On a Degenerate Zakharov System

F. Linares, G. Ponce and J-C. Saut

Abstract. We establish a local well-posedness result for an initial value problem associated to a Zakharov system arising in the study of laser-plasma interactions. We called this system degenerate due to the lack of dispersion presented in one of the spatial variables. One of the key tools to obtain our results is the presence of appropriate global versions of the so called "local smoothing effects" inherent to the dispersive character of the model.

Keywords: Zakharov system, Smoothing effects, Nonlinear Schrödinger equation.

1. Introduction

Consider the initial value problem (IVP) associated to the "degenerate" Zakharov system

$$\begin{cases} i(\partial_t+\partial_z)E+\Delta_\perp E=n\,E, \qquad (x,y,z)\in\mathbb{R}^3,\ \ t>0,\\ (\partial_t^2-\Delta_\perp)n=\Delta_\perp(|E|^2),\\ E(x,y,z,0)=E_0(x,y,z),\\ n(x,y,z,0)=n_0(x,y,z),\\ \partial_t n(x,y,z,0)=n_1(x,y,z) \end{cases} \tag{1.1}$$

where $\Delta_\perp=\partial_x^2+\partial_y^2$, E is a complex valued function and n is a real valued function.

The equation arises as a model in laser propagation when the paraxial approximation is used and the effect of the group velocity is negligible, see [4], [5], [7]. Note that the system is "degenerate" in the sense that there is no dispersive term in the space variable z in the first equation. Thus the existence results for the classical Zakharov system, i.e. $\Delta_\perp$ replaced by Δ, do not apply (see [7] and references therein).

1991 *Mathematics Subject Classification.* 35Q55, 35Q60, 35B65.
FL was partially supported by CNPq-Brazil.
GP was partially supported by NSF grant DMS-0140023.

In [1] Colin and Colin derived an alternative model to (1.1) and posed the question of the well-posedness of the IVP (1.1). No result seems to be available so far. Our goal here is to give a positive answer to this question. We will establish a local well-posedness theory for the IVP (1.1) in a suitable function space.

To describe our result we first reduce the IVP (1.1) into an IVP associated to a single equation, that is,

$$\begin{cases} i(\partial_t+\partial_z)E+\Delta_\perp E = n\,E, & (x,y,z)\in\mathbb{R}^3,\ \ t>0,\\ E(x,y,z,0)=E_0(x,y,z) \end{cases} \tag{1.2}$$

where

$$n(t)=\mathcal{N}'(t)n_0+\mathcal{N}(t)n_1+\int_0^t \mathcal{N}(t-t')\,\Delta_\perp(|E(t')|^2)\,dt', \tag{1.3}$$

with

$$\mathcal{N}(t)f=(-\Delta_\perp)^{-1/2}\,\sin((-\Delta_\perp)^{1/2}t)f \tag{1.4}$$

and

$$\mathcal{N}'(t)f=\,\cos((-\Delta_\perp)^{1/2}t)f \tag{1.5}$$

where $(-\Delta_\perp)^{1/2}f=((\xi_1^2+\xi_2^2)^{1/2}\widehat{f}\,)^\vee$.

Then we consider the integral equivalent formulation of IVP (1.2), that is,

$$\begin{aligned} E(t)=\mathcal{E}(t)E_0+&\int_0^t \mathcal{E}(t-t')(\mathcal{N}'(t')n_0+\mathcal{N}(t')n_1)E(t')\,dt'\\ &+\int_0^t \mathcal{E}(t-t')\Big(\int_0^{t'}\mathcal{N}(t'-s)\,\Delta_\perp(|E(s)|^2)\,ds\Big)\,E(t')\,dt'. \end{aligned} \tag{1.6}$$

where

$$\mathcal{E}(t)E_0=(e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\widehat{E}_0(\xi))^\vee. \tag{1.7}$$

To prove local well-posedness for the IVP (1.2) we will explore the smoothing effect associate to the operator $\mathcal{E}(t)$. We observe that the linear equation in (1.2) is almost a linear Schrödinger equation but not quite due to the propagation on the z-direction. However, we are able to prove similar smoothing effects for the operator $\mathcal{E}(t)$ as those of the Schrödinger propagator. We shall recall that the homogeneous local smoothing effect which provides a gain of 1/2 derivatives respect to the data was established by Constantin and Saut [2], Sjölin [6] and Vega [8]. Its inhomogeneous version which gives a gain of 1 derivative was proved by Kenig, Ponce and Vega [3]. Here we shall use a global version which is more appropriate for the problems we are dealing with. In particular we will show (see Proposition 2.1) that one has that

$$\|D_x^{1/2}\mathcal{E}(t)f\|_{L^\infty_xL^2_{yzt}}\le c\,\|f\|_{L^2_{xyz}} \tag{1.8}$$

and the same inequality with x and y exchanged. These estimates are one of the key points in our analysis.

The use of this type of estimate and properties of the operators $\mathcal{N}(t)$ and $\mathcal{N}'(t)$ will allow us to prove that an integral operator associated to (1.6) is a contraction in a certain function space that we will define next.

The function space $\widetilde{H}^{2j+1}(\mathbb{R}^3)$ is defined as

$$\widetilde{H}^{2j+1}(\mathbb{R}^3)=\{f \in H^{2j+1}(\mathbb{R}^3),\ D_x^{1/2}\partial^\alpha f,\ D_y^{1/2}\partial^\alpha f \in L^2(\mathbb{R}^3),\ |\alpha| \leq 2j+1,\ j \in \mathbb{Z}^+\} \quad (1.9)$$

where ∂^α denotes any derivative in (x, y, z) of order α. Thus initial data will be considered as being

$$\begin{cases} E_0 \in \widetilde{H}^{2j+1}(\mathbb{R}^3), \\ n_0 \in H^{2j+1}(\mathbb{R}^3),\ n_1 \in H^{2j}(\mathbb{R}^3),\ \partial_z n_1 \in H^{2j}(\mathbb{R}^3),\ j \in \mathbb{Z}^+. \end{cases} \quad (1.10)$$

In what follows we will use ∂^{2j+1} to denote any derivative in (x, y, z) of order less or equal than $2j+1$.

Now we are ready to give the statement of our main result.

Theorem 1.1. *For initial data (E_0, n_0, n_1) in (1.10), $j \geq 2$, there exist $T > 0$ and a unique solution E of the integral equation (1.6) such that*

$$E \in C([0,T] : \widetilde{H}^{2j+1}(\mathbb{R}^3)) \quad (1.11)$$

$$\|D_x^{1/2}\partial^{2j+1} E\|_{L_x^\infty L_{yzT}^2} < \infty \quad (1.12)$$

and

$$\|D_y^{1/2}\partial^{2j+1} E\|_{L_y^\infty L_{xzT}^2} < \infty. \quad (1.13)$$

Moreover, for $T' \in (0,T)$, the map $(E_0, n_0, n_1) \mapsto E(t)$ from $\widetilde{H}^{2j+1} \times H^{2j+1} \times H^{2j}$ into the class defined by (1.11)–(1.13) is Lipschitz.

