STATISTICAL STABILITY OF SADDLE-NODE ARCS

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ABSTRACT. We study the dynamics of generic unfoldings of saddle-node circle local diffeomorphisms from the measure theoretical point of view, obtaining statistical and stochastic stability results for deterministic and random perturbations in this kind of one-parameter families.

CONTENTS

1. INTRODUCTION

The study of the modifications of the long term behavior of a dynamical system undergoing perturbations of the parameters has been one of the main themes of Bifurcation Theory. In the last decades the measure theoretical point of view has been intensively developed emphasizing the understanding of the asymptotic behavior of almost all orbits. The main notions associated to this point of view are those of *physical measure* and of *stochastic or statistical stability*.

2000 *Mathematics Subject Classification.* Primary: 37A10, 37G25. Secondary: 37A35, 37H99.

Key words and phrases. statistical and stochastic stability, equilibrium states, entropy formula, random perturbations, generic saddle-node unfolding.

V.A. was partially supported by CMUP-FCT (Portugal), post-doc grant SFRH/BPD/16082/2004/UX2U from FCT (Portugal) and CNPq (Brazil). Part of this work was done while enjoying a leave from CMUP at PUC-Rio/IMPA. M.J.P. was partially supported by CNPq-Brazil/Faperj-Brazil/Pronex Dyn. Systems.

Let M be a circle and $f_0 : M \to M$ be a C^2 local diffeomorphism. An f_0-invariant probability measure μ is *physical* if the *ergodic basin*

$$B(\mu) = \left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_0^j(x)) \to \int \varphi \, d\mu \text{ for all continuous } \varphi : M \to \mathbf{R} \right\}$$

has positive Lebesgue (length) measure in M. This means that the asymptotic behavior of "most points" is observable in a "physical sense" and determined by the measure μ.

Given a smooth family $(f_t)_{t\in[0,1]}$ of local diffeomorphisms of M admitting physical measures μ_t for every t, we say that f_0 is *statistically stable* if μ_t tends to μ_0 when $t \to 0$ in a suitable topology. This corresponds to stability of the long term dynamics of most orbits under deterministic perturbations of f_0.

In this setting a straightforward consequence of the Ergodic Theorem is that *every ergodic f_0-invariant probability measure μ_0 absolutely continuous with respect to Lebesgue measure m is a physical measure*.

A *random perturbation* of f_0 is defined by a family of probability measures $(\theta_\varepsilon)_{\varepsilon>0}$ on $[0,1]$ and the random sequence of maps

$$f_\omega^n = f_{t_n} \circ \cdots \circ f_{t_1}, \quad n \geq 1 \quad \text{and} \quad f_\omega^0 = Id,$$

where $Id : M \to M$ is the identity transformation, for a sequence $\omega = (t_1, t_2, \dots) \in \operatorname{supp}(\theta_\varepsilon)^{\mathbf{N}}$ and a given fixed $\varepsilon > 0$. An invariant measure in this setting is said a ε-*stationary measure*, which is a probability measure μ such that for each continuous function $\varphi : M \to \mathbf{R}$

$$\int \varphi \, d\mu = \int \int \varphi(f_t(x)) \, d\mu(x) \, d\theta_\varepsilon(t).$$

Ergodicity in this setting needs an extension of the notion of invariant set. We say that a subset E is ε-*invariant* when it satisfies

if $x \in E$ then $f_t(x) \in E$ for θ_ε-almost every t, and
if $x \in M \setminus E$ then $f_t(x) \in M \setminus E$ for θ_ε-almost every t.

We say that a ε-stationary measure μ is *ergodic* if $\mu(E)$ equals 0 or 1 for every ε-invariant set E. In this setting a point x belongs to the *ergodic basin* $B(\mu)$ if for all continuous $\varphi : M \to \mathbf{R}$ and $\theta_\varepsilon^{\mathbf{N}}$-almost every ω we have

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x)) \to \int \varphi \, d\mu \quad \text{when} \quad n \to \infty.$$

A stationary measure is *physical* if the Lebesgue measure of its ergodic basin is positive. We again have that *an absolutely continuous ergodic stationary probability measure is physical*.

Assuming that the family $(\theta_\varepsilon)_{\varepsilon>0}$ satisfies $\operatorname{supp}(\theta_\varepsilon) \to \{0\}$ when $\varepsilon \to 0$ and there exist physical stationary measures μ^ε for every small enough $\varepsilon > 0$, we say that f_0 is *stochastically stable* if every limit point of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ is a physical measure for f_0. This corresponds to stability of the asymptotic dynamics under random perturbations of f_0.

In this paper we study the dynamics of generic unfoldings $(f_t)_{t\in[0,1]}$ of a saddle-node circle local diffeomorphism f_0 from the measure theoretical point of view, obtaining statistical stability results for deterministic and random perturbations in this kind of one-parameter families. In particular we show that the map is uniformly expanding for all parameters close enough to the parameter of the saddle-node and has positive Lyapunov exponent uniformly bounded away from zero.

This kind of results in the particular case of saddle-node circle homeomorphisms might have applications to the mathematical modeling of neuron firing, see [17].

Our results can be seen as an extension of the work in [5] where maps which are expanding everywhere except at finitely many points were studied. Moreover these results open the way into further study of the unfolding of critical saddle-node circle maps considered in [7]. In addition, piecewise smooth families unfolding a saddle-node as in [15] were used to build new kinds of chaotic attractors for flows, and the statistical properties of this kind of attractors can possibly be obtained through suitable extensions of the techniques we present below.

1.1. **Statements of the results.** Let $f_0 : M \to M$ be a C^2 local diffeomorphism having a unique saddle-node fixed point that we call 0.

The fixed point 0 is a *saddle-node* if $f'(0) = 1$ and $f''(0) \neq 0 (> 0$ say). A *generic unfolding* of 0 (or f) is a one-parameter family of maps $f_t : M \to M$ with $t \in [0,t_0]$, so that $f_0 = f$ and if $f(x,t) = f_t(x)$, then $f(0,0) = 0$, $\partial_x f(0,0) = 1$, $\partial_x^2 f(0,0) > 0$ and $\partial_t f(0,0) > 0$. The family $(f_t)_{t\in[0,t_0]}$ is called a *saddle-node arc* in [8].

Let $B(\{0\})$ be the basin of attraction of the saddle-node fixed point 0 for f_0, i.e.

$$B(\{0\}) = \{x \in M : f_0^k(x) \to 0 \quad \text{as} \quad k \to \infty\},$$

and let the *immediate basin* W_0 of 0 be the connected component of $B(\{0\})$ containing 0.

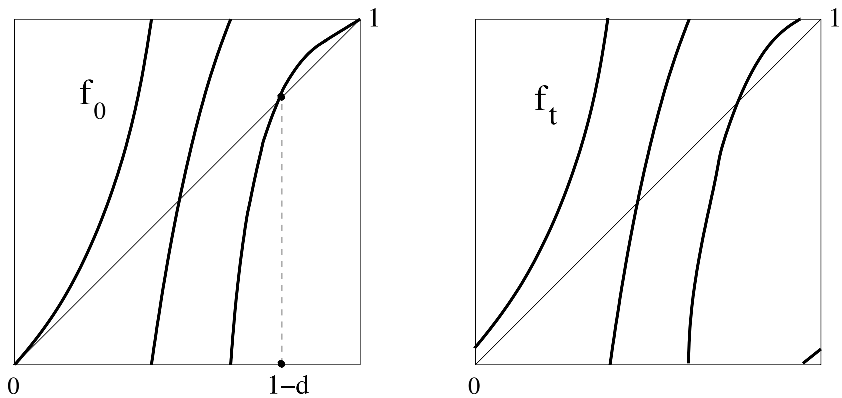

FIGURE 1. A saddle-node circle map.

