*\mathcal{X}\cdot\mathcal{K}_{i-1})|G_1=\mathcal{K}_i|G_1$. In addition,

$$(\gamma_i^*\mathcal{X}\otimes\mathcal{K}_{i-1})|G_1\xrightarrow{\sim}(\gamma_i^*\mathcal{X}\cdot\mathcal{K}_{i-1})|G_1,$$

and therefore

$$\mathcal{K}_i\big|G_1=\big(\gamma_i^*\mathcal{X}\otimes\mathcal{K}_{i-1}\big)\big|G_1.$$

Combining the expression above with (3.3.4) for $i-1$, we get (3.3.4) for i.

Since $\psi^{(m-1)}=(\psi_{m-1,m-1},\dots,\psi_{m-1,1})$, Formula (3.3.4) is equivalent to

$$\mathcal{K}_i\big|G_1=((1\times\psi^{(m-1)})^*\mathcal{G}_i)|G_1. \tag{3.3.6}$$

For $i = 1, \dots, m-1$, set $\widetilde{H}_i := (\psi_{m-1,i}, 1)^{-1}(G_1)$. As the second map in (3.3.5) is an isomorphism on G_1, we have $(\psi_{m-1,i}, 1)^* \mathcal{K}_{i-1} | \widetilde{H}_i = \mathcal{N}_i | \widetilde{H}_i$. Now, note that

$$(1 \times \psi^{(m-1)}) \circ (\psi_{m-1,i}, 1) = (u_{m-i}, 1) \circ \psi^{(m-1)}; \tag{3.3.7}$$

whence $(\psi^{(m-1)})^{-1} H_i = \widetilde{H}_i$. Therefore, (3.3.6) yields

$$\mathcal{N}_i | \widetilde{H}_i = (\psi^{(m-1)})^* \widetilde{\mathcal{G}}_{i-1} | \widetilde{H}_i. \tag{3.3.8}$$

Consider a geometric point x of a fiber of H/G, and view x as well as a point of X_m. Suppose, at x, the ideal $p_{1,m-j+1}^* \mathcal{X}$ is not invertible. Then $x \in X_m \cap X_j$. And $p_{1,m-j+1}^* \mathcal{X}$ is generated by two elements owing to Nakayama's Lemma, since the fibers of X/T are locally planar. Furthermore, $x \notin X_m \cap X_k$ for any $k \neq j, m$ since $X_m \cap X_j \cap X_k$ is T-smooth by hypothesis; whence $p_{1,m-k+1}^* \mathcal{X}$ is invertible at x. Now, $\mathcal{I}$ is invertible. Therefore, it follows from (3.3.6) that $\mathcal{K}_i | G_1$ is everywhere locally generated by two sections.

Since $F_m = \mathbf{P}(\mathcal{K}_{m-1})$, Lemma (3.1) yields

$$(\phi_m, \tau_m)_* \mathcal{O}_{F_m}(1) \big| G_1 = \mathcal{K}_{m-1} \big| G_1.$$

Therefore, it follows from (3.3.3) that

$$(\phi_m, \tau_m)_* \left(\bigotimes_{i=1}^{m} \mathcal{M}_i \right) \Bigg| \, G_1 = \left(q_{m-1}^* \left(\bigotimes_{i=1}^{m-1} \mathcal{N}_i \right) \otimes \mathcal{K}_{m-1} \right) \Bigg| \, G_1.$$

Let $r \colon X^{\times m} \to X^{\times m-1}$ be the projection onto the product of the last factors. To complete the proof, it is now enough to prove that

$$(1 \times \psi^{(m-1)})_* \left(q_{m-1}^* \left(\bigotimes_{i=1}^{m-1} \mathcal{N}_i \right) \otimes \mathcal{K}_k \right) \Bigg| \, H = (r^* \mathcal{G} \otimes \mathcal{G}_k) \; \big| \; H \tag{3.3.9}$$

for $k = 0, \dots, m-1$. Again, we proceed by induction.

First off, $\mathcal{K}_0 = \mathcal{I}_{F_{m-1}}$, and $\mathcal{I}$ is invertible. So the projection formula yields

$$(1 \times \psi^{(m-1)})_* \left(q_{m-1}^* \left(\bigotimes_{i=1}^{m-1} \mathcal{N}_i \right) \otimes \mathcal{K}_0 \right) \Bigg| \, H = \left(p_1^* \mathcal{I} \otimes r^* \psi_*^{(m-1)} \left(\bigotimes_{i=1}^{m-1} \mathcal{N}_i \right) \right) \Bigg| \, H.$$

But $\mathcal{G}_0 := p_1^* \mathcal{I}$. Thus (3.3.2) yields (3.3.9) when $k = 0$.

