SEMICONTINUITY OF ENTROPY, EXISTENCE OF EQUILIBRIUM STATES AND OF PHYSICAL MEASURES

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Abstract. We obtain some results of existence and continuity of physical measures through equilibrium states and apply these to non-uniformly expanding transformations on compact manifolds with non-flat critical sets, obtaining sufficient conditions for continuity of physical measures and, for local diffeomorphisms, necessary and sufficient conditions for stochastic stability.

1. Introduction and statement of results

The statistical viewpoint on Dynamical Systems is one of the cornerstones of most recent developments in dynamics. Given a map f_0 from a manifold M into itself, a central concept is that of *physical measure*, a f_0-invariant probability measure μ whose *ergodic basin*

$$B(\mu) = \Big\{x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_0^j(x)) \to \int \varphi \, d\mu \text{ for all continuous } \varphi : M \to \mathbf{R}\Big\}$$

has positive *volume* or *Lebesgue measure*, which we write m and take as the measure associated with any non-vanishing volume form on M.

This kind of measures provides asymptotic information on a set of trajectories that one hopes is large enough to be observable in real-world models.

Here we present recent developments on the relation between the existence of physical measures and of *equilibrium states* for smooth endomorphisms $f_0 : M \to M$ of a compact boundaryless finite dimensional Riemannian manifold. We obtain some results of existence and continuity of physical measures through equilibrium states and apply these to non-uniformly expanding transformations on compact manifolds, obtaining sufficient conditions for continuity of physical measures and necessary and sufficient conditions for stochastic stability.

The stability of physical measures under small variations of the map allows for small errors along orbits not to disturb too much the long term

2000 *Mathematics Subject Classification.* Primary: 37D25. Secondary: 37D35.

Trabalho parcialmente financiado por CMUP. O CMUP é financiado por FCT, no âmbito de POCTI-POSI do Quadro Comunitário de Apoio III (2000-2006), com fundos comunitários (FEDER) e nacionais. The author was also partially supported by CNPq (Brazil). Part of this work was done while enjoying a post-doc period at PUC (Rio de Janeiro) and IMPA whose facilities and research atmosphere are gratefully acknowledged.

behavior, as measured by the most basic statistical data provided by asymtotic time averages of continuous functions along orbits. In principle every realistic mathematical model should exhibit these stability features to be able to cope with unavoidable uncertainty about the "correct" parameter values, observed initial states and even the specific mathematical formulation involved.

1.1. **Pressure and Equilibrium States.** The physical measures are related to equilibrium states of a certain potential function. Let $\phi : M \to \mathbf{R}$ be a continuous function. Then a f_0-invariant probability measure μ is a *equilibrium state for the potential* ϕ if

$$P_{f_0}(\phi) = h_\mu(f_0) + \int \phi \, d\mu, \quad \text{where} \quad P_{f_0}(\phi) = \sup_{\nu \in \mathcal{M}} \left\{ h_\nu(f_0) + \int \phi \, d\nu \right\},$$

and $\mathcal{M}$ is the set of all f_0-invariant probability measures. The quantity $P_{f_0}(\phi)$ is called the *topological pressure* and the identity on the right hand side is a consequence of the *variational principle*, see e.g. [19] for definitions of entropy $h_\mu(f_0)$ and topological pressure $P_{f_0}(\phi)$.

For uniformly expanding maps it turns out that physical measures are the equilibrium states for the potential function $\phi(x) = -\log|\det Df_0(x)|$. It is a remarkable fact that *for uniformly hyperbolic and uniformly expanding systems these two classes of measures (physical and equilibrium states) coincide.*

1.2. **The Entropy Formula.** Pesin's Entropy Formula [16, 12, 13, 14, 17] ensures, in particular, that for $C^{1+\alpha}$ maps, $\alpha > 0$, the metric entropy with respect to an invariant measure μ *with positive Lyapunov exponents in every direction for μ almost all points* satisfies the relation

$$h_\mu(f_0) = \int \log|\det Df_0(x)| \, d\mu(x) \tag{1.1}$$

if, and only if, μ is absolutely continuous with respect to the reference measure m. In general we integrate the sum of the positive Lyapunov exponents, see [14] for a proof in the C^2 setting. In our setting of the proof that $\mu \ll m$ implies the Entropy Formula (1.1) is an exercise using the bounded distortion provided by the Hölder condition on the derivative.

We recall that by the Ergodic Theorem any ergodic absolutely continuous f_0-invariant measure μ is a physical measure.

The relation between equilibrium states and physical measures provided by the Entropy Formula, among many other facts, motivates the search for conditions guaranteeing existence of equilibrium states. We obtain sufficient conditions for existence and continuous variation of equilibrium states in what follows, but first we present some applications of these results to non-uniformly expanding systems with criticalities, and to non-uniformly expanding local diffeomorphisms.

1.3. **Maps with critical sets.** Let $f_0 : M \to M$ be a continuous map of the compact manifold M such that f_0 is a C^2 local diffeomorphism outside a critical set $\mathcal{C} \subset M$ with zero Lebesgue measure. We assume that f_0 is *non-flat* near $\mathcal{C}$: there exist $B > 1$ and $\beta > 0$ for which

(S1) $\displaystyle \frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta} \le \frac{\|Df_0(x)v\|}{\|v\|} \le B\operatorname{dist}(x,\mathcal{C})^{-\beta}$;

(S2) $\displaystyle \left|\log\|Df_0(x)^{-1}\| - \log\|Df_0(y)^{-1}\|\,\right| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{C})^{\beta}}$;

(S3) $\displaystyle \left|\log|\det Df_0(x)^{-1}| - \log|\det Df_0(y)^{-1}|\,\right| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{C})^{\beta}}$;

for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ and $v \in T_xM$. Given $\delta > 0$ we define the *δ-truncated distance* from $x \in M$ to $\mathcal{C}$ by

$$\operatorname{dist}_\delta(x,\mathcal{C}) = \begin{cases} 1 & \text{if } \operatorname{dist}(x,\mathcal{C}) \ge \delta, \\ \operatorname{dist}(x,\mathcal{C}) & \text{otherwise.} \end{cases}$$

We observe that if M is one-dimensional (either the interval or the circle) and $\mathcal{C}$ is discrete, then (S1)-(S3) amount to the zeroes of f_0' being non-flat, see [10].