We also assume the following global conditions on f_0,

H1: $|f'(x)| > 1$ for all $x \in M \setminus W_0$,

see Figure 1 for an example of such a map where $W_0 = [1-d, 1]$.

Remark 1.1. *We note that since f_0 is a local diffeomorphism, there must be a fixed source s ($s = 1-d$ in Figure 1) linked to the saddle-node, that is, a connected component of $W^u(s) \setminus \{s\}$ is contained in W_0.*

Theorem A. *Let f_0 be as above satisfying hypothesis (H1). Then the Dirac mass δ_0 concentrated at 0 is the unique physical measure of f_0.*

The proof of this result in in Section 2, where it is shown that $B(\{0\}) = M$ except for a zero Lebesgue measure subset of points.

Theorem B. *Let f_0 be as above satisfying hypothesis (H1). Then every f_0-invariant probability measure μ satisfying the Entropy Formula*

$$h_\mu(f_0) = \int \log|f_0'|\,d\mu, \tag{1.1}$$

must coincide with the Dirac mass δ_0 at the saddle-node point 0.

The proof of this theorem is in Section 3.

1.2. **Statistical stability.** The source linked to the saddle-node, see Remark 1.1, prevents the existence of either sinks or nonhyperbolic period points in the unfolding of the saddle-node. Using this we obtain the following statistical stability result.

Theorem C. *Let $f_t : M \to M$ be a generic unfolding of f_0 satisfying hypothesis (H1) above. Then*

(1) *for every $t > 0$ there exist $e_0 = e_0(t) > 0$ such that*
   (a) *f_t is uniformly expanding and there exists a unique absolutely continuous physical measure μ_t whose basin equals M except for a zero Lebesgue measure subset of points;*
   (b) *the Lyapunov exponent of Lebesgue almost every point is bigger than $e_0(t)$.*

(2) *$\mu_t \to \delta_0$ when $t \to 0$ in the weak* topology.*

We recall that item (2) means that f_0 *is statistically stable* with respect to the unfolding given by $(f_t)_{t\geq 0}$.

The proof of Theorem C is in Section 4.

1.3. **Stability under random perturbations.** Now we consider random perturbations of f_0 along the family $f_t(x) = f_0(x) + t, x, t \in M$, which generically unfolds the saddle-node at 0, with a family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on M such that $\mathrm{supp}(\theta_\varepsilon) \to \{0\}$ when $\varepsilon \to 0$.

Theorem D. *Let f_0 satisfy hypothesis (H1) and let $f_t : M \to M$ be the family defined above. Then*

(1) *for every family $(\theta_\varepsilon)_{\varepsilon>0}$ as above satisfying additionally*
   (a) *$\theta_\varepsilon \ll m$;*
   (b) *$\mathrm{int}(\mathrm{supp}(\theta_\varepsilon)) \neq \emptyset$;*
   (c) *$\mathrm{supp}(\theta_\varepsilon) \subset [0, t_0]$;*

   for every $\varepsilon > 0$, there exists a unique absolutely continuous stationary and ergodic probability μ^ε.

(2) *$\mu^\varepsilon \to \delta_0$ when $\varepsilon \to 0$ in the weak* topology.*

The above property (2) means that f_0 *is stochastically stable* under absolutely continuous random perturbations.

1.4. **Statistical and stochastical stability for saddle-node circle homeomorphisms.** Considering circle homeomorphisms with saddle-node points we easily achieve the same results as we now explain.

We say that a homeomorphism $f_0 : M \to M$ is a *saddle-node circle homeomorphism* if it satisfies (see Figure 2):

(1) $f_0(0) = 0$ and $f_0(x) \neq x$ for all $x \neq 0$;
(2) $f_0(x) > x$ for all $x \in V \setminus \{0\}$ for some open neighborhood V of 0.

Note that if f_0 were C^2 differentiable then conditions (1) and (2) above would imply that 0 was a usual C^2 saddle-node fixed point [16].

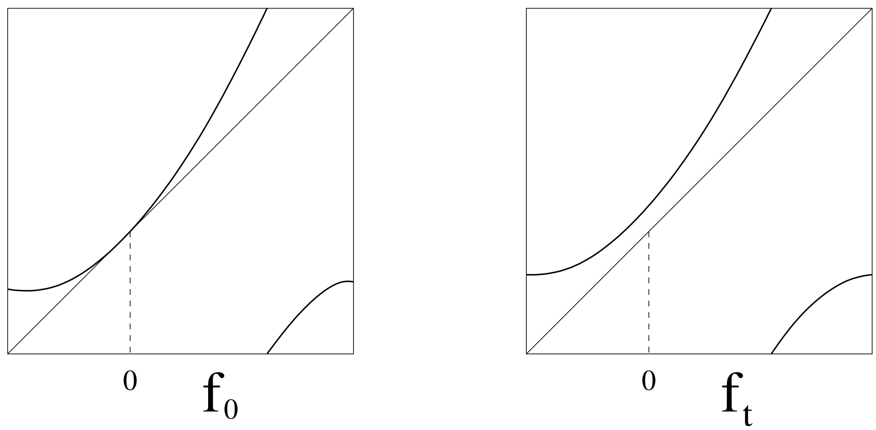

FIGURE 2. A saddle-node circle homeomorphism and a one-parameter family.

Since these kind of maps are *uniquely ergodic* with measure δ_0 (note that $f_0^n(x) \to 0$ when $n \to +\infty$ for all $x \in M$) the following two stability results follow from the fact that weak* accumulation measures of stationary or invariant measures are invariant measures for the limit map.

Theorem E. *Let $f_0 : M \to M$ be a saddle-node circle homeomorphism and let $(f_t)_{t\in[0,1]}$ be a continuous one-parameter family of circle homeomorphisms. If we choose for every t close to 0 a f_t-invariant probability measure μ_t, then every weak* accumulation point μ of the family $(\mu_t)_{t\in[0,1]}$, when $t \to 0$, is equal to the Dirac mass δ_0 concentrated at the saddle-node.*

This means that saddle-node circle homeomorphisms are *statistically stable*, i.e., the invariant probability measure always vary continuously with the unfolding parameter t near the saddle-node parameter 0.

Now let $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on M for each $\varepsilon > 0$ such that $\theta_\varepsilon \to \delta_0$ when $\varepsilon \to 0$ in the weak* topology.

Theorem F. *Let $f_0 : M \to M$ be a saddle-node circle homeomorphism, $(f_t)_{t\in[0,1]}$ be a continuous one-parameter family of circle homeomorphisms and $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on M as above.*

If we choose for every ε *close to* 0 *a stationary probability measure* μ^ε*, then every weak** *accumulation point* μ *of the family* $(\mu^\varepsilon)_{\varepsilon>0}$*, when* $\varepsilon\to 0$*, is equal to the Dirac mass* δ_0 *concentrated at the saddle-node.*

Acknowledgment. We thank Y. Cao for pointing out to us a mistake in the first version of this paper and for suggesting a possible way to improve the statements of our results.

2. Basins of attraction of sinks or saddle-node points

Here we prove Theorem A. For this it is enough to show that the basin $B(\{0\})$ of the saddle-node 0 has full Lebesgue measure in M.

Theorem 2.1. $m(M\setminus B(\{0\}))=0.$

In what follows we set $g=f_0$ Clearly to prove Theorem 2.1 it is sufficient to obtain

Proposition 2.2.
$$m\left(I\cap\bigcap_{n\geq 0} g^{-n}(M\setminus W_0)\right)=0$$
for every interval $I\subset M\setminus W_0$ *whose length is small enough.*

To prove this proposition we show that for any given interval $I\subset M\setminus W_0$ there exists the first iterate k such that $g^k(I)\not\subset M\setminus W_0$ and the relative measure of the subinterval G of points in I that fall into W_0 is a fixed proportion of the measure of I. For this we proceed as follows.