Suppose (3.3.9) holds for $k-1$ with $k < m$. Now, owing to the discussion surrounding (3.3.5), the exact sequence

$$0 \to \mathcal{G}_k \big| H \to \mathcal{G}_{k-1} \big| H \to (\mathcal{G}_{k-1} \otimes p_{1,m-k+1}^* \mathcal{O}_\Delta) \big| H \to 0$$

pulls back under $1 \times \psi^{(m-1)}$ to the exact sequence

$$0 \to \mathcal{K}_k \big| G_1 \to \mathcal{K}_{k-1} \big| G_1 \to \mathcal{O}_{\Gamma_k}(1) \big| G_1 \to 0.$$

Hence (3.3.9) for k follows from (3.3.9) for $k-1$ provided

$$(1 \times \psi^{(m-1)})_* \left(q_{m-1}^* \left(\bigotimes_{i=1}^{m-1} \mathcal{N}_i \right) \otimes \mathcal{O}_{\Gamma_k}(1) \right) \Bigg| \, H = (r^* \mathcal{G} \otimes \mathcal{G}_{k-1} \otimes p_{1,m-k+1}^* \mathcal{O}_\Delta) \; \big| \; H.$$

Here, the right-hand side is equal to $(u_{m-k},1)_*(\mathcal{G}\otimes\widetilde{\mathcal{G}}_{k-1})\big|H$. On the other hand, since $\mathcal{O}_{\Gamma_k}(1) := (\psi_{m-1,k},1)_*\mathcal{N}_k$, the left-hand side is equal to

$$(1\times\psi^{(m-1)})_*(\psi_{m-1,k},1)_*\left(\left(\bigotimes_{i=1}^{m-1}\mathcal{N}_i\right)\otimes\mathcal{N}_k\right)\,\bigg|\,H,$$

or because of (3.3.7), to

$$(u_{m-k},1)_*\psi_*^{(m-1)}\left(\left(\bigotimes_{i=1}^{m-1}\mathcal{N}_i\right)\otimes\mathcal{N}_k\right)\,\bigg|\,H.$$

Finally, observe that

$$\psi_*^{(m-1)}\left(\left(\bigotimes_{i=1}^{m-1}\mathcal{N}_i\right)\otimes\mathcal{N}_k\right)\,\bigg|\,H_k = (\mathcal{G}\otimes\widetilde{\mathcal{G}}_{k-1})\big|H_k;$$

Indeed, this formula results from (3.3.8) and from the projection formula, since, as noted above, $\widetilde{\mathcal{G}}_{k-1}\big|H_k$ is invertible. The proof is now complete.

4. Isomorphism

Theorem (4.1). *Let C/S be a flat projective family of integral curves with, at worst, ordinary nodes and cusps. Then the autoduality isomorphism $\mathrm{Pic}^0_{J/S} \xrightarrow{\sim} J$ extends uniquely to an isomorphism of compactifications $\mathrm{Pic}^-_{\bar{J}/S} \xrightarrow{\sim} \bar{J}$.*

Proof. Set $U := \mathrm{Pic}^0_{J/S}$ and $\overline{U} := \mathrm{Pic}^-_{\bar{J}/S}$. By Theorem (2.6), the autoduality isomorphism extends uniquely to a map, say $\eta\colon\overline{U}\to\bar{J}$. By descent theory, it suffices to prove η is an isomorphism after a faithfully flat base change. So we may assume the smooth locus of C/S admits a section σ. Set $\mathcal{L} := \mathcal{O}_C(\sigma(S))$. Then the autoduality isomorphism is simply $A^*_{\mathcal{L}}$. Also, $C\times\bar{J}/\bar{J}$ carries a universal sheaf $\mathcal{I}$, which is of degree 0 and rigidified along σ. Finally, we may assume as well that S is the spectrum of a Henselian local ring with algebraically closed residue field.