Furthermore, from (1.11)–(1.13) one also has that

$$n \in C([0,T] : H^{2j+1}(\mathbb{R}^3)). \quad (1.14)$$

The proof of this local well-posedness result is based on the contraction principle (in the space adapted to the system), which guarantees that the map data–solution is Lipschitz, but since the nonlinearity is smooth the Implicit Function Theorem shows that this map is in fact smooth.

This note is organized as follows. We will obtain a series of estimates regarding the smoothing properties of the operator $\mathcal{E}(t)$, key in the present analysis, in Section 2. In Section 3 we will establish some estimates involving the nonlinear term that allow us to simplify the exposition of the proof of the main result. Theorem 1.1 will be proved in Section 4.

2. Linear Estimates

In this section we study the smoothing properties of solutions of the associated linear problems.

We begin with the solutions of the linear problem

$$\begin{cases}(\partial_t+\partial_z)E-i\Delta_\perp E=0\\ E(x,y,z,0)=E_0(x,y,z)\end{cases} \tag{2.15}$$

where $\Delta_\perp=\partial_x^2+\partial_y^2$.

The solution of the linear IVP (2.15) is given by

$$\mathcal{E}(t)E_0=E(x,y,z,t)=\big(e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\widehat{E}_0(\xi)\big)^{\vee}. \tag{2.16}$$

Proposition 2.1. *The solution of the linear problem* (2.15) *satisfies*

$$\|D_x^{1/2}\mathcal{E}(t)f\|_{L^\infty_xL^2_{yzt}}\le c\|f\|_{L^2_{xyz}}, \tag{2.17}$$

$$\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')\,G(t')\,dt'\|_{L^\infty_tL^2_{xyz}}\le c\|G\|_{L^1_xL^2_{yzt}}, \tag{2.18}$$

and

$$\|\partial_x\int_0^t\mathcal{E}(t-t')\,G(t')\,dt'\|_{L^\infty_xL^2_{yzt}}\le c\|G\|_{L^1_xL^2_{yzt}}. \tag{2.19}$$

The same estimates hold exchanging x and y. Here $D_x^{1/2}f=(|\xi|^{1/2}\widehat{f}\,)^{\vee}$.

Proof. We first prove (2.16). Denoting $\mathbf{x}=(x,y,z)$, $\xi=(\xi_1,\xi_2,\xi_3)$, $\bar{\mathbf{x}}=(y,z)$ and $\bar{\xi}=(\xi_2,\xi_3)$ we have

$$\|D_x^{1/2}\mathcal{E}(t)f\|_{L^2_{\bar{\mathbf{x}}}L^2_t}=\|\int_{\mathbb{R}^3}e^{i\mathbf{x}\cdot\xi}|\xi_1|^{1/2}e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\widehat{f}(\xi)\,d\xi\|_{L^2_{\bar{\mathbf{x}}}L^2_t}. \tag{2.20}$$

Introducing the change of variables

$$(\xi_1,\xi_2,\xi_3)=(\xi,\bar{\xi})\to(-\xi_1^2-\xi_2^2-\xi_3,\bar{\xi})=(r,\bar{\xi})$$

$$d\xi_1d\bar{\xi}=\begin{vmatrix}\frac{\partial r}{\partial\xi_1}&\frac{\partial r}{\partial\xi_2}&\frac{\partial r}{\partial\xi_3}\\0&1&0\\0&0&1\end{vmatrix}^{-1}dr\,d\bar{\xi}=(2|\xi_1|)^{-1}dr\,d\bar{\xi}$$

we obtain

$$\begin{aligned}\|D_x^{1/2}\mathcal{E}(t)f\|_{L^2_{\bar{\mathbf{x}}}L^2_t} &= \Big\| \int_{\mathbb{R}^3} e^{i(\bar{\mathbf{x}}\cdot\bar{\xi}+rt)}\,|\xi_1|^{1/2}\,e^{ix\sqrt{-r-\xi_2^2-\xi_3}}\,\widehat{f}(r,\bar{\xi})\,\frac{dr\,d\bar{\xi}}{2|\xi_1|}\,\Big\|_{L^2_{\bar{\mathbf{x}}}L^2_t}\\ &= c\Big(\int_{\mathbb{R}^3}\frac{1}{|\xi_1|}|\widehat{f}(r,\bar{\xi})|^2\,dr\,d\bar{\xi}\Big)^{1/2} = c\,\|\widehat{f}\,\|_{L^2_\xi} = c\,\|f\|_{L^2_{\mathbf{x}}}.\end{aligned}$$

This proves (2.17).

The inequality (2.18) follows using a duality argument.

Next we show (2.19). A simple computation shows that

$$\begin{aligned}\partial_x\int_0^t \mathcal{E}(t-t')\,G(t')\,dt' &= \int_{\mathbb{R}^4} e^{i\mathbf{x}\cdot\xi}\big(e^{it\tau}-e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\big)\frac{\xi_1}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,\widehat{G}(\xi,\tau)\,d\xi d\tau\\ &= \int_{\mathbb{R}^4} e^{i\mathbf{x}\cdot\xi}\,e^{it\tau}\frac{\xi_1}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,\widehat{G}(\xi,\tau)\,d\xi d\tau \qquad (2.21)\\ &\quad-\int_{\mathbb{R}^4} e^{i\mathbf{x}\cdot\xi}\,e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\frac{\xi_1}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,\widehat{G}(\xi,\tau)\,d\xi d\tau\\ &= \partial_x E_1(\mathbf{x},t)+\partial_x E_2(\mathbf{x},t).\end{aligned}$$

Then

$$\begin{aligned}\|\partial_x E_1(\mathbf{x},t)\|_{L^2_{\bar{\mathbf{x}}t}} &= \|\int_{\mathbb{R}^4} e^{i(\mathbf{x}\cdot\xi+t\tau)}\frac{\xi_1}{\tau+\xi_1^2+\xi_3^2+\xi_2}\,\widehat{G}(\xi,\tau)\,d\xi d\tau\|_{L^2_{\bar{\mathbf{x}}t}}\\ &= \|\int_{\mathbb{R}} e^{ix\xi_1}\frac{\xi_1}{\tau+\xi_1^2+\xi_3^2+\xi_2}\,\widehat{G}(\xi_1,\bar{\xi},\tau)\,d\xi_1\|_{L^2_{\bar{\xi}\tau}} \qquad (2.22)\\ &= \|\int_{\mathbb{R}} K(x-x',\bar{\xi},\tau)\,\widehat{G}^{(\bar{\xi},\tau)}(x',\bar{\xi},\tau)\,dx_1'\|_{L^2_{\bar{\xi}\tau}}\end{aligned}$$

where

$$K(x-x',\bar{\xi},\tau) = c\int_{\mathbb{R}} e^{i(x-x')\xi_1}\,\frac{\xi_1}{\tau+\xi_1^2+\xi_3^2+\xi_2}\,d\xi_1. \qquad (2.23)$$

To obtain (2.19) it is enough to show that $K\in L^\infty(\mathbb{R}^4)$. To prove this we write

$$\frac{\xi_1}{\tau+\xi_1^2+\xi_2^2+\xi_3} = \frac{\xi_1}{\alpha+\xi_1^2} \qquad (2.24)$$

and distinguish three cases:

(i) $\alpha > 0$. In this case we have that K is just $c(\alpha)\text{sign}(\xi_1)e^{-a|\xi_1|}$, therefore it is bounded.
(ii) $\alpha = 0$. It is clear that $K = \widehat{\text{p.v.}\,\frac{1}{\xi_1}}$, that is, the kernel of the Hilbert transform which is bounded.
(iii) $\alpha < 0$. Here we have that

$$\frac{\xi_1}{\alpha+\xi_1^2} = \frac{1}{2(|\alpha|^{1/2}-\xi_1)} - \frac{1}{2(|\alpha|^{1/2}+\xi_1)}. \tag{2.25}$$

Thus K is roughly a sum of translated of the Hilbert transform kernel. Therefore $K \in L^\infty(\mathbb{R}^4)$.