Let $I\subset M\setminus W_0$ be a given fixed interval and denote by $\ell(I)$ its length. We observe that the boundary ∂W_0 of the immediate basin consists of a source s. This means that in a neighborhood outside W_0 we always have some expansion.

For $\eta>0$ small we define the following compact subset
$$W(\eta)=\{x\in W_0 : d(x,M\setminus W_0)\geq\eta\}.$$
We assume that $\ell(I)\leq 1/4$ (recall that $\ell(M)=1$). Let us choose $\eta_0>0$ small enough such that
$$\int_{[r,r+\ell(I)]}|g'|>1 \tag{2.1}$$
for every $r\in\partial W(\eta_0)$. Then there exists $\sigma>1$ such that
$$\int_J |g'|\geq\sigma \tag{2.2}$$
for every interval $J\subset M\setminus W(\eta_0)$ such that $\ell(J)\geq\ell(I)$.

Remark 2.3. *The value of* σ *depends on* η *but if* $0<\eta<\eta_0$ *then* $\sigma(\eta_0)=\sigma(\eta)$*.*

This uniform rate of expansion ensures the following.

Lemma 2.4. *For any* $0<\eta<\eta_0$ *there exists* k_1 *such that*
$$g^k(I)\subset M\setminus W(\eta),\quad k=0,\dots,k_1-1\quad\textit{and}\quad g^{k_1}(I)\not\subset M\setminus W(\eta).$$

Proof. We define

$$L_0 = \max\{\ell(C) : C \text{ is a connected component of } M\setminus W(\eta)\}.$$

If $g^k(I)\subset M\setminus W(\eta),\quad k=0,\dots,k_0-1$ for some $k_0>0$, we obtain $\ell(g^k(I))\geq \sigma\ell(g^{k-1}(I))$ for all $1\leq k\leq k_0$. Thus $\ell(g^{k_0}(I))\geq \sigma^{k_0}\ell(I)$.

If k_0 were arbitrarily large, then we would have

$$\ell(g^{k_0}(I))\geq \sigma^{k_0}\ell(I) > L_0. \tag{2.3}$$

Thus by definition of L_0 we must have $g^{k_0}(I)\not\subset M\setminus W(\eta)$ as stated. □

Now it is easy to see that after a finite number of iterates either $g^{k_1}(I)$ is completely inside the basin of the saddle-node, or it contains a piece of the basin of uniform size η.

Lemma 2.5. *If k_1 is given by Lemma 2.4 then*

- *either $g^{k_1}(I)\subset W_0$*
- *or $g^{k_1}(I)\cap(M\setminus W_0)\neq\emptyset$ and $g^{k_1}(I)\cap W(\eta)\neq\emptyset$.*

Moreover in the last case we have $\ell(g^{k_1}(I)\cap(W\setminus W(\eta)))\geq\eta$.

Proof. The lemma follows from the fact that $g^{k_1}(I)$ is connected. □

Now we use the following bounded distortion result to estimate the size of the piece of I which is sent into $W_0\setminus W(\eta)$.

Lemma 2.6 (Bounded distortion)**.** *For $I\subset M\setminus W(\eta)$, k_1 given by Lemma 2.4 and $x,y\in I$ it holds*

$$\log\left|\frac{(g^{k_1})'(x)}{(g^{k_1})'(y)}\right|\leq C_0 \quad \textit{where} \quad C_0=\sup\left|\frac{g''}{g'}\right|\cdot\frac{1}{1-\sigma^{-1}}. \tag{2.4}$$

Proof. Since g is a local diffeomorphism , if $g^k(I)\subset M\setminus W(\eta),\quad k=0,\dots,k_1-1$ for some $k_1>0$ given by Lemma 2.4, then by the definition of σ we get $\ell(g^k(I))\geq\sigma\ell(g^{k-1}(I))$ for all $1\leq k\leq k_1$. We have

$$\begin{aligned}
\log\left|\frac{(g^{k_1})'(x)}{(g^{k_1})'(y)}\right| &= \sum_{j=0}^{k_1-1}\left|\log g'(g^j(x))-\log g'(g^j(y))\right| = \sum_{j=0}^{k_1-1}|(\log g')'(z_j)|\,\ell([g^j(x),g^j(y)])\\
&\leq \sup\left|\frac{g''}{g'}\right|\sum_{j=0}^{k_1-1}\ell(g^j(I))\leq\sup\left|\frac{g''}{g'}\right|\sum_{j=0}^{k_1-1}\sigma^{-(k_1-j)}\ell(g^{k_1}(I))\\
&\leq \sup\left|\frac{g''}{g'}\right|\cdot\ell(M)\cdot\frac{1}{1-\sigma^{-1}}=\sup\left|\frac{g''}{g'}\right|\cdot\frac{1}{1-\sigma^{-1}}.
\end{aligned}$$

□

Corollary 2.7. *There exists a constant $C>0$ such that for every interval $G\subset I$ and for k_1 given by Lemma 2.4 we have*

$$\frac{1}{C}\cdot\frac{m(G)}{m(I)}\leq\frac{m(g^{k_1}(G))}{m(g^{k_1}(I))}\leq C\cdot\frac{m(G)}{m(I)}.$$

Proof. It is straightforward to write for some $z \in I$

$$\frac{m(g^{k_1}(G))}{m(g^{k_1}(I))} = \frac{\int_G [(g^{k_1})'(x)/(g^{k_1})'(z)]\,dm(x)}{\int_I [(g^{k_1})'(x)/(g^{k_1})'(z)]\,dm(x)} \le C_0^2 \cdot \frac{m(G)}{m(I)}.$$

Analogously we get

$$\frac{m(g^{k_1}(G))}{m(g^{k_1}(I))} \ge \frac{1}{C_0^2} \cdot \frac{m(G)}{m(I)}$$

showing that the corollary holds with $C = C_0^2$. □

Now we are ready to exclude from I the points that fall into the basin of the saddle-node in a controlled way.

Let $G = (g^{k_1} \mid I)^{-1}(W_0) \subseteq I$. On the one hand, if $G \neq I$ then by Lemma 2.5 and Corollary 2.7 we obtain

$$\frac{m(G)}{m(I)} \ge \frac{1}{C}\frac{m(g^{k_1}(G))}{m(g^{k_1}(I))} \ge \frac{m(g^{k_1}(I) \cap (W_0 \setminus W(\eta)))}{C \cdot \sup|g'| \cdot m(M \setminus W(\eta))} \ge \frac{\eta}{C \cdot \sup|g'| \cdot m(M \setminus W(\eta))} \tag{2.5}$$

where by definition of k_1 we have

$$m(g^{k_1}(I)) \le \sup|g'| \cdot m(g^{k_1-1}(I)) \le \sup|g'| \cdot m(M \setminus W(\eta)).$$

Taking $\eta > 0$ small enough (see also Remark 2.3) (2.5) gives

$$\begin{aligned} m(I \setminus G) &= m(I) - m(G) \le m(I)\left(1 - \frac{\eta}{C \cdot \sup|g'| \cdot m(M \setminus W(\eta))}\right) \\ &\le \gamma_0 \cdot m(I), \end{aligned}$$

where $\gamma_0 \in (0,1)$ does not depend on I nor on k_1.

On the other hand, if $G = I$ then the last inequality is trivially true and we are done.

Otherwise, if a positive Lebesgue measure set remains in $I \setminus G$, we proceed by induction to conclude the proof of Proposition 2.2.

In what follows we set $I_0 = I$ and $I_1 = I \setminus G$. Let us assume that we have already constructed a nested collection of sets $I_0 \supset I_1 \supset \cdots \supset I_n$ such that

(1) for each $i = 1, \ldots, n$, I_i is a collection of intervals $J_{i,j}$ contained in I_{i-1} and
(2) to each $J_{i-1,j}$ there corresponds an integer $k = k(i-1,j) \in \mathbf{N}$ and a value $\eta = \eta(i-1,j) \in (0,\eta_0)$ satisfying (recall (2.1))

$$g^l(J_{i-1,j}) \subset M \setminus W(\eta) \text{ for all } l = 0, \ldots, k-1 \text{ and } g^k(J_{i-1,j} \setminus I_i) \subset W_0.$$

The previous lemmas show that the following result is true.