Let $\beta\colon J\to U$ be the inverse of $A^*_{\mathcal{L}}$, and set $U^= := \mathrm{Pic}^=_{\bar{J}/S}$. It suffices to extend β to a map $\overline{\beta}\colon\bar{J}\to U^=$. Indeed, $\bar{J}$ is the schematic closure of J; so $\overline{\beta}$ factors through $\overline{U}\subseteq U^=$. Also, $\eta\overline{\beta}\big|J = 1$, so $\eta\overline{\beta}=1$. And $\overline{\beta}\eta\big|U=1$; so $\overline{\beta}\eta=1$. Hence η is an isomorphism. Thus it suffices to construct $\overline{\beta}\colon\bar{J}\to U^=$.

First, recall from [**8**, Prop. 2.2, p. 595], or rather from [**8**, Rmk. 2.4, p. 597], the definition of $\beta\colon J\to U$. Let q_1 and q_2 be the projections of $C\times\bar{J}$ onto the indicated factors. Form the product $\mathcal{K} := \mathcal{I}\otimes q_1^*\mathcal{L}^{\otimes(g-1)}$ on $C\times\bar{J}$ and its determinant of cohomology $\mathcal{P} := \mathcal{D}_{q_2}(\mathcal{K})$ on $\bar{J}$. Let $r_1\colon\bar{J}\times J\to\bar{J}$ be the projection, and $\mu\colon\bar{J}\times J\to\bar{J}$ the multiplication map. Set

$$\mathcal{T} := r_1^*\mathcal{P}\otimes\mu^*\mathcal{P}^{-1},$$

Then $\mathcal{T}$ is invertible and defines β.

(Note that there are two canceling sign errors in [**8**, Rmk. 2.4, p. 597]: first, the theta divisor is the zero scheme of the canonical regular section of $\mathcal{P}^{-1}$, not $\mathcal{P}$; second, there $\mathcal{T}$ is the inverse of what it should be. With these corrections, the discussion in [**8**, Rmk. 2.4] goes through.)

Let $C_0 \subseteq C$ be the smooth locus of C/S. Set $Z := C - C_0$ and equip Z with its induced reduced subscheme structure. Let $s \in S$ be the closed point, and $P_1, \ldots, P_m \in C(s)$ the singularities. For $i = 1, \ldots, m$, there exists, by [**11**, IV$_4$ 18.5.11, p. 130], a decomposition $Z = Z_i \cup \widetilde{Z}_i$ in which Z_i and $\widetilde{Z}_i$ are disjoint closed subschemes such that one of them, say Z_i, meets $C(s)$ only in P_i.

Set $C_i := C - \widetilde{Z}_i$ for $i = 1, \ldots, m$. Then C_i is open, and $C_i \supset C_0$. Also $P_i \in C_i$. So the closed fiber $C(s)$ is covered by the C_i; whence, $C = C_1 \cup \cdots \cup C_m$. If $i \neq j$, then $C_i \cap C_j$ does not contain any point of Z in the closed fiber $C(s)$; whence, $C_i \cap C_j \subseteq C_0$. Therefore, $C_i \cap C_j = C_0$.

View the bigraded Abel map as a rational map $A: C \times \bar{J} \to \bar{J}^{-1}$. Set $\nu := \mu_{\mathcal{L}} \circ A$. Viewing $C^{\times(i+1)}$ as $C^i \times C$, form $(1 \times \nu): C^{\times(i+1)} \times \bar{J} \to C^i \times \bar{J}$ and

$$C^{\times m} \times \bar{J} \to C^{\times m-1} \times \bar{J} \to \cdots \to C \times \bar{J} \to \bar{J}.$$

By [**7**, Cor. 2.6, p. 5969], the rational map A is smooth where A is defined; whence, the composition is smooth where it is defined. Set

$$H_m := C_m \times \cdots \times C_2 \times C_1 \subset C^{\times m} \text{ and } W := H_m \times J.$$

Then the composition is defined on W because $C_i \cap C_j = C_0$ if $i \neq j$. Thus there is a well-defined smooth map $\zeta: W \to \bar{J}$. Again since $C_i \cap C_j = C_0$ if $i \neq j$, it follows that $\zeta^{-1} J = C_0^{\times m} \times J$.

Not only is $\zeta: W \to \bar{J}$ smooth, but also ζ is surjective. Indeed, since $P_1, \ldots, P_m$ are ordinary nodes or cusps, every torsion-free rank-1 sheaf on $C(s)$ is of the form $\mathcal{J} \otimes \mathcal{L}$ where $\mathcal{L}$ is invertible and $\mathcal{J}$ is the ideal of a reduced subscheme of $\bigcup P_i$; hence, $\zeta(W) \supset \bar{J}(s)$. But ζ is smooth, so open. Therefore, $\zeta(W) = \bar{J}$.

