A NOTE ON M. SOARES' BOUNDS

EDUARDO ESTEVES* AND ISRAEL VAINSENCHER†

Abstract. We give an intersection theoretic proof of M. Soares' bounds for the Poincaré–Hopf index of an isolated singularity of a foliation of $\mathbb{CP}^n$.

1. Introduction

Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ be a holomorphic map germ such that $f^{-1}(0)=\{0\}$ in some neighborhood of 0. The *Poincaré–Hopf index* i of f at 0 is the topological degree of f; cf. [6, p. 88]. Algebraically,

$$i = \dim_{\mathbb{C}} \mathbb{C}[|x_1,\dots,x_n|]/\langle f_1,\dots,f_n\rangle,$$

where $x_1,\dots,x_n$ are local coordinates for $\mathbb{C}^n$ at 0; see [5, Ch. 5, §2]. In [2] M. Soares addresses the question of finding bounds for i. Using Mather's theory, he replaces the germ by a suitable polynomial map of an appropriate degree k. The latter map is reinterpreted in terms of a certain foliation of degree δ (equal to k or $k-1$) of $\mathbb{CP}^n$. By this one means a global holomorphic section σ of $T\mathbb{CP}^n(\delta-1)=T\mathbb{CP}^n\otimes\mathcal{O}(\delta-1)$. The foliation has an isolated singularity at $0\in\mathbb{C}^n\subset\mathbb{CP}^n$ of the same index as the original map germ. When all the singularities of σ are isolated, the bound is deduced from Baum–Bott's formula, which expresses the sum of indices as the degree, $\sum_0^n \delta^i$, of the top Chern class $c_n(T\mathbb{CP}^n(\delta-1))$. Soares then proves the general case by means of a clever analytic deformation argument in order to get rid of possible wrong dimensional components in the singular locus of the foliation.

Our purpose here is to replace the deformation argument by a direct application of Fulton's intersection theory. Our main result, Theorem 3.1, gives bounds for the Poincaré–Hopf index of an isolated singularity of a foliation of $\mathbb{CP}^n$. There is a general bound, and special bounds for foliations that admit smooth invariant hypersurfaces not passing through the singularity. The case of nonlinear hypersurfaces was not contemplated in [2].

Date: March 12, 2004.

1991 *Mathematics Subject Classification.* 32S65 (Primary); 14C17, 37F75 (Secondary).

Key words and phrases. intersection theory, singularities, foliations.

* Partially supported by PRONEX, Convênio CNPq 41/96/0883/00, and by CNPq, Proc. 300004/95-8.

† Partially supported by CNPq.

2. Intersection Theory

We start giving a proof of the following lemma (cf. [7, Ex. (4), p. 16]).

2.1. Lemma. *Let Z be an irreducible component of a scheme $\mathcal{Z}$. Let $Z_1, \ldots, Z_n$ be subvarieties of $\mathcal{Z}$ distinct from Z. Then any relation $mZ + \sum m_i Z_i = 0$ in the Chow group $A_\star(\mathcal{Z})$ implies $m = 0$.*

Proof. The relation means that there are subvarieties $V_j \subseteq \mathcal{Z}$ and rational functions $r_j \in R(V_j)$ such that the corresponding divisors of zeros and poles $[r_j]$ add up to $mZ + \sum m_i Z_i$. If $m \neq 0$ then Z must appear as zero or pole of some r_j. In particular, $Z \subseteq V_j \subseteq \mathcal{Z}$. Since Z is an irreducible component of $\mathcal{Z}$, we must have $Z = V_j$. This is absurd since zeros and poles are of codimension one. □

Let Y be an algebraic scheme over $\mathbb{C}$, and let $E \to Y$ be a vector bundle of rank r generated by its sections. Let s_E be the zero section. Let $V \subseteq E$ be a m-dimensional subvariety. Let $\mathcal{Z} = {s_E}^{-1}(V)$. We recall the construction and some properties of the Gysin class $s_E^\star[V]$ for the specific case at hand. We form the cartesian diagram,

$$\begin{array}{cccc} \mathcal{Z} = & {s_E}^{-1}(V) & \hookrightarrow & V \\ & \downarrow & \square & \downarrow \\ & Y & \underset{s_E}{\hookrightarrow} & E \end{array}$$