We assume that f_0 is a non-uniformly expanding map, that is there is $c > 0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df_0(f_0^j(x))^{-1}\| \le -c < 0 \tag{1.2}$$

for Lebesgue almost every $x \in M$ (recall that we are taking $\mathcal{C}$ with zero Lebesgue measure). Moreover, we suppose that the orbits of f_0 have *slow approximation to the critical set*, i.e., given small $\gamma > 0$ there is $\delta > 0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1} -\log\operatorname{dist}_\delta(f_0^j(x),\mathcal{C}) \le \gamma \tag{1.3}$$

for Lebesgue almost every $x \in M$. These asymptotic conditions are motivated by the following result.

Theorem 1.1 (Theorem C in [3])**.** *Let f_0 be a C^2-non-uniformly expanding maps whose orbits have slow approximation to the critical set. Then there is a finite number of ergodic absolutely continuous (*SRB*) f_0-invariant probability measures $\mu_1, \ldots, \mu_p$ whose basins cover a full Lebesgue measure subset of M, i.e.*

$$B(\mu_1) \cup \cdots \cup B(\mu_p) = M,\ m \bmod 0.$$

Moreover, every absolutely continuous f_0-invariant probability measure μ may be written as a convex linear combination of $\mu_1, \ldots, \mu_p$: there are non-negative real numbers $w_1, \ldots, w_p$ with $w_1 + \cdots + w_p = 1$ for which $\mu = w_1 \cdot \mu_1 + \cdots + w_p \cdot \mu_p$.

1.3.1. *Random perturbations and stationary measures.* Let $\hat{f} = \{f_t : Y \to Y, t \in X\}$ be a parameterized family of maps where X, Y are connected compact metric spaces, which we assume are subsets of finite dimensional manifolds. We assume that $\hat{f} : X \times Y \to Y, (t, x) \mapsto f_t(x)$ is continuous. We consider the *random iteration of* f

$$f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_1}$$

for any sequence $\omega = (\omega_1, \omega_2, \dots)$ of parameters in X and for all $n \geq 1$. We let also $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of *non-atomic* probability measures on X such that $\text{supp}(\theta_\varepsilon) \to \{0\}$ when $\varepsilon \to 0$.

We set $\Omega = X^{\mathbf{N}}$ with the standard infinite product topology, which makes Ω a compact metrizable space, and also take the standard product probability measure $\theta^\varepsilon = \theta_\varepsilon^{\mathbf{N}}$, which makes $(\Omega, \mathcal{B}, \theta^\varepsilon)$ a probability space, where $\mathcal{B}$ is the σ-algebra of Ω generated by cylinder sets. The following skew-product map is the natural setting for many definitions connecting random with standard dynamical systems

$$F : \Omega \times Y \to \Omega \times Y \quad (\omega, x) \mapsto (\sigma(\omega), f_{\omega_1}(x))$$

where σ is the left shift on Ω. A probability measure μ^ε on Y is a *stationary measure for the random system* $(\hat{f}, \theta_\varepsilon)$ if $\theta^\varepsilon \times \mu^\varepsilon$ on $\Omega \times Y$ is F-invariant. We say that μ^ε is ergodic if $\theta^\varepsilon \times \mu^\varepsilon$ is F-ergodic.

In this setting ($\hat{f}$ continuous) it is well known that there always exist an ergodic stationary probability measure μ^ε for all $\varepsilon > 0$, see e.g. [9]. Moreover every weak* accumulation point μ of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ is a f_0-invariant probability measure, see e.g. [5].

This suggests the notion of stochastic stability: we say that a map f_0 having physical measures (at most countably many by definition of ergodic basin) is *stochastically stable under the perturbation* $(\hat{f}, (\theta_\varepsilon)_{\varepsilon>0})$ if *every weak* accumulation point* μ *of* $(\mu^\varepsilon)_{\varepsilon>0}$ *when* $\varepsilon \to 0$ *is a convex linear combination of the physical measures of* f_0.

1.3.2. *Non-uniform expansion and slow approximation on random orbits.* We study perturbations of maps with critical sets by considering families $(f_t)_{t\in X}$ of maps with the same critical set $\mathcal{C}$ such that

$$Df_t(x) = Df_0(x) \quad \text{for every } x \in M \setminus \mathcal{C} \text{ and } t \in X, \tag{1.4}$$

where X is a compact connected subset of an Euclidean space and $(t, x) \mapsto f_t(x)$ is a C^2-map. This may be implemented, e.g in parallelizable manifolds (with an additive group structure: tori $\mathbf{T}^d$ or cylinders $\mathbf{T}^{d-k} \times \mathbf{R}^k$) by considering $X = \{t \in \mathbf{R}^d \colon \|t\| \leq \varepsilon_0\}$ for some $\varepsilon_0 > 0$, θ_ε the normalized Lebesgue measure on the ball of radius $\varepsilon \leq \varepsilon_0$, and taking $f_t = f + t$; that is, adding a jump t to the image of f_0, which we call *additive random perturbations.*

We consider an analog of condition (1.2) for random orbits. We say that the map f_0 is *non-uniformly expanding for random orbits* if there exists $c > 0$

such that for $\varepsilon > 0$ small enough and for $\theta^\varepsilon \times m$ almost every $(\omega, x) \in \Omega \times M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_0(f_\omega^j(x))^{-1}\| \le -c < 0. \tag{1.5}$$

We also consider an analog of condition (1.3) for random orbits; we assume *slow approximation of random orbits to the critical set*, i.e. given any small $\gamma > 0$ there is $\delta > 0$ such that for $\theta^\varepsilon \times m$ almost every $(\omega, x) \in \Omega \times M$ and small $\varepsilon > 0$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_\omega^j(x), \mathcal{C}) \le \gamma. \tag{1.6}$$

Under these conditions we are able to obtain a result on the existence of finitely many physical measures for the randomly perturbed system. In the setting of random perturbations, a stationary measure μ^ε for $(\hat{f}, \theta_\varepsilon)$ is a *physical measure* if its *ergodic basin* $B(\mu^\varepsilon)$ has positive Lebesgue measure, where

$$B(\mu^\varepsilon) = \Big\{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x)) \to \int \varphi\, d\mu \qquad \text{for all continuous } \varphi : M \to \mathbf{R}$$
$$\text{and } \theta^\varepsilon\text{-almost every } \omega \in \Omega\Big\}.$$