Lemma 2.8. *The sequence I_n is well defined for all $n \ge 1$ (it can be empty from some value of n onward) and*

$$m(I_{n+1}) \le \gamma_0 \cdot m(I_n).$$

We conclude that $m(\cap_{n\geq 0} I_n) = 0$. We now show that this implies Proposition 2.2.

Let us take $x \in \cap_{n\geq 0} I_n$. Then there exists a sequence $0 = k_0 < k_1 < k_2 < \dots$ of integers and $\eta_1, \eta_2, \dots$ of reals in $(0, \eta_0)$ such that

$$g^j(x) \in M \setminus W(\eta_i) \text{ for } k_i \leq j < k_{i+1} \text{ and } i \geq 0.$$

Moreover $M \setminus W(\eta_i) \supset M \setminus W(\eta_0)$ for all $i \geq 0$. Hence $x \in g^{-j}(M \setminus W(\eta_i)) \supset g^{-j}(M \setminus W(\eta_0))$.

We deduce that if $g^j(y) \in M \setminus W(\eta_0)$ and $y \in I$, then $y \in g^{-j}(M \setminus W(\eta_0)) \subset g^{-j}(M \setminus W(\eta_i))$ and thus $y \in \cap_{n\geq 0} I_n$. This means that

$$I \cap \bigcap_{n\geq 0} g^{-n}(M \setminus W) \subset I \cap \left(\bigcap_{n\geq 0} g^{-n}\big(M \setminus W(\eta_0)\big) \right) \subset \bigcap_{n\geq 0} I_n.$$

Since we already know that $m(\cap_{n\geq 0} I_n) = 0$, this ends the proof of Proposition 2.2.

3. Invariant measures satisfying the Entropy Formula

Here we prove Theorem B. Let μ_0 be a f_0-invariant probability measure satisfying the Entropy Formula (1.1), i.e., μ_0 is a equilibrium state for the potential $-\log|f_0'|$. The following result shows that we can assume without loss that μ_0 is ergodic.

Lemma 3.1. *Almost every ergodic component of an equilibrium state for $-\log|f_0'|$ is itself an equilibrium state for this same function.*

Proof. Let μ be an f_0-invariant measure satisfying $h_\mu(f_0) = \int \log|f_0'|\,d\mu$. On the one hand, the Ergodic Decomposition Theorem (see e.g Mañé [13]) ensures that

$$\int \log|f_0'|\,d\mu = \int\!\!\int \log|f_0'|\,d\mu_z\,d\mu(z) \quad \text{and} \quad h_\mu(f_0) = \int h_{\mu_z}(f_0)\,d\mu(z). \tag{3.1}$$

On the other hand, Ruelle's inequality guarantees for a μ-generic z that

$$h_{\mu_z}(f_0) \leq \int \log|f_0'|\,d\mu_z. \tag{3.2}$$

By (3.1) and (3.2), and because μ is an equilibrium state, we conclude that we have equality in (3.2) for μ-almost every z. □

Now we have two cases.

(1) If $\mu_0(\{0\}) > 0$ then $\mu_0 = \delta_0$ because μ_0 is ergodic and 0 is fixed.
(2) Else if $\mu_0(\{0\}) = 0$ then we let x be a μ_0-generic point, that is

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f_0^j(x)} \rightharpoonup \mu_0 \quad \text{when} \quad n \to \infty,$$

and we subdivide the argument in two more cases.
   (a) Either $x \in B(\{0\})$ or
   (b) $x \notin B(\{0\})$.

In case (a) since x is a μ_0-generic point we conclude that $\mu_0 = \delta_0$ also.

In case (b) the positive orbit $O^+_{f_0}(x)$ of x is contained in the region of M where $|f_0'| > 1$, thus the integral in the Entropy Formula is positive and so $h_{\mu_0}(f_0) > 0$.

It is known [19] that measures satisfying the Entropy Formula with positive entropy for endomorphisms of one-dimensional manifolds must be absolutely continuous (with respect to Lebesgue (length) measure).

Finally, since by Theorem 2.1 we have $B(\{0\}) = M, m$ mod 0, the absolute continuity of μ_0 implies that there exits a μ_0-generic point x in $B(\{0\})$, thus $\mu_0 = \delta_0$ as we wanted, proving Theorem B.

4. Statistical stability

Here we prove Theorem C. First we recall some properties of the generic unfolding of saddle-node arcs, which can be found in [8, 16].

4.1. **Transition maps for saddle-node unfoldings.** In what follows we let f_0 be a saddle-node local diffeomorphism and perform a local analysis of the dynamics near the saddle-node point 0. In this setting the map f_0 is a C^2 diffeomorphism in a neighborhood of 0.

Given a saddle-node arc $(f_t)_t$ of one dimensional maps, as defined in Section 1.1, there is what is called *an adapted arc of saddle-node vector fields* $(X(t,.))_t$, which has the form

$$X(t,x) = t + \alpha x^2 + \beta xt + \gamma t^2 + O(|t|^3 + |x|^3), \text{ with } \alpha > 0, \tag{4.1}$$

and describes the local dynamics of $(f_t)_t$: the arc $(f_t)_t$ embeds as the time-one of $(X(t,.))_t$. That is, if $X_s(t,.)$ denotes the time-s map induced by $(X(t,.))_t$ then $f_t(x) = X_1(t,x)$ for every t and every x. For $a < 0 < b$ fixed close enough to 0, $k \in \mathbf{N}$ and $t > 0$ sufficiently small, if $\sigma_k(t) \in [0,1]$ is defined by the relation

$$X_{k+\sigma_k(t)}(t,a) = b,$$

then it is proved in [8] that for $k \geq 1$ large enough, there is a unique $t_k^* > 0$ such that $\sigma_k(t_k^*) = 0$, and

$$\sigma_k : [t_{k+1}^*, t_k^*] \to [0,1]$$

is a C^∞ decreasing diffeomorphism onto $[0,1]$. Set t_k the inverse of σ_k.

For each $k \geq 1$ large enough, define $T_k : [0,1] \times [f_0^{-1}(a), f_0(a)] \to \mathbf{R}$ by $T_k(\sigma, x) = f^k_{t_k(\sigma)}(x)$. Note that T_k depends on both a and b. For $f_0^{-1}(a) < x < f_0(a)$ and t small, define $t_a(t,x)$ by $X_{t_a(t,x)}(t,x) = a$. The sequence $(T_k)_k$ converges in the C^∞ topology to the *transition map*

$$T_\infty : [0,1] \times [f_0^{-1}(a), f_0(a)] \mapsto \mathbf{R},$$

defined by $T_\infty(\sigma, x) = X_{t_a(0,x)-\sigma}(0,b)$. Note that T_∞ depends also on both a and b.

Observe also that $\partial_x T_\infty(\sigma, x)$ is bounded away from zero by a constant which does not depend on (σ, x). With b fixed, and taking a sufficiently close to 0, we can assume that this constant is arbitrarily large, since the number of iterates needed to take a to b increases without limit if t is small enough and a close enough to 0.

4.2. **Uniform expansion.** Now we present the arguments proving statistical stability of saddle-node arcs.