Below, we construct a torsion-free rank-1 sheaf $\mathcal{Q}$ on $\bar{J} \times W/W$ such that $\mathcal{Q}$ coincides with $(1 \times \zeta)^* \mu^* \mathcal{P}^{-1}$ on $\bar{J} \times C_0^{\times m} \times J$. Using $\mathcal{Q}$, we can complete the proof as follows. First, let $\overline{r}_1: \bar{J} \times W \to \bar{J}$ denote the projection, and set $\mathcal{T}_\zeta := \overline{r}_1^* \mathcal{P} \otimes \mathcal{Q}$. Then $\mathcal{T}_\zeta$ is also a torsion-free rank-1 sheaf on $\bar{J} \times W/W$. So $\mathcal{T}_\zeta$ induces a map $\overline{\beta}': W \to U^{=}$. But $\mathcal{T}_\zeta$ and $(1 \times \zeta)^* \mathcal{T}$ coincide on $\bar{J} \times C_0^{\times m} \times J$. Hence $\overline{\beta}'$ and $\beta \circ \zeta$ coincide on $C_0^{\times m} \times J$.

Set $V := W \times_{\bar{J}} W$, and let ζ_1 and ζ_2 be the projections. Set $V_0 := \zeta_1^{-1} \zeta^{-1}(J)$. Then $V_0 = \zeta_2^{-1} \zeta^{-1}(J)$. So $\overline{\beta}' \circ \zeta_1 \big| V_0 = \overline{\beta}' \circ \zeta_2 \big| V_0$. Now, ζ is flat, and $J \supset \operatorname{Ass}(\bar{J})$; so $V_0 \supset \operatorname{Ass}(V)$. Hence $\overline{\beta}' \circ \zeta_1 = \overline{\beta}' \circ \zeta_2$. Therefore, descent theory yields a unique map $\overline{\beta}: \bar{J} \to U^{=}$ such that $\overline{\beta}' = \overline{\beta} \circ \zeta$. Since

$$\overline{\beta} \circ \zeta \;\big|\; C_0^{\times m} \times J = \overline{\beta}' \;\big|\; C_0^{\times m} \times J = \beta \circ \zeta \;\big|\; C_0^{\times m} \times J,$$

and ζ is faithfully flat, $\overline{\beta} | J = \beta$. In other words, $\overline{\beta}$ is the desired extension of β.

It remains to construct $\mathcal{Q}$. To lighten the notation, given S-schemes U and V and coherent sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$ on $C \times U$ and $C \times V$, denote by $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ the tensor product on $C \times U \times V$ of the pulled-back sheaves. Let $\mathcal{X}$ denote the ideal of the diagonal of $C \times C$, and for each $n > 0$ and $i = 1, \ldots, n$, set

$$\mathcal{E}_i^{(n)} := p_{1,n-i+2,\ldots,n+1}^* \big(\mathcal{X} \boxtimes \cdots \boxtimes \mathcal{X} \big)$$

where $p_{1,n-i+2,\ldots,n+1}: C^{n+1} \to C^{i+1}$ is the projection onto the indicated factors.

Given any $n \geq 0$, let F_n be the n-flag scheme of $\mathcal{I}/(C \times \bar{J})/\bar{J}$, and let

$$\mathcal{I}_n^{(n)} \subset \mathcal{I}_{n-1}^{(n)} \subset \cdots \subset \mathcal{I}_1^{(n)} \subset \mathcal{I}_0^{(n)} := \mathcal{I}_{F_n}$$

be the universal flag. Form the natural map $\psi^{(n)}: F_n \to C^{\times n} \times \bar{J}$, and set

$$F_n' := (\psi^{(n)})^{-1}(C_0^{\times n} \times \bar{J}).$$

Also, let $\iota: C^{\times n} \times \bar{J} \to \bar{J} \times C^{\times n}$ be the switch map, and set $\widehat{\psi}^{(n)} := \iota \circ \psi^{(n)}$.

Note that $\psi^{(n)} \big| F_n'$ is an isomorphism, whose inverse is defined by the n-flag

$$(\mathcal{E}_n^{(n)}|(C \times C_0^{\times n})) \boxtimes \mathcal{I} \subset \cdots \subset (\mathcal{E}_1^{(n)}|(C \times C_0^{\times n})) \boxtimes \mathcal{I} \subset \mathcal{I}_{C_0^{\times n} \times \bar{J}};$$

this flag is well-defined since $\mathcal{X}$ is invertible on $C_0 \times C_0$.

