On codimension one foliations with Morse singularities

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Abstract

We study codimension one smooth foliations with Morse singularities. A characterization of the three-sphere in terms of the number of centers and saddles is obtained and generalizations of the classical theorems of Reeb, Haefliger and Novikov are given.

Our purpose is to study the effect of the presence of singularities of Morse type on the global topology of a codimension one foliation defined on a compact manifold of dimension $n \geq 2$. After some preliminaries in § 1 and § 2 we present in § 3 a method of elimination of singularities, via an isotopy of the foliation, in a domain where the sum of the indices is zero. This is based in an index theorem for foliations with isolated singularities that will be used throughout the paper and can be found in the Appendix § 7.

In § 4 we prove generalizations of the Reeb and Milnor topological characterizations of the n-sphere. A sample of our results reads as (Theorem 4.3):

Theorem. *Let $\mathcal{F}$ be a C^∞ codimension one Morse foliation on a closed, connected 3-manifold M^3. Suppose that in the singular set the number k of centers and the number ℓ of saddles satisfy $k \geq \ell + 1$. Then M is homeomorphic to S^3.*

One of the fundamental theorems in codimension one foliation theory on compact manifolds with finite fundamental group is the existence of a leaf with nontrivial holonomy. This is due to Haefliger ([C-LN],[G]). The proof consists of two main steps. First, a closed transverse path to the foliation is found, thus inducing in a 2-disc a pull back foliation by lines with Morse singularities, transverse to the boundary. Then, using Poincaré-Bendixson theorem, one finds a leaf with one-sided nontrivial holonomy. Taking this as a model we consider in § 5 codimension one foliations with Morse singularities, defined on closed manifolds, and transverse to the boundary. We show a generalization of Haefliger's argument to this situation, using strongly the stability theorem of Reeb, as a substitute for the Poincaré-Bendixson arguments. Along the same line of reasoning we present in § 6 a generalization of Novikov's compact leaf theorem for the case of foliations with Morse singularities on 3-manifolds.

1 Preliminaries

Let $\mathcal{F}$ be a codimension one C^∞ foliation on a manifold M and $p \in M$ an isolated singularity of $\mathcal{F}$. We say that $p \in \operatorname{sing}(\mathcal{F})$ is a *Morse type singularity* if there is a neighborhood $p \in U \subset M$ where it is defined a C^∞ function $f \colon U \to \mathbb{R}$ such that $\mathcal{F}\big|_U$ is given by $df = 0$ and p is a non-degenerate critical point of f. We can assume that p is the only singularity of f in U and, by Morse Lemma, we know that we have a coordinate system $(y_1, \ldots, y_n) \in U$ for M such that $y_j(p) = 0$, $\forall\, j$ and $f = f(p) - (y_1^2 + \cdots + y_r^2) + y_{r+1}^2 + \cdots + y_n^2$. The number $r \in \{0, \ldots, n\}$ is the Morse index of f at p. We say that p is a *center* singularity if $r = 0$ or n and p is called a *saddle* singularity otherwise. In a neighborhood of a center the nonsingular leaves of $\mathcal{F}$ are diffeomorphic to $(n-1)$-spheres. Given a saddle singularity $p \in \operatorname{sing}(\mathcal{F})$ we have cone leaves given by the expressions $y_1^2 + \cdots + y_r^2 = y_{r+1}^2 + \cdots + y_n^2 \neq 0$ in a neighborhood of the singular point $p = (0, \ldots, 0)$. These leaves will be called *separatrices* of $\mathcal{F}$ through p. A *saddle connection* for a foliation $\mathcal{F}$ is a leaf L of $\mathcal{F}$ that contains separatrices of two distinct saddle singularities of $\mathcal{F}$. We say that a saddle singularity $p \in \operatorname{sing}(\mathcal{F})$ is *self-connected* if there is a leaf L of $\mathcal{F}$ containing two distinct local branches of separatrices of $\mathcal{F}$ through p.

A codimension one C^∞ foliation $\mathcal{F}$ with isolated singularities on M will be called a *Morse foliation* if each singularity of $\mathcal{F}$ is of Morse type and $\mathcal{F}$ has no saddle connections. The first example of a Morse foliation is given by the levels of a Morse function $f\colon M \to \mathbb{R}$. These functions approximate any given C^∞ function [M] and in this same spirit we can prove that if $\mathcal{F}_0$ is a C^∞ codimension one foliation with isolated singularities on a compact manifold M and suppose that each singularity of $\mathcal{F}_0$ admits a C^∞ local first integral then $\mathcal{F}_0$ can be arbitrarily approximated by Morse foliations on M.

A foliation $\mathcal{F}$ of codimension one and isolated singularities on M will be called *orientable* if there exists a one-form Ω of class C^∞ on M such that $\operatorname{sing}(\mathcal{F}) = \operatorname{sing}(\Omega)$, Ω is integrable in the sense that $\Omega \wedge d\Omega = 0$ everywhere, and $\mathcal{F}$ coincides with the foliation $\Omega = 0$ outside the singular set. The choice of such a one-form Ω is called an *orientation* for $\mathcal{F}$ and two such one-forms Ω and Ω' define the *same* orientation for $\mathcal{F}$ if $\Omega' = h \cdot \Omega$ for some positive function h on M. The foliation $\mathcal{F}$ is called *locally orientable* if each (singular) point $p \in M$ admits a neighborhood where $\mathcal{F}$ is orientable, i.e., given by a one-form Ω_p as above. Clearly a foliation with Morse type singularities on a simply-connected manifold is always orientable.

2 Singular Seifert fibrations

Let M be a compact manifold of dimension $n \geq 3$, connected, possibly with non-empty boundary ∂M. Let $\mathcal{F}$ be a codimension one C^∞ foliation with isolated singularities on M and if $\partial M \neq \emptyset$ we suppose ∂M is either a union of leaves of $\mathcal{F}$ or is everywhere transverse to $\mathcal{F}$. We shall say that the foliation $\mathcal{F}$ is a (singular) *Seifert fibration* of M if its leaves are compact with trivial holonomy group. Assume now that the singularities of $\mathcal{F}$ are of Morse type. We introduce the set $\mathcal{C}(\mathcal{F})$ union of all centers and leaves diffeomorphic to S^{n-1} in M^n. Given any center singularity $p \in \operatorname{sing}(\mathcal{F})$ we denote by $\mathcal{C}_p(\mathcal{F})$ the connected component of $\mathcal{C}(\mathcal{F})$ that contains p.

Remark 2.1. (i) If $q \in \operatorname{sing}(\mathcal{F}) \cap \partial\mathcal{C}(\mathcal{F})$ then q must be a saddle. (ii) $\mathcal{C}(\mathcal{F})$ is open in M as a consequence of Reeb local stability theorem. (iii) $\mathcal{C}_p(\mathcal{F})$ is open in M and $\mathcal{C}_p(\mathcal{F}) \cap \mathcal{C}_q(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{C}_p(\mathcal{F}) = \mathcal{C}_q(\mathcal{F})$. (iv) Since $\mathcal{C}_p(\mathcal{F})$ is open in M we have $\mathcal{C}_p(\mathcal{F}) = M$ if and only if $\partial\mathcal{C}_p(\mathcal{F}) = \emptyset$. In this case the singularities of $\mathcal{F}$ are centers and the leaves diffeomorphic to S^{n-1}. The classification of these foliations is given below:

Proposition 2.1. *Let $\mathcal{F}$ be a Seifert fibration on M tangent to the boundary ∂M if non-empty. Suppose that the singularities of $\mathcal{F}$ are of Morse type. Then we have the following possibilities:*
(i) *$\partial M = \emptyset$, M is homeomorphic to S^n, $\mathcal{F}$ is given by a function $f\colon M^n \to [0,1] \subset \mathbb{R}$ with exactly two critical points and nonsingular levels diffeomorphic to S^{n-1}.*
(ii) *∂M is diffeomorphic to S^{n-1}, M is homeomorphic to closed ball $\overline{B^n}$, $\mathcal{F}$ is given by a function $f\colon M^n \to [0,1] \subset \mathbb{R}$ with exactly one critical point and nonsingular levels diffeomorphic to S^{n-1}.*
(iii) *∂M is diffeomorphic to the disjoint union of two spheres S^{n-1}, M^n is homeomorphic to $S^{n-1} \times [0,1]$, $\mathcal{F}$ is nonsingular given by a function $f\colon M^n \to [0,1] \subset \mathbb{R}$ with no critical points and levels diffeomorphic to S^{n-1}.*
(iv) *$\mathcal{F}$ is nonsingular, $\partial M = \emptyset$ and $\mathcal{F}$ is given by a fibration $f\colon M \to S^1$ with typical fiber a leaf L of $\mathcal{F}$. In particular, if $\mathcal{F}$ has some leaf diffeomorphic to S^{n-1} then M is homeomorphic to $S^{n-1} \times S^1$ and $\mathcal{F}$ is the trivial foliation by spheres $S^{n-1} \times \{y\}, y \in S^1$.*

In the proof we shall use:

Lemma 2.1. *Let $\mathcal{F}$ be a foliation with Morse singularities on M^n, connected, compact with boundary ∂M invariant by $\mathcal{F}$ if non-empty. Suppose $\mathcal{F}$ has all leaves compact then we have the following possibilities:*
(i) *$\partial M = \emptyset$ and $\mathcal{F}$ has exactly two singularities;* (ii) *$\partial M \neq \emptyset$ and $\mathcal{F}$ has exactly one singularity;* (iii) *The foliation $\mathcal{F}$ is nonsingular.*

Proof. Assume that $\partial M = \emptyset$ and $\operatorname{sing}(\mathcal{F}) \neq \emptyset$. Clearly any singularity of $\mathcal{F}$ must be a center for $\mathcal{F}$ has all leaves compact. Take $p \in \operatorname{sing}(\mathcal{F})$ a center singularity. Define $A(p;\mathcal{F})$ as the union of leaves $L \in \mathcal{F}$ such that L is diffeomorphic to S^{n-1} and bounds a region $R(L)$ homeomorphic to the ball $\overline{B^n}$ and such that $p \in R(L)$. Notice that all leaves of $\mathcal{F}$ are diffeomorphic to S^{n-1}, indeed $M = \cup_{p\in\operatorname{sing}(\mathcal{F})}\mathcal{C}_p(\mathcal{F})$ and $A(p;\mathcal{F}) \subseteq \mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$. We claim that:

Claim 2.1. *$A(p;\mathcal{F}) = \mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$.*

Proof. Indeed, $A(p;\mathcal{F})$ is open in $\mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$ by the Local Stability Theorem. To see that $A(p;\mathcal{F})$ is closed in $\mathcal{C}_p(\mathcal{F})\setminus\operatorname{sing}(\mathcal{F})$ we take a sequence of points $x_j \in A(p;\mathcal{F})$ say $x_j \in L_j \subset A(p;\mathcal{F})$ with $x_j \to x_o \in \mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$. Since the leaf $L_o \ni x_o$ is also homeomorphic to S^{n-1} the Local Stability Theorem again shows that necessarily $L_j \to L_o$ and also by the local product structure of $\mathcal{F}$ near L_o we have that L_o bounds a ball $R(L_o)$ containing p and its interior, indeed $R(L_o) = \lim R(L_j)$. Thus $x_o \in L_o \subset A(p;\mathcal{F})$. This implies that $A(p;\mathcal{F}) = \mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$ (recall that by definition $\mathcal{C}_p(\mathcal{F})$ is connected and therefore $\mathcal{C}_p(\mathcal{F}) \setminus \operatorname{sing}(\mathcal{F})$ is also connected).