As in the previous subsection we fix $a < 0 < b$ with a close enough to 0 in order to get $\partial_x T_a(\sigma, x) \geq 2c_0 > 1$, for every $\sigma \in [0,1]$ and for all $x \in [f_0^{-1}(a), f_0(a)]$. For small $t > 0$ there exists $k \geq 1$ such that $t \in [t^*_{k-1}, t^*_k]$ and

$$T_{k+\sigma_k(t)} : [f_0^{-1}(a), f_0(a)] \to [f_0^{-2}(b), \infty), \quad x \mapsto f^k_{t_k(t)}(x)$$

has derivative bigger than $c_0 > 1$, i.e.

$$(T_{k+\sigma_k(t)})' \geq c_0 > 1. \tag{4.2}$$

Remark 4.1. *We have $f_t(x) = f(x) + t > f(x)$ for all $t > 0$ and x in a small neighborhood of* 0. *Since $f_0''(0) > 0$ this ensures that if x is near* 0 *and $\omega_1, \dots, \omega_k \in [t_0^-, t_0]$ with $t_0^- > 0$, then*

$$\big(f^k_\omega\big)'(x) = \prod_{i=1}^{k} f_0'\big(f_{\omega_i} \circ \cdots \circ f_{\omega_1}(x)\big) \geq \big(f^k_{t_0^-}\big)'(x).$$

Hence the derivative of the transition maps $T_{k+\sigma_k(t)}$ can be used as a lower bound for the derivative along random orbits near 0.

Theorem 4.2. *There exist $t_0 > 0$ small enough such that for every probability measure θ supported in $[0, t_0]$ with $\theta(\{0\}) < 1$ and for every $x \in \mathbf{S}^1$*

$$\liminf_{n\to\infty} \frac{1}{n} \log \left| \big(f^n_\omega\big)'(x) \right| \geq 0 \quad \textit{for} \quad \theta^{\mathbf{N}} - \textit{a.e. } \omega \in [0, t_0]^{\mathbf{N}}.$$

Moreover for every $0 < t_0^- < t_0$ there exists $e_0 = e_0(t_0^-, t_0) > 0$ such that for all $\omega \in [t_0^-, t_0]^{\mathbf{N}}$ and every $x \in \mathbf{S}^1$

$$\liminf_{n\to+\infty} \frac{1}{n} \log |(f^n_\omega)'(x)| \geq e_0. \tag{4.3}$$

Proof. To obtain such result we note that since we are assuming that f_t is expanding outside the immediate basin of the saddle-node, it is enough to analyse the dynamical behavior near 0.

We fix $d_0 \in W_0$ very close to the source s connected to the saddle-node, where W_0 is the immeadiate basin of attraction of the saddle-node 0 for f_0 (see Section 2 and Remark 1.1). We note that for all $t \in \mathbf{S}^1$ we have

$$\left| f_t'(x) \right| \geq \sigma_0 \quad \text{for all} \quad x \in \mathbf{S}^1 \setminus [d_0, f_0^{-1}(a)], \tag{4.4}$$

for some $\sigma_0 > 1$. Now we fix $x \in \mathbf{S}^1$ and define for any given $\omega \in [0, t_0]^{\mathbf{N}}$

$$R = R(\omega, x) = \{i \geq 0 : f^i_\omega(x) \in [d_0, f_0^{-1}(a)]\}.$$

If $R = \emptyset$, then by (4.4) we have $\left| \big(f_{\omega_i}\big)'(f^i_\omega(x)) \right| \geq \sigma_0$ for every $i \geq 1$ and thus (4.3) holds with $e_0 = \log \sigma_0$. Otherwise $R \neq \emptyset$ and we set $k = \min R$.

If $k > 0$, then $\left| \big(f^k_{\omega_i}\big)'(x) \right| \geq \sigma_0^k$ by construction. Otherwise $k = 0$ and so $x \in [d_0, f_0^{-1}(a)]$. In this case we set

$$\ell = \ell(\omega, x) = \min\{i > 0 : f^i_\omega(x) \in [f_0^{-1}(b), f_0(b)]\}.$$

We consider first the case $\omega \in [t_0^-, t_0]^{\mathbf{N}}$ with $0 < t_0^- < t_0$. In this setting there is $\ell_0 = \ell_0(t_0^-)$ such that $\ell(\omega, x) \le \ell_0$ for all $x \in [d_0, f_0^{-1}(a)]$. We note that $\ell_0(t_0^-) \to +\infty$ when $t_0^- \to 0^+$.

The transition map and the geometry near the saddle-node (see Remark 4.1) ensure that there is $c_\ell > 1$ such that

$$\left|\big(f_\omega^\ell\big)'(x)\right| \ge c_\ell > 1 \quad \text{for} \quad x \in [d_0, f_0^{-1}(a)], \omega \in [t_0^-, t_0]^{\mathbf{N}}, \ell = \ell(\omega, x).$$

If we define

$$\beta = \beta(\ell_0) = \min\left\{\frac{1}{\ell}\log c_\ell : \ell = 1, \ldots, \ell_0\right\},$$

then $\left|\big(f_\omega^\ell\big)'(x)\right| \ge (e^\beta)^\ell$ and we set $\sigma_1 = e^\beta$.

We have shown that for all $(\omega, x) \in [t_0^-, t_0]^{\mathbf{N}} \times \mathbf{S}^1$ there is a sequence $n_1 < n_2 < n_3 < \ldots$ satisfying

$$\left|\big(f_\omega^{s_k}\big)'(x)\right| \ge \sigma^{s_k} \quad \text{with} \quad s_k = n_1 + \cdots + n_k \quad \text{and} \quad \sigma = \min\{\sigma_0, \sigma_1\}. \tag{4.5}$$

Here σ depends on t through ℓ_0 and β. Hence (4.3) holds with $e_0 = \sigma$, finishing the proof of the second part of the statement.

For the first part of the statement, we may assume that a $\theta^{\mathbf{N}}$-generic ω does not contain an infinite sequence of coordinates arbitrarily near 0, which is enough to deduce that $\ell(\omega, x)$ is finite for every $x \in \mathbf{S}^1$ and $\theta^{\mathbf{N}}$-a.e. ω. Let us be more precise.

The assumption $\theta(\{0\}) < 1$ ensures that for every $\varepsilon > 0$ there exists $0 < \delta_0 < \varepsilon$ such that $\theta([0,\delta]) < 1$ for all $0 \le \delta < \delta_0$. Hence $X_0 = [0,\delta]^{\mathbf{N}}$ and $X_n = [0,t_0]^n \times X_0, n \ge 1$ satisfy $\theta^{\mathbf{N}}(X_n) = 0$ for all $n \ge 0$ and thus $Y = [0,t_0]^{\mathbf{N}} \setminus \cup_{n\ge 0} X_n$ has full $\theta^{\mathbf{N}}$-measure. In particular, letting $\delta = 0$ we get that a $\theta^{\mathbf{N}}$-generic ω has no zeroes.

This shows that a $\theta^{\mathbf{N}}$-generic sequence ω admits $\delta > 0$ and a subsequence $n_1 < n_2 < n_3 < \ldots$ such that

$$\omega_{n_k} > \delta \quad \text{for all} \quad k \ge 1 \quad \text{and} \quad \omega_n > 0 \quad \text{for all} \quad n \ge 1.$$

We conclude that $\ell(\omega, x) < +\infty$ (although it may be arbitrarily big) for all $x \in \mathbf{S}^1$ and $\theta^{\mathbf{N}}$-a.e. ω.

Finally, going back to the initial argument, we assume that $R \ne \emptyset$ and $k = 0$, and set

$$\alpha = \min\{\omega_1, \ldots, \omega_{\ell(\omega,x)}\} > 0.$$

Again by Remark 4.1 we see that there is $c = c(\alpha)$ such that

$$\left|\big(f_\omega^{\ell(\omega,x)}\big)'(x)\right| \ge c > 1 \quad \text{for} \quad x \in [d_0, f_0^{-1}(a)], \omega \in [\alpha, t_0]^{\mathbf{N}}.$$

Since $\ell(\omega, x)$ can be arbitrarily big, the exponent $\ell(\omega, x)^{-1} \cdot \log c$ can be arbitrarily close to zero. Therefore the value of σ in (4.5) must be 1, finishing the proof of the theorem.