Hence

$$\begin{aligned} \sup_x \|\partial_x E_1\|_{L^2_{\bar{\mathbf{x}}t}} &\le c \int_{\mathbb{R}} \|\widehat{G}^{(\bar{\xi},\tau)}(x')\|_{L^2_{\bar{\xi}\tau}}\, dx' \\ &= c\int_{\mathbb{R}} \|G(x)\|_{L^2_{\bar{\mathbf{x}}t}}\, dx_1 = c\,\|G\|_{L^1_xL^2_{\bar{\mathbf{x}}t}}. \end{aligned} \tag{2.26}$$

On the other hand, we have that

$$\partial_x E_2(\mathbf{x},t) = D_x^{1/2}\mathcal{E}(t)g(\mathbf{x}) \tag{2.27}$$

where

$$\widehat{g}^{(\xi)}(\xi) = c\int_{-\infty}^{\infty} \frac{\text{sign}(\xi_1)|\xi_1|^{1/2}}{\tau+\xi_1^2+\xi_2^2+\xi_3}\widehat{G}(\xi,\tau)\,d\tau. \tag{2.28}$$

An easy computation shows that

$$\Big(\text{p.v.}\frac{1}{\tau+\xi_1^2+\xi_2^2+\xi_3}\Big)^{\wedge(\tau)} = \int_{-\infty}^{\infty} \frac{e^{-it\tau}}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,d\tau = c\,\text{sign}(t)\,e^{it(\xi_1^2+\xi_2^2+\xi_3)}. \tag{2.29}$$

By (2.27) and (2.17) we obtain

$$\sup_x \Big(\int_{\mathbb{R}^3} |\partial_x E_2(x,\bar{\mathbf{x}},t)|^2\,d\bar{\mathbf{x}}dt\Big)^{1/2} \le c\,\|\int_{-\infty}^{\infty} \frac{\text{sign}(\xi_1)|\xi_1|^{1/2}}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,\widehat{G}(\xi,\tau)\,d\tau\|_{L^2(\xi)}. \tag{2.30}$$

The identity (2.29) and Plancherel's theorem imply then that

$$\begin{aligned} & c\,\| \int_{-\infty}^{\infty} \frac{\operatorname{sign}(\xi_1)|\xi_1|^{1/2}}{\tau+\xi_1^2+\xi_2^2+\xi_3}\,\widehat{G}(\xi,\tau)\,d\tau\|_{L^2(\xi)} \\ & = c\,\|\Big(\int_{-\infty}^{\infty} \operatorname{sign}(\xi_1)|\xi_1|^{1/2}\operatorname{sign}(t)\,e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\,\widehat{G}^{(\xi)}(\xi,t)\,dt\Big)^{\vee(\xi)}\|_{L^2(\mathbf{x})} \\ & = c\,\|D_x^{1/2}\int_{-\infty}^{\infty} \mathcal{E}(t)(\operatorname{sign}(t)G(\cdot,t))dt\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{2.31}$$

So we can apply (2.18) in the last term of (2.31) to get the desired estimate. In fact, defining $\mathcal{E}(t)=e^{-itP}$ and noticing that by (2.18) we obtain

$$\begin{aligned} & \|D_x^{1/2}\int_0^t e^{-t'P}G(t')\,dt'\|_{L_t^\infty L_{\mathbf{x}}^2} \le \|D_x^{1/2}e^{itP}\int_0^t e^{-t'P}G(t')\,dt'\|_{L_t^\infty L_{\mathbf{x}}^2} \\ & \le \|D_x^{1/2}\int_0^t e^{-(t-t')P}G(t')\,dt'\|_{L_t^\infty L_{\mathbf{x}}^2} \le c\,\|G\|_{L_x^1 L_{\bar{\mathbf{x}}t}^2} \end{aligned} \tag{2.32}$$

where in the first two inequalities we used that e^{itP} is a unitary group in L^2. We have then that

$$\|D_x^{1/2}\int_0^\infty e^{-t'P}G(t')\,dt'\|_{L^2(\mathbb{R}^3)} \le c\,\|G\|_{L_x^1 L_{\bar{\mathbf{x}}t}^2}. \tag{2.33}$$

Therefore (2.30), (2.31) and (2.33) give

$$\sup_x \Big(\int_{\mathbb{R}^3} |\partial_x E_2(x,\bar{\mathbf{x}},t)|^2\,d\bar{\mathbf{x}}dt\Big)^{1/2} \le c\,\|G\|_{L_x^1 L_{\bar{\mathbf{x}}t}^2}. \tag{2.34}$$

Combining (2.26) and (2.34) inequality (2.19) follows. □

Lemma 2.2.

$$\|\mathcal{E}(t)E_0\|_{L_x^2 L_{yzT}^\infty} \le c\,(1+T)\,\|E_0\|_{H^4(\mathbb{R}^3)}. \tag{2.35}$$

Proof. Since

$$f(x,y,z,t) = f(x,y,z,0) + \int_0^t \partial_t f(x,y,z,s)\,ds, \tag{2.36}$$

the Sobolev embedding gives

$$\begin{aligned}\sup_{t\in[0,T],y,z} |f(x,y,z,t)| &\leq \sup_{y,z} \Big\{|f(x,y,z,0)| + \int_0^T |\partial_t f(x,y,z,s)|\,ds\Big\} \\ &\leq \|f(x,\cdot,\cdot,\cdot)\|_{H^2(\mathbb{R}^2_{yz})} + T^{1/2}\|\partial_t f(x,\cdot,\cdot,\cdot)\|_{L^2_T H^2(\mathbb{R}^2_{yz})}.\end{aligned} \tag{2.37}$$

Now taking the L^2_x–norm we get

$$\|f\|_{L^2_x L^\infty_{yzT}} \leq \|f(\cdot,\cdot,\cdot,0)\|_{L^2_x H^2(\mathbb{R}^2_{yz})} + T^{1/2}\|\partial_t f\|_{L^2_{xT} H^2(\mathbb{R}^2_{yz})}. \tag{2.38}$$

Taking $f(x,y,z,t) = \mathcal{E}(t)E_0$ and using equation (2.15) and group properties we obtain (2.35). □

Next we establish some estimates associated to solutions of the linear problem

$$\begin{cases}(\partial_t^2 - \Delta_\perp)\, n = 0 \\ n(x,0) = n_0(x) \\ \partial_t n(x,0) = n_1(x),\end{cases} \tag{2.39}$$

where $\Delta_\perp$ was defined in (1.1). The solution of problem (2.39) can be written as

$$n(x,t) = \mathcal{N}'(t)n_0 + \mathcal{N}(t)n_1 \tag{2.40}$$

where $\mathcal{N}(t)$ and $\mathcal{N}'(t)$ where defined in (1.4) and (1.5).