Theorem 1.2 (Theorem C in [2]). *Let $f_0 : M \to M$ be a C^2-non-uniformly expanding map non-flat near $\mathcal{C}$, and whose orbits have slow approximation to $\mathcal{C}$. If f_0 is non-uniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$, then for sufficiently small $\varepsilon > 0$ there are physical measures $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$ (with l not depending on ε) such that:*

1. *for each $x \in M$ and θ^ε almost every $\omega \in \Omega$, the average of Dirac measures $\delta_{f_\omega^n(x)}$ converges in the weak* topology to some μ_i^ε with $1 \le i \le l$;*
2. *for each $1 \le i \le l$ we have*
$$\mu_i^\varepsilon = \lim \frac{1}{n} \sum_{j=0}^{n-1} \int \left(f_\omega^j\right)_* \big(m \mid B(\mu_i^\varepsilon)\big)\, d\theta^\varepsilon(\omega),$$
where $m \mid B(\mu_i^\varepsilon)$ is the normalization of the Lebesgue measure restricted to $B(\mu_i^\varepsilon)$;
3. *if f_0 is topologically transitive, then $l = 1$.*

Using Theorem 1.2 together with the general results from Section 2 provides the following.

Theorem 1.3. *Let $f_0 : M \to M$ be non-uniformly expanding C^2 map away from a non-flat critical set $\mathcal{C}$ and whose orbits have slow approximation to $\mathcal{C}$.*

Let us suppose there is a continuous family $\hat{f} = (f_t)_{t\in X}$ of C^2-maps (with $0 \in X$) and a family $\hat{\theta} = (\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on X such that

$(\hat{f},\hat{\theta})$ is non-uniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$. Then every weak accumulation point μ of any family $(\mu^\varepsilon)_{\varepsilon>0}$ of stationary measures given by Theorem 1.2 is an absolutely continuous f_0-invariant probability measure. In particular f_0 admits an absolutely continuous invariant measure.*

We remark that if μ is an absolutely continuous f_0-invariant measure as in Theorem 1.3, then $\phi_0 = -\log|\det Df_0|$ is μ-integrable since f_0 is a C^2 endomorphism. Hence the Entropy Formula ensures that μ is an equilibrium state for f_0 with respect to the potential ϕ_0.

In this setting of non-uniformly expanding maps for random orbits with $m(\mathcal{C})=0$ we have that μ^ε satisfies a similar Entropy Formula, see e.g. [7]

$$h_{\mu^\varepsilon} = \int \phi_\varepsilon \, d\mu^\varepsilon \quad \text{where} \quad \phi_\varepsilon(x) = \int \log|\det Df_\omega(x)| \, d\theta^\varepsilon(\omega) = \phi_0(x),$$

by the choice of f_t satisfying (1.4). Putting this together with the abstract results from Section 2 enables us to prove the following weak* continuity result.

Theorem 1.4. *Let $\mathcal{W}\subset \mathrm{Diff}^2(M)$ be an open subset of non-uniformly expanding C^2 maps f, with the same exponent bound c, non-flat near its critical set $\mathcal{C}(f)$ and whose orbits have slow approximation to $\mathcal{C}(f)$.*

If we assume that every $f\in\mathcal{W}$

1. *there exists a unique ergodic absolutely continuous f-invariant measure μ_f, i.e., $m(M\setminus B(\mu_f))=0$, and*
2. *there exits a family $(\hat{f},\hat{\theta})$ of maps and probability measures defining a random perturbation of f which is non-uniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}(f)$,*

then

- *μ_f is stochastically stable for $f\in\mathcal{W}$, and*
- *μ_f varies continuously with $f\in\mathcal{W}$ in the weak* topology.*

1.3.3. *A non-trivial example.* As an application of the previous theorem to the class of maps on the cylinder $\mathbf{S}^1\times\mathbf{R}$ introduced in [18], we obtain with different proofs a version of results in subsequent works [1] and [4] where it was shown that such maps have a unique physical measure which varies continuously with the map. Here we only provide continuous variation in the weak* topology, while the abovementioned works (much harder and longer) prove the continuous variation of the density of the phyical measure in the L^1 topology.

The class of non-uniformly expanding maps with critical sets introduced by M. Viana can be described as follows. Let $a_0\in(1,2)$ be such that the critical point $x=0$ is preperiodic for the quadratic map $Q(x)=a_0-x^2$. Let $\mathbf{S}^1=\mathbf{R}/\mathbf{Z}$ and $b:\mathbf{S}^1\to\mathbf{R}$ be a Morse function, for instance, $b(s)=\sin(2\pi s)$. For fixed small $\alpha>0$, consider the map

$$\tilde{f}:\mathbf{S}^1\times\mathbf{R}\to\mathbf{S}^1\times\mathbf{R},\quad (s,x)\mapsto\big(\hat{g}(s),\hat{q}(s,x)\big)$$

where $\tilde{g}$ is the uniformly expanding map of the circle defined by $\tilde{g}(s) = ds$ (mod $\mathbf{Z}$) for some $d \geq 16$, and $\tilde{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$. It is not difficult to check that for small enough $\alpha > 0$ there is an interval $I \subset (-2, 2)$ such that $\tilde{f}(\mathbf{S}^1 \times I) \subset \operatorname{int}(S^1 \times I)$. Hence every map f C^0-close to $\tilde{f}$ has $\mathbf{S}^1 \times I$ as a forward invariant region. We consider these maps f close to $\tilde{f}$ restricted to $\mathbf{S}^1 \times I$. By the expression of $\tilde{f}$ it is not difficult to verify that $\tilde{f}$, and any map f C^2-close to it, is non-flat near the critical set.

Theorem 1.5 (Theorem A in [18] & Theorem C in [4] & Theorem E in [2]). *If f is sufficiently close to $\tilde{f}$ in the C^3 topology then f is topologically mixing, non-uniformly expanding and its orbits have slow approximation to the critical set. Moreover if the noise level ε of an additive random perturbation $(\hat{f}, \theta_\varepsilon)$ of f is sufficiently small, then f is non-uniformly expanding for random orbits and random orbits have slow approximation to the critical set.*

As an immediate consequence of Theorems 1.2 and 1.3 and Theorem 1.5 we have that *there is a unique physical measure μ_f for $f \in \mathcal{W}$* because the transformations f near $\tilde{f}$ are topologically mixing. Moreover by Theorem 1.4 we conclude that *μ_f varies continuously with f near $\tilde{f}$ in the weak* topology.*

1.4. **Local diffeomorphisms.** Let $f_0 : M \to M$ be a C^2 local diffeomorphism of the manifold M and assume that f_0 satisfies condition (1.2) for Lebesgue almost every $x \in M$. We are in the setting of maps "with empty critical set $\mathcal{C} = \emptyset$" so Theorems 1.1, 1.2, 1.3 and 1.4 also hold since (1.3) and (1.6) are vacuous.