Claim 2.2. *$\mathcal{C}_p(\mathcal{F}) = M$.*

Proof. Clearly $\mathcal{C}_p(\mathcal{F})$ is open in M. Let now $x_j \in \mathcal{C}_p(\mathcal{F})$ with $x_j \to x_o \in M$. If $x_o \in \operatorname{sing}(\mathcal{F})$ then clearly $x_o \in \mathcal{C}_p(\mathcal{F})$. Assume that $x_o \notin \operatorname{sing}(\mathcal{F})$. Then L_{x_o} is homeomorphic to S^{n-1} and by the Local Stability Theorem we must have $L_{x_j} \to L_{x_o}$ in the Hausdorff topology. This implies that for $j >> 1$ the leaves L_{x_o} and L_{x_j} belong to the same connected component of $\mathcal{C}(\mathcal{F})$ and therefore $L_{x_o} \subset \mathcal{C}_p(\mathcal{F})$ as well. Thus $\mathcal{C}_p(\mathcal{F})$ is also closed in M and since M is connected we have $\mathcal{C}_p(\mathcal{F}) = M$. Thus $A(p;\mathcal{F}) = M \setminus \operatorname{sing}(\mathcal{F})$ and $\operatorname{sing}(\mathcal{F}) = M \setminus A(p;\mathcal{F}) = \partial A(p;\mathcal{F}) \ni p$.

Claim 2.3. *$\partial A(p;\mathcal{F}) = \{p, q\}$ for some point $q \in M \setminus \{p\}$.*

Proof. Suppose that $\partial A(p;\mathcal{F}) = \{p\}$. Then $\mathcal{F}$ has only one singularity on M and is defined by a function $f\colon M \to \mathbb{R}$ with a single critical point. This cannot happen because M is compact with empty boundary. Therefore $\sharp\partial A(p;\mathcal{F}) \geq 2$. Suppose by contradiction that there exist $q_1 \neq q_2$ in $\partial A(p;\mathcal{F}) \setminus \{p\}$. Let $\mathcal{A}(p;\mathcal{F})$ be the collection of leaves that belong to $A(p;\mathcal{F})$. In $\mathcal{A}(p;\mathcal{F})$ we define an order $<$ as follows: Given $L_1, L_2 \in \mathcal{A}(p;\mathcal{F})$ we say that $L_1 < L_2$ if $R(L_1) \subseteq R(L_2)$. By the definition of $A(p;\mathcal{F})$ this order is total (notice that if $L_1 \neq L_2$ then $L_1 \cap L_2 = \emptyset$ and thus $R(L_1), R(L_2)$ are two balls containing p as interior point and therefore either $R(L_1) \subset R(L_2)$ or $R(L_2) \subset R(L_1)$. Thus given $q_j \in \partial A(p;\mathcal{F})$, $q_j \neq p$ $(j = 1, 2)$ we have a sequence $\{L^j_\nu\}_\nu$ such that $L^j_\nu \subset L^j_{\nu+1}$ and $q_j = \lim_{\nu\to\infty} x^j_\nu$ for some points $x^j_\nu \in L^j_\nu$. But we can always compare terms $\{L^1_\nu\}$ with terms $\{L^2_\nu\}$, therefore we may assume that $L^1_\nu < L^2_{\nu+1} < L^1_{\nu+2} < L^2_{\nu+2}$ up to passing to subsequences. On the other hand, since q_j is a center it follows that $L^j_\nu \to q_j$ in the Hausdorff topology and therefore necessarily $q_1 = q_2$. □

Now we assume that $\partial M \neq \emptyset$ and $\operatorname{sing}(\mathcal{F}) \neq \emptyset$. Any singularity of $\mathcal{F}$ is a center and therefore all leaves are diffeomorphic to S^{n-1}. In particular any connected component of the boundary ∂M is diffeomorphic to S^{n-1}. Let $\partial M = L_1 \cup \cdots \cup L_r$ where L_j is a leaf of $\mathcal{F}$ diffeomorphic to S^{n-1} and $L_i \cap L_j = \emptyset$ if $i \neq j$. We attach to M a closed ball B_j diffeomorphic to $\overline{B^n}$, to the boundary

$L_j = B_j$ diffeomorphic to S^{n-1}, and obtain a compact manifold $\overline{M}$, connected and with empty boundary. On B_j we consider a foliation $\mathcal{F}_j$ with a single center singularity p_j and tangent to the boundary ∂B_j. Then $\overline{M}$ is equipped with a codimension one foliation $\overline{\mathcal{F}}$ such that $\overline{\mathcal{F}}\big|_{B_j}$ is conjugate to $\mathcal{F}_j, \forall j$ and $\overline{\mathcal{F}}\big|_M$ is conjugate to $\mathcal{F}$. Using the empty boundary case we conclude that $\sharp\operatorname{sing}(\overline{\mathcal{F}}) = 2$ and therefore since $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ we have $1 \leq r = \sharp\{B_j\} \leq 1$, that is $r = 1$ and $\partial M \simeq S^{n-1}$ and $\sharp\operatorname{sing}(\mathcal{F}) = 1$. This ends the proof of the lemma. □

Proof of Proposition 2.1. We assume $\operatorname{sing}(\mathcal{F}) \neq \emptyset$. Suppose $\partial M = \emptyset$. We have seen above that $\sharp\operatorname{sing}(\mathcal{F}) = 2$, $\operatorname{sing}(\mathcal{F})$ consists of two centers p, q and $M = A(p; \mathcal{F}) \cup \{p, q\}$. Also we have $M = A(q; \mathcal{F}) \cup \{p, q\}$ and therefore $A(p; \mathcal{F}) = A(q; \mathcal{F})$. This shows that M is homeomorphic to S^n and that $\mathcal{F}$ is given by a function $f\colon M^n \to \mathbb{R}$ with exactly two critical points, its maximum and minimum value points in M. Let $[a, b] = f(M)$ then we can assume that $a = 0, b = 1$. The critical values of f are 0 and 1 and we have $\{p, q\} = f^{-1}\{0, 1\}$. This gives case (i).

Suppose now that $\partial M \neq \emptyset$. We have seen that $\operatorname{sing}(\mathcal{F})$ consists of a single center singularity $p \in M$ and ∂M is diffeomorphic to S^{n-1}. We can double $\mathcal{F}$ through the boundary S^{n-1} and apply (i) to obtain (ii). Alternatively we may repeat the argumentation of (i) to conclude this case (ii).

Suppose now that $\mathcal{F}$ is nonsingular and ∂M contains some sphere S^{n-1}. In this case by the classical Global Stability Theorem of Reeb we obtain that $\mathcal{F}$ is a foliation by spheres and ∂M is a union of spheres. The same argument as in the last part of the proof of Lemma 2.1, embedding $(M, \mathcal{F})$ into a pair $(\overline{M}, \overline{\mathcal{F}})$ where $\overline{M}$ is compact with empty boundary, implies that ∂M is a union of two spheres and $\mathcal{F}$ is given by a submersion $f\colon M \to [0, 1]$. The conclusion of (iii) follows. Now, (iv) follows from the Global Stability Theorem. □

Corollary 2.1 ([R2]). *If a closed orientable, connected, manifold M^n of dimension $n \geq 3$ admits a foliation $\mathcal{F}$ such that:* (i) *$\mathcal{F}$ has only Morse singularities and* $\operatorname{sing}(\mathcal{F}) \neq \emptyset$. (ii) $\operatorname{sing}(\mathcal{F})$ *consists of centers or, equivalently, $\mathcal{F}$ is a foliation by compact leaves. Then M^n is homeomorphic to the sphere S^n.*

Regarding the case when the boundary is transverse to the foliation, we have,

Proposition 2.2. *Let $M^n, n \geq 2$, be a compact, connected manifold with connected boundary ∂M. Let $\mathcal{F}$ be a C^∞ codimension one foliation on M, transverse to ∂M. Suppose* $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ *consists only of center singularities. Then:* (i) *$\mathcal{F}$ is a Seifert fibration for $n \geq 3$.* (ii) *The restriction $\mathcal{F}\big|_{\partial M}$ defines a* (nonsingular) *fibration $\partial M \to S^1$. In particular, if M is the closed ball $\overline{B^n}$ then $n = 2$.*

Proof. We may assume that $n \geq 3$, otherwise the manifold is the closed 2-disc by Proposition 2.1. Given a center $p_0 \in \operatorname{sing}(\mathcal{F}) \subset M \setminus \partial M$ the leaves of $\mathcal{F}$ in a punctured neighborhood of p_0 in B^n are diffeomorphic to $(n-1)$-spheres. For $n > 2$ these leaves are simply-connected. Applying Reeb's Global Stability Theorem for nonsingular foliations transverse to the boundary of a compact manifold ([G]) we conclude that all the leaves of $\mathcal{F}$ are compact with finite holonomy group and in particular the restriction $\mathcal{F}\big|_{\partial M}$ is a foliation by compact leaves with finite holonomy and therefore it is given by a submersion $f_0\colon \partial M \to S^1$. By Ehresmann's theorem f_0 must be a fibration. On the other hand, if $n \geq 3$ and ∂M is diffeomorphic to S^{n-1} then Haefliger's Theorem implies the existence of a leaf with nonperiodic holonomy, a contradiction. □

3 Dead branches, pairing and elimination of pairs of singularities

In this section we shall see how to perform modifications on foliations, under suitable conditions, in order to eliminate certain arrangements of singularities.

3.1 Trivial center-saddle pairings

In dimension two our basic picture is the following:

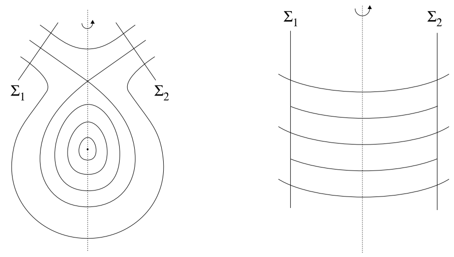

Figure 1

We have a pair center-saddle that is replaced by a trivial foliation. The replacement of a pairing center-saddle as above does not change the holonomy of the foliation. In dimension $n = 3$ we have the same construction which can be obtained from the two-dimensional case by rotation as in the figure. The final result is a pairing center-saddle called *trivial center-saddle pairing*.