Set $\mathcal{N} := \mathcal{I}|(C \times J)$. Then $\mathcal{I}_n^{(n)} \boxtimes \mathcal{N} \boxtimes \mathcal{L}^{\otimes n}$ is a torsion-free rank-1 sheaf on $C \times F_n \times J/F_n \times J$; so it defines a map $\gamma_n: F_n \times J \to \bar{J}$.

Take $n := m$. Set $\widetilde{F}_m := (\psi^{(m)})^{-1}(H_m \times \bar{J})$ and $\widetilde{F}_m' := (\psi^{(m)})^{-1}(W)$. Note that $\psi^{(m)} \big| \widetilde{F}_m'$ is an isomorphism, whose inverse is defined by the m-flag

$$(\mathcal{E}_m^{(m)}|(C \times H_m)) \boxtimes \mathcal{N} \subset \cdots \subset (\mathcal{E}_1^{(m)}|(C \times H_m)) \boxtimes \mathcal{N} \subset \mathcal{I}_W;$$

this flag is well-defined since $\mathcal{N}$ is invertible and $C_i \cap C_j = C_0$ if $i \neq j$.

Next, let us see that the following square is commutative:

$$\begin{array}{ccc} F_m' \times J & \xrightarrow{\gamma_m} & \bar{J} \\ {\scriptstyle \widehat{\psi}^{(m)} \times 1}\downarrow & & \uparrow{\scriptstyle \mu} \\ \bar{J} \times C_0^{\times m} \times J & \xrightarrow{1 \times \zeta} & \bar{J} \times J. \end{array}$$

Indeed, here $\widehat{\psi}^{(m)} \times 1$ is an isomorphism; in addition, $\gamma_m \circ (\widehat{\psi}^{(m)} \times 1)^{-1}$ and $\mu \circ (1 \times \zeta)$ coincide, because they are defined by the same sheaf, namely,

$$\mathcal{I} \boxtimes (\mathcal{E}_m^{(m)}|(C \times C_0^{\times m})) \boxtimes \mathcal{N} \boxtimes \mathcal{L}^{\otimes m} \text{ on } C \times \bar{J} \times C_0^{\times m} \times J.$$

Since the square is commutative and $\widehat{\psi}^{(m)} \times 1$ is an isomorphism, $(1 \times \zeta)^* \mu^* \mathcal{P}^{-1}$ and $(\widehat{\psi}^{(m)} \times 1)_* \gamma_m^* \mathcal{P}^{-1}$ coincide on $\bar{J} \times C_0^{\times m} \times J \subset \bar{J} \times W$. So set

$$\mathcal{Q} := (\widehat{\psi}^{(m)} \times 1)_* \gamma_m^* \mathcal{P}^{-1} \;\big|\; \bar{J} \times W. \tag{4.1.1}$$

It remains to show that $\mathcal{Q}$ is torsion-free rank-1 on $\bar{J} \times W/W$. Since ζ is faithfully flat, it suffices to form the map $\zeta \times 1: W \times W \to \bar{J} \times W$ and show that $(\zeta \times 1)^* \mathcal{Q}$ is torsion-free rank-1 on $W \times W/W$.

Given any $p, q \geq 0$, take $n := p + q$, and let us construct the following square:

$$\begin{array}{ccc} F_n & \xrightarrow{\theta} & F_p \\ {\scriptstyle \tau_{n,q}}\downarrow & & \downarrow{\scriptstyle \tau_{p,0}} \\ F_q & \xrightarrow{\omega} & \bar{J}. \end{array} \tag{4.1.2}$$

Here $\tau_{n,q}$ and $\tau_{p,0}$ are the maps defined in (3.2). Let θ be defined by the p-flag

$$\mathcal{I}_n^{(n)} \boxtimes \mathcal{L}^{\otimes q} \subset \cdots \subset \mathcal{I}_q^{(n)} \boxtimes \mathcal{L}^{\otimes q}$$

on $C \times F_n \times \bar{J}/F_n \times \bar{J}$; here $\mathcal{I}_q^{(n)} \boxtimes \mathcal{L}^{\otimes q}$ is torsion-free, rank-1, and of degree 0 since $\mathcal{I}_0^{(n)}$ is so, since each quotient $\mathcal{I}_{i-1}^{(n)}/\mathcal{I}_i^{(n)}$ is flat of relative length 1, and since $\mathcal{L}$ is invertible of degree 1. Let ω be defined by the torsion-free, rank-1, and degree-0 sheaf $\mathcal{I}_q^{(q)} \boxtimes \mathcal{L}^{\otimes q}$. Clearly, the square is Cartesian.

