The eddy viscosity for time reversing waves in an irreversible environment

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(Dated: May 5, 2004)

We present new results for the time-reversal of weakly nonlinear pulses traveling in a random irreversible environment. We consider long water waves propagating in the presence of a spatially random depth. A viscous shallow water model is considered. We demonstrate that weakly nonlinear waves can still be time-reversed under weak dissipation. Incoherently scattered signals are recompressed, both, for time-reversal in transmission as well as in reflection. Under the weakly nonlinear, weakly dissipative regime dissipation only affects the refocused pulse profile regarding its amplitude, but not its shape. Numerical experiments are presented. Moreover we also describe a new theory for calculating the *eddy viscosity* for weakly nonlinear waves propagating over a random surface.

PACS numbers: 47.35.+i,42.25.Dd,43.25.Ts,43.60.Cg

Time-reversal refocusing for waves propagating in inhomogeneous media has been extensively studied in various contexts, as for example in ultrasound and underwater acoustics [1]. Applications include telecommunication [2] and imaging [3]. In a recent Letter [4] Fouque et al. considered the time-reversal of one-dimensional waves in three new (related) regimes: (i) dispersive waves [5], (ii) weakly nonlinear waves, with the theory presented in detail in [6], and (iii) solitary waves, with detailed numerics provided in [7]. Dissipation is usually expected to be a serious limitation to time-reversal recompression efficiency [8]. In this Letter we consider one-dimensional weakly nonlinear waves in the unexplored regime of time-reversal in an irreversible environment. It is shown for the first time the time-reversal of weakly nonlinear waves in a viscous fluid. Also, for the first time, we describe the derivation of the *eddy (or turbulent) viscosity* [9] obtained by propagating weakly nonlinear waves over a random (turbulent) surface. More specifically we consider time-reversal, both in transmission (TRT) and in reflection (TRR), for waves propagating in a dissipative (hence irreversible) model: the viscous shallow water equations

$$u_t + \alpha u u_x + \eta_x = \mu u_{xx}, \quad \eta_t + ((1 + \varepsilon h + \alpha \eta) u)_x = 0.$$

In all cases the initial setup is given by a derivative of a Gaussian exactly as in [4]. $u(x,t)$ is the particle velocity and $\eta(x,t)$ is the wave elevation. This dimensionless system is obtained as a one dimensional simplification of the Navier-Stokes equations under the hypothesis of an incompressible flow with a hydrostatic pressure due to the long wave regime (first equation; i.e. momentum balance). Mass balance is built into the free surface condition for the evolution of the wave elevation $\eta(x,t)$ (second equation) [10, 11]. The characteristic wave amplitude is a_o and the typical depth is h_o. The nonlinearity parameter is $\alpha = a_o/h_o$ and the reference speed is $U = g a_o / c_o$ following the scaling in [11] where the linear shallow water speed is given by $c_o = (g h_o)^{1/2}$. The kinematic viscosity is denoted by μ. The parameter ε is the ratio of the standard deviation of the fluctuations of the depth over the average depth. It is assumed to be small. The fluctuations are modeled by the stationary zero-mean random process $h(x)$.

Let a pulse shaped wave propagate over a random region. Due to multiple scattering the transmitted coherent pulse will be followed by an incoherent coda. One will also observe a reflected incoherent signal. The following result is shown for the first time: even in the presence of weak dissipation (due to a small viscosity) time-reversing the corresponding data still leads to the recompression (i.e. refocusing) of its incoherent component. Another striking feature regarding refocusing is that, in the weakly nonlinear, weakly diffusive regime, the pulse shape is effectively preserved in the sense that the initial pulse profile can still be recovered through time-reversal. For large values of the nonlinearity the initial profile shape can be only approximately recovered. This has been shown in [6] for nonlinear (inviscid) shallow water waves in the presence of a random topography and also for a dissipative KdV equation in a homogeneous medium [12]. In this Letter we describe the physics behind these results together with the main ingredients from the mathematical theory which, in the future, will be published in full detail.