In [2] sufficient conditions and necessary conditions were obtained for stochastic stability of non-uniformly expanding local diffeomorphisms. Using results from Subsection 2.1 on zero-noise limits of random equilibrium states we obtain a necessary and sufficient condition for stochastic stability in this setting.

Theorem 1.6. *Let $f_0 : M \to M$ be a nonuniformly expanding C^2 local diffeomorphism. Then f_0 is stochastically stable if, and only if, f_0 is nonuniformly expanding for random orbits.*

1.4.1. *Equilibrium states for potentials of low variation.* We consider the following class $\mathcal{U}$ of C^2 local diffeomorphisms $f : M \to M$ which may be seen as small deformations of uniformly expanding maps. We assume that for positive constants $\delta_0, \beta, \delta_1, \sigma_1$ and integers p, q there exists a covering $B_1, \ldots, B_{p+q}$ of M such that $f \mid B_i$ is injective for all $i = 1, \ldots, p+q$ and

1. f expands uniformly at $x \in B_1 \cup \cdots \cup B_p$: $\|Df(x)^{-1}\| \leq (1+\delta_1)^{-1}$;
2. f never contracts too much: $\|Df(x)^{-1}\| \leq 1 + \delta_0$ for all $x \in M$;
3. f is volume expanding: $|\det Df(x)| \geq \sigma_1$ for all $x \in M$ with $\sigma_1 > p$;
4. there exists a set W such that
   (a) $V = \{x \in M : \|Df(x)^{-1}\| \geq (1+\delta_1)^{-1}\} \subset W \subset B_{p+1} \cup \cdots \cup B_{p+q}$;
   (b) $\inf \log \|Df \mid M \setminus W\| > \sup \log \|Df \mid V\|$; and
5. $\sup \log \|Df \mid V\| - \inf \log \|Df \mid V\| < \beta$.

We observe that $\mathcal{U}$ contains an open set of C^2 local diffeomorphisms on tori $\mathbf{T}^n, n \geq 2$, see e.g.[3, 2].

Given a continuous function $\phi : M \to \mathbf{R}$ and $\rho > 0$ we say that ϕ *has ρ-low variation* if

$$\sup \phi \leq P_f(\phi) - \rho \cdot \mathrm{h}_{\mathrm{top}}(f),$$

where $\mathrm{h}_{\mathrm{top}}(f)$ is the *topological entropy* of f which coincides (through the variational principle) with the pressure $P_f(0)$ for any constant potential.

Theorem 1.7 ([15]). *For δ_0 and β small enough there exists $\rho_0 > 0$ such that, for all $f \in \mathcal{U}$ and $0 < \rho < \rho_0$, every $\phi : M \to \mathbf{R}$ of ρ-low variation admits an ergodic equilibrium state μ_ϕ. Moreover $\mu_\phi(\log \|(Df)^{-1}\|) \leq c = c(\delta_1, \sigma_1, p, q) < 0$, that is, every Lyapunov exponent of μ_ϕ is positive.*

We note that *the notion of low variation potential includes the constant potentials.* Hence for this C^2-open class $\mathcal{U}$ of maps there are measures of maximal entropy, which are equilibrium states for the potential $\phi \equiv 0$. We may apply to these maps the abstract Theorem 2.1 from Section 2 to deduce the following.

Theorem 1.8. *When restricted to maps in $\mathcal{U}$, topological entropy $\mathrm{h}_{\mathrm{top}} : \mathcal{U} \to \mathbf{R}, f \mapsto \mathrm{h}_{\mathrm{top}}(f)$ is an upper semicontinuous function.*

In Section 2 we present the abstract results used to prove Theorems 1.3, 1.4, 1.5 and 1.6. In Section 3 we prove the abstract results. Finally in Section 4 we show how to derive the abovementioned theorems from the results in Section 2.

2. Semicontinuity of pressure, entropy and equilibrium states

Now we state the main technical results. In the following statements X, Y denote compact metric spaces.

Given a map $f : Y \to Y$ and a Borel probability measure μ we say that a μ mod 0 partition ξ of Y is a *generating partition* if

$$\bigvee_{i=0}^{+\infty} (f^i)^{-1}\xi = \mathcal{A}, \quad \mu \bmod 0,$$

where $\mathcal{A}$ is the Borel σ-algebra of Y. We denote by $\partial\xi$ the set of topological boundaries of all elements of ξ.

Theorem 2.1 (Upper semicontinuity of topological pressure). *Let $f : X \times Y \to Y$ define a family of continuous maps $f_t : Y \to Y, y \in Y \mapsto f_t(y) = f(t, y)$ and $(\phi_t)_{t \in X}$ a family of continuous functions (potentials) $\phi_t : Y \to \mathbf{R}$ satisfying the following conditions.*

1. *f_t admits some equilibrium state for ϕ_t, i.e. there exists $\mu_t \in \mathcal{P}_{f_t}(Y)$ such that $P_{f_t}(\phi_t) = h_{\mu_t}(f_t) + \int \phi_t \, d\mu_t$ for all $t \in X$.*
2. *Given a weak* accumulation point μ_0 of μ_t when $t \to 0 \in X$, let $t_k \to 0$ when $k \to \infty$ be such that $\mu_k = \mu_{t_k} \to \mu_0$. We write $f_k = f_{t_k}$, $\phi_k = \phi_{t_k}$ and assume also that*

(a) $f_k(y) \to f_0(y)$ *when* $k \to \infty$ *for all* $y \in Y$.
(b) *there exists a finite* μ_k*-modulo zero partition* ξ *of* Y *which is generating for* (Y, f_k, μ_k), $k \geq 1$, *and* $\mu_0(\partial\xi) = 0$.
(c) $\limsup_{k\to\infty} \int \phi_k \, d\mu_k \leq \int \phi_0 \, d\mu_0$.

Then $\limsup_{k\to\infty} P_{f_k}(\phi_k) \leq P_{f_0}(\phi_0)$.

Theorem 2.1 is a simple consequence of the next result.