3.2 Non-trivial center-saddle pairings

This is an example in $\mathbb{R}^3$ of a combination of a center-saddle pairing where the saddle is also accumulated by spherical leaves from a third singularity, of center type. We begin with a foliation given by a quadratic center and by an inverse modification we introduce in a regular part a pair center-saddle as depicted below:

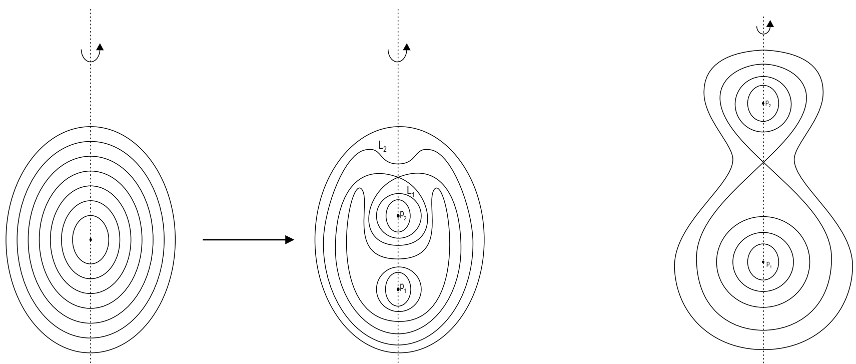

Figure 2

The separatrix of the saddle has the topology of two spheres with a unique intersection point. All other leaves are diffeomorphic to spheres and if we consider only the annular region bounded by one internal leaf L_1 and one external leaf L_2 as in the figure, then we have a non-trivial center-saddle pairing. This example can be completed to S^3 by putting a center at infinity. Another example can be obtained in $\mathbb{R}^3$ by taking the center p_1 to infinity in S^3 and infinity to a finite point p_2'. Figure 2 shows this example.

3.3 Singular Reeb foliations

We shall now construct two analogous of the Reeb foliation on the solid torus. The first one $\mathcal{F}_1$ exhibits two Morse singularities in a center-saddle combination. We begin with a quadratic center at $0 \in \mathbb{R}^3$ defined on the ball $\mathbb{B}^3$. Then we pick two different points $q_1, q_2, \in S^2 = \partial\mathbb{B}^3$ and identify them so as to become a singularity q of saddle type.

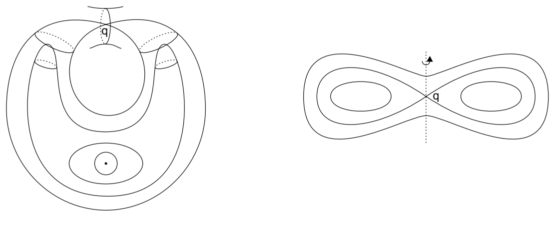

Figure 3

The 2-sphere becomes a self connecting separatrix homeomorphic to a 2-torus where a meridian was identified to a point q. We can extend the foliation to the exterior of the singular solid torus with trivial holonomy and leaves diffeomorphic to $S^1 \times S^1$. We suppose the center of $\mathcal{F}_1$ is at $0 \in \mathbb{R}^3$. The second example, $\mathcal{F}_2$, is obtained by rotation of a mirror symmetric figure eight (see Fig. 3). This generates a singular surface T homeomorphic to a 2-torus with a parallel identified to a point p. This surface T bounds a solid torus $\mathbb{T}^3$. In a neighborhood of $T \subset \mathbb{T}^3$ we define a trivial foliation whose leaves are 2-torus bounding a standard Reeb foliation. Outside $\mathbb{T}^3$ all leaves are homeomorphic to S^2. Adding a center at infinity we can assume that this example is defined on S^3. It is clear that $\overline{\mathcal{C}_0(\mathcal{F}_1)}$ and $\overline{\mathcal{C}_\infty(\mathcal{F}_2)}$ are homeomorphic.

3.4 Dead branches

Motivated by the examples we have given above we define:

Definition 3.1. Let $\mathcal{F}$ be a codimension one foliation with isolated singularities on a manifold M^n. By a *dead branch* of $\mathcal{F}$ we mean a region $R \subset M$ homeomorphic to the ball B^n and which is a manifold with corners whose boundary ∂R is the union of connected invariant components (pieces of leaves of $\mathcal{F}$) and of totally transverse curves (segments transverse to $\mathcal{F}$) say $\partial R = P_1 \cup P_2 \cup \Sigma_1 \cup \Sigma_2$ where P_j is $\mathcal{F}$-invariant, Σ_j is transverse to $\mathcal{F}$ as in the picture.

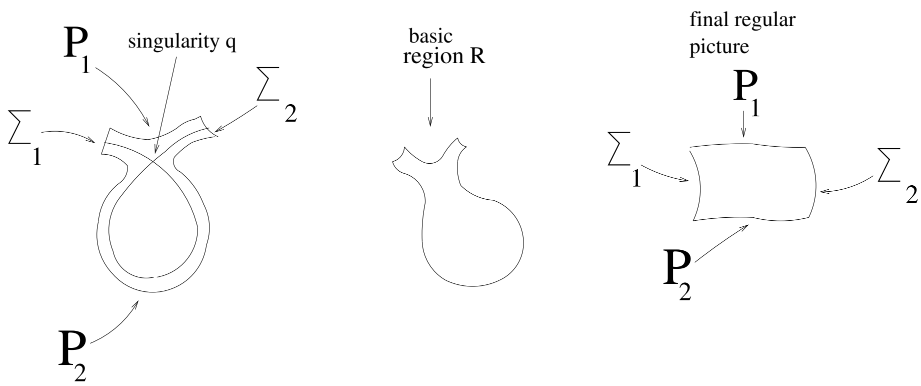

Figure 4

We do not exclude the possibility $P_1 = P_2$ a priori. Moreover we also assume that the holonomy form Σ_1 to Σ_2 is *trivial* in the sense that $\mathcal{F}\big|_{\Sigma_1}$ and $\mathcal{F}\big|_{\Sigma_2}$ are conjugate by a diffeomorphism $h\colon \Sigma_1 \to \Sigma_2$ such that $L_{h(p)} = L_p \quad \forall\, p \in \Sigma_1$ except if p belongs to a leaf which is a separatrix of some singularity of $\mathcal{F}$ in R, in which case we ask the image of p to be another point $h(p)$ belonging to a leaf which is a separatrix of the same singularity of $\mathcal{F}$ in R.

Proposition 3.1. *Let $\mathcal{F}$ be given on M having a dead branch $R \subset M$. Then there is a foliation $\widetilde{\mathcal{F}}$ on M such that:* (i) *$\widetilde{\mathcal{F}}$ and $\mathcal{F}$ agree on $M \backslash R$.* (ii) *$\widetilde{\mathcal{F}}$ is nonsingular in a neighborhood of R; indeed $\widetilde{\mathcal{F}}\big|_R$ is conjugate to a trivial fibration.* (iii) *The holonomy of $\widetilde{\mathcal{F}}$ is conjugate to the holonomy of $\mathcal{F}$ in the following sense: given any leaf L of $\mathcal{F}$ such that $L \cap (M \backslash R) \neq p$ then the corresponding leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ satisfies* $\mathrm{Hol}(\widetilde{\mathcal{F}}, \widetilde{L})$ *is conjugate to* $\mathrm{Hol}(\mathcal{F}, L)$.

We shall refer to $\widetilde{\mathcal{F}}$ as a *direct modification* of $\mathcal{F}$ by elimination of the dead branch R. If a foliation $\mathcal{F}$ is obtained from a foliation $\widetilde{\mathcal{F}}$ by introduction of a dead branch then we shall say $\mathcal{F}$ is an *inverse modification* of $\widetilde{\mathcal{F}}$ by introduction of the dead branch R. Two singularities p, q of a foliation $\mathcal{F}$ on M are said to be in *trivial coupling* or *trivial pairing* if they belong to a dead branch R of $\mathcal{F}$ and $\mathcal{F}$ has no other singularities in R.

The examples above are not the unique examples of pairings of singularities in a dead branch, indeed we can construct pairings of two saddles of complementary indices.

3.5 Regularization of singular Reeb foliations

Now we show how to transform a singular Reeb foliation into a regular foliation. Given a singular Reeb foliation $\mathcal{F}$ we can assume that the center and the saddle are close as in the picture below. In a small box B around these two singularities we have the following picture. We may therefore replace the fibration $\mathcal{F}$ in the box B by a regular foliation as indicated. This corresponds to the following foliation leading to a Reeb component.

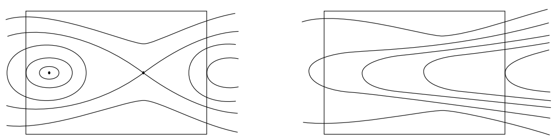

Figure 5

3.6 Pairings in dimension three

For the rest of this section we assume $\dim M = 3$.

Lemma 3.1 (Topology of separatrices). *Let $\mathcal{F}$ be a Morse foliation on a compact 3-manifold M^3. If $p \in \mathrm{sing}(\mathcal{F})$ is a center and $\partial C_p(\mathcal{F}) \neq \emptyset$, then $\mathrm{sing}(\mathcal{F}) \cap \partial \mathcal{C}_p(\mathcal{F}) = \{q\}$ is a saddle point. Moreover we have the following possibilities for $\mathcal{C}_p(\mathcal{F})$ and $\partial \mathcal{C}_p(\mathcal{F})$:*
(i) *$\partial \mathcal{C}_p(\mathcal{F}) \setminus \{q\}$ is connected. Then*
(a) *$\partial \mathcal{C}_p(\mathcal{F})$ is homeomorphic to a sphere S^2 with a pinch at q and the pair $q - p$ belongs to a dead branch pairing, i.e. it can be modified to a trivial foliation; or*
(b) *$\partial \mathcal{C}_p(\mathcal{F})$ is homeomorphic to a singular torus obtained by pinching an sphere at two points and joining these points., equivalently $\mathcal{C}_p(\mathcal{F})$ is homeomorphic to a singular torus with a pinching along a meridian. The foliation has a singular Reeb component.*
(ii) *$\mathcal{C}_p(\mathcal{F})$ has two connected components. Then $\partial \mathcal{C}_p(\mathcal{F})$ is the union of two spheres S^2 with a common point q. In this case $\mathcal{C}_p(\mathcal{F})$ is homeomorphic to the example in figure 2.*

Proof. $\partial C_p(\mathcal{F})$ is compact, connected and invariant, i.e. a union of leaves and singularities. By the Stability theorem of Reeb no leaf in $\partial C_p(\mathcal{F})$ can be compact; otherwise it would be homeomorphic to S^2 and therefore contained in $\mathcal{C}_p(\mathcal{F})$. Thus $\mathrm{sing}(\mathcal{F}) \cap \partial C_p(\mathcal{F}) \neq \emptyset$ and, as there are no saddle connections, then $\mathrm{sing}(\mathcal{F}) \cap \partial \mathcal{C}_p(\mathcal{F}) = \{q\}$ is a singular point of saddle type. If $\partial \mathcal{C}_p(\mathcal{F}) \setminus \{q\}$ is connected then it is homeomorphic to a disc or to a cylinder. In other words, $\partial \mathcal{C}_p(\mathcal{F})$ is obtained by taking a disc and identifying its boundary to a point q (case (i)-(a)) or by taking a cylinder and identifying its two ends to a point q (case (i)-(b)).