Note that the following square is commutative:

$$\begin{array}{ccc} F_n & \xrightarrow{\quad\theta\quad} & F_p \\ {\scriptstyle\psi^{(n,p)}}\big\downarrow & & \big\downarrow{\scriptstyle\psi^{(p)}} \\ C^{\times p} \times F_q & \xrightarrow{1\times\omega} & C^{\times p} \times \bar{J}. \end{array} \tag{4.1.3}$$

It follows formally that this square is Cartesian since (4.1.2) is so.

Take $p = q := m$. Recall that $\psi^{(m)}$ restricts to the isomorphism $\widetilde{F}'_m \xrightarrow{\sim} W$. Set $\widetilde{F}_{2m} := (\psi^{(2m)})^{-1}(H_m \times W)$. Then Square (4.1.3) yields this one:

$$\begin{array}{ccc} \widetilde{F}_{2m} & \xrightarrow{\quad\theta\quad} & \widetilde{F}_m \\ {\scriptstyle\psi^{(2m)}}\big\downarrow & & \big\downarrow{\scriptstyle\psi^{(m)}} \\ H_m \times W & \xrightarrow{1\times\zeta} & H_m \times \bar{J}. \end{array} \tag{4.1.4}$$

Interchange the two factors in the lower left and right corners of Square (4.1.4), and multiply on the right by J. The result is the following Cartesian square:

$$\begin{array}{ccc} \widetilde{F}_{2m} \times J & \xrightarrow{\theta\times 1} & \widetilde{F}_m \times J \\ {\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle\widehat{\psi}^{(m)}\times 1} \\ W \times W & \xrightarrow{\zeta\times 1} & \bar{J} \times W, \end{array} \tag{4.1.5}$$

which introduces ξ. In fact, $F_{2m} \times J$ can be viewed as the $2m$-flag scheme of $(\mathcal{I} \boxtimes \mathcal{N})/C \times \bar{J} \times J/\bar{J} \times J$, and ξ is, up to interchanging factors, the restriction of the natural map $F_{2m} \times J \to C^{\times 2m} \times \bar{J} \times J$ over $H_m \times H_m \times J \times J$.

Since the lower map $\zeta \times 1$ in Square (4.1.5) is flat, Equation (4.1.1) yields

$$(\zeta \times 1)^* \mathcal{Q} = \xi_*(\theta \times 1)^* \gamma_m^* \mathcal{P}^{-1}.$$

So it remains to show $\xi_*(\theta \times 1)^* \gamma_m^* \mathcal{P}^{-1}$ is torsion-free rank-1 on $W \times W/W$.

Recall $\mathcal{P} := \mathcal{D}_{q_2}(\mathcal{K})$ where $\mathcal{K} := \mathcal{I} \boxtimes \mathcal{L}^{\otimes(g-1)}$. Now, on $C \times F_{2m} \times J$, we have

$$(1 \times \theta \times 1)^*(1 \times \gamma_m)^* \mathcal{K} = \mathcal{I}_{2m}^{(2m)} \boxtimes \mathcal{N} \boxtimes \mathcal{L}^{\otimes 2m}.$$

Let $q_{2,3}\colon C \times F_{2m} \times J \to F_{2m} \times J$ be the projection. Since forming the determinant of cohomology commutes with changing the base, it remains to see that

$$\xi_* D_{q_{2,3}}\big(\mathcal{I}_{2m}^{(2m)} \boxtimes \mathcal{N} \boxtimes \mathcal{L}^{\otimes 2m}\big) \bigm| \big(\widetilde{F}_{2m} \times J\big)$$

is torsion-free rank-1 on $W \times W/W$. But it is clearly so in view of Lemma (3.3) applied with $C \times J \times J/J \times J$ for X/T, with $\mathcal{N} \boxtimes \mathcal{N} \boxtimes \mathcal{L}^{\otimes 2m}$ for $\mathcal{I}$, and with $X_i := C_i \times J \times J$ for $i = 1, \ldots, m$ and $X_i := C_{i-m} \times J \times J$ for $i = m+1, \ldots, 2m$.

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