In the next lemmas we list a series of useful estimates involving the operators $\mathcal{N}'(t)$ and $\mathcal{N}(t)$.

Lemma 2.3. *For $f \in L^2(\mathbb{R}^3)$ we have*

$$\|\mathcal{N}(t)f\|_2 \leq |t|\,\|f\|_2, \tag{2.41}$$

$$\|\mathcal{N}'(t)f\|_2 \leq \|f\|_2, \tag{2.42}$$

and

$$\|(-\Delta_\perp)^{1/2}\,\mathcal{N}(t)f\|_2 \leq \|f\|_2. \tag{2.43}$$

Lemma 2.4.

$$\|\mathcal{N}'(t)n_0\|_{L^2_x L^\infty_{yzT}} \leq \|n_0\|_{H^2(\mathbb{R}^3)} \tag{2.44}$$

and

$$\|\mathcal{N}(t)n_1\|_{L^2_x L^\infty_{yzT}} \leq T\,\|n_1\|_{H^2(\mathbb{R}^3)}. \tag{2.45}$$

These estimates remain valid when x and y are exchanged.

Proof. Use of the Sobolev embedding and the definition of $\mathcal{N}'(t)$ yield

$$\|\mathcal{N}'(t)n_0\|_{L^2_x L^\infty_{yzT}} \leq \|\cos((-\Delta_\perp)^{1/2}t)n_0\|_{L^2_x L^\infty_T H^2(\mathbb{R}^2_{yz})} \leq \|n_0\|_{H^2(\mathbb{R}^3)}. \tag{2.46}$$

Similarly we obtain (2.45).

□

3. Nonlinear Estimates

In this section we will find estimates for the nonlinear terms in our analysis. Consider

$$E(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')\,dt' + \int_0^t \mathcal{E}(t-t')(EH)(t')\,dt' \tag{3.47}$$

where

$$F(t) = \mathcal{N}'(t)n_0 + \mathcal{N}(t)n_1 \tag{3.48}$$

and

$$H(t) = \int_0^t \mathcal{N}(t-t')\Delta_\perp(|E|^2)(t')\,dt'. \tag{3.49}$$

Lemma 3.1.

$$\|H\|_{L^2_x L^\infty_{yzT}} \le cT^2 \|E\|^2_{L^\infty_T H^4(\mathbb{R}^3)} \tag{3.50}$$

and

$$\|\partial^{2j+1} H\|_{L^2_{xyzT}} \le cT\,\|\partial_x \partial^{2j+1} E\|_{L^\infty_x L^2_{yzT}} \|E\|_{L^2_x L^\infty_{yzT}} + cT^{3/2}\,\|E\|^2_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}, \tag{3.51}$$

where ∂^{2j+1} denotes any derivative in (x, y, z) of order $\le 2j+1$. The estimates also hold true when x and y are exchanged.

Proof. To prove (3.50) we use the Sobolev embedding and the properties of the operator $\mathcal{N}(t)$, i.e.

$$\begin{aligned} \|H\|_{L^2_x L^\infty_{yzT}} &\le c\|H\|_{L^2_x L^\infty_T H^2(\mathbb{R}^2_{yz})} \\ &\le c\,\Big\| \int_0^t (t-t')\|\Delta_\perp(|E|^2)\|_{H^2(\mathbb{R}^2_{yz})}\,dt' \Big\|_{L^2_x L^\infty_T} \\ &\le c\,\Big\| \int_0^T (T-t')\|\Delta_\perp(|E|^2)\|_{H^2(\mathbb{R}^2_{yz})}\,dt' \Big\|_{L^2_x} \\ &\le c\,T^{3/2}\,\|\Delta_\perp(|E|^2)\|_{L^2_x L^2_T H^2(\mathbb{R}^2_{yz})} \le c\,T^2 \|E\|^2_{L^\infty_T H^4(\mathbb{R}^3)}. \end{aligned} \tag{3.52}$$

To obtain the estimate (3.51) we use inequality (2.43) and Sobolev's embedding to yield

$$\begin{aligned} \|\partial^{2j+1} H\|_{L^2_{xyzT}} &\le T\,\|\partial_x \partial^{2j+1}(|E|^2)\|_{L^2_{xyzT}} \\ &\le 2T\,\|\partial_x \partial^{2j+1} E\|_{L^\infty_x L^2_{yzT}} \|E\|_{L^2_x L^\infty_{yzT}} + T \sum_{\substack{k+l\le 2j+1 \\ k,l\ne 2j+1}} \|\partial_x(\partial^k E\,\partial^l E)\|_{L^2_{xyzT}} \\ &\le c\,T\,\|\partial_x \partial^{2j+1} E\|_{L^\infty_x L^2_{yzT}} \|E\|_{L^2_x L^\infty_{yzT}} + c\,T^{3/2} \|E\|^2_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}. \end{aligned}$$

□

We shall observe that the important terms to handle in the estimates below are those nonlinear terms with highest derivatives. The other ones, where the derivatives are split and lower order derivatives arising in the nonlinear terms, can generally be treated by interpolation between extreme cases.

Lemma 3.2. *Let ∂^{2j+1} be as above. Then*

$$\begin{aligned}&\|\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}+\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}\\ &\le c\,T^2\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}+c\,T^{3/2}\,\|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}}\,\|E\|_{L^\infty_T H^4(\mathbb{R}^3)}.\end{aligned} \tag{3.53}$$

Proof. Group properties and Minkowski's inequality give

$$\begin{aligned}&\|\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}\le c\,\|\partial^{2j+1}(EH)\|_{L^1_T L^2_{xyz}}\\ &\le c\,T\,\|\partial^{2j+1}E\|_{L^\infty_T L^2_{xyz}}\|H\|_{L^\infty_{xyzT}}+c\,T^{1/2}\|E\|_{L^\infty_{xyzT}}\|\partial^{2j+1}H\|_{L^2_{xyzT}}\\ &\quad+c\sum_{\substack{k+l\le 2j+1\\ k,l\neq 2j+1}}\|\partial^k E\,\partial^l H\|_{L^1_T L^2_{xyz}}\\ &\le A_1+A_2+A_3.\end{aligned} \tag{3.54}$$

The term A_1 can be estimated by using Sobolev embedding to control $\|H\|_{L^\infty_{xyzT}}$. Thus

$$A_1\le cT^2\,\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^4(\mathbb{R}^3)}. \tag{3.55}$$