□

Remark 4.3. *The proof of Theorem 4.2 shows, in particular, (discarding the iterates near* 0*) that for all $x \in \mathbf{S}^1$ and $\theta^{\mathbf{N}}$-a.e. $\omega \in [0,t_0]^{\mathbf{N}}$ there exists a sequence $n_1 < n_2 < n_3 < \ldots$ such that for all $k \ge 1$*

a: $\left|\big(f_\omega^{n_{2k-1}}\big)'(x)\right| \ge 1$*; and*

b: $\left|\left(f_{\sigma^{n_{2k-1}}\omega}^{n_{2k}-n_{2k-1}}\right)'\left(f_{\omega}^{n_{2k-1}}(x)\right)\right| \geq \sigma_0^{n_{2k}-n_{2k-1}}$.

Since Theorem 4.2 ensures, in particular, that for every $x \in M$ there exists $n \geq 1$ such that $|(f_t^n)'(x)| > 1$, then we conclude that f_t is uniformly expanding, i.e., there are constants $C > 0$ and $\sigma > 1$ satisfying $|(f_t^n)'(x)| \geq C\sigma^n$ for all $x \in M$, $n \geq 1$ and $t \in (0,t_0)$, see e.g. [2].

Theorem 4.4. *Let $t_0 > 0$ be given by Theorem 4.2. Then for all $t \in (0,t_0)$ there exists a unique absolutely continuous ergodic probability measure μ_t for f_t such that*

$$0 < h_{\mu_t}(f_t) = \int \log|f_t'|\,d\mu_t. \tag{4.6}$$

Proof. The conclusion of Theorem 4.2 is enough to guarantee that f_t is uniformly expanding, for $t \in (0,t_0)$, by [2, Theorem A]. It is well known that uniformly expanding maps admit a unique absolutely continuous ergodic invariant measure satisfying the Entropy Formula (4.6), see e.g. [13]. □

Another consequence of uniform expansion is the following.

Theorem 4.5. *Let μ_0 be a weak* accumulation point of μ_t when $t \to 0$. Then there exists a finite partition ξ of M which is a μ_t mod 0 generating partition for f_t, for all $t \in (0,t_0)$, and also that $\mu_0(\partial\xi) = 0$, i.e., the μ_0 measure of the boundary of the atoms of ξ is zero.*

Proof. Any finite partition of M Lebesgue modulo zero is a μ_t mod 0 partition of M (since $\mu_t \ll m$) and also a generating partition, by the uniform expansion of f_t for $t \in (0,t_0)$, see e.g. [13].

A finite partition Lebesgue modulo zero whose boundary has also zero measure with respect to μ_0 may be obtained as follows. For any fixed $\delta > 0$ we may find a finite open cover of M by δ-balls: $\{B(x_i,\delta), i = 1,\ldots,k\}$. We observe that since μ_0 is a finite measure, there exist arbitrarily small values $\eta > 0$ such that $\mu_0(\partial B(x_i,\delta+\eta)) = 0$ for all $i = 1,\ldots,k$. Moreover we automatically have $m(\partial B(x_i,\delta+\eta)) = 0$ also. Let us fix such a η. Then the partition $\xi = \{B(x_1,\delta+\eta), M\setminus B(x_1,\delta+\eta)\} \vee \cdots \vee \{B(x_k,\delta+\eta), M\setminus B(x_k,\delta+\eta)\}$ is as stated. □

Theorem 4.6. *In this setting, for all weak* accumulation point μ_0 of μ_t when $t \to 0^+$ we have*

$$\limsup_{t\to 0^+} h_{\mu_t}(f_t) \leq h_{\mu_0}(f_0). \tag{4.7}$$

This result together with Ruelle's inequality will show that every weak* accumulation point μ_0 of μ_t when $t \to 0^+$ satisfies the Entropy Formula.

Proof. Let us fix a weak* accumulation point μ_0 of μ_t when $t \to 0^+$ and a partition ξ as in Theorem 4.5. Then by the Kolmogorov-Sinai Theorem [13] and setting $\xi_n = \vee_{j=0}^{n-1} f_t^{-j}\xi$ we have for any given fixed $n \geq 1$

$$h_{\mu_t}(f_t) = h_{\mu_t}(f_t,\xi) = \inf_{k\geq 1} \frac{1}{k} H_{\mu_t}(\xi_k) \leq \frac{1}{n}\int -\log\mu_t(\xi_n(x))\,d\mu_t(x).$$

Now since the boundary of every element of ξ has μ_0 measure zero, then we have the following convergence

$$\frac{1}{n}\int -\log\mu_t(\xi_n(x))\,d\mu_t(x) \to \frac{1}{n}\int -\log\mu_0(\xi_n(x))\,d\mu_0(x) = \frac{1}{n}H_{\mu_0}(\vee_{j=0}^{n-1} f_0^{-j}\xi).$$

Since this holds for all $n \geq 1$, we have

$$\limsup_{t\to 0^+} h_{\mu_t}(f_t) \leq h_{\mu_0}(f_0),$$

completing the proof. □

From Theorem 4.6 we conclude that the Entropy Formula (1.1) holds for every weak* accumulation point μ_0 of $(\mu_t)_{t>0}$ when $t \to 0^+$, since as already observed the opposite inequality in (4.7) is always true by [20].

Finally, by Theorem B, we see that every weak* accumulation point μ_0 as above is the Dirac mass δ_0, which ends the proof of Theorem C.

5. Stochastic stability

Here we prove Theorem D. We consider the family $f_t(x) = f_0(x) + t$, where f_0 satisfies (H1), which is a generic unfolding of the saddle-node at 0. Hence for all $t > 0$ close enough to 0 the map f_t is uniformly expanding, by Theorem 4.2.

5.1. **Uniqueness of stationary probability measures.** We note that by the choice of the family $(f_t)_{t\in[0,t_0]}$, generically unfolding the saddle-node at 0, we have that there exists $\zeta > 0$ such that for all $x \in M$

$$\{f_t(x) : t \in \operatorname{supp}(\theta_\varepsilon)\} \supset B(f_{t^*}(x), \zeta), \tag{5.1}$$

for some fixed $t^* \in \operatorname{supp}(\theta_\varepsilon)$, where $B(z,\zeta)$ is the ball of radius ζ centered at z. This holds just because $\operatorname{supp}(\theta_\varepsilon)$ has nonempty interior and the map $t \mapsto f_t(x)$ is continuous (in fact C^2) for every fixed x.

Let us define $f_x : [0,t_0] \to M, t \mapsto f_t(x) = f_0(x) + t$ for any given fixed $x \in M$. The condition $\theta_\varepsilon \ll m$ ensures that for every $x \in M$ we have $(f_x)_*(\theta_\varepsilon^{\mathbf{N}}) \ll m$, where $(f_x)_*(\theta_\varepsilon)$ is the probability measure defined by

$$\int \varphi(f_t(x))\, d\theta_\varepsilon(t)$$

for every bounded measurable function $\varphi : M \to \mathbf{R}$. Indeed, if E is a Borel subset of M such that $m(E) = 0$, then

$$(f_x)_*(\theta_\varepsilon)(E) = \int 1_E(f_0(x) + t)\, d\theta_\varepsilon(t) = \int 1_{E - f_0(x)}\, d\theta_\varepsilon = \theta_\varepsilon(E - f_0(x)) = 0,$$

because $m(E - f_0(x)) = m(E) = 0$. The definition of stationary measure shows that every ε-stationary measure μ^ε is such that

$$\int \varphi\, d\mu^\varepsilon = \int\int \varphi(f_t(x))\, d\theta_\varepsilon(t)\, d\mu^\varepsilon(x) = \int [(f_x)_*\theta_\varepsilon]\varphi\, d\mu^\varepsilon(x),$$

hence $\mu^\varepsilon \ll m$ also. Since $\mu^\varepsilon(\operatorname{supp}(\mu^\varepsilon)) = 1$ we get $m(\operatorname{supp}(\mu^\varepsilon)) > 0$ using the absolute continuity.