Theorem 2.2 (Upper semicontinuity of measure-theoretic entropy). *Let* $f_t : Y \to Y$ *be a family of continuous maps as above and* μ_t *a family of* f_t*-invariant probability measures for* $t \in X$. *Given a weak* accumulation point* μ_0 *of* μ_t *when* $t \to 0 \in X$, *we let* $t_k \to 0$ *when* $k \to \infty$ *be such that* $\mu_k = \mu_{t_k} \to \mu_0$ *and write* $f_k = f_{t_k}$.

If there exists a finite μ_k*-modulo zero partition* ξ *of* Y *which is generating for* (Y, f_{t_k}, μ_k), $k \geq 1$, *and* $\mu_0(\partial\xi) = 0$, *then* $\limsup_{k\to\infty} h_{\mu_k}(f_k) \leq h_{\mu_0}(f, \xi)$.

From this we easily deduce the following.

Theorem 2.3 (Continuity of equilibrium states). *Let* $f_t : Y \to Y$ *be a family of continuous maps and* $\phi_t : Y \to \mathbf{R}$ *a family of continuous functions (potentials) satisfying conditions 1 and 2 on Theorem 2.1, for* $t \in X$.

If $P_{f_k}(\phi_k) \to P_{f_0}(\phi_0)$ *for a sequence* $t_k \to 0 \in X$, *then every weak* accumulation point* μ *of* $(\mu_k)_{k\geq 1}$ *when* $k \to \infty$ *is a equilibrium state for* f_0 *and the potential* ϕ_0.

2.1. **Upper semicontinuity of random measure-theoretic entropy.** We need the notion of metric entropy for random dynamical systems which may be defined as follows.

Theorem 2.4 (Theorem 1.3 in [11]). *For any finite measurable partition* ξ *of* Y *the limit*

$$h_{\mu^\varepsilon}((\hat{f}, \theta_\varepsilon), \xi) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu^\varepsilon}\left(\bigvee_{k=0}^{n-1} (f_\omega^k)^{-1}\xi \right) d\theta^\varepsilon(\omega)$$

exists. This limit is called the entropy of the random dynamical system with respect to ξ *and to* μ^ε.

As in the deterministic case the above limit can be replaced by the infimum. The *metric entropy of the random dynamical system* $(\hat{f}, \theta_\varepsilon)$ is given by $h_{\mu^\varepsilon}(\hat{f}, \theta_\varepsilon) = \sup h_{\mu^\varepsilon}((\hat{f}, \theta_\varepsilon), \xi)$, where the supremum is taken over all measurable partitions.

Kolmogorov-Sinai's result about generating partitions is also available for random maps. We say that ξ is a μ^ε-random generating partition if ξ is a finite partition of Y such that

$$\bigvee_{k=0}^{+\infty} (f_\omega^k)^{-1}\xi = \mathcal{A}, \ \mu^\varepsilon \bmod 0 \quad \text{for} \quad \theta^\varepsilon - \text{almost all } \omega \in \Omega.$$

Theorem 2.5 (Corollary 1.2 in [11]). *If* ξ *is a* θ_ε*-random generating partition, then we have* $h_{\theta^\varepsilon \times \mu^\varepsilon}(\hat{f}, \theta_\varepsilon) = h_{\mu^\varepsilon}((\hat{f}, \theta_\varepsilon), \xi)$.

Now we can state the following upper-semicontinuity property.

Theorem 2.6 (Upper semicontinuity of random measure-theoretic entropy)**.** *Let μ be the weak* limit of $(\mu^{\varepsilon_k})_{k\geq 1}$ when $k\to\infty$ for a sequence $\varepsilon_k\to 0$. Let us assume that there exists a finite partition ξ of Y which is θ_{ε_k}-generating for random orbits, for every $k\geq 1$, and such that $\mu(\partial\xi)=0$. Then*

$$\limsup_{k\to\infty} h_{\mu^{\varepsilon_k}}((\hat f,\theta_{\varepsilon_k}),\xi)\leq h_\mu(f,\xi).$$

As a consequence of this we deduce a result which provides a way to obtain equilibrium states as zero-noise limits.

Theorem 2.7 (Continuity of random equilibrium states)**.** *Let μ be the weak* limit of $(\mu_k=\mu^{\varepsilon_k})_{k\geq 1}$ when $k\to\infty$ for a sequence $\varepsilon_k\to 0$ when $k\to\infty$. Let us assume that there exists a finite partition ξ of Y $\mu_k \bmod 0$ which is θ_{ε_k}-generating for random orbits for all $k\geq 1$.*

Moreover we suppose that $h_{\mu_k}(\hat f,\theta_{\varepsilon_k})=\int\phi_k\,d\mu_k$ for all $k\geq 1$ where $\phi_k:Y\to\mathbf{R}$ is a sequence of functions such that $\phi_k\to\phi_0$ pointwisely when $k\to\infty$ and $P_{f_0}(-\phi_0)\leq 0$. Then μ is an equilibrium state for $-\phi_0$, that is $h_\mu(f_0)=\int\phi_0\,d\mu$.

3. Proof of semicontinuity of measure-theoretic entropy and equilibrium states

3.1. **The random setting.** Here we prove Theorem 2.6 and Theorem 2.7.

Let $(\hat f,(\theta_\varepsilon)_{\varepsilon>0})$ be a random perturbation of $f_0:Y\to Y$, μ^0 be the weak* limit of $(\mu^{\varepsilon_k})_{k\geq1}$ when $k\to\infty$ for a sequence $\varepsilon_k\to0$ and let ξ be a finite θ_{ε_k}-generating partition for random orbits, for all $k\geq1$, as in the statement of Theorem 2.6, that is $\mu(\partial\xi)=0$.

We first construct a sequence of partitions of Ω according to the following result. For a partition $\mathcal{P}$ and $y\in\Omega$ we denote by $\mathcal{P}(y)$ the element (atom) of $\mathcal{P}$ containing y. We set $\omega_0=(0,0,0,\dots)\in\Omega$ in what follows.