To estimate A_2 we use (3.51) in Lemma 3.1 and the Sobolev lemma to obtain

$$A_2\le cT^{3/2}\,\|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}}\|E\|_{L^\infty_T H^4(\mathbb{R}^3)}+cT^2\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}. \tag{3.56}$$

To bound A_3 we use Hölder's inequality, the Sobolev lemma and Lemma 2.3. Indeed, let us consider the case $k+l=2j+1$ with $k\le l$.

$$\begin{aligned}\|\partial^k E\partial^l H\|_{L^1_T L^2_{xyz}}&\le\|\partial^k E\|_{L^\infty_{xyzT}}\|\partial^l H\|_{L^1_T L^2_{xyz}}\\ &\le c\,T^{1/2}\,\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\|\partial^l H\|_{L^2_{xyzT}}\\ &\le c\,T^2\,\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^{l+1}(\mathbb{R}^3)}\\ &\le c\,T^2\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\end{aligned} \tag{3.57}$$

whenever $2j+1\ge 3$. Thus

$$A_3\le T^2\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}. \tag{3.58}$$

On the other hand, Minkowski's inequality, group properties and the smoothing effect (2.17) yield

$$\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_x L^2_{yzT}} \leq \|\partial^{2j+1}(EH)\|_{L^1_T L^2_{xyz}}. \tag{3.59}$$

Hence the previous argument can be applied to obtain the result. □

The next estimate is the most delicate one in our argument. Here the smoothing effects obtained in Section 2 play an important role.

Lemma 3.3. *With the notation in the previous lemma we have that*

$$\begin{aligned}
&\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}} + \|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}\\
&\quad\leq c\,T^2(1+T^{1/2})\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\\
&\quad\quad + c\,T\,\|E\|_{L^2_x L^\infty_{yzT}}\big(\|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}} + T^{1/2}\|E\|^2_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\big)\\
&\quad\quad + c\,T^{3/2}\,\|E\|_{L^2_x L^\infty_{yzT}}\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\big(\|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}} + \|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\big).
\end{aligned} \tag{3.60}$$

A similar estimate follows exchanging x and y in (3.60).

Proof. We first use Leibniz's rule and then separate the highest order derivatives and the lower order ones.

$$\begin{aligned}
&\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}\\
&\qquad\leq \|D_x^{1/2}\int_0^t \mathcal{E}(t-t')(\partial^{2j+1}E\,H + E\partial^{2j+1}\,H)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}\\
&\qquad\quad + \sum_{\substack{k+l\leq 2j+1\\ k,l\neq 2j+1}}\|D_x^{1/2}\int_0^t \mathcal{E}(t-t')(\partial^k E\partial^l H)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}\\
&\qquad = D_1 + D_2
\end{aligned} \tag{3.61}$$

The estimate (2.18) implies that

$$D_1 \leq \|\partial^{2j+1}E\,H + E\partial^{2j+1}\,H\|_{L^1_x L^2_{yzT}} \leq c\,\|\partial^{2j+1}E\,H\|_{L^1_x L^2_{yzT}} + c\,\|E\partial^{2j+1}\,H\|_{L^1_x L^2_{yzT}}. \tag{3.62}$$

Using Lemma 3.1 we obtain

$$\begin{aligned}D_1 &\le c\|\partial^{2j+1}E\|_{L^2_{xyzT}}\|H\|_{L^2_xL^\infty_{yzT}} + c\|E\|_{L^2_xL^\infty_{yzT}}\|\partial^{2j+1}H\|_{L^2_{xyzT}}\\ &\le c\,T^{5/2}\,\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^4(\mathbb{R}^3)} + c\,T\|E\|^2_{L^2_xL^\infty_{yzT}}\|\partial_x\partial^{2j+1}E\|_{L^\infty_xL^2_{yzT}}\\ &\quad + cT^{3/2}\|E\|_{L^2_xL^\infty_{yzT}}\|E\|^2_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}.\end{aligned} \tag{3.63}$$

To show how to estimate D_2 let us take, for instance, the case $k+l=2j+1$ with $k\le l$. So, the Minkowski inequality and group properties yield

$$\|D_x^{1/2}\int_0^t \mathcal{E}(t-t')(\partial^kE\partial^lH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}} \le \|D_x^{1/2}(\partial^kE\partial^lH)\|_{L^1_TL^2_{xyz}}. \tag{3.64}$$

Using that

$$\|D_x^{1/2}(fg)\|_{L^2_{xyz}} \le c\,\|D_x^{1/2}f\|_{L^4_{xyz}}\|g\|_{L^4_{xyz}} + c\,\|f\|_{L^\infty_{xyz}}\|D_x^{1/2}g\|_{L^2_{xyz}} \tag{3.65}$$

we have that

$$\begin{aligned}&\|D_x^{1/2}(\partial^kE\partial^lH)\|_{L^1_TL^2_{xyz}}\\ &\le c\,\|\|D_x^{1/2}\partial^kE\|_{L^4_{xyz}}\|\partial^lH\|_{L^4_{xyz}}\|_{L^1_T} + \|\|\partial^kE\|_{L^\infty_{xyz}}\|D_x^{1/2}\partial^lH\|_{L^2_{xyz}}\|_{L^1_T}\end{aligned} \tag{3.66}$$

Let $l\neq 2j$. Using the Sobolev inequality and Lemma 2.3 we obtain

$$\begin{aligned}\|\|D_x^{1/2}\partial^kE\|_{L^4_{xyz}}\|\partial^lH\|_{L^4_{xyz}}\|_{L^1_T} &\le c\,T^{1/2}\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\|\partial^lH\|_{L^2_TH^1(\mathbb{R}^3)}\\ &\le c\,T^2\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^{l+2}(\mathbb{R}^3)}\\ &\le c\,T^2\,\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}.\end{aligned} \tag{3.67}$$

On the other hand, the Sobolev inequality, (3.65) and Lemma 2.3 imply that

$$\begin{aligned}\|\|\partial^kE\|_{L^\infty_{xyz}}\|D_x^{1/2}\partial^lH\|_{L^2_{xyz}}\|_{L^1_T} &\le c\,T^{1/2}\,\|\partial^kE\|_{L^\infty_{xyzT}}\|D_x^{1/2}\partial^lH\|_{L^2_{xyzT}}\\ &\le c\,T^2\,\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^{l+2}(\mathbb{R}^3)}\\ &\le c\,T^2\,\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}.\end{aligned} \tag{3.68}$$