Time-reversing in a homogeneous dissipative environment.—— The time-reversal of nonlinear acoustic waves in a homogeneous medium has been investigated experimentally by Tanter *et al.* [13]. They analyzed the nonlinear mechanism for energy transfer to higher harmonic components during forward propagation of a sinusoidal wave. The main goal of their experiments was to check for the reversibility of this energy transfer. The energy reversibility among harmonics is broken only for propagation longer than the shock formation distance. This issue was addressed in [4, 6] for a broadband pulse in the presence of randomness. It was shown how the random medium regularizes the problem allowing for propagation beyond the shock distance. Just as in the super-resolution case [14], this is another instance where randomness helps in a dramatic fashion.

We repeat these homogeneous medium experiments, but now, in the presence of dissipation. In Dean and Dalrymple [15] (page 262) normal ocean waves are in a regime where the Reynolds number is of the order $Re^{-1} = O(10^{-7} \to 10^{-8})$. By setting the dimensionless viscosity as $\mu = 10^{-4}$ we are considering stronger dissipation or equivalently dissipation over long propagation distances. Time-reversed water wave refocusing is a potential tool for waveform inversion, as for example in the cases of tsunamis [16]. Hence time-reversal is most likely to be performed on a computer with a water wave model. This allows for considering two strategies for the time-reversed dynamics: (i) as usual, we perform it with the exact same model considered for the forward analysis; (ii) we switch off the dissipative mechanism, which can be easily done with a numerical model. By doing so we expect to have a better estimate of the initial profile's amplitude. This can be seen in Fig. 1 and its detailed version Fig. 2. In both cases we obtain the same inverse cascading of energy among higher harmonics as was seen in the reversible (inviscid) analysis presented in [6, 13]. This means that the initial pulse *shape* (and therefore Fourier content) is effectively recovered, the only difference being a reduction in the amplitudes. This is due to dissipation, obviously acting over all wavenumbers. When the viscosity is switched off, only for the (backward) time-reversed experiment, the inverse cascading is more efficient since no energy is lost in the reversed propagation. This can been seen in Fig. 1 and the detail presented in Fig. 2. Moreover if nonlinearity is increased beyond a certain level inverse cascading of energy through harmonics still takes place but the original wavenumber distribution is modified. This has been observed in [6] in the inviscid regime and also for a dissipative KdV equation [12].

Time-reversing in a random dissipative environment.—— The time-reversal of the corresponding inviscid (weakly nonlinear) system was analyzed in [4, 6]. It was shown that, to leading order, the transmitted nonlinear pulse is governed by a viscous Burgers equation. The "apparent viscosity" depends on statistical properties of the random medium, namely the integral of the autocorrelation function of the fluctuations. This single parameter controls the level of coherent energy converted into incoherent energy for both the forward and back scattered signals. Hence stronger fluctuations increase the refocused amplitude in reflection and therefore decrease the refocused amplitude in transmission [6, 17].

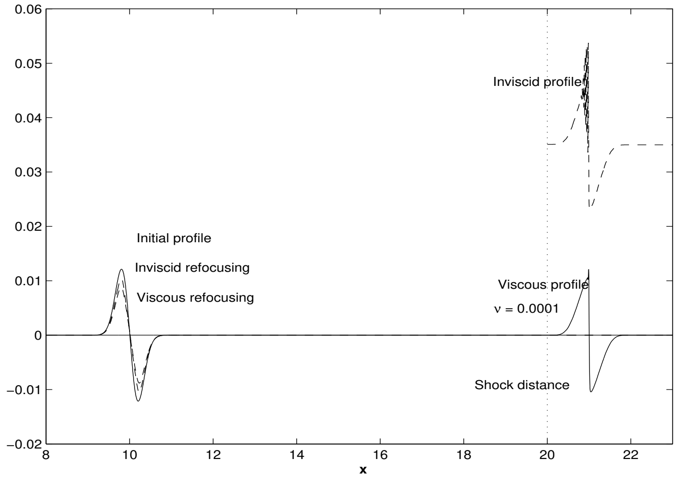

FIG. 1: Propagation in a homogeneous medium. To the right we have the transmitted pulse. To the left we have the initial profile together with refocusing in two regimes: in the presence of viscosity smaller profile) and with viscosity switched off (intermediate profile). The upper right profile shows spurious numerical oscillations due to the shock. The regularizing effect, due to the small viscosity, is clearly seen in the time-reversed profile.