Lemma 3.1. *There exists an increasing sequence of measurable partitions $(\mathcal{B}_n)_{n\geq1}$ of Ω such that*

1. $\omega_0\in\operatorname{int}(\mathcal{B}_n(\omega_0))$ *for all* $n\geq1$*;*
2. $\mathcal{B}_n\nearrow\mathcal{B}$, $\theta^{\varepsilon_k} \bmod 0$ *for all* $k\geq1$ *when* $n\to\infty$*;*
3. $\lim_{n\to\infty}H_\rho(\xi\mid\mathcal{B}_n)=H_\rho(\xi\mid\mathcal{B})$ *for every measurable finite partition* ξ *of* Ω *and any* F*-invariant probability measure* ρ*.*

Proof. For the first two items we let $\mathcal{E}_n$ be a finite $\theta_{\varepsilon_k} \bmod 0$ partition of X such that $0\in\operatorname{int}(\mathcal{E}_n(t_0))$ with $\operatorname{diam}(\mathcal{E}_n)\to0$ when $n\to\infty$. Example: take a cover $(B(t,1/n))_{t\in X}$ of X by $1/n$-balls and take a subcover $U_1,\dots,U_k$ of $X\setminus B(t_0,2/n)$ together with $U_0=B(t_0,3/n)$; then let $\mathcal{E}_n=\{U_0,M\setminus U_0\}\vee\cdots\vee\{U_k,M\setminus U_k\}$.

We observe that we may assume that the boundary of these balls has null θ_{ε_k}-measure for all $k\geq1$, since $(\theta_{\varepsilon_k})_{k\geq1}$ is a denumerable family of non-atomic probability measures on X and X may be taken as a subset of some Euclidean space. Now we set

$$\mathcal{B}_n = \mathcal{E}_n \times \overset{n}{\cdots} \times \mathcal{E}_n \times \Omega \quad \text{for all } n \geq 1.$$

Then since $\operatorname{diam} \mathcal{E}_n \leq 2/n$ for all $n \geq 1$ we have that $\operatorname{diam} \mathcal{B}_n \leq 2/n$ also and so tends to zero when $n \to \infty$. Clearly $\mathcal{B}_n$ is an increasing sequence of partitions. Hence $\vee_{n \geq 1} \mathcal{B}_n$ generates the σ-algebra $\mathcal{B}$, θ^{ε_k} mod 0 (see e.g. Lemma 3 of Chapter 2 in [8]) for all $k \geq 1$. This proves items (1) and (2).

Item 3 of the statement of the lemma is Theorem 12.1 of Billingsley [8]. $\square$

Now we use some known properties of conditional entropy to derive the right inequalities. First we recall that

$$\begin{aligned} h_{\mu^{\varepsilon_k}}(\hat{f}, \theta_{\varepsilon_k}) &= h_{\mu^{\varepsilon_k}}((\hat{f}, \theta_{\varepsilon_k}), \xi) = h^{\mathcal{B} \times Y}_{\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}}(F, \Omega \times \xi) \\ &= \inf \frac{1}{n} H_{\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}} \left(\bigvee_{j=0}^{n-1} (F^j)^{-1}(\Omega \times \xi) \mid \mathcal{B} \times Y \right) \end{aligned}$$

where the first equality comes from the Kolmogorov-Sinai Theorem 2.5 and the assumption that ξ be generating, while the second one can be found in Theorem 1.4 of Chapter II in [11], with $\Omega \times \xi = \{\Omega \times A : A \in \xi\}$. Here $h^{\mathcal{B} \times Y}_{\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}}(F, \Omega \times \xi)$ is the *conditional entropy of* $\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}$ *with respect to the* σ*-algebra* $\mathcal{B} \times Y$ *on the partition* $\Omega \times \xi$, whose definition is translated in the second line of the above formula and whose basic properties can be found in [8, 11].

The last expression shows that for arbitrary fixed $N \geq 1$ and for any $l \geq 1$

$$\begin{aligned} h_{\mu^{\varepsilon_k}}(\hat{f}, \theta_{\varepsilon_k}) &\leq \frac{1}{N} H_{\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}} \left(\bigvee_{j=0}^{N-1} (F^j)^{-1}(\Omega \times \xi) \mid \mathcal{B} \times Y \right) \\ &\leq \frac{1}{N} H_{\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}} \left(\bigvee_{j=0}^{N-1} (F^j)^{-1}(\Omega \times \xi) \mid \mathcal{B}_l \times Y \right) \end{aligned}$$

because $\mathcal{B}_l \times Y \subset \mathcal{B} \times Y$. Now we fix N and l, let $k \to \infty$ and note that since $\mu^0(\partial \xi) = 0 = \delta_{\omega_0}(\partial \mathcal{B}_l)$ it must be that

$$(\delta_{\omega_0} \times \mu^0)(\partial(B_i \times \xi_j)) = 0 \quad \text{for all } B_i \in \mathcal{B}_l \text{ and } \xi_j \in \xi,$$

where δ_{ω_0} is the Dirac mass concentrated at $\omega_0 \in \Omega$. Thus we get by weak* convergence of $\theta^{\varepsilon_k} \times \mu^{\varepsilon_k}$ to $\delta_{\omega_0} \times \mu^0$ when $k \to \infty$

$$\limsup_{k \to \infty} h_{\mu^{\varepsilon_k}}(\hat{f}, \theta_{\varepsilon_k}) \leq \frac{1}{N} H_{\delta_{\omega_0} \times \mu^0} \left(\bigvee_{j=0}^{N-1} (F^j)^{-1}(\Omega \times \xi) \mid \mathcal{B}_l \times M \right). \tag{3.7}$$

Here it is easy to see that the conditional entropy on the right hand side of (3.7) (involving only finite partitions) equals

$$\frac{1}{N}H_{\mu^0}\Big(\bigvee_{j=0}^{N-1} f^{-j}\xi\Big) = \frac{1}{N}\sum_i \mu^0(P_i)\log\mu^0(P_i), \tag{3.8}$$

with $P_i = \xi_{i_0}\cap f^{-1}\xi_{i_1}\cap\cdots\cap f^{-(N-1)}\xi_{i_{N-1}}$ ranging over every possible sequence of $\xi_{i_0},\dots,\xi_{i_{N-1}}\in\xi$.

Finally, since N was an arbitrary integer, Theorem 2.6 follows from (3.7) and (3.8).

Now to prove Theorem 2.7 we assume in addition that for each μ_k there exists a continuous potential $\phi_k : Y\to\mathbf{R}$ such that $h_{\mu_k}(\hat f,\theta_{\varepsilon_k}) = \int\phi_k\,d\mu_k$, for $k\ge 1$. Moreover $\phi_k\to\phi_0$ pointwisely to a continuous potential ϕ_0 when $k\to\infty$ and $P_{f_0}(-\phi_0)\le 0$. Then by the previous arguments

$$\int\phi_0\,d\mu_0 = \limsup_{k\to\infty} h_{\mu^{\varepsilon_k}}(\hat f,\theta_{\varepsilon_k}) \le h_{\mu_0}(f_0,\xi)\le h_{\mu_0}(f_0)\le\int\phi_0\,d\mu_0$$

concluding the proof of Theorem 2.7.