If $\partial \mathcal{C}_p(\mathcal{F}) \setminus \{q\}$ is not connected then it must have two connected components. Each one of these components is homeomorphic to a disc, thus $\partial \mathcal{C}_p(\mathcal{F})$ is a region bounded by two spheres touching at one point q. □

4 Variations on a Theorem of Reeb

One of the very first applications of Morse theory is the following theorem of Reeb:

Theorem 4.1 (Reeb, [R1]). *Let M^n be a n-dimensional compact manifold admitting a C^∞ function having only two critical points, both of which are non-degenerate. Then M is homeomorphic to the n-sphere.*

Remark 4.1. (i) This result also holds if the critical points are degenerate ([M1]), however with a difficult proof. (ii) The manifold M is not necessarily diffeomorphic to a sphere with the original differentiable structure. (iii) The function f defines a foliation with exactly two center singularities and compact leaves on M.

Reeb's Theorem above can be generalized as follows:

Proposition 4.1. *Let $\mathcal{F}$ be a codimension one foliation with Morse singularities on a compact manifold M^n. Suppose* $\mathrm{sing}(\mathcal{F})$ *consists of two center type singularities and M is oriented. Then M is homeomorphic to a n-sphere.*

Proof. If $n = 1$ then M is homeomorphic to S^1 since it is compact. If $n = 2$ then $\mathcal{F}$ has exactly two singularities of index $+1$ and therefore by the Index Theorem we have $\chi(M) = +2$. So M is homeomorphic to S^2. If $n \geq 3$ this result follows immediately from Proposition 2.1. □

In order to present other generalizations of Reeb's Theorem we shall first introduce a new concept. Let $\mathcal{F}$ be a C^∞ codimension one foliation on M and $p \in \mathrm{sing}(\mathcal{F})$ an isolated singularity of $\mathcal{F}$. We shall say that p is a *stable singularity* of $\mathcal{F}$ if there is a neighborhood U of p in M where $\mathcal{F}$ is given by the levels of a C^∞ function $f\colon U \to \mathbb{R}$ and such that $f(p) = 0$ and the level hypersurfaces $\{f = a\}$ are compact if $|a| > 0$ is small enough. In other words, p has a fundamental system of neighborhoods bounded by compact leaves of $\mathcal{F}$. The first examples of these singularities are centers, however we can give other kind of stable singularities.

Example 4.1. (i) Let $f = \sum\limits_{j=1}^{n} x_j^{m_j}$ in $\mathbb{R}^n$. Then the origin is a stable singularity if, and only if, $m_j \geq 2$ is even $\forall\, j \in \{1, \ldots, n\}$. (ii) The function $f = \exp\left(\frac{-1}{x^2+y^2}\right)$ defines a stable singularity and its Taylor polynomial at the origin $0 \in \mathbb{R}^2$ is identically zero.

A simpler characterization of stable singularities is as follows:

Lemma 4.1. *An isolated singularity p of a function $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ defines a stable singularity if, and only if, there exists a neighborhood $p \in V \subset U$ such that either $\forall\, x \in V$ we have $\omega(x) = \{p\}$ or $\alpha(x) = \{p\}$, where $\omega(x)$ respectively $\alpha(x)$ is the ω-limit respectively the α-limit set of the orbit of the vector field* $\mathrm{grad}(f)$ *through the point x.*

As an immediate consequence we obtain the following well-known result:

Lemma 4.2. *If a function $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ has an isolated local maximum or minimum at $p \in U$ then p is a stable singularity for df.*

Example 4.2. Let us consider perturbations of the Monkey saddle singularity $f_0 = x^3 - 3xy^2$. For $\varepsilon \in \mathbb{R}$ we set $f = f_0 + \varepsilon(x^2 + y^2)$ obtaining $\operatorname{grad} f = (3x^2 - 3y^2 + 2\varepsilon x, -6xy + 2\varepsilon y)$. Thus f has 3 singular points $(0,0)$, $\left(\frac{\varepsilon}{3}, \pm\frac{\varepsilon}{\sqrt{3}}\right)$. The origin is a center and the other two singularities are saddles. This does not occur when perturbing stable singularities:

Lemma 4.3. *Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with a stable singularity at the point $p \in U$. Then we can perturb f to obtain a function $\tilde{f}$ with a Morse center type singularity and no other singularity in a neighborhod of p.*

Proof. We can assume that $p = 0$ is the origin and $f(0) = 0$. There are two cases to consider:
$1^{\textbf{st}}$ **case**. The Taylor polynomial of f at $0 \in \mathbb{R}^n$ is non-trivial. In this case we have a first non-trivial jet for f at 0 and since the origin is stable we must have this jet of the form $\pm \sum\limits_{j=1}^{n} a_j\, x_j^{2m_j}$ where $a_j > 0$, $\forall\, j \in \{1, \dots, n\}$ and $m_j \in \mathbb{N}$. If $m_j = 1$, $\forall\, j$ then the origin is already a Morse singularity. By adding to f terms of the form $\pm\varepsilon \sum\limits_{j=1}^{n} x_j^2$ for $\varepsilon > 0$ small we may perturb f obtaining a Morse center at $0 \in \mathbb{R}^n$ and no new singularities.
$2^{\textbf{nd}}$ **case**. f is flat at the origin, that is, all derivatives of f vanish at 0. In this case we put $\tilde{f} = f + \varepsilon \sum\limits_{j=1}^{n} x_j^2$. Since $\lim\limits_{x_j \to 0} \frac{f(x_1,\dots,x_n)}{x_j^k} = 0$, $\forall\, k \in \mathbb{N}$ the function $\tilde{f}$ has a unique singular point in a neighborhood of the origin and also this is a Morse center for $\tilde{f}$. □

Using the preceding results we obtain:

Proposition 4.2. *Let $\mathcal{F}$ be a C^∞ codimension one foliation on a compact manifold M. Suppose the singularities of $\mathcal{F}$ are stable. Then $\mathcal{F}$ can be arbitrarily approximated by Morse foliations $\mathcal{F}'$ having only center singularities and such that $\#\operatorname{sing}(\mathcal{F}) = \#\operatorname{sing}(\mathcal{F}')$.*

Theorem 4.2. *Let M^n be a closed n-dimensional manifold, $n \geq 3$. Suppose M supports a C^∞ codimension one foliation $\mathcal{F}$ with non-empty singular set all of whose singularities are stable. Then M is homeomorphic to the sphere S^n.*

Proof. We perturb $\mathcal{F}$ into a foliation $\mathcal{F}'$ having only center singularities and apply Corollary 2.1. □

In particular we reobtain Milnor's generalization of Reeb theorem: Given a C^∞ function $f\colon M \to \mathbb{R}$ with only two critical points. Since they are a maximum and a minimum of $\mathcal{F}$ they are stable singularities. The foliation $\mathcal{F}$ induced by f has therefore exactly two singularities both of which are stable. Now we can easily conclude by applying Theorem 4.2 above. □

4.1 The Center-Saddle Theorem

Now we proceed to search for further generalizations of Reeb theorem in the presence of saddles or, more precisely, in terms of comparisons between the number of centers and saddles for a given foliation on the manifold. Let M^n be a compact manifold supporting a nonsingular C^∞ codimension one foliation $\mathcal{F}$ (e.g., if M is odd-dimensional). Then by our standard modification procedure we can obtain a foliation $\widetilde{\mathcal{F}}$ on M having as singular set $k \geq 1$ centers and $k \geq 1$ saddles. Nevertheless

M is not necessarily homeomorphic to S^n. This indicates that, a priori, the topology of M is not determined by the equality $\#\{\text{centers}\} = \#\{\text{saddles}\}$ for a given foliation on M. An example of a manifold that admits a C^∞ function $f\colon M \to \mathbb{R}$ having only three critical points of indices 0, 4 and 2 is the complex projective plane $\mathbb{C}P(2)$. Therefore $M^4 = \mathbb{C}P(2)$ admits a foliation with exactly two centers and one saddle, though M^4 is not homeomorphic to S^4. Thus, in general, the inequality $\#\{\text{centers}\} \geq \#\{\text{saddles}\} + 1$ also does not imply M is homeomorphic to a sphere. Notice that if $\dim M = 4$ then $\chi(M) = \#\{\text{centers}\}+\#\{\text{saddles}\}$, hence the inequality $\#\{\text{centers}\} \geq \#\{\text{saddles}\} + 1$ implies $\chi(M) \geq 2\#\{\text{saddles}\} + 1 \geq 3$ if $\mathcal{F}$ has some saddle. Therefore in this case M^4 is *never* homeomorphic to S^4.

However, despite the above remarks, we can prove for dimension three a generalization of Reeb theorem as follows:

Theorem 4.3 (Center-Saddle Theorem). *Let $\mathcal{F}$ be a C^∞ codimension one Morse foliation on a closed 3-manifold M^3. Suppose that the number k of centers and the number ℓ of saddles in $\operatorname{sing}(\mathcal{F})$ satisfy $k \geq \ell + 1$. Then M is homeomorphic to S^3.*

For the proof of this theorem we need some preliminary results besides Lemma 3.1:

Lemma 4.4. *Let $\mathcal{F}$ be a codimension one foliation of Morse type on a 3-manifold M^3. Let $q \in \operatorname{sing}(\mathcal{F})$ be a saddle such that $q \in \partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \partial\mathcal{C}_{p_2}(\mathcal{F})$ for two distinct centers $p_1, p_2 \in \operatorname{sing}(\mathcal{F})$. Then the union of the separatrices of $\mathcal{F}$ through q with $\{q\}$ is compact with each branch homeomorphic to S^2 and q belongs to a dead branch with a pairing $q - p_1$ or $q - p_2$.*

Proof. Let $q \in \partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \partial\mathcal{C}_{p_2}(\mathcal{F})$ be a saddle where $p_1 \neq p_2$ are centers. We first prove that $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is not a singular torus. Suppose on the contrary that $\Gamma_q = \partial\mathcal{C}_{p_1}(\mathcal{F})$ is a singular torus. Then $L_q = \Gamma_q \setminus \{q\}$ is a leaf homeomorphic to a cylinder. We fix a small closed disc Σ transverse to $\mathcal{F}$ and such that $\Sigma \cap \Gamma_q = \{q\}$, with boundary $\gamma = \partial\Sigma$ diffeomorphic to S^1. The existence of this disc is a consequence of the local normal form of $\mathcal{F}$ close to q. Notice that, since Γ_q is accumulated on both sides by compact leaves (spheres) it follows that the leaf L_q has trivial holonomy. We can assume that γ is contained in a leaf L_0 of $\mathcal{F}$ diffeomorphic to S^2 and that Σ is arbitrarily small. Since L_0 is diffeomorphic to S^2 the loop γ bounds a disc D_{L_0} in L_0. By triviality of the holonomy D_{L_0} projects normally (i.e., along the gradient vector field of the 1-form defined by the foliation, cf.§ 7) into a disc D_{L_q} in L_q. By the local description of $\mathcal{F}$ around q and by the choice of Σ and γ, the boundary of D_{Γ_0} is a meridian ∂D_{L_q} in the torus Γ_q.