Let $l=2j$. The Sobolev lemma and Lemma 3.1 imply that

$$\begin{aligned}&\|\|D_x^{1/2}\partial E\|_{L^4_{xyz}}\|\partial^lH\|_{L^4_{xyz}}\|_{L^1_T}\\ &\le c\,T^{1/2}\|D_x^{1/2}\partial E\|_{L^\infty_T H^1(\mathbb{R}^3)}\|\partial^lH\|_{L^2_TH^1(\mathbb{R}^3)}\\ &\le c\,T^{1/2}\,\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}\|\partial^{2j+1}H\|_{L^2_{xyzT}}\\ &\le cT^{3/2}\,\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\|\partial_x\partial^{2j+1}E\|_{L^\infty_xL^2_{yzT}}\|E\|_{L^2_xL^\infty_{yzT}} + c\,T^2\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}.\end{aligned} \tag{3.69}$$

and

$$\begin{aligned}
&\| \|\partial E\|_{L^\infty_{xyz}} \|D_x^{1/2}\partial^l H\|_{L^2_{xyz}}\|_{L^1_T}\\
&\le c\,T^{1/2}\|\partial E\|_{L^\infty_T H^2(\mathbb{R}^3)}\|D_x^{1/2}\partial^l H\|_{L^2_{xyzT}}\\
&\le c\,T^{3/2}\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}\|D_x^{1/2}\partial^{2j}\partial_x(|E|^2)\|_{L^2_{xyzT}}\\
&\le cT^{3/2}\,\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}\|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}} + cT^2\,\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}.
\end{aligned} \tag{3.70}$$

Thus

$$\begin{aligned}
D_2 \le\ & c\,T^{3/2}\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\|E\|_{L^\infty_x L^2_{yzT}}\big(\|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}} + \|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\big)\\
&+ c\,T^2\,\|E\|^3_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}
\end{aligned} \tag{3.71}$$

Hence combining (3.61), (3.63) and (3.71) it follows that

$$\begin{aligned}
&\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_T L^2_{xyz}}\\
&\qquad\le c\,T\,(1+T^{1/2})\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}\|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}}\\
&\qquad+ T^{3/2}\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}\|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{yzT}}\\
&\qquad+ c\,T^{3/2}(1+T^{1/2}+T)\|E\|_{L^\infty_T H^3(\mathbb{R}^3)}.
\end{aligned} \tag{3.72}$$

Finally to estimate

$$\|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_x L^2_{yzT}},$$

we first apply the Leibniz rule and split the highest derivative terms and lower derivative terms as above. Then we use the smoothing effects (2.18) and (2.17) to obtain

$$\begin{aligned}
&\|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EH)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}\\
&\le \|\partial_x\int_0^t \mathcal{E}(t-t')(\partial^{2j+1}E\,H + E\partial^{2j+1}\,H)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}
\end{aligned} \tag{3.73}$$

$$+\sum_{\substack{k+l\le 2j+1\\ k,l\ne 2j+1}}\|D_x^{1/2}\mathcal{E}(t)\Big(\int_0^t \mathcal{E}(-t')D_x^{1/2}(\partial^k E\partial^l H)(t')\,dt'\Big)\|_{L^\infty_x L^2_{yzT}}$$
$$\le c\,\|\partial^{2j+1}EH\|_{L^1_xL^2_{yzT}}+c\,\|E\partial^{2j+1}H\|_{L^1_xL^2_{yzT}}+c\sum_{\substack{k+l\le 2j+1\\ k,l\ne 2j+1}}\|D_x^{1/2}(\partial^k E\partial^l H)(t')\,dt'\|_{L^1_TL^2_{xyz}}.$$

From this point on the analysis is similar to the previous one. □

Lemma 3.4. *Let F be as in* (3.48). *Then*

$$\begin{aligned}\|\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_TL^2_{xyz}}+\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_xL^2_{yzT}}\\ \le c\,T\|E\|_{L^\infty_TH^{2j+1}(\mathbb{R}^3)}\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)}+(1+T)\,\|n_1\|_{H^{2j}(\mathbb{R}^3)}\big)\end{aligned}\tag{3.74}$$

and

$$\begin{aligned}\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_TL^2_{xyz}}+\|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_xL^2_{yzT}}\\ \le c\,T^{1/2}\|E\|_{L^\infty_TH^{2j+1}(\mathbb{R}^3)}\,(\|n_0\|_{H^2(\mathbb{R}^3)}+T\|n_1\|_{H^2(\mathbb{R}^3)})\\ +c\,T^{1/2}\|E\|_{L^2_xL^\infty_{yzT}}\,(\|n_0\|_{H^{2j}(\mathbb{R}^3)}+T\|n_1\|_{H^{2j}(\mathbb{R}^3)})\\ +c\,T\,\|E\|_{L^\infty_TH^{2j+1}(\mathbb{R}^3)}\,(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)}+(1+T)\|n_1\|_{H^{2j}(\mathbb{R}^3)}).\end{aligned}\tag{3.75}$$

Proof. By Minskowski's inequality and group properties we have

$$\|\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_TL^2_{xyz}}\le c\|\partial^{2j+1}(EF)\|_{L^1_TL^2_{xyz}}.\tag{3.76}$$

On the other hand, the inequality (2.17) and group properties imply that

$$\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_xL^2_{yzT}}\le c\|\partial^{2j+1}(EF)\|_{L^1_TL^2_{xyz}}.\tag{3.77}$$

So to obtain (3.74) it is enough to estimate $\|\partial^{2j+1}(EF)\|_{L^1_TL^2_{xyz}}$. First we have that

$$\|\partial^{2j+1}(EF)\|_{L^1_TL^2_{xyz}}\le T\,\|\partial^{2j+1}E\|_{L^\infty_TL^2_{xyz}}\|F\|_{L^\infty_{xyzT}}+T\,\|E\|_{L^\infty_{xyzT}}\|\partial^{2j+1}F\|_{L^\infty_TL^2_{xyz}}$$
$$+\text{ lower order terms}$$

By Sobolev's Lemma, (2.41) and (2.42) it follows that

$$\|F\|_{L^\infty_{xyzT}}\le\|F\|_{L^\infty_TH^2(\mathbb{R}^2)}\le c\,(\|n_0\|_{H^2(\mathbb{R}^3)}+T\|n_1\|_{H^2(\mathbb{R}^3)}).$$

Using (2.41), (2.42) and (1.10) we obtain

$$\|\partial^{2j+1}F\|_{L^\infty_T L^2_{xyz}} \le c\,(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + \|n_1\|_{H^{2j}(R^3)}).$$

Similarly, we can estimated the lower order terms. Thus

$$\begin{aligned}\|\partial^{2j+1}(EF)\|_{L^1_T L^2_{xyz}} &\le c\,T\,\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}(\|n_0\|_{H^{2j}(\mathbb{R}^3)} + T\,\|n_1\|_{H^{2j}(\mathbb{R}^3)})\\ &\quad + cT\,\|E\|_{L^\infty_T H^2(\mathbb{R}^3)}(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + \|n_1\|_{H^{2j}(\mathbb{R}^3)}).\end{aligned} \tag{3.78}$$

This inequality combined with (3.76) and (3.77) gives (3.74).