3.2. **The deterministic setting.** Here we prove Theorems 2.1, 2.2 and 2.3.

Let $f_k : Y\to Y$ be a sequence of continuous maps such that μ_k is f_k-invariant for all $k\ge 1$, $f_0 : Y\to Y$ is continuous with $f_k\to f_0$ pointwisely and $\mu_k\to\mu_0$ in the weak* topology when $k\to\infty$. Let ξ be a finite μ_k-modulo zero partition ξ of Y which is generating for (Y, f_{t_k},μ_k), $k\ge 1$, and $\mu_0(\partial\xi)=0$.

Following the same reazoning as in Subsection 3.1 we have for any given fixed $N\ge 1$ that

$$h_{\mu_k}(f_k) = h_{\mu_k}(f_k,\xi) = \inf_{n\ge 1}\frac{1}{n}H_{\mu_k}\big(\vee_{j=0}^{n-1}(f_k^j)^{-1}\xi\big)\le\frac{1}{N}H_{\mu_k}(\xi_k^N),$$

since ξ is generating, where $\xi_k^N = \vee_{j=0}^{N-1}(f_k^j)^{-1}\xi$. But $\mu_0(\partial\xi)=0$ so for any given $N\ge 1$ we have $\mu_0(\partial\xi_0^N)=0$ also because μ_0 is f_0-invariant. Moreover the weak* convergence and f_k-invariance ensures that $(f_k^i)_*\mu_k = \mu_k\to\mu_0$ for all $i\ge 0$, hence $\mu_k(\xi_k^N(z))\to\mu_0(\xi_0^N(z))$ when $k\to\infty$ for μ_0-almost every $z\in Y$. In particular we get for arbitrary $N\ge 1$

$$\limsup_{k\to+\infty} h_{\mu_k}(f_k)\le\frac{1}{N}H_{\mu_0}(\xi_0^N)\quad\text{and so}\quad\limsup_{k\to+\infty} h_{\mu_k}(f_k)\le h_{\mu_0}(f_0,\xi)$$

concluding the proof of Theorem 2.2.

To prove Theorem 2.1 we assume in addition that for each $k\ge 1$ there exists a potential ϕ_k and a probability measure μ_k such that $P_{f_k}(\phi_k) = h_{\mu_k}(f_k)+\int\phi_k\,d\mu_k$. If we assume also condition (2b) from the statement of Theorem 2.1, then the result follows using Theorem 2.2 since

$$\limsup_{k\to\infty} P_{f_k}(\phi_k)\le\limsup_{k\to\infty} h_{\mu_k}(f_k)+\limsup_{k\to\infty}\int\phi_k\,d\mu_k\le h_{\mu_0}(f_0)+\int\phi_0\,d\mu_0.$$

Moreover if we assume that $P_{f_k}(\phi_k) \to P_{f_0}(\phi_0)$ when $k \to \infty$, then the same argument above gives

$$P_{f_0}(\phi_0) \leq \limsup_{k\to\infty} h_{\mu_k}(f_k) + \limsup_{k\to\infty} \int \phi_k \, d\mu_k \leq h_{\mu_0}(f_0) + \int \phi_0 \, d\mu_0 \leq P_{f_0}(\phi_0),$$

showing that μ_0 is an equilibrium state for f_0 with respect to the potential ϕ_0, thus proving Theorem 2.3.

4. Statistical stability for non-uniformly expanding maps

Here we prove the results in Subsections 1.3 and 1.4.

4.1. **Maps with critical sets.** Here we prove Theorem 1.3.

Let $f_0 : M \to M$ be a nonuniformly expanding C^2 away from the non-flat critical set $\mathcal{C}$ whose orbits have slow approximation to $\mathcal{C}$. Let also $\hat{f} = (f_t)_{t\in X}$ be a continuous family in $C^2(M, M)$ satisfying (1.4), and $\hat{\theta} = (\theta_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on X such that $(\hat{f}, \hat{\theta})$ is nonuniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$.

According to Theorem 1.2, for every small $\varepsilon > 0$ there exists an absolutely continuous stationary probability measure μ^ε. Since every f_t is a C^2 endomorphism, the random version of the Entropy Formula ensures that (see e.g. [14]) μ^ε is an equilibrium state for $\phi_0 = -\log|\det Df_0|$:

$$h_{\mu^\varepsilon} = \int\int \log|\det Df_t(x)| \, d\theta_\varepsilon(t) \, d\mu^\varepsilon(x) v = \int \log|\det Df_0| \, d\mu^\varepsilon \geq c \cdot \dim(M),$$

since every Lyapunov exponent of the random system is positive and the sum of all Lyapunov exponents is given by the above integral.

Now we choose a stationary measure μ^{ε_k} for a sequence $\varepsilon_k \to 0$ and take any weak* accumulation point μ_0 of $(\mu^{\varepsilon_k})_k$ when $k \to \infty$.

If we assume that a uniform random generating partition exists, then by Theorem 1.3 μ_0 is f_0-invariant and satisfies

$$h_{\mu_0}(f_0) = \int \phi_0 \, d\mu_0 \geq c \cdot \dim(M) > 0. \tag{4.9}$$

But the characterization of measures satisfying the Entropy Formula for endomorphisms, see e.g. [13], ensures that μ_0 is absolutely continuous.

This finishes the proof of Theorem 1.3 except for the existence of a uniform random generating partition, which is the content of the following subsection.

4.1.1. *Uniform generating partitions for equilibrium states.* To build a uniform random generating partition for equilibrium measures we make use of the following notion: given $0 < \alpha < 1$ and $\delta > 0$, we say that $n \in \mathbf{N}$ is a (α, δ)*-hyperbolic time for* $(\omega, x) \in \Omega \times M$ if

$$\prod_{j=n-k}^{n-1} \|Df_{\omega_{j+1}}(f_\omega^j(x))^{-1}\| \leq \alpha^k \quad \text{and} \quad \operatorname{dist}_\delta(f_\omega^{n-k}(x), \mathcal{C}) \geq \alpha^{bk}$$

for every $1 \le k \le n$, where $\Omega = X^{\mathbf{N}}$ was defined in Subsection 1.3.1. The following results ensures the existence of hyperbolic times in our setting.