The bottom embedding is regular with normal bundle E. Take the normal cone $C = C_{\mathcal{Z}}V$ to $\mathcal{Z}$ in V. This cone is a pure $m-$dimensional subscheme of the restricted bundle $E_{\mathcal{Z}}$. The corresponding cycle $[C]$ lives in the Chow group $A_m(E_{\mathcal{Z}})$, which is isomorphic to $A_{m-r}(\mathcal{Z})$ via the pullback associated to the structure map $\pi\colon E_{\mathcal{Z}} \to \mathcal{Z}$. By construction, $s_E^\star[V]$ is the unique class in $A_{m-r}(\mathcal{Z})$ such that $\pi^\star(s_E^\star[V]) = [C]$ holds.

If $Z \subseteq \mathcal{Z}$ is an irreducible component of dimension $m - r$, then, as explained in [4, p. 95 and §7.1, p. 120], we may write $s_E^\star[V] = i[Z] + \sum i_{Z'}[Z']$ in $A_{m-r}(\mathcal{Z})$, where i is positive and the Z' are subvarieties of dimension $m - r$ of $\mathcal{Z}$ distinct from Z. Furthermore, i is at most the length of $\mathcal{O}_{Z,\mathcal{Z}}$, with equality if V is smooth. (In fact, equality holds if $\mathcal{O}_{Z,\mathcal{Z}}$ is Cohen–Macaulay.)

2.2. Proposition. *Let Y be an algebraic scheme over $\mathbb{C}$, and let $E \to Y$ be a vector bundle of rank r generated by its sections. Let s_E be the zero section. Let $V \subseteq E$ be a m-dimensional subvariety. Let $\mathcal{Z} = {s_E}^{-1}(V)$. Then $s_E^\star[V]$ can be represented by a nonnegative cycle in $\mathcal{Z}$. Furthermore, suppose $Z \subseteq \mathcal{Z}$ is an irreducible component of dimension $m - r$ and V is smooth. Then*

$$s_E^\star[V] = i[Z] + \sum i_{Z'}[Z'] \text{ in } A_{m-r}(\mathcal{Z}),$$

where the Z' are subvarieties of dimension $m-r$ of $\mathcal{Z}$ distinct from Z, the $i_{Z'}$ are nonnegative and i is the length of $\mathcal{O}_{Z,\mathcal{Z}}$.

Proof. We show first that $s_E^\star[V]$ can be represented by a nonnegative cycle in $\mathcal{Z}$.

Let $F \to Y$ be a trivial bundle endowed with a surjection of bundles $\rho\colon F \to E$ over Y. The existence is assured by the hypothesis of global generation. We have $s_E = \rho \circ s_F$. Hence, we may write

$$\mathcal{Z} = s_E{}^{-1}(V) = s_F{}^{-1}\rho^{-1}(V).$$

In view of general properties of the intersection class [4, Prop. 6.5, p. 110], we have

$$s_E^\star[V] = s_F^\star\rho^\star[V] = s_F^\star[\rho^{-1}(V)] \ \text{ in } \ A_{m-r}(\mathcal{Z}).$$

Thus we may assume E is trivial. Now proceed by induction on the rank r of E. If $r=1$ then $s_E^\star[V]$ is the same as the intersection class of V by the principal Cartier divisor $Y = Y\times\{0\}$ of $E = Y\times\mathbb{C}$ [4, prop. 6.1(c), p. 94]. Now, if $V \not\subset s_E(Y)$ then $s_E^\star(V)$ is represented by the effective Cartier divisor $Y\cdot V \in A_{m-1}(\mathcal{Z})$, *i.e.*, we have $s_E^\star(V) \in A^{\geq}(\mathcal{Z})$ as asserted. If $V \subseteq s_E(Y)$ then $Y \cdot V = 0$ in $A_{m-1}(\mathcal{Z})$ [4, Def. 2.3, p. 33], because the normal bundle to $Y\times\{0\}$ in $Y\times\mathbb{C}$ is trivial. Next, for the inductive step, we write $E = G \oplus \mathcal{O}$, with G trivial of rank $r-1$ and $\mathcal{O} = Y\times\mathbb{C}$. Let j denote the inclusion of G in E, so $j(g) = (g,0)$. Then $s_E = j\circ s_G$. Now, on one hand $s_E^\star[V] \in A_{m-r}(\mathcal{Z})$, and on the other hand $s_E^\star[V] = s_G^\star j^\star[V]$ by [4, Thm. 6.5, p. 108]. The bundle E is a rank-1 bundle over G, under the natural surjection, and j is the zero section. Hence the case $r=1$ allows us to write