A standard property of ε-stationary measures is that $f_t(\operatorname{supp}(\mu^\varepsilon)) \subset \operatorname{supp}(\mu^\varepsilon)$ for every $t \in \operatorname{supp}(\theta_\varepsilon)$, see e.g. [4].

This invariance property together with (5.1) show that there exist $\zeta > 0$ and $x \in \mathrm{supp}(\mu^\varepsilon)$ such that $\mathrm{supp}(\mu^\varepsilon) \supset B(f_{t^*}(x), \zeta)$. Thus the support of μ^ε has nonempty interior. By the uniform expansion we know that f_t is transitive (even topologically mixing, see e.g.[13]) for all $t \in \mathrm{supp}(\theta_\varepsilon)$, hence we conclude that $\mathrm{supp}(\mu^\varepsilon) = M$ *for every* ε*-stationary probability measure* μ^ε.

Under the conditions assumed in the statement of Theorem D together with property (5.1) it is known (see e.g. [4]) that there are at most finitely many ε-stationary ergodic absolutely continuous probability measures with pairwise disjoint supports. Since we have shown that any ε-stationary measure has full support in M, we conclude that for every $\varepsilon > 0$ there is a unique ε-stationary absolutely continuous and ergodic measure μ^ε, as stated in item (1) of Theorem D.

5.2. **Entropy and random generating partitions.** Let μ^ε be a ε-stationary measure as defined above. Here we give two equivalent definitions of the entropy of μ^ε to be used in what follows.

Theorem 5.1. [9, Thm. 1.3] *For any finite measurable partition* ξ *of* M *the limit*

$$h_{\mu^\varepsilon}(\xi) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu^\varepsilon}\Big(\bigvee_{k=0}^{n-1} (f_\omega^k)^{-1}\xi\Big)\, d\theta_\varepsilon^{\mathbf{N}}(\omega)$$

exists.

This limit is called the *entropy of the random dynamical system with respect to* ξ *and to* μ^ε. As in the deterministic case the above limit can be replaced by the infimum.

The *metric entropy* of the random dynamical system is defined as

$$h_{\mu^\varepsilon} = \sup_\xi h_{\mu^\varepsilon}(\xi),$$

where the supremum is taken over all finite measurable partitions.

It seems natural to define the entropy of a random system by $h_{\theta_\varepsilon^{\mathbf{N}}\times\mu^\varepsilon}(S)$ where S is the skew-product map $S : [0,t_0]^{\mathbf{N}} \times M \to [0,t_0]^{\mathbf{N}} \times M,\ (\omega, x) \mapsto (\sigma(\omega), f_{t_1}(x))$, and $\sigma : [0,t_0]^{\mathbf{N}} \to [0,t_0]^{\mathbf{N}}$ is the left shift on sequences. However (see e.g. [9, Thm. 1.2]) under some mild conditions the value of this function is infinite. But the conditional entropy of $\theta_\varepsilon^{\mathbf{N}} \times \mu^\varepsilon$ with respect to a suitable σ-algebra of subsets coincides with the entropy as defined above.

Let $\mathcal{B} \times M$ denote the minimal $\sigma-$algebra containing all products of the form $A \times M$ with $A \in \mathcal{B}$. In what follows we denote by $h^{\mathcal{B}\times M}_{\theta_\varepsilon^{\mathbf{N}}\times\mu^\varepsilon}(S)$ the conditional metric entropy of S with respect to the σ-algebra $\mathcal{B} \times M$. (See e.g. [6] for definition and properties of conditional entropy.)

Theorem 5.2. [9, Thm. 1.4] *Let* μ^ε *be a* ε*-stationary probability measure. Then*

$$h_{\mu^\varepsilon} = h^{\mathcal{B}\times M}_{\theta_\varepsilon^{\mathbf{N}}\times\mu^\varepsilon}(S).$$

The Kolmogorov-Sinai result about generating partitions is also available in a random version. We denote by $\mathcal{A} = \mathcal{B}(M)$ the Borel σ-algebra of M and say that for a given fixed $\varepsilon > 0$, a finite partition ξ is a *random generating partition for* $\mathcal{A}$ if

$$\bigvee_{k=0}^{+\infty} (f_\omega^k)^{-1}\xi = \mathcal{A} \quad \text{for} \quad \theta_\varepsilon^{\mathbf{N}} - \text{almost all } \omega \in [0,t_0]^{\mathbf{N}}. \tag{5.2}$$

Theorem 5.3. [9, Cor. 1.2] *If* ξ *is a random generating partition for* $\mathcal{A}$*, then* $h_{\mu^\varepsilon} = h_{\mu^\varepsilon}(\xi)$.

5.2.1. *Entropy Formula for random perturbations.* We want to show that μ^ε satisfies an Entropy Formula analogous to (1.1) in the random setting. The absolute continuity and ergodicity of μ^ε gives that μ^ε satisfies the Entropy Formula in the following form (see [11]):

$$h_{\mu^\varepsilon} = \lim_{n\to+\infty} \frac{1}{n} \log|(f^n_\omega)'(x)| = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|f_0'(f^j_\omega(x))| = \int \log|f_0'|\, d\mu^\varepsilon, \tag{5.3}$$

for $\theta_\varepsilon^{\mathbf{N}} \times \mu^\varepsilon$ almost every $(\omega, x) \in \Omega \times M$, as long as the random Lyapunov exponent given by the above limit is non-negative. (This limit does not depend on (ω, x) by the Ergodic Theorem.) Since by Theorem 4.2 we have that the random Lyapunov exponent is non-negative for all $x \in \mathbf{S}^1$ and $\theta_\varepsilon^{\mathbf{N}}$-a.e. ω, then the Entropy Formula (5.3) holds.

5.2.2. *Constructing the generating partition.* Here we use the previous results to prove the following theorem analogous to Theorem 4.5.

Theorem 5.4. *Let μ_0 be a weak* accumulation point of μ^ε when $\varepsilon \to 0$. Then there exists a finite partition ξ of M which is a μ^ε* mod 0 *generating partition for all small enough $\varepsilon > 0$, and also that $\mu_0(\partial\xi) = 0$, i.e., the μ_0 measure of the boundary of the atoms of ξ is zero.*

Proof. A finite partition Lebesgue modulo zero whose boundary has also zero measure with respect to μ_0 and with arbitrarily small diameter $\delta > 0$ may be obtained as already explained in the proof of Theorem 4.5.

Now we show that if the diameter δ of ξ satisfies $0 < \delta < \delta_1$, where δ_1 is the *injectivity radius* of f_t for all $t \in [0, t_0]$, i.e.,

$$f_t \mid B(x, \delta_1) \quad \text{is a diffeomorphism onto its image,} \quad t \in [0, t_0], x \in \mathbf{S}^1.$$

(since f_t is a family of local diffeomorphisms, this value $\delta_1 > 0$ is guaranteed to exist), then ξ is a random generating partition for the Borel σ-algebra as in (5.2) for all small enough $\varepsilon > 0$.

Indeed, let $x, y \in \mathbf{S}^1$ be given and let ω be a $\theta_\varepsilon^{\mathbf{N}}$-generic sequence such that

$$\operatorname{dist}\left(f^n_\omega(x), f^n_\omega(y)\right) \le \delta \quad \text{for every} \quad n \ge 1, \tag{5.4}$$

where $0 < \delta < \delta_1$. Let $n_1 < n_2 < n_3 < \dots$ be given by Theorem 4.2 and Remark 4.3. Then we have for all $n \ge 1$

$$\operatorname{dist}(x, y) \le \sigma_0^{-\sum_{k=1}^n (n_{2k} - n_{2k-1})} \cdot \operatorname{dist}\left(f^{n_{2k}}_\omega(x), f^{n_{2k}}_\omega(y)\right)$$

because, by assumption (5.4), $f^{n_{2k}}_\omega(x), f^{n_{2k}}_\omega(y)$ are always in a region where f_t is invertible.