Proposition 4.1 (Proposition 2.3 in [2]). *If $(\hat{f}, \theta^\varepsilon)$ is non-uniformly expanding for random orbits and random orbits have slow approximation to the critical set $\mathcal{C}$, then there are $\delta > 0$ and $\alpha \in (0,1)$ such that $\theta^\varepsilon \times m$-almost every $(\omega, x) \in \Omega \times M$ has infinitely many (α, δ)-hyperbolic times.*

Remark 1. When $\mathcal{C} = \emptyset$ the second condition on the definition of hyperbolic time is vacuous and in this case we just write δ-hyperbolic time. Moreover setting $\omega_t = (t, t, t, \dots)$ then a hyperbolic time for x with respect to a map f_t is just the same as a hyperbolic time for (ω_t, x), $t \in X$.

Now we state the main properties of hyperbolic times.

Proposition 4.2 (Proposition 2.6 in [2]). *There is $\delta_1 = \delta_1(f_0) > 0$ such that for every small enough $\varepsilon > 0$, if n is (α, δ)-hyperbolic time for $(\omega, x) \in$ $\operatorname{supp}(\theta^\varepsilon) \times M$, then there is a neighborhood $V_n(\omega, x)$ of x in M such that*

1. *f^n_ω maps $V_n(\omega, x)$ diffeomorphically onto the ball of radius δ_1 around $f^n_\omega(x)$;*
2. *for every $1 \le k \le n$ and $y, z \in V_k(\omega, x)$*

$$\operatorname{dist}(f^{n-k}_\omega(y), f^{n-k}_\omega(z)) \le \alpha^{k/2} \operatorname{dist}(f^n_\omega(y), f^n_\omega(z)).$$

The uniform value of δ_1 in Proposition 4.2 is the crucial point to get a uniform random generating partition. Indeed, let $B_1, \dots, B_k$ be a finite open cover of M by $\delta_1/2$-balls and let us take ξ to be the partition induced by this cover, i.e.

$$\xi = \{B_1, M \setminus B_1\} \vee \cdots \vee \{B_k, M \setminus B_k\}.$$

Lemma 4.3 (Lemma 6.6 in [6]). *If for a stationary measure μ we have that $\theta^\varepsilon \times \mu$-almost all $(\omega, x) \in \Omega \times M$ have infinitely many (α, δ)-hyperbolic times, then $\lim_{k\to\infty} \operatorname{diam}(\vee_{j=0}^{k-1} (f^j_\omega)^{-1} \xi(x)) = 0$ for $\theta^\varepsilon \times \mu$-almost every (ω, x).*

By standard arguments, this ensures that if a stationary measure μ is nonuniformly expanding and $\theta^\varepsilon \times \mu$-almost all random orbits have slow approximation to $\mathcal{C}$, then ξ is a random generating partition for μ, see e.g. Lemma 6.7 in [6].

4.1.2. *Continuous variation of physical measures.* Here we prove Theorem 1.4.

We start by observing that since we are assuming that each $f \in \mathcal{W}$ has a physical measure μ_f satisfying $m(M \setminus B(\mu_f)) = 0$, then by the proof of Theorem 1.3 we see that every weak* accumulation point μ_0 of the stationary measures (μ^{ε_k}) must equal μ_f, since $\mu_0 \ll m$. Thus μ_f is stochastically stable for the random perturbations $(\hat{f}, \hat{\theta})$ we are considering.

In addition, since the exponent bound c is uniform in $\mathcal{W}$, by the arguments in Subsection 4.1.1 there is a uniform generating partition ξ for all (f, μ_f) with $f \in \mathcal{W}$. Moreover for every $f \in \mathcal{W}$ each μ_f is an equilibrium state for

$\phi_f = -\log|\det Df|$, $P_f(\phi_f) = 0$ and $\Phi : \mathcal{W} \subset C^2(M, M) \to C^0(M, \mathbf{R}), f \mapsto \phi_f$ is continuous. Then by Theorem 2.3 if we take any sequence $f_k \in \mathcal{W}$ converging to $f_0 \in \mathcal{W}$ when $k \to \infty$, we know that every weak* accumulation point μ_0 of $(\mu_{f_k})_k$ satisfies (4.9) with $\phi_0 = \phi_{f_0}$. Hence $\mu_0 \ll m$ and by uniqueness of the physical measure of f_0 we get $\mu_0 = \mu_{f_0}$.

This finishes the proof of Theorem 1.4.

4.2. **Local diffeomorphisms.** Here we prove the results in Subsection 1.4.

In [2] it was shown that for the random perturbations provided by Theorem 1.2 the stochastic stability of f_0 implies non-uniform expansion for random orbits. Here we prove the converse without the extra technical condition used in [2].

Let $(\hat{f}, \hat{\theta})$ be a family of C^2 local diffeomorphisms and of probability measures defining a random perturbation of f_0 such that f_0 is nonuniformly expanding and is nonuniformly expanding for random orbits.

First we observe that by Remark 1 we may use the results in Subsection 4.1.1 also for local diffeomorphisms. Hence we can assume that there exists a uniform random generating partition. We can also use Theorem 1.3 to conclude that any weak* accumulation point μ of stationary measure when $\varepsilon \to 0$ is absolutely continuous.

Using now Theorem 1.1 we see that μ is a linear convex combination of stationary measures. This means that f_0 is stochastically stable whenever it is nonuniformly expanding for random orbits and ends the proof of Theorem 1.6.

Finally, to prove Theorem 1.8 we just have to use Theorem 2.1 with $\phi_f \equiv 0$ for all $f \in \mathcal{U}$. This can be done since uniform generating partitions exists for maximal entropy measures.

Proposition 4.4 (Lemma 4.8 in [15]). *There is $\delta > 0$ satisfying, for $f \in \mathcal{U}$ (as defined in Subsection 1.4.1) and every equilibrium state μ_ϕ for a low-variation potential ϕ (as given by Theorem 1.7), that μ_ϕ-almost every point $x \in M$ has infinitely many δ-hyperbolic times.*

This proposition together with Proposition 4.2 and Lemma 4.3 ensure the existence of a fixed generating partition for every $f \in \mathcal{U}$.

This concludes the proof of Theorem 1.8.

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