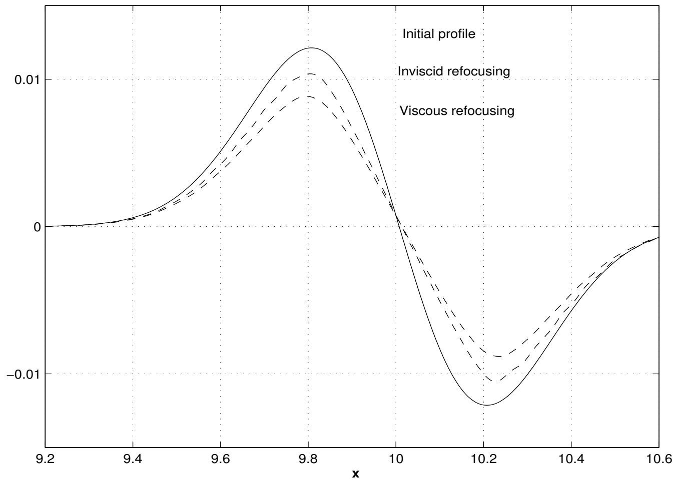

FIG. 2: A detail of Fig. 1.

Therefore, in contrast with the kinematic viscosity, the *apparent viscosity* does not remove energy from the system: it converts coherent wave energy into incoherent fluctuations. In the linear regime this mechanism leads to pulse spreading and attenuation and is known as the O'Doherty-Anstey (ODA) approximation [5, 11]. Linear Fourier analysis shows that, to leading order, the ODA "apparent attenuation" is diffusive like and expressed through a Gaussian kernel. The transmitted wave is given by the convolution of the Gaussian kernel with the initial pulse's Fourier transform. Bearing in mind the discussion above we perform the time-reversal in a random dissipative environment, by considering the viscous

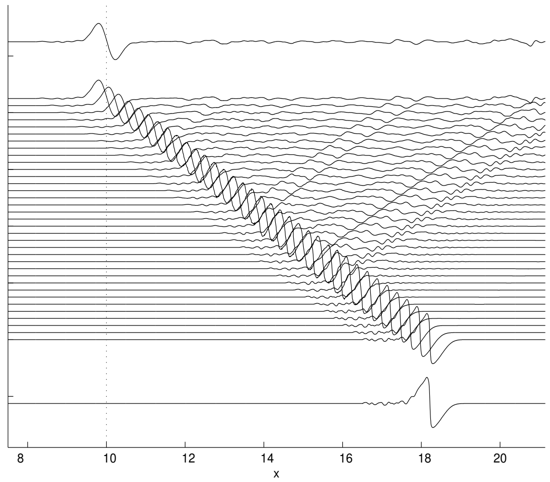

FIG. 3: Similar experiment to Fig. 1 but now in the presence of a (piecewise-linear) random medium located in [11, 16.5]. Evolution is upward with the TRT profile as the bottom trace. The correlation length is $\ell = 0.1$ with $\varepsilon = 0.5$.

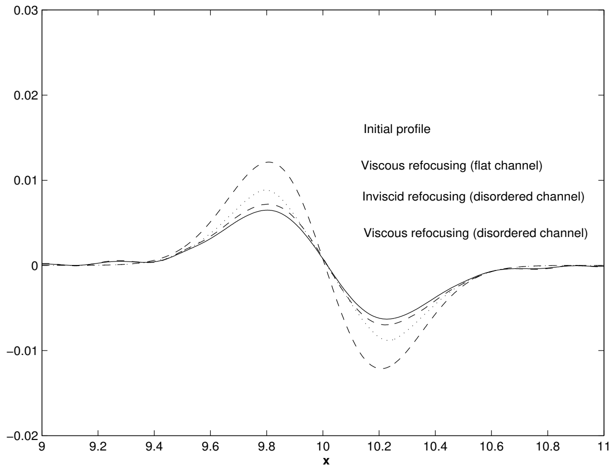

FIG. 4: A detail of Fig. 3 with additional TRT options.