$$j^\star[V] = \sum m_i[W_i] \in A^{\geq}_{m-1}(j^{-1}(V)),$$

with $m_i \geq 0$ and the W_i denoting subvarieties of dimension $m-1$ of $j^{-1}(V)$. By induction, each $s_G^\star[W_i]$ lies in $A^{\geq}_{m-1-(r-1)}(s_G{}^{-1}(W_i))$, hence maps to $A^{\geq}_{m-r}(s_E{}^{-1}(V))$. We are done with the first statement by linearity.

As for the last statement, the class $s_E^\star[V]$ can be represented by a cycle of $\mathcal{Z}$ where $[Z]$ appears with coefficient i equal to the length of $\mathcal{O}_{Z,\mathcal{Z}}$. By Lemma 2.1, $[Z]$ has the same coefficient in any representation of $s_E^\star[V]$ by a cycle of $\mathcal{Z}$. In particular, by the first statement just proved, i is the coefficient of $[Z]$ in a representation of $s_E^\star[V]$ by a nonnegative linear combination of prime cycles of $\mathcal{Z}$. □

2.3. Remark. In order to prove the first statement of Proposition 2.2, we could have invoked Fulton's treatment of positivity, to the effect that $s_E^\star[V]$ has a representation as a nonnegative linear combination of prime cycles [4, Thm. 12.1(a), p. 212]. Though the result only asserts that $s_E^\star[V]$ is in $A^{\geq}_{m-r}(Y)$, the proof actually gives that $s_E^\star[V]$ lies in $A^{\geq}_{m-r}(\mathcal{Z})$; this refinement is necessary for our proof of the second statement. Though we could have quoted the more general [4, Thm. 12.2, p. 218], we reproduced above a small variation of the proof given to [4, Thm. 12.1(a), p. 212] for the reader's convenience.

3. Foliations of $\mathbb{CP}^n$

3.1. Theorem. *Let $P \in \mathbb{CP}^n$ be an isolated singularity with Poincaré–Hopf index i of a one-dimensional foliation of degree δ of $\mathbb{CP}^n$. Then*

$$i \leq \sum_{j=0}^{n} \delta^j.$$

Furthermore, if the foliation admits a smooth invariant hypersurface of degree d not passing through P then

$$i \leq \sum_{j=0}^{n} \delta^j (1-d)^{n-j}.$$

Proof. Let σ be the section of the bundle $T\mathbb{CP}^n(\delta - 1)$ corresponding to the foliation. For the first bound, we apply Proposition 2.2 taking

(1) $Y = \mathbb{CP}^n$,
(2) $E = T\mathbb{CP}^n(\delta - 1)$,
(3) and V as the image of $\mathbb{CP}^n$ under the section σ.

The bundle E is globally generated. Indeed, tensoring the Euler exact sequence,

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T\mathbb{P}^n \to 0$$

by $\mathcal{O}(\delta - 1)$, we obtain the following exact sequence:

$$0 \to \mathcal{O}(\delta - 1) \to \mathcal{O}(\delta)^{n+1} \to E \to 0. \tag{1}$$

Since $\mathcal{O}(\delta)$ is globally generated, so is E.

Furthermore, $\mathcal{Z} = {s_E}^{-1}(V)$ is precisely the singular set of the foliation. Since P is isolated, $Z = \{P\}$ is a component of $\mathcal{Z}$ of the dimension stipulated by the proposition. So we may write

$$s_E^{\star}[V] = i_P[P] + \sum i_{P'}[P'] \quad \text{in} \quad A_0(\mathcal{Z})$$

for suitable points $P' \in \mathcal{Z}$, distinct from P, and nonnegative $i_{P'}$. The coefficient i_P is the length of $\mathcal{O}_{P,\mathcal{Z}}$. This length is the Poincaré–Hopf index i of P. So i is at most the degree of $s_E^{\star}[V]$. Now,

$$\int_{\mathbb{P}^n} s_E^{\star}[V] = \int_{\mathbb{P}^n} c_n(E) = \sum_{j=0}^{n} \delta^j,$$

the last equality following from the exactness of (1), through the Whitney formula,