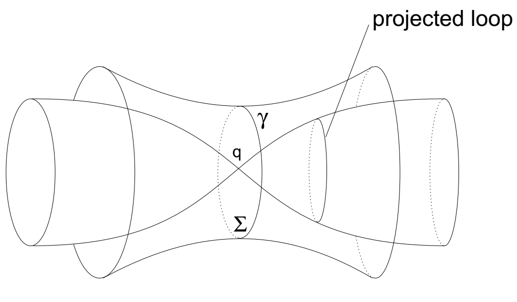

Figure 6

This gives a contradiction because L_q is a cylinder and ∂D_{L_q} is simultaneously a meridian in L_q and bounds a disc in L_q. Therefore the only possibility is to have $\partial\mathcal{C}_{p_1}(\mathcal{F})$ and $\partial\mathcal{C}_{p_2}(\mathcal{F})$ homeomorphic to S^2 and a pairing $q - p_1$ or $q - p_2$ in a dead branch. $\square$

Proof of Theorem 4.3. We will proceed by induction on the number ℓ of saddle singularities. If $\ell = 0$ then $\mathcal{F}$ has only centers and the result follows from Corollary 2.1. Assume now that $\ell \geq 1$

and that the result has been proved for foliations with at most $\ell - 1$ singularities of saddle type. By hypothesis $\mathcal{F}$ has some center type singularity, say $p_1 \in \operatorname{sing}(\mathcal{F})$. If $\mathcal{C}_{p_1}(\mathcal{F}) = M$ then the theorem follows as remarked in §2. Thus we may assume that $\partial\mathcal{C}_{p_1} \neq \emptyset$ and by Reeb Stability Theorem necessarily we must have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \operatorname{sing}(\mathcal{F}) \neq \emptyset$, indeed any leaf $L \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$ must be a separatrix of some saddle singularity $q_1 \in \operatorname{sing}(\mathcal{F})$. This singularity is unique for any fixed leaf L because $\mathcal{F}$ has no saddle connections. According to Lemma 4.4 either $q_1 \notin \partial\mathcal{C}_{p_1'}(\mathcal{F})$ for any center singularity $p_1' \neq p_1$ or q_1 belongs to a dead branch associated to a pairing $q_1 - p_1'$ for some center $p_1' \in \operatorname{sing}(\mathcal{F})$ possibly $p_1' = p_1$. In the first case we call p_1 *single*. In this last case we can perform a modification of $\mathcal{F}$ eliminating two singularities, one center and the saddle q_1. On the other hand, since the number of centers is greater than the number of saddles, then not all the centers are single. Thus necessarily we have the last case above occurring for some suitable choice of the center p_1. Therefore we can always perform the modification and, since for $\mathcal{F}$ the number of centers is greater than the number of saddles, we conclude that the same holds for the modification of $\mathcal{F}$. By the induction hypothesis the manifold M is homeomorphic to S^3. $\square$

5 Variations on a Theorem of Haefliger

Let us study the existence and properties of foliations transverse to spheres. We shall begin with the most simple situation: Let $\mathcal{F}$ be a C^∞ codimension one foliation of Morse type defined in a neighborhood W of the closed ball $\overline{B^n} = \overline{B^n(0;1)}$ in $\mathbb{R}^n$ and transverse to the boundary sphere $S^{n-1} = \partial\overline{B}^n = S^{n-1}(0;1)$. Since $\overline{B^n}$ is simply-connected we can obtain a one-form Ω that defines $\mathcal{F}$ in W fixing the orientation of $\mathcal{F}$. Given any singularity $p \in \operatorname{sing}(\mathcal{F}) \subset B^n$ we have local coordinates $(y_1, \ldots, y_n) \in U_p \subset B^n$ such that $\Omega(y_1, \ldots, y_n) = h_p\, d(-y_1^2 - \cdots - y_{r_p}^2 + y_{r_p+1}^2 + \cdots + y_n^2)$, for a C^∞ function $h_p > 0$ in U_p. We have defined the *index* of $\mathcal{F}$ at p with respect to the orientation defined by Ω as $\operatorname{Ind}_\Omega(\mathcal{F}; p) = (-1)^{r_p} \in \{+1, -1\}$. By the Index Theorem (see § 7-Appendix) we have $\sum\limits_{p \in \operatorname{sing}(\mathcal{F})} \operatorname{Ind}_\Omega(\mathcal{F}; p) = +1$, in particular $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ and $\mathcal{F}$ has an odd number of singularities in the ball. Since the boundary sphere admits a transverse foliation we have $\chi(S^{n-1}) = 0$ and therefore n is an even number. In particular, in this case, the index $\operatorname{Ind}_\Omega(\mathcal{F}; p)$ does not depend on the orientation fixed for $\mathcal{F}$. Thus a center singularity always has index $+1$, however a saddle may have index $+1$. If $n = 2$ then $\mathcal{F}$ has some center singularity because in dimension two a saddle has index -1. From Proposition 2.2, if $n > 2$ then $\operatorname{sing}(\mathcal{F})$ must contain a saddle. The following example illustrates this last situation:

Example 5.1 (a 2-2 saddle in the closed 4-ball). The following is an example of a codimension one C^∞ foliation in the ball $\mathbb{B}^4$, of radius one centered at $0 \in \mathbb{R}^4$, with only one singularity of saddle type $2-2$ at $0 \in \mathbb{B}^4$ and transverse to the boundary $S^3 = \partial\mathbb{B}^4$. Consider in $\mathbb{R}^4$ the function $f(x) = -x_1^2 - x_2^2 + x_3^2 + x_4^2$. The level zero of this function, $C = f^{-1}(0)$, is a cone over a 2-torus. This can easily be seen by taking the intersection $T = C \cap S^3$ which is clearly a 2-torus, intersection of the cylinders $x_1^2 + x_2^2 = 1/2$ and $x_3^2 + x_4^2 = 1/2$. Given $\varepsilon > 0$, $f^{-1}([-\varepsilon, \varepsilon])$ is a neighborhood of C and $\mathbb{R}^4 \backslash f^{-1}([-\varepsilon, \varepsilon])$ is the union of two connected components R_1 and R_2, $R_1 \cap \{x_3 = x_4 = 0\} \neq \emptyset$, $R_1 \cap \{x_1 = x_2 = 0\} = \emptyset$. Moreover R_1 and R_2 are diffeomorphic to $\mathbb{B}^3 \times S^1$. For $\varepsilon > 0$ small enough $R_1 \cap \mathbb{B}^4$ and $R_2 \cap \mathbb{B}^4$ are nonempty and $S^3 \backslash f^{-1}((-\varepsilon, \varepsilon))$ is a union of two solid tori, i.e., diffeomorphic to $\mathbb{B}^2 \times S^1$. We define a new domain $\mathbb{D} = f^{-1}((-\varepsilon, \varepsilon)) \cup S_1 \cup S_2$ where $S_1 \subset R_1$ and $S_2 \subset R_2$ are diffeomorphic to $\mathbb{B}^3 \times S^1$ and such that $\partial S_1 \cap \mathbb{B}^4 = \partial R_1 \cap \mathbb{B}^4$ and similarly $\partial S_2 \cap \mathbb{B}^4 = \partial R_2 \cap \mathbb{B}^4$. We define on $\mathbb{D}$ a foliation $\mathcal{F}$ that on $f^{-1}((-\varepsilon, \varepsilon))$ has as leaves the levels of f. On S_1 we plug in a Reeb component on $\mathbb{B}^3 \times S^1$, having as sections on each $\mathbb{B}^3 \times \{\theta\}$ a foliation by 2-spheres, taking as axis of the solid torus the circle $(x_3 = x_4 = 0) \cap S^3$. Similarly on

S_2 we introduce a Reeb component taking as axis of the torus $\mathbb{B}^3 \times S^1$ the circle $(x_1 = x_2 = 0) \cap S^3$. Clearly the leaves of $\mathcal{F}$ are transverse to the 3-sphere S^3. We finally take the restriction $\mathcal{F}\big|_{\mathbb{B}^4}$.

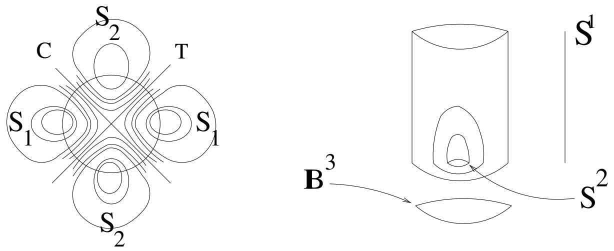

Figure 7

5.1 Haefliger's Theorem for the disc

The classical Haefliger's theorem for the disc states that if a C^1-vector field X defined in a neighborhood of a disc $D \subset \mathbb{R}^2$ points inward the disc from the boundary and has only Morse singularities without saddle-connections in D then there is a compact invariant one dimensional subset $\Gamma \subset D$ (Γ is either a periodic orbit or a graph of X) whose corresponding holonomy map is conjugate to a germ of diffeomorphism $h\colon \mathbb{R}, 0 \to \mathbb{R}, 0$ such that $h\big|_{(-\varepsilon,0]}$ is the identity and $h\big|_{(0,\varepsilon)}$ is not the identity for some $\varepsilon > 0$. This implies the following:

Theorem 5.1 (Haefliger, [H1],[H2]). *A codimension one regular foliation $\mathcal{F}$ of class C^2 on a manifold M has some leaf with one-sided holonomy provided that it has some null-homotopic closed transversal. This is always the case if M is compact with finite fundamental group.*

Let us show how our notions of dead-branch and modification can be used to simplify slightly the proof of Haefliger's theorem. We consider the following situation: $\mathcal{F}$ is a C^2 Morse foliation in a neighborhood W of the *closed* disc $D \subset \mathbb{R}^2$ and transverse to the boundary $\partial D \simeq S^1$. As we have seen $\mathcal{F}$ is orientable and we can choose a C^2 vector field X in W that is tangent to $\mathcal{F}$, $\operatorname{sing}(X) = \operatorname{sing}(\mathcal{F})$ and X points inward D from the boundary ∂D. By the classical Poincaré-Hopf Index Theorem we have $\sum\limits_{p\in \operatorname{sing}(\mathcal{F})} \operatorname{Ind}(X,p) = +1$ and therefore $\mathcal{F}$ must have some center singularity. Indeed we have $k = \ell + 1$ where $k = \#\{\text{centers}\}$ and $\ell\#\{\text{saddles}\}$. Denote by $\mathcal{C}(\mathcal{F})$ the union of all centers and periodic orbits *having trivial holonomy* for the vector field X in D. Notice that $\mathcal{C}(\mathcal{F})$ is an open subset of D and $\mathcal{C}(\mathcal{F})$ avoids a certain neighborhood of ∂D in $\mathbb{R}^2$ (compare with the beginning of § 2). Denote by $\mathcal{C}_p(\mathcal{F})$ the connected component of $\mathcal{C}(\mathcal{F})$ that contains the center $p \in \operatorname{sing}(\mathcal{F}) \subset D$. Then $\mathcal{C}_p(\mathcal{F})$ is a non-empty open subset of $\mathcal{C}(\mathcal{F})$ and of D. Let us proceed again by induction on ℓ. If $\ell = 0$ then $\mathcal{F}$ has only one center singularity $p_1 \in D$. In particular, $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \operatorname{sing}(\mathcal{F}) = \emptyset$. In this case $\partial\mathcal{C}_{p_1}(\mathcal{F})$ must be a periodic orbit with non-trivial holonomy map: indeed, by Poincaré-Bendixson theorem $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is a periodic orbit of X because it contains no singularity of $\mathcal{F}$. We claim that the holonomy of the periodic orbit $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is not trivial and therefore unilateral. Otherwise in a small neighborhood of $\partial\mathcal{C}_{p_1}(\mathcal{F})$ in D^2 every orbit of X is periodic with trivial holonomy and therefore $\partial\mathcal{C}_{p_1}(\mathcal{F})$ must be contained in $\mathcal{C}_{p_1}(\mathcal{F})$, absurd. Assume now that $\ell \geq 1$ and that the result has been proved for foliations with $\ell - 1$ saddles. Take a center $p_1 \in \operatorname{sing}(\mathcal{F})$. We have two cases to consider:

1st case. $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \operatorname{sing}(\mathcal{F}) = \emptyset$. According to what we have seen above, $\partial\mathcal{C}_{p_1}(\mathcal{F})$ must be a periodic orbit with unilateral holonomy.