shallow water system. In this case we (effectively) have an *eddy viscosity* [9] which arises from the combination of the kinematic viscosity together with the apparent viscosity. To leading order both components of the eddy viscosity removes energy from the coherent wavefront in a diffusive like manner. Loosely speaking we can think of two Gaussian filters: one due to the physical viscosity (say G_μ) and the other due to the apparent viscosity (say G_{μ_a}). The energy filtered by G_μ is lost forever and can not be recovered. As discussed above, the energy filtered through G_{μ_a} can be recovered along the coherent wavefront by time-reversal recompression. Eventhough these two filters are of different nature, they are formally similar (i.e., both Gaussian). This leads to the striking fact that we can still recover the initial pulse *shape*, time-reversing waves in an irreversible environment. To leading order the dissipative mechanism does not destroy/interfere on the waveform inversion promoted by *incoherence recompression* and *inverse energy cascading* from higher harmonics. A TRT example is given in Fig. 3. We again explore the two strategies of time-reversing *with* and *without* the fluid viscosity (c.f. Fig 4). All forward experiments performed have viscous effects. We finally present a TRR experiment where, in the absence of the wavefront, we see more clearly the recompression of the incoherent reflected wave. Note in Fig. 5 that as soon as refocusing takes place a traveling pulse emerges propagating to the left. See detail in Fig. 6.

Mathematical theory.—– The time-reversal theory for nonlinear hyperbolic waves makes use of the right and left Riemann invariants $A(x,t) = (\alpha u - 2c + 2)/\alpha$ and $B(x,t) = (\alpha u + 2c - 2)/\alpha$, corresponding to the underlying ($\varepsilon = 0$) nonlinear inviscid ($\mu = 0$) shallow water system [18]. The local propagation speed is $c = \sqrt{1 + \alpha\eta}$.

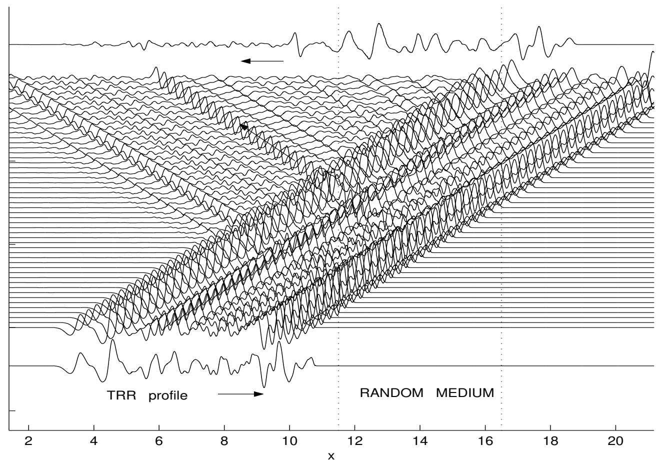

FIG. 5: Recompression of the incoherent reflected signal for a viscous fluid. The bottom trace is the time-reversed reflected signal. The top trace is the wave field at the proper refocusing time. The refocused pulse is a traveling wave propagating to the left.

The random fluid body is given by $H(x,t) = 1 + \varepsilon h(x) + \alpha\eta(x,t)$. The shallow water system is rewritten in terms of these Riemann invariants

$$A_t + C^- A_x = \frac{\varepsilon[h(A+B)]_x}{2 + \alpha(B-A)/2} + \frac{\mu}{2}(A+B)_{xx},$$

$$B_t + C^+ B_x = -\frac{\varepsilon[h(A+B)]_x}{2 + \alpha(B-A)/2} + \frac{\mu}{2}(A+B)_{xx},$$

with $C^- \equiv -1 + \alpha(3A+B)/4, \quad C^+ \equiv 1 + \alpha(A+3B)/4$. In the presence of stochastic forcing (i.e. $\varepsilon \neq 0$) they are not invariant along characteristics, being actually coupled due to the random scattering. We apply a convenient change of variables to rewrite the system in the framework moving with the right Riemann invariant, and in a new spatial variable related to the travel time. The