$$c(\mathcal{O}(\delta))^{n+1} = c(E)c(\mathcal{O}(\delta - 1)).$$

Assume now that there is a smooth invariant hypersurface $H \subset \mathbb{CP}^n$ of degree d such that $P \notin H$. Let $T\mathbb{CP}^n(\log H)$ be the elementary transformation of $T\mathbb{CP}^n$ at H by the normal bundle. In terms of sheaves, $T\mathbb{CP}^n(\log H)$ is the kernel of the natural homomorphism $h\colon T\mathbb{CP}^n \to N_H\mathbb{CP}^n$, where $N_H\mathbb{CP}^n$ is the normal bundle to H in $\mathbb{CP}^n$. (Also, $T\mathbb{CP}^n(\log H)$ is the dual of the bundle of logarithmic forms on $\mathbb{CP}^n$ with poles along H.)

Since H is invariant, σ factors through a section σ' of $T\mathbb{CP}^n(\log H)(\delta - 1)$. To prove the second bound, we apply Proposition 2.2 taking

(1) $Y = \mathbb{CP}^n$,
(2) $E = T\mathbb{CP}^n(\log H)(\delta - 1)$,
(3) and V as the image of $\mathbb{CP}^n$ under the section σ'.

The sections σ and σ' coincide off H. So $\mathcal{Z} = {s_E}^{-1}(V)$ coincides off H with the singular set of the foliation. Since $P \notin H$, the set $Z = \{P\}$ is a component of $\mathcal{Z}$ of the dimension stipulated by the proposition, and of multiplicity i. As with the first bound, to be able to apply Proposition 2.2, and derive the second bound for i, we need only show that E is globally generated and

$$\int_{\mathbb{P}^n} c_n(E) = \sum_{j=0}^{n} \delta^j (1-d)^{n-j}. \tag{2}$$

For this purpose, consider the homomorphism $g : \mathcal{O}(1)^{n+1} \to \mathcal{O}(d)$ obtained from the partial derivatives of the equation defining H. We claim that $T\mathbb{P}^n(\log H)$ is also the kernel of g. Indeed, the following diagram commutes,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(1)^{n+1} & \longrightarrow & T\mathbb{P}^n & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle g}\Big\downarrow & & {\scriptstyle h}\Big\downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(d) & \longrightarrow & N_H\mathbb{P}^n & \longrightarrow & 0,
\end{array}$$

where at the top is the Euler sequence and at the bottom is the natural sequence associated to the identification $N_H\mathbb{P}^n = \mathcal{O}(d)|H$. Since both sequences are exact, the snake lemma yields the claim.

Now, since H is smooth, g is surjective, and the following piece of the Koszul complex is exact,

$$\mathcal{O}(2-d)^{\binom{n+1}{2}} \longrightarrow \mathcal{O}(1)^{n+1} \xrightarrow{\ g\ } \mathcal{O}(d) \longrightarrow 0.$$

Thus E is a quotient of a sum of copies of $\mathcal{O}(\delta + 1 - d)$, and hence is globally generated if $d \le \delta + 1$. This inequality holds, as proved by Soares in [1]. (The inequality $d \le \delta + 1$ was basically known to Zariski; see [3, Thms. 7 and 8].) In addition, E sits in an exact sequence,

$$0 \to E \to \mathcal{O}(\delta)^{n+1} \to \mathcal{O}(\delta - 1 + d) \to 0.$$

The Whitney formula,

$$c(\mathcal{O}(\delta))^{n+1} = c(E)c(\mathcal{O}(\delta - 1 + d)),$$

yields (2). $\square$

4. Soares' bounds

We describe here the reasoning of Soares in [2] and reobtain his bounds.

Let $f = (f_1, \dots, f_n)\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ. Let

$$i = \dim_{\mathbb{C}} \mathbb{C}[|x_1, \dots, x_n|]/\langle f_1, \dots, f_n\rangle.$$

Then i is finite if and only if $f^{-1}(0) = 0$ in some neighborhood of 0. If $i < \infty$ then i is the Poincaré–Hopf index of f at 0.