2$^{\text{nd}}$ case. $\partial\mathcal{C}_{p_1}(\mathcal{F})\cap\text{sing}(\mathcal{F}) \neq \emptyset$. By the argument of the first case we can assume that every leaf in $\partial\mathcal{C}_{p_1}(\mathcal{F})$ accumulates on $\text{sing}(\mathcal{F})$. Since there are no saddle connections we have $\text{sing}(\mathcal{F})\cap\partial\mathcal{C}_{p_1}(\mathcal{F}) = \{q_1\}$ which is a saddle point. We consider the separatrices of $\mathcal{F}$ at q_1 . At least one separatrix S_1 is contained in $\partial\mathcal{C}_{p_1}(\mathcal{F})$ and since there are no saddle-connections we conclude that S_1 is also a self-connection separatrix of q_1 so that necessarily $S_1 \cup \{q_1\}$ is a graph of $\mathcal{F}$. If S_1 is the only separatrix contained in $\partial\mathcal{C}_{p_1}(\mathcal{F})$ then we have a trivial dead-branch pairing $p_1 - q_1$. Suppose therefore that $\partial\mathcal{C}_{p_1}(\mathcal{F})$ consists of the union $S_1 \cup \{q_1\} \cup S_2$ of q_1 with two separatrices of q_1. In this case $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is a graph for which there is a well-defined holonomy. If the holonomy of the graph $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is not trivial then it is unilateral and we are done. If the holonomy is trivial then we have an annular region $A \subset D$ bounded by circle leaves L_1 and L_2 (see figure 2) where we can perform a modification of $\mathcal{F}$ to a trivial foliation by circles. In the trivial holonomy case we obtain by direct modification of $\mathcal{F}$ a Morse foliation $\mathcal{F}_1$ with the same holonomy than $\mathcal{F}$, also transverse to the boundary ∂D^2 and such that $\#\{\text{centers of } \mathcal{F}_1\} = \#\{\text{center of } \mathcal{F}\} - 1$ and $\#\{\text{saddles of } \mathcal{F}_1\} = \#\{\text{saddles of } \mathcal{F}\} - 1$. Now by the induction hypothesis the modification $\mathcal{F}_1$ has some compact leaf of graph with unilateral holonomy and the theorem follows.

5.2 Haefliger's Theorem for the three sphere

In this section we prove the following version of Haefliger's Theorem:

Theorem 5.2. *Let $\mathcal{F}$ be C^∞ codimension one Morse foliation on the 3-sphere S^3. Suppose that the number of centers k and the number of saddles ℓ satisfy the inequality $k \geq \ell$. Either $\mathcal{F}$ has some compact codimension one invariant set whose holonomy group is one-sided or $\mathcal{F}$ is an inverse modification of a Seifert fibration of S^3, i.e., a singular foliation by spheres S^2.*

Indeed, we shall prove that there is either a compact leaf L or a leaf L such that $L \cup \{q\}$ is compact for some singularity $q \in \text{sing}(\mathcal{F})$ and such that the holonomy group of L is one-sided, or $\mathcal{F}$ is an inverse modification of a Seifert fibration on S^3.

Proof. We fix an orientation for $\mathcal{F}$. Let us proceed by induction on ℓ. First we consider the case $\ell = 0$. If also $k = 0$ then $\mathcal{F}$ is a nonsingular foliation on S^3 and by Novikov theorem $\mathcal{F}$ has some Reeb component and therefore $\mathcal{F}$ has a toral leaf $L \simeq S^1 \times S^1$ with one-sided holonomy group. Assume now that $k \geq 1$ and $\ell = 0$. In this case $\mathcal{F}$ has only center singularities and therefore it is a Seifert fibration by Proposition 2.1. Assume now that $k \geq \ell \geq 1$, and that the result has been proved for foliations with $\ell - 1$ saddles. Then $\mathcal{F}$ has some center singularity p_1 in S^3. Denote by $\mathcal{C}_{p_1}(\mathcal{F})$ the connected component of $\mathcal{C}(\mathcal{F})$ that contains p_1 , where $\mathcal{C}(\mathcal{F})$ is the union of all centers and leaves diffeomorphic to S^2 of the foliation $\mathcal{F}$. If $\partial\mathcal{C}_{p_1}(\mathcal{F}) = \emptyset$ then by Corollary 2.1 $\mathcal{C}_{p_1}(\mathcal{F}) = S^3$ and all leaves of $\mathcal{F}$ are compact diffeomorphic to S^2 with trivial holonomy. In other words, $\mathcal{F}$ is a singular Seifert fibration of S^3. Suppose therefore that $\partial\mathcal{C}_{p_1}(\mathcal{F}) \neq \emptyset$. In this case we must have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \text{sing}(\mathcal{F}) \neq \emptyset$, indeed any leaf $L \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$ is the separatrix of some saddle singularity $q_1 \in \text{sing}(\mathcal{F})$, which is necessarily unique for $\mathcal{F}$ has no saddle-connections. On the other hand we cannot have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \subset \text{sing}(\mathcal{F})$ because if a leaf accumulates on some saddle singularity q_1 then it accumulates on a separatrix of this singularity. Thus we can find a leaf L_0 of $\mathcal{F}$ such that L_0 is a separatrix of a saddle q_1 with $\Gamma_{q_1} = L_0 \cup \{q_1\} \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$. Notice that since Γ_{q_1} is accumulated by spherical leaves if it has nontrivial holonomy then it has unilateral holonomy and the theorem follows. Assume therefore that Γ_{q_1} has trivial holonomy. According to Lemma 3.1 we have the following possibilities (for some suitable choice of Γ_{q_1}):

(i) We have a trivial center-saddle pairing for $p_1 - q_1$.

(ii) Γ_{q_1} is homeomorphic to a torus with a meridian reduced to a point.

(iii) The saddle q_1 is not self-connected and Γ_{q_1} is homeomorphic to S^2 diffeomorphic to a sphere pinched at one point and with a nontrivial pairing with p_1.

In the first case we eliminate both singularities p_1 and q_1 obtaining a foliation $\mathcal{F}_1$ in S^3 with same holonomy than $\mathcal{F}$ and with less one saddle and one center singularity. Notice that $\#\{\text{centers of } \mathcal{F}_1\} \geq \#\{\text{saddles of } \mathcal{F}_1\}$. By the induction hypothesis $\mathcal{F}_1$ has unilateral holonomy and therefore the same holds for $\mathcal{F}$. In case (ii), since Γ_{q_1} has trivial holonomy, Γ_{q_1} is surrounded by leaves diffeomorphic to the torus. In particular we can isolated a region $R \subset S^3$ containing $\partial\mathcal{C}_{p_1}(\mathcal{F})$, invariant by $\mathcal{F}$ and diffeomorphic to a solid torus. In this region we perform a modification of $\mathcal{F}$, replacing it by a standard Reeb foliation. We obtain in this way a foliation $\mathcal{F}_1$ on S^3 with same holonomy than $\mathcal{F}$ and one center and one saddle singularity less. Now the induction hypothesis applies to conclude the existence of unilateral holonomy. Assume that we are in case (iii). In this case the other separatrix of q_1 is also diffeomorphic to a pinched sphere at q_1. Since the holonomy of Γ_{q_1} is trivial these two separatrices are surrounded by spherical leaves. Thus we can fix an invariant region R diffeomorphic to $[0,1] \times S^2$, containing the union of separatrices and with invariant boundary. In this region we perform a modification of $\mathcal{F}$ by a trivial foliation by spheres. We obtain this in way a foliation $\mathcal{F}_1$ on S^3 with same holonomy than $\mathcal{F}$ and one center and one saddle singularity less. Now the induction hypothesis applies to conclude the existence of unilateral holonomy.

6 A Novikov type theorem for singular foliations

The well-known theorem of Novikov states that a (nonsingular) codimension one C^2 foliation on S^3 must have a Reeb component and, in particular, a torus leaf with one-sided holonomy. This holds indeed for foliations on simply-connected closed 3-manifolds. On the other hand if we consider foliations with singularities then a construction due to Rosenberg and Roussarie [R-R] gives a C^∞ foliation with Morse singularities on S^3 and no compact leaf. Nevertheless restrictions on the singularity types can lead to a singular version of Novikov theorem.

Definition 6.1. Let $\mathcal{F}$ be a C^∞ foliation of codimension one on a compact 3-manifold M with singular set $\operatorname{sing}(\mathcal{F}) \neq \emptyset$. We shall say that $\operatorname{sing}(\mathcal{F})$ is *regular* if its connected components are either isolated points or smoothly embedded curves diffeomorphic to S^1. We say that a connected component Γ of $\operatorname{sing}(\mathcal{F})$ is *topologically stable* or C^0*-stable* if Γ has a fundamental system of neighborhoods in M bounded by compact leaves of $\mathcal{F}$.

Theorem 6.1. *Let $\mathcal{F}$ be a C^∞ codimension one foliation on a simply-connected closed* 3*-manifold M^3. Assume that the singular set* $\operatorname{sing}(\mathcal{F})$ *is discrete and all the singularities are topologically-stable. Then either* (i) *all leaves of $\mathcal{F}$ are compact or* (ii) *$\mathcal{F}$ has a Reeb component.*

Proof. By Novikov's theorem we may assume that $\mathcal{F}$ has a non-empty singular set. We divide the proof in two cases:
1^{st} **case**. The leaves of $\mathcal{F}$ are closed outside $\operatorname{sing}(\mathcal{F})$. In this case, since the singularities are C^0-stable, the leaves of $\mathcal{F}$ are all compact and we are in situation (i).
2^{nd} **case**. $\mathcal{F}$ has some leaf which is not closed in $M \setminus \operatorname{sing}(\mathcal{F})$. In this case there exists a leaf L_0 of $\mathcal{F}$ accumulating at some point $p \in \overline{L}_0 \setminus L_0$, which is not a singular point of $\mathcal{F}$. Since by hypothesis the singularities of $\mathcal{F}$ are C^0-stable it follows that there exists an open invariant neighborhood V of $\operatorname{sing}(\mathcal{F})$ in M such that $\overline{L} \cap V = \emptyset$ for any noncompact leaf L of $\mathcal{F}$ and in particular $\overline{L}_0 \cap V = \emptyset$ and $p \notin V$. By standard arguments the existence of a non-closed leaf L_0 as above implies the existence of a smooth embedding $a_0 \colon S^1 \to M^3$ transverse to $\mathcal{F}$ and, since $\overline{L}_0 \cap V = \emptyset$ we may (shrinking V if necessary) assume that $a_0(S^1) \cap V = \emptyset$. The manifold $M' = M \setminus \operatorname{sing}(\mathcal{F})$ is simply-connected because $\operatorname{sing}(\mathcal{F})$ is discrete $a_0(S^1)$ is homotopic to zero in M' and we may extend the map A_0 to a C^0 map $A_0 \colon D^2 \to M'$ from the closed disc $D^2 \subset \mathbb{R}^2$ to M' such that $A_0\big|_{\partial D^2} = a_0$. We may

indeed assume that A_0 is of class C^∞ and that $A_0(D^2) \cap V = \emptyset$ (notice that $A_0(D^2)$ and $\text{sing}(\mathcal{F})$ are disjoint compact sets).