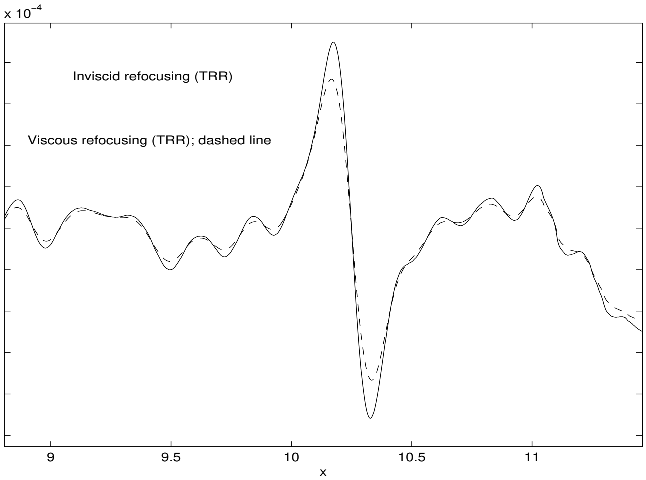

FIG. 6: A comparison of the refocused pulse from Fig. 5 (dashed line) with an experiment where viscosity was switched off during time-reversion (solid line).

system then reads as an upper triangular (two-by-two) system of partial differential equations. The equation for the leading order transmitted quantity is integrated in a Lagrangian fashion (with $\tau = t - x$) over the corresponding right characteristic. Assuming that $\varepsilon \ll 1$, $\mu \sim \varepsilon^2$, and $\alpha \sim \varepsilon^2$, we can apply a limit theorem for differential equations with random coefficients [19]. This yields that the transmitted pulse front is governed by a Burgers-like equation to leading order as $\varepsilon \downarrow 0$

$$B_x = \mathcal{L}B + \alpha_e B B_\tau$$

where the effective nonlinearity is $\alpha_e = 3\alpha/4$. The linear pseudo-differential operator $\mathcal{L}$ can be written in the Fourier domain as the sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_i$, with

$$\begin{aligned} \int \mathcal{L}_r B(\tau) e^{i\omega\tau} d\tau &= -\frac{[b_r(2\omega) + 2\mu]\omega^2}{4} \int B(\tau) e^{i\omega\tau} d\tau, \\ \int \mathcal{L}_i B(\tau) e^{i\omega\tau} d\tau &= -\frac{i b_i(2\omega)\omega^2}{4} \int B(\tau) e^{i\omega\tau} d\tau, \end{aligned}$$

b_r and b_i are respectively the real and imaginary parts of b_0 defined as the Fourier transform of the positive lag part of the autocorrelation function of $\varepsilon h(x)$:

$$b_0(\omega) = \varepsilon^2 \int_0^\infty \langle h(0)h(x)\rangle\, e^{i\omega x} dx.$$

$\mathcal{L}_r$ results from the action of the kinematic viscosity and the apparent viscosity imposed by random forcing. The contribution $-b_r(2\omega)\omega^2/4$ is proportional to the power spectral density of the random process h that is nonnegative. It behaves like a pure diffusion for small frequencies, but it decays to zero for high-frequencies. $\mathcal{L}_r$ can thus be seen as an effective *diffusion* operator. $\mathcal{L}_i$ preserves the energy and it is an effective *dispersion* operator. It vanishes for small and high frequencies.

If the typical wavelength of the pulse is larger than the correlation radius of the medium, then b_0 can be considered as constant over the spectral range of the input pulse: $b_0(\omega) \equiv \mu_a$. As a result the early steps of the evolution are governed by the viscous Burgers equation $B_x = \mu_e B_{\tau\tau} + \alpha_e B B_\tau$ where the eddy viscosity is $\mu_e = \mu/2 + \mu_a/4$. However new frequencies generated by the nonlinear term may fall in the tail of the function b_0. Then the last equation may eventually fail describing the front pulse dynamics, and the equation with the pseudo-differential operator should be considered.

In conclusion, we have presented in this paper the time-reversed refocusing of weakly nonlinear waves in an irreversible environment. The main result is that dissipation does not destroy the recompression of the incoherent coda and the inverse cascading from high harmonics, but only affects the amplitude of the refocused pulse. This conclusion holds true for a spatially uniform dissipation, and extension to localized dissipation should reveal new features leading to original ways to image the dissipation distribution of a sample.

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