Assume from now on that i is finite. Then $\mathcal{M}^k \subseteq \langle f_1, \dots, f_n\rangle$ for some integer k, where $\mathcal{M} = \langle x_1, \dots, x_n\rangle$ denotes the maximal ideal. For each $i = 1, \dots, n$ let $f_i^{(k)}$ be a polynomial of degree at most k such that $f_i^{(k)} - f_i \in \mathcal{M}^{k+1}$. Then

$$\langle f_1, \dots, f_n\rangle = \langle f_1^{(k)}, \dots, f_n^{(k)}\rangle,$$

and hence the Poincaré–Hopf index of f is the same as that of the polynomial map $f^{(k)} = (f_1^{(k)}, \dots, f_n^{(k)})$. So, in order to bound i, we may and will assume from now on that f is a polynomial map of degree k.

(The map germs f and $f^{(k)}$ are said to be $\mathcal{K}$-equivalent, in the sense of Mather; see [2]. It might be that f is $\mathcal{K}$-equivalent to a polynomial map of degree smaller than the minimum k such that $\mathcal{M}^k \subseteq \langle f_1, \dots, f_n\rangle$. If two map germs are $\mathcal{K}$-equivalent, they have the same Poincaré–Hopf index. The converse might not be true, though.)

If $k = 1$, then $i = 1$. Suppose $k > 1$. Consider on $\mathbb{C}^n$ the vector field

$$X = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}.$$

Since i is finite, 0 is an isolated singularity of X. Extend X to a foliation of $\mathbb{CP}^n$. The homogeneization process is as follows. Write $X = Y_0 + \cdots + Y_k$, where Y_j is a homogeneous vector field of degree j. Then form

$$X' = Y_k + x_0 Y_{k-1} + \cdots + x_0^k Y_0.$$

The homogeneous vector field X' on $\mathbb{C}^{n+1}$ induces a section σ' of $T\mathbb{CP}^n(k-1)$, *i.e.*, a foliation $\mathcal{F}'$ of $\mathbb{CP}^n$ of degree k. The index i is also the Poincaré–Hopf index of $\mathcal{F}'$ at $P = (1 : 0 : \cdots : 0)$.

Note that the hyperplane H at infinity, given by $x_0 = 0$, is left invariant by $\mathcal{F}'$, as $X'(x_0) = 0$. However, H might be in the singular locus of $\mathcal{F}'$. This is the case when Y_k is a multiple of the radial vector field,

$$R = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}.$$

In fact, since the Euler vector field,

$$E = x_0\frac{\partial}{\partial x_0} + \cdots + x_n\frac{\partial}{\partial x_n},$$

induces the zero section of $T\mathbb{CP}^n$, if $Y_k = GR$ then also

$$X'' = -Gx_0\partial/\partial x_0 + x_0Y_{k-1} + \cdots + x_0^kY_0$$

induces σ'. In this case, let $X^h = X''/x_0$; otherwise let $X^h = X'$.

Let $\mathcal{F}$ be the foliation of $\mathbb{CP}^n$ induced by X^h. Note that $\mathcal{F}$ coincides with $\mathcal{F}'$ off H. Thus P is an isolated singularity of $\mathcal{F}$ with Poincaré–Hopf index i. So we may apply Theorem 3.1. If Y_k is a multiple of the radial vector field, then $\mathcal{F}$ has degree $k-1$, and hence

$$i \leq \sum_{j=0}^{n}(k-1)^j.$$

If Y_k is not a multiple of the radial vector field, then $\mathcal{F}$ has degree k and leaves invariant the hyperplane at infinity. Thus

$$i \leq k^n.$$

These were the bounds obtained by Soares in [2].

References

[1] M. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*, Invent. math. 128 (1997), 495–500.

[2] __________, *Bounding Poincaré-Hopf indices and Milnor numbers*, preprint, 2002. To appear in Math. Nachrichten.

[3] E. Esteves, *The Castelnuovo–Mumford regularity of an integral variety of a vector field on projective space*, Math. Res. Lett. 9 (2002), 1–15.

[4] W. Fulton, *Intersection theory*, Springer, New York, 1985.

[5] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York, 1978.

[6] T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations*, Hermann, Paris, 1998.

[7] I. Vainsencher, *Classes características em geometria algébrica*, IMPA, Rio de Janeiro, 1985.

IMPA
Estrada D. Castorina 110
Jardim Botânico
22460-320 Rio de Janeiro RJ
Brasil
esteves@impa.br

UFMG
Departamento de Matemática
Cidade Universitária
30123-970 Belo Horizonte MG
Brasil
israel@mat.ufmg.br