To show (3.75) we proceed as in the proof of Lemma 3.3. We first use the argument developed in (3.61) and (3.73) to obtain

$$\begin{aligned}&\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_T L^2_{xyz}} + \|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}\\ &\le \|\partial^{2j+1}EF\|_{L^1_x L^2_{yzT}} + \|E\partial^{2j+1}F\|_{L^1_x L^2_{yzT}} + \sum_{\substack{k+l\le 2j+1\\ k,l\neq 2j+1}} \|D_x^{1/2}(\partial^k E\partial^l F)\|_{L^1_T L^2_{xyz}}.\end{aligned}$$

Applying Lemma 2.4 we obtain

$$\begin{aligned}\|\partial^{2j+1}EF\|_{L^1_x L^2_{yzT}} &\le \|\partial^{2j+1}E\|_{L^2_{xyzT}}\|F\|_{L^2_x L^\infty_{yzT}}\\ &\le c\,T^{1/2}\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\big(\|n_0\|_{H^2(\mathbb{R}^2)} + T\|n_1\|_{H^2(\mathbb{R}^2)}\big).\end{aligned} \tag{3.79}$$

From Lemma 2.3 it follows that

$$\begin{aligned}\|E\partial^{2j+1}F\|_{L^1_x L^2_{yzT}} &\le \|E\|_{L^2_x L^\infty_{yzT}}\|\partial^{2j+1}F\|_{L^2_{xyzT}}\\ &\le cT^{1/2}\|E\|_{L^2_x L^\infty_{yzT}}\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + \|n_1\|_{H^{2j}(\mathbb{R}^3)}\big).\end{aligned} \tag{3.80}$$

Using the estimate (3.65), the Sobolev lemma and Lemma 2.4 we obtain

$$\begin{aligned}\|D_x^{1/2}(\partial^k E\partial^l F)\|_{L^1_T L^2_{xyz}} &\le c\,T^{1/2}\,\|D_x^{1/2}\partial^k E\|_{L^\infty_T H^1(\mathbb{R}^3)}\|\partial^l F\|_{L^2_T H^1(\mathbb{R}^3)}\\ &\quad + c\,T^{1/2}\,\|\partial^k E\|_{L^\infty_{xyzT}}\|D_x^{1/2}\partial^l F\|_{L^2_{xyzT}}\\ &\le c\,T\,\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\big(\|n_0\|_{H^{2j}(\mathbb{R}^3)} + T\|n_1\|_{H^{2j}(\mathbb{R}^3)}\big)\\ &\quad + c\,T\,\|E\|_{L^\infty_T H^{k+2}(\mathbb{R}^3)}\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + \|n_1\|_{H^{2j}(\mathbb{R}^3)}\big).\end{aligned} \tag{3.81}$$

Therefore,

$$\begin{aligned}&\|D_x^{1/2}\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_T L^2_{xyz}} + \|\partial_x\partial^{2j+1}\int_0^t \mathcal{E}(t-t')(EF)(t')\,dt'\|_{L^\infty_x L^2_{yzT}}\\ &\le cT^{1/2}(1+T^{1/2})\|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)}\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + (1+T)\|n_1\|_{H^{2j}(\mathbb{R}^3)}\big)\\ &\quad + c\,T^{1/2}\|E\|_{L^2_x L^\infty_{yzT}}\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + \|n_1\|_{H^{2j}(\mathbb{R}^3)}\big).\end{aligned}$$

□

In the next lemma the restriction $j \geq 2$ in the statement of Theorem 1.1 arises.

Lemma 3.5.

$$\begin{aligned} &\|\int_0^t \mathcal{E}(t-t')(EF+EH)(t')\,dt'\|_{L^2_xL^\infty_{yzT}} \\ &\leq c\,(1+T)T\|E\|_{L^\infty_T H^4(\mathbb{R}^3)}(\|n_0\|_{H^4(\mathbb{R}^3)}+T\|n_1\|_{H^4(\mathbb{R}^3)}) \\ &\quad + c\,(1+T)T^2(\|E\|^3_{L^\infty_T H^4(\mathbb{R}^3)}+\|E\|^2_{L^\infty_T H^2(\mathbb{R}^3)}\|E\|_{L^\infty_T H^5(\mathbb{R}^3)}). \end{aligned} \tag{3.82}$$

The same estimate holds with x and y exchanged.

Proof. Applying group properties and Lemma 2.2 it follows that

$$\begin{aligned} &\|\int_0^t \mathcal{E}(t-t')(EF+EH)(t')\,dt'\|_{L^2_xL^\infty_{yzT}} \\ &\quad+\int_0^T (1+T)\,\big(\|E(t')F(t')\|_{H^4(\mathbb{R}^3)}+\|E(t')H(t')\|_{H^4(\mathbb{R}^3)}\big)\,dt' \\ &\leq c\,(1+T)\|E_0\|_{H^4(\mathbb{R}^3)} \\ &\quad + c\,(1+T)\{\|E(t')F(t')\|_{L^1_T H^4(\mathbb{R}^3)}+\|E(t')H(t')\|_{L^1_T H^4(\mathbb{R}^3)}\} \\ &= c\,(1+T)\|E_0\|_{H^4(\mathbb{R}^3)}+c\,(1+T)\{B_1+B_2\}. \end{aligned} \tag{3.83}$$

One has that

$$B_2 \leq T\|\partial^4 E(t)\|_{L^\infty_T L^2_{xyz}}\|H\|_{L^\infty_{xyzT}}+T^{1/2}\|E\|_{L^\infty_{xyzT}}\|\partial^4 H\|_{L^2_{xyzT}}+\text{low order terms}.$$

Using (2.43) and Sobolev's inequality we get

$$\|H\|_{L^\infty_{xyzT}} \leq c\,\|H\|_{L^\infty_T H^2(\mathbb{R}^3)} \leq c\,T\|E\|^2 L^\infty_T H^3(\mathbb{R}^3).$$

Applying (2.43) yields that

$$\|\partial^4 H\|_{L^2_{xyzT}} \leq c\,T^{3/2}\|E\|^2_{L^\infty_T H^5(\mathbb{R}^3)}.$$

The lower order terms can be estimated using the Sobolev lemma and (2.43) as above. Hence

$$B_2 \leq cT^2\|E\|^3_{L^\infty_T H^4(\mathbb{R}^3)}+c\,T^2\|E\|_{L^\infty_T H^2(\mathbb{R}^3)}\|E\|^2_{L^\infty_T H^5(\mathbb{R}^3)}. \tag{3.84}$$

On the other hand,

$$B_1 \leq cT\|E\|_{L^\infty_T H^4(\mathbb{R}^3)}\|F\|_{L^\infty_{xyzT}}+cT^{1/2}\|E\|_{L^\infty_{xyzT}}\|F\|_{L^2_T H^4(\mathbb{R}^3)}+\text{lower order terms}.$$

By (2.41) and (2.42) it follows that

$$\|F\|_{L^\infty_{xyzT}} \le c\,(\|n_0\|_{H^2(\mathbb{R}^3)} + T\|n_1\|_{H^2(\mathbb{R}^3)})$$

and

$$\|F\|_{L^2_T H^4(\mathbb{R}^3)} \le c\,T^{1/2}\big(\|n_0\|_{H^4(\mathbb{R}^3)} + T\|n_1\|_{H^4(\mathbb{R}^3)}\big).$$

A similar argument can be used to estimate the lower order terms. Hence

$$B_1 \le c\,T\,\|E\|_{L^\infty_T H^4(\mathbb{R}^3)}(\|n_0\|_{H^4(\mathbb{R}^3)} + T\|n_1\|_{H^4(\mathbb{R}^3)}). \tag{3.85}$$

Combining (3.83), (3.84) and (3.85) the result follows.