Applying now the classical arguments from Haefliger Theorem we obtain a *vanishing cycle* for $\mathcal{F}' = \mathcal{F}\big|_M$, in M' which is a map $H\colon S^1 \times [0,\varepsilon] \to M'$ of class C^∞ such that if we denote $H_t(x) = H^x(t) = H(x,t)$, $\forall (x,t) \in S^1 \times [0,\varepsilon]$, then the following properties hold:

(a) $H_t(S^1)$ is a closed curve contained in a leaf $A(t)$ of $\mathcal{F}$, $\forall\, t$

(b) $H_t(S^1)$ is null homotopic in (t), $\forall\, t > 0$ and is not null homotopic in $A(t)$ for $t = 0$.

(c) $H^x([0,\varepsilon])$ is transverse to $\mathcal{F}$, $\forall\, x \in S^1$.

We can also suppose that $H(S^1 \times [0,\varepsilon]) \cap V = \emptyset$.

As in Novikov's proof we can obtain a positive coherent normal and simple extension of H_0 also denoted H. This means that: fixed a transverse orientation for $\mathcal{F}'$ in M' and a Riemannian metric on M', the curves H^x are normal to $\mathcal{F}$ and positively oriented. The map H is simple means that the lift of the curve $H_t(S^1)$ to the universal covering $\widehat{A}(t)$ of the leaf $A(t)$, is a simple closed curve in $\widehat{A}(t)$, $\forall\, t > 0$.

Following the classical proof of Novikov we need to reprove the following lemma:

Lemma 6.1. *Let $H\colon S^1 \times [0,\varepsilon] \to M \backslash V$ be as above then the exists an immersion $F\colon D^2 \times [0,\varepsilon'] \to M\backslash V$ for some $0 < \varepsilon' < \varepsilon$ satisfying the following conditions:*

(i) $F_t\big|_{\partial D^2} = H_t$, $\forall\, t$

(ii) $F(D^2 \times \{t\}) \subset A(t)$*,* $\forall\, t$

(iii) $F^x([0,\varepsilon'])$ *is normal to* $\mathcal{F}$*,* $\forall\, t$

(iv) *If* $U = \{x \in D^2; \lim\limits_{t\to 0} F^x(t) \text{exists}\}$ *then* $\partial D^2 \subset U$ *and* $U \neq D^2$ *is an open subset.*

Proof. We only prove the part which represents a difficulty due to the existence of singularities. Denote by $\pi(t)\colon \widehat{A}t) \to A(t)$ the universal covering of $A(t)$. It is enough to prove:

Claim 6.1. $\widehat{L}_t \simeq \mathbb{R}^2$ *for all* $t \geq 0$ *small enough.*

Proof. Suppose the claim is not true. There exists then a sequence $t_n \searrow 0$ such that $\widehat{A}(t_n) \simeq S^2$: notice that $\widehat{A}(0)$ is not homeomorphic to S^2 because $A(0)$ is not simply-connected since it contains a vanishing cycle. The foliation $\mathcal{F}$ is assumed to be oriented, therefore $A(t_n)$ is compact and covered by S^2 so that $A(t_n)$ is homeomorphic to S^2. We shall use the following lemma:

Lemma 6.2. *If a foliation $\mathcal{F}$ on a compact manifold M has only C^0-stable singularities and has some compact leaf L_0 with finite fundamental group then all leaves of $\mathcal{F}$ are compact.*

Proof. By the local stability theorem of Reeb the set $\mathcal{L}(L_0)$ of points $p \in M; p \notin \text{sing}(\mathcal{F})$ such that the leaf L_p is diffeomorphic to L_0 is an open non-empty subset of M. We want to prove that $\mathcal{L}(L_0) = M\backslash \text{sing}(\mathcal{F})$. It is enough to prove that $\partial\mathcal{L}(L_0) \subset \text{sing}(\mathcal{F})$. Suppose by contradiction that $\exists\, p \in \partial\mathcal{L}(L_0)$, $\;p \notin \text{sing}(\mathcal{F})$. Then $L_p \subset \partial\mathcal{L}(L_0)$ and, since the singularities are C^0-stable, we obtain $L_p \cap V = \emptyset$ for an invariant neighborhood V of $\text{sing}(\mathcal{F})$ as above. The leaf L_p therefore is accumulated by leaves which are diffeomorphic to L_0 compact and having finite fundamental group. Since L_p and these leaves avoid the neighborhood U of $\text{sing}(\mathcal{F})$, $\;V$ invariant, we can conclude from the classical arguments in the proof of Reeb Global Stability Theorem that L_p is also compact and has finite fundamental group. The holonomy group of L_p is finite and since $\mathcal{F}$ is oriented and has codimension one this group is trivial. Therefore L_p is diffeomorphic to L_0 . In particular we cannot have $L_p \subset \partial\mathcal{L}(L_0)$. This proves the lemma. ☐

Thus, since we are assuming that $\mathcal{F}$ has a vanishing cycle, we must have $\widehat{A}(t_n) \simeq \mathbb{R}^2$, $\forall\, t_n$ proving the claim.

Once Lemma 6 is proved the next step, also according to the classical proof of Novikov, is:

Lemma 6.3. *Given $\alpha > 0$ there are $0 < t' < t'' < \alpha$ and an embedding $h\colon D^2 \to \mathrm{Int}(D^2)$ such that $F(t', h(x)) = F(t'', x)$, $\forall\, x \in D^2$. In particular there exists a sequence $\tau_n \searrow 0$, $\tau_n > 0$ such that:*
(a) *$A(\tau_n) = A(\tau_{n+1}) = A \in \mathcal{F}$, $\forall\, n$*
(b) *$D(\tau_{n+1}) \supset D(\tau_n)$, $\forall\, n$ where $D(t) \subset A(t)$ is the projection of the disc in $\widehat{A}(t)$ bounded by the Jordan curve $\widehat{F}_t(S^1)$, $t \in (0, \varepsilon]$.*
(c) *For each $n \geq 1$, there exists a transformation $g_n\colon D^2 \to D^2$ such that $g_n\colon D^2 \to g_n(D^2)$ is a diffeomorphism and $F_{\tau_n} = F_{\tau_{n+1}} \circ g_n$.*

Proof. Again we only prove the part in the original proof which represents a difficulty arising from the existence of singularities for $\mathcal{F}$. This part is as follows:

Claim 6.2. *Let $x_0 \in D^2 - U$ in the above lemma. Then there exists a sequence $s_n \searrow 0$, $x_n > 0$ such that $p_n = F(s_n, x_0)$ converges to some point $p_0 \in M \backslash \mathrm{sing}(\mathcal{F})$ such that $F(s_n, x_0) \in L_{p_0}$ and $L_{p_0} \neq A(0)$.*

Proof of the Claim. Since M is compact there exists a sequence $s_n \searrow 0$ such that $p_n = F(s_n, x_0)$ converges to a point $p_0 \in M$. It could happen, a priori, that p_0 is a singular point for $\mathcal{F}$. However this does not happen because by construction the map F avoids an invariant neighborhood V of $\mathrm{sing}(\mathcal{F})$ in M, $F(S^1 \times [0, \varepsilon]) \cap V = \emptyset$. Therefore necessarily $p_0 \in \overline{F(S^1 \times [0, \varepsilon])}$ and thus $p_0 \notin \mathrm{sing}(\mathcal{F})$. We can indeed assume that $p_o \notin V$. The rest now follows as in the classical nonsingular case. This finishes the proof of the lemma. □

From now on there are no major difficulties in repeating the classical argumentation of Novikov. □

We have also obtained:

Theorem 6.2. *Let $\mathcal{F}$ be a C^∞ foliation with C^0-stable and discrete singular set on the closed* 3*-manifold M^3. We have the following possibilities:* (i) *$\mathcal{F}$ has all leaves compact.* (ii) *$\mathcal{F}$ has a Reeb component.* (iii) *Every closed transversal to $\mathcal{F}$ represents a non trivial element in $\pi_1(M)$ in particular $\pi_1(M)$ is not finite.*

For the case $\mathrm{sing}(\mathcal{F})$ is not discrete we can state:

Theorem 6.3. *Let $\mathcal{F}$ be a C^∞ codimension one foliation on a simply-connected closed* 3*-manifold M^3. Assume that the singular set is regular and C^0-stable. There are three possibilities:* (i) *$\mathcal{F}$ has all leaves compact.* (ii) *$\mathcal{F}$ has a Reeb component.* (iii) *Each closed transversal to $\mathcal{F}$ gives a non trivial element in $\pi_1(M \backslash \mathrm{sing}_1(\mathcal{F}))$ where $\mathrm{sing}_1(\mathcal{F})$ is the union of compact curves in $\mathrm{sing}(\mathcal{F})$.*

7 Appendix - The Index theorem

We shall consider the following situation: M is a C^∞ manifold and $S \hookrightarrow M$ is a submanifold. Given a differential one-form Ω in M we denote by $\mathrm{Ker}(\Omega)$ the distribution in M given by $\mathrm{Ker}(\Omega)(p) = \{v \in T_p(M); \Omega(p) \cdot v = 0\}$, for every $p \in M$. We shall say that $\mathrm{Ker}(\Omega)$ is *transverse to S* if for every $p \in S$ we have $\Omega(p) \neq 0$ and $\mathrm{Ker}(\Omega)(p) + T_p(S) = T_p(M)$ as real vector spaces. Denote by $\mathrm{sing}(\Omega) = \{p \in M; \Omega(p) = 0\}$ the singular set of Ω. Let $p \in \mathrm{sing}(\Omega)$ be an isolated singularity of Ω. We assume that M is an oriented manifold and choose a local chart $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^m$ of M such that $p \in U$, $\varphi(p) = 0$, $\mathrm{sing}(\Omega) \cap U = \{p\}$ and φ belongs to the positive atlas of M. Let $\omega = \varphi_*(\Omega) = (\varphi^{-1})^*(\Omega) \in \Lambda^1(\varphi(U))$. Write $\omega = \sum\limits_{j=1}^{m} f_j dx_j$ with $f_j \in C^\infty(\varphi(U), \mathbb{R})$ and $f_j(0) = 0, j = 1, ..., m$. Let $\mathrm{grad}(\omega) := \sum\limits_{j=1}^{m} f_j \frac{\partial}{\partial x_j}$ be the *gradient vector field* of ω. We define the *index* of Ω at p by $\mathrm{Ind}(\Omega;p) = \mathrm{Ind}(\mathrm{grad}(\omega);0)$, where $\mathrm{Ind}(\mathrm{grad}(\omega);0)$ is the ordinary Poincaré-Hopf index of the smooth vector field $\mathrm{grad}(\omega)$ at the singular point $0 \in \mathbb{R}^m$ (cf. [M]). Notice that the definition of $\mathrm{Ind}(\Omega;p)$ does not depend on the positive chart $\varphi\colon U \to \varphi(U)$ chosen as above.