Hence for a partition ξ with $\operatorname{diam}\xi < \delta_1$ and $\mu_0(\partial\xi) = 0$, setting $\xi_{n,\omega} = \vee_{j=0}^{n-1} (f^j_\omega)^{-1}\xi$ we have that for every $x \in \mathbf{S}^1$

$$\operatorname{diam}\xi_{n_{2k},\omega}(x) \to 0 \quad \text{when} \quad k \to \infty$$

for $\theta_\varepsilon^{\mathbf{N}}$-a.e. ω. This implies that $\bigvee_{n\ge1} \xi_{n,\omega} = \mathcal{A}$, μ^ε mod 0, finishing the proof. □

5.3. **Accumulation measures and Entropy Formula.** Now we prove that every weak* accumulation measure μ_0 of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ satisfies the Entropy Formula.

We start by fixing a weak* accumulation point μ_0 of μ^ε when $\varepsilon \to 0$: there exists $\varepsilon_k \to 0$ when $k \to \infty$ such that $\mu = \lim_k \mu^{\varepsilon_k}$. We also fix a uniform random generating partition ξ as in the previous subsection.

We need to construct a sequence of partitions of $\Omega \times M$ according to the following result. We set $\omega_0 = (0,0,0,\dots) \in \Omega$ in what follows.

Lemma 5.5. *There exists an increasing sequence of measurable partitions $(\mathcal{B}_n)_{n\geq 1}$ of Ω such that*

1. $\omega_0 \in \operatorname{int}(\mathcal{B}_n(\omega_0))$ *for all* $n \geq 1$*;*
2. $\mathcal{B}_n \nearrow \mathcal{B}$*,* θ^{ε_k} mod 0 *for all* $k \geq 1$ *when* $n \to \infty$*;*
3. $\lim_{n\to\infty} H_\rho(\xi \mid \mathcal{B}_n) = H_\rho(\xi \mid \mathcal{B})$ *for every measurable finite partition* ξ *and any* S*-invariant probability measure* ρ.

Proof. For the first two items we let $\mathcal{C}_n$ be a finite θ_{ε_k} mod 0 partition of Ω such that $t_0 \in \operatorname{int}(\mathcal{C}_n(t_0))$ with $\operatorname{diam} \mathcal{C}_n \to 0$ when $n \to \infty$. Example: take a cover $(B(t,1/n))_{t\in X}$ of Ω by $1/n$-balls and take a subcover $U_1,\dots,U_k$ of $\Omega \setminus B(t_0,2/n)$ together with $U_0 = B(t_0,3/n)$; then let $\mathcal{C}_n = \{U_0, M\setminus U_0\} \vee \cdots \vee \{U_k, M\setminus U_k\}$.

We observe that we may assume that the boundary of these balls has null θ_{ε_k}-measure for all $k \geq 1$, since $(\theta_{\varepsilon_k})_{k\geq 1}$ is a denumerable family of non-atomic probability measures on Ω. Now we set

$$\mathcal{B}_n = \mathcal{C}_n \times \overset{n}{\cdots} \times \mathcal{C}_n \times \Omega \quad \text{for all } n \geq 1.$$

Then since $\operatorname{diam} \mathcal{C}_n \leq 2/n$ for all $n \geq 1$ we have that $\operatorname{diam} \mathcal{B}_n \leq 2/n$ also and so tends to zero when $n \to \infty$. Clearly $\mathcal{B}_n$ is an increasing sequence of partitions. Hence $\vee_{n\geq 1} \mathcal{B}_n$ generates the σ-algebra $\mathcal{B}$, θ^{ε_k} mod 0 (see e.g. [6, Lemma 3, Chpt. 2]) for all $k \geq 1$. This proves items (1) and (2).

Item (3) of the statement of the lemma is Theorem 12.1 of Billingsley [6]. □

Now we use some properties of conditional entropy to obtain the right inequalities. We start with

$$\begin{aligned} h_{\mu^{\varepsilon_k}} &= h_{\mu^{\varepsilon_k}}(\xi) = h^{\mathcal{B}\times M}_{\theta^{\mathbf{N}}_{\varepsilon_k}\times\mu^{\varepsilon_k}}(S, \Omega\times\xi) \\ &= \inf_{n\geq 1} \frac{1}{n} H_{\theta^{\mathbf{N}}_{\varepsilon_k}\times\mu^{\varepsilon_k}}\left(\bigvee_{j=0}^{n-1} (S^j)^{-1}(\Omega\times\xi) \mid \mathcal{B}\times M \right) \end{aligned}$$

where the first equality comes from the random Kolmogorov-Sinai Theorem 5.3 and the second one can be found in Kifer [9, Thm. 1.4, Chpt. II], with $\Omega \times \xi = \{\Omega \times A : A \in \xi\}$. Hence for arbitrary fixed $N \geq 1$ and for any $m \geq 1$

$$\begin{aligned} h_{\mu^{\varepsilon_k}} &\leq \frac{1}{N} \cdot H_{\theta^{\mathbf{N}}_{\varepsilon_k} \times \mu^{\varepsilon_k}} \left(\bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid \mathcal{B} \times M \right) \\ &\leq \frac{1}{N} \cdot H_{\theta^{\mathbf{N}}_{\varepsilon_k} \times \mu^{\varepsilon_k}} \left(\bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid \mathcal{B}_m \times M \right) \end{aligned}$$

because $\mathcal{B}_m \times M \subset \mathcal{B} \times M$. Now we fix N and m, let $k \to \infty$ and note that since $\mu_0(\partial\xi) = 0 = \delta_{\omega_0}(\partial\mathcal{B}_m)$ it must be that

$$(\delta_{\omega_0} \times \mu_0)(\partial(B_i \times \xi_j)) = 0 \quad \text{for all } B_i \in \mathcal{B}_m \text{ and } \xi_j \in \xi,$$

where δ_{ω_0} is the Dirac mass concentrated at $\omega_0 \in \Omega$. Thus we get by weak* convergence of $\theta^{\mathbf{N}}_{\varepsilon_k} \times \mu^{\varepsilon_k}$ to $\delta_{\omega_0} \times \mu_0$ when $k \to \infty$

$$\limsup_{k\to\infty} h_{\mu^{\varepsilon_k}} \leq \frac{1}{N} \cdot H_{\delta_{\omega_0} \times \mu_0} \left(\bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid \mathcal{B}_m \times M \right) = \frac{1}{N} \cdot H_{\mu_0}\big(\bigvee_{j=0}^{N-1} f^{-j}\xi \big). \tag{5.5}$$

Here it is easy to see that the middle conditional entropy of (5.5) (involving only finite partitions) equals

$$\frac{1}{N} \sum_i \mu_0(P_i) \log \mu_0(P_i),$$

with $P_i = \xi_{i_0} \cap f^{-1}\xi_{i_1} \cap \cdots \cap f^{-(N-1)}\xi_{i_{N-1}}$ ranging over all possible sequences $\xi_{i_0}, \ldots, \xi_{i_{N-1}} \in \xi$. Finally, since N was an arbitrary integer, it follows from (5.3), (5.5) and the Ruelle Inequality that

$$\int \log |f_0'| \, d\mu_0 \leq \limsup_{k\to\infty} h_{\mu^{\varepsilon_k}} \leq h_{\mu_0}(f_0) \leq \int \log |f_0'| \, d\mu_0,$$

showing that μ_0 satisfies the Entropy Formula.

To conclude the proof of Theorem D we observe that μ_0 is f_0-invariant by construction and since it satisfies the Entropy Formula, Theorem B ensures that $\mu_0 = \delta_0$ the Dirac mass at the saddle-node 0.

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