□

4. Proof of Theorem 1.1

We define

$$X_{a,T} = \{E \in C([0,T] : \widetilde{H}^{2j+1}(\mathbb{R}^3)) : \|\!|E|\!\|_T \le a\} \tag{4.86}$$

where

$$\begin{aligned} \|\!|E|\!\| := {}& \|E\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)} + \|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_T L^2_{xyz}} + \|D_y^{1/2}\partial^{2j+1}E\|_{L^\infty_T L^2_{xyz}} \\ &+ \|D_x^{1/2}\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}} + \|\partial_x\partial^{2j+1}E\|_{L^\infty_x L^2_{yzT}} \\ &+ \|E\|_{L^2_x L^\infty_{yzT}} + \|E\|_{L^2_y L^\infty_{xzT}} \\ &+ \|D_y^{1/2}\partial^{2j+1}E\|_{L^\infty_y L^2_{xzT}} + \|\partial_y\partial^{2j+1}E\|_{L^\infty_y L^2_{xzT}} \end{aligned} \tag{4.87}$$

and the integral operator on $X_{a,T}$,

$$\begin{aligned} \Psi(E)(t) ={}& \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')\,E(t')(\mathcal{N}'(t')n_0 + \mathcal{N}(t')n_1) \\ &+ \int_0^t \mathcal{E}(t-t')\,E(t')\Big(\int_0^{t'} \mathcal{N}(t'-s)\Delta_\perp(|E|^2)(s)\,ds\Big)\,dt' \\ ={}& \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')\,(EF)(t') + \int_0^t \mathcal{E}(t-t')\,(EH)(t')\,dt'. \end{aligned} \tag{4.88}$$

We will show that for appropriate a and T the operator $\Psi(\cdot)$ defines a contraction on $X_{a,T}$.

From Proposition 2.15, Lemmas 3.2, 3.4 and 3.5 we deduce that

$$\begin{aligned} &\|\Psi(E)\|_{L^\infty_T H^{2j+1}(\mathbb{R}^3)} + \|D_x^{1/2}\partial^{2j+1}\Psi(E)\|_{L^\infty_x L^2_{yzT}} \le c\,\|E_0\|_{H^{2j+1}(\mathbb{R}^3)} \\ &\quad + c\,T^{3/2}(1+T^{1/2})\,\|\!|E|\!\|^3 + c\,T\,\|\!|E|\!\|\,(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + (1+T)\|n_1\|_{H^{2j}(\mathbb{R}^3)}). \end{aligned} \tag{4.89}$$

On the other hand, Proposition 2.15, Lemmas 3.3, 3.4 and 3.5 yield the following inequality

$$\begin{aligned}\|\partial_x\partial^{2j+1}\Psi(E)\|_{L^\infty_x L^2_{yzT}} + \|D_x^{1/2}\partial^{2j+1}\Psi(E)\|_{L^\infty_T L^2_{xyz}} &\le c\,\|D_x^{1/2}\partial^{2j+1}\,E_0\|_{L^2(\mathbb{R}^3)}\\ &+ cT^{1/2}(1+T^{1/2})\,\|\!|E|\!\|\big(\|n_0\|_{H^{2j+1}(\mathbb{R}^3)} + (1+T)\|n_1\|_{H^{2j}(\mathbb{R}^3)}\big)\\ &+ c\,T\,(1+T^{1/2}+T+T^{3/2})\,\|\!|E|\!\|^3.\end{aligned} \tag{4.90}$$

Similarly, we obtain estimates for

$$\|D_y^{1/2}\partial^{2j+1}\Psi(E)\|_{L^\infty_y L^2_{xyz}} \;\text{ and }\; \|D_y^{1/2}\partial^{2j+1}\Psi(E)\|_{L^\infty_T L^2_{xyz}} + \|\partial_y\partial^{2j+1}\Psi(E)\|_{L^\infty_y L^2_{xzT}}. \tag{4.91}$$

Finally, from Lemmas 2.2 and 3.5 it follows that

$$\begin{aligned}\|\Psi(E)\|_{L^\infty_x L^2_{yzT}} + \|\Psi(E)\|_{L^\infty_y L^2_{xzT}} &\le c\,(1+T)\,\|E_0\|_{H^4(\mathbb{R}^4)}\\ +c\,(1+T)T\|\!|E|\!\|(\|n_0\|_{H^4(\mathbb{R}^3)} + T\|n_1\|_{H^4(\mathbb{R}^3)}) &+ c\,(1+T)T^2\|\!|E|\!\|^3.\end{aligned} \tag{4.92}$$

Hence, choosing $a = 2c(1+T)\|E_0\|_{\widetilde{H}^{2j+1}}$ and T sufficiently small depending on $\|n_0\|_{H^{2j+1}(\mathbb{R}^3)}$, $\|n_1\|_{H^{2j}(\mathbb{R}^3)}$ and $\|\partial_z n_1\|_{H^{2j}(\mathbb{R}^3)}$, we see that Ψ maps $X_{a,T}$ into $X_{a,T}$. To show that Ψ is a contraction we follow a similar argument as the one described above. The remainder of the proof uses a standard procedure so we will omit it. This completes the proof of Theorem 1.1.

Remark 4.1. *The smoothness requirement on the data in Theorem 1.1 could probably be weakened. Note however that since no conservation law (energy), except the conservation of the L^2 norm of E, seems to be available for* (1.1)*, such a weakening would not obviously lead to global existence of solutions.*

References

[1] M. Colin and T. Colin, *On a quasilinear Zakharov system describing laser-plasma interactions*, preprint.

[2] P. Constantin and J. C. Saut, *Local smoothing properties of dispersive equations*, Journal A.M.S. **1** (1989), 413–446.

[3] C. E. Kenig, G. Ponce, and L. Vega, *Small solutions to non-linear Schrödinger equation*, Ann. IHP Analyse Nonlineare **10** (1993), 255–280.

[4] G. Riazuelo, *Etude théorique et numérique de l'influence du lussage optique sur la filamentation des faisceaux lasers dans les plasmas sous-critiques de fusion inertielle*, Thèse de l'Université Paris XI.

[5] H. A. Rose and D. F. Dubois, *Initial development of ponderomotive filaments in plasma from intense hot spots produced by random phase*, Phys. Fluids B **9** (1993), 3337–3356.

[6] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), 699–715.

[7] C. Sulem and P-L. Sulem, *The nonlinear Schrödinger Equation. Self-focusing and collapse*, Applied Mathematical Sciences **139**. Springer (1999).
[8] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc A.M.S. **102** (1988), 874–878.

Felipe Linares
IMPA
Estrada Dona Castorina 110
Rio de Janeiro, 22460-320
Brazil
E-mail: linares@impa.br

Gustavo Ponce
Department of Mathematics
University of California
Santa Barbara, CA 93106
USA
E-mail: ponce@math.ucsb.edu

Jean-Claude Saut
Département de Mathématiques
Université de Paris XI (Paris-Sud)
91405 Orsay
FRANCE
E-mail: Jean-Claude.SAUT@math.u-psud.fr