Theorem 7.1 (Index Theorem). *Let M^m be an oriented manifold and $D \Subset M$ a domain with connected regular boundary of class C^2. Given a C^∞ differential one-form Ω in M such that the singular set* $\mathrm{sing}(\Omega)$ *is discrete and the distribution* $\mathrm{Ker}(\Omega)$ *is either everywhere transverse to ∂D or everywhere tangent to ∂D we have $\sum\limits_{p\in \mathrm{sing}(\Omega)\cap D} \mathrm{Ind}(\Omega;p) = \chi(D)$ where $\chi(D)$ is the Euler-Poincaré-Hopf characteristic of D.*

Proof of Theorem 7.1. Fixed a Riemannian metric $\langle,\rangle$ on M we denote by $\mathrm{grad}(\Omega)$ the only C^∞ vector field orthogonal to $\mathrm{Ker}(\Omega)$ and such that $\Omega\cdot\mathrm{grad}(\Omega) > 0$ everywhere in $M\setminus \mathrm{sing}(\Omega)$ and which is given in local coordinates $(x_1,...,x_m)$ by $\Omega = \sum\limits_{j=1}^m f_j dx_j$ and $\mathrm{grad}(\Omega) = \sum\limits_{j=1}^m f_j \frac{\partial}{\partial x_j}$. Let also $\rho\colon M \to \mathbb{R}$ be a C^∞ global defining function for $D\cup\partial D$ in the following sense: $D = \rho^{-1}(-\infty,0)$, $\partial D = \rho^{-1}(0)$ and $0\in\mathbb{R}$ is a regular value of ρ. We may assume that ρ is a Morse function in $\overline{D}$ and in particular its critical points in D are non-degenerate. Therefore the vector field $\vec{n} = \mathrm{grad}(\rho)$ has only simple singularities in D and thus its singular set in D is finite. This is done by replacing ρ and D by a Morse function $\rho_1\colon M\to\mathbb{R}$ of class C^∞ close enough to ρ in the strong Whitney topology and taking $D_1 := \rho_1^{-1}(-\infty,0)$ which is a domain with boundary $\partial D_1 = \rho^{-1}(0)$, in such a way that the pair (D_1,ρ_1) is close to the pair $(D,\partial D)$ as manifolds with boundaries in M. We may therefore assume that the pairs $(D,\partial D)$ and $(D_1,\partial D_1)$ are in homotopy equivalence so that we have $\chi(D) = \chi(D_1)$. On the other hand $\mathrm{Ker}(\Omega)$ is transverse to ∂D which is compact so that, for ρ_1 close enough to ρ, we also have $\mathrm{Ker}(\Omega)$ transverse to ∂D_1. Finally, clearly, we may assume that $\mathrm{sing}(\Omega)\cap D = \mathrm{sing}(\Omega)\cap D_1$ so that $\sum\limits_{p\in\mathrm{sing}(\Omega)\cap D}\mathrm{Ind}(\Omega;p) = \sum\limits_{p\in\mathrm{sing}(\Omega)\cap D_1}\mathrm{Ind}(\Omega;p)$. Hence we may assume that $\vec{n} = \mathrm{grad}(\rho)$ has only simple singularities in D. Notice that $\vec{n}$ points outward D from ∂D. We introduce the auxiliary set $\Sigma_\Omega = \{p\in M; \Omega(p)\cdot\vec{n}(p) = 0\}$. There are two cases to consider:

Case 1. If $\Sigma_\Omega\cap D = \emptyset$. In this case $\mathrm{sing}(\vec{n})\cap D = \emptyset$ and $\mathrm{sing}(\Omega)\cap D = \emptyset$. By the classical Poincaré-Hopf index theorem ([M]) we have $\chi(D) = \sum\limits_{p\in\mathrm{sing}(\vec{n})\cap D}\mathrm{Ind}(\vec{n};p) = 0$. Also clearly we have $\sum\limits_{p\in\mathrm{sing}(\Omega)\cap D}\mathrm{Ind}(\Omega;p) = 0$. Thus Theorem 7.1 is true in this case.

Case 2. $\Sigma_\Omega\cap D \neq\emptyset$. Notice that not necessarily we have $\Sigma_\Omega\cap\partial D\neq\emptyset$. If $\Sigma_\Omega\cap\partial D = \emptyset$ then $0\neq\Omega(p)\cdot\vec{n}(p) = \langle\mathrm{grad}(\Omega)(p),\vec{n}(p)\rangle, \forall p\in\partial D$. This implies that $\mathrm{grad}(\Omega)$ is transverse to ∂D and therefore we may apply Poincaré-Hopf index theorem to obtain $\sum\limits_{p\in\mathrm{sing}(\mathrm{grad}(\Omega))\cap D}\mathrm{Ind}(\mathrm{grad}(\Omega);p) = \chi(D)$. Since by definition $\mathrm{sing}(\Omega) = \mathrm{sing}(\mathrm{grad}(\Omega))$ and $\mathrm{Ind}(\mathrm{grad}(\Omega);p) = \mathrm{Ind}(\Omega;p)$ for every isolated singularity of Ω we obtain again the formula stated in Theorem 7.1. Thus we may assume that $\Sigma_\Omega\cap\partial D\neq\emptyset$. Given $p\in\Sigma_\Omega\cap\partial D$ we have $\Omega(p)\cdot\vec{n}(p) = 0$ but $\Omega(p)\cdot\mathrm{grad}(\Omega)(p)\neq 0$ because p is not a singularity of Ω. Therefore, $\vec{n}(p)$ and $\mathrm{grad}(\Omega)(p)$ are not collinear.

For a point $p\in\partial D$ we have $p\in\Sigma_\Omega \iff \langle\mathrm{grad}(\Omega)(p),\vec{n}(p)\rangle = 0 \iff \mathrm{grad}(\Omega)(p)\perp\vec{n}(p) \iff \vec{n}(p)\in\mathrm{Ker}(\Omega)(p) \iff \Omega(p)\cdot\vec{n}(p) = 0$. Let $p\in\partial D\setminus(\Sigma_\Omega\cap\partial D)$ then $\mathrm{grad}(\Omega)(p)$ is not orthogonal to $\vec{n}(p)$ and therefore $\mathrm{grad}(\Omega)(p)$ is transverse to $T_p(\partial D)$.

Let V be a tubular neighborhood of ∂D given by the trajectories of the vector field $\vec{n}$ in a neighborhood of ∂D. We may assume that V is fibred by discs of radius $\epsilon > 0$ given by a projection $\pi\colon V\to\partial D$.

Notice that if $p\in\partial D$ is such that $\mathrm{grad}(\Omega)(p)$ and $\vec{n}(p)$ are collinear then $\vec{n}(p)$ is orthogonal to

$\mathrm{Ker}(\Omega)(p)$ and therefore $\mathrm{Ker}(\Omega)(p) = T_p(\partial D)$. Since by hypothesis $\mathrm{Ker}(\Omega)$ is transverse to ∂D this cannot occur, that is, *the vector fields* $\mathrm{grad}(\Omega)$ *and* $\vec{n}$ *are never collinear in* ∂D.

By continuity of Ω and compactness of ∂D we may choose the neighborhood V such that $\mathrm{grad}(\Omega)$ and $\vec{n}$ are never collinear in V. In particular, for any continuous function $\varphi\colon V \to \mathbb{R}$ we have $\mathrm{grad}(\Omega) + \varphi.\vec{n} \neq 0$ in V. Let now $C > 0$ be a constant such that

$$C.\inf_{\overline{V}} ||\vec{n}||^2 > \sup_{\overline{V}} |\langle \vec{n}, \mathrm{grad}(\Omega)\rangle|.$$

Then the vector field $\mathrm{grad}(\Omega) + C.\vec{n}$ is transverse to ∂D because $\langle \mathrm{grad}(\Omega)+C.\vec{n}, \vec{n}\rangle = C.||\vec{n}||^2 + \langle \mathrm{grad}(\Omega), \vec{n}\rangle > 0$ in ∂D by the choice of C. Also, it points outward D from ∂D. Choose now a C^∞ *bump-function* $\varphi\colon M \to [0, C]$ such that: (i) $\varphi = C$ in a neighborhood V' of ∂D with $\partial D \subset V' \Subset V$. (ii) $\varphi = 0$ outside V in M. (iii) $0 < \varphi < C$ in V.

We introduce the C^∞ vector field Z in M by setting $Z := \mathrm{grad}(\Omega) + \varphi \cdot \vec{n}$. Then Z is transverse to ∂D, points outward D from ∂D and also we have

Lemma 7.1. $\mathrm{sing}(Z) \cap D = \mathrm{sing}(\Omega) \cap D$ *indeed,* $D \cap \mathrm{sing}(\Omega) \subset D \setminus (V \cap D)$.

Proof. Clearly Z coincides with $\mathrm{grad}(\Omega)$ in $D \setminus (V \cap D)$, thus it remains to show that $\mathrm{sing}(Z) \cap V = \emptyset$. This is clear because $\mathrm{grad}(\Omega)$ and $\vec{n}$ are linearly independent in V. □

Given any singularity $p \in \mathrm{sing}(Z) \cap D$ we have $Z = \mathrm{grad}(\Omega)$ in a neighborhood of p in M and therefore $\mathrm{Ind}(Z; p) = \mathrm{Ind}(\mathrm{grad}(\Omega); p)$. Applying now Poincaré-Hopf index Theorem to $\tilde{\xi}$ we obtain $\sum\limits_{p \in \mathrm{sing}(Z) \cap D} \mathrm{Ind}(Z; p) = \chi(D)$ therefore $\chi(D) = \sum\limits_{p \in \mathrm{sing}(\Omega) \cap D} \mathrm{Ind}(\mathrm{grad}(\Omega); p) = \sum\limits_{p \in \mathrm{sing}(\Omega) \cap D} \mathrm{Ind}(\Omega; p)$ and the first part of Theorem 7.1 is proved. Now we prove the second part. Consider the vector field $\mathrm{grad}(\Omega)$ in M as above. Since $\mathrm{Ker}(\Omega)(p) = T_p(\partial D)$, $\forall p \in \partial D$ we conclude that $\mathrm{grad}(\Omega)(p) \perp T_p(\partial D)$, $\forall p \in \partial D$ and therefore $\mathrm{grad}(\Omega)$ is transverse to ∂D. Since ∂D is connected we have either $\mathrm{grad}(\Omega)$ always points inward D from ∂D or it always points outward D from ∂D. In any case, applying Poincaré-Hopf index theorem we conclude that $\sum\limits_{p \in \mathrm{sing}(\Omega) \cap D} \mathrm{Ind}(\mathrm{grad}(\Omega); p) = \chi(D)$.

Therefore $\sum\limits_{p \in \mathrm{sing}(\Omega) \cap D} \mathrm{Ind}(\Omega; p) = \chi(D)$. □

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