Restricted Overlapping Balancing Domain Decomposition Methods and Restricted Coarse Problems for the Helmholtz Problem

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Abstract

Overlapping balancing domain decomposition methods and their combination with restricted additive Schwarz methods are proposed for the Helmholtz equation. These new methods also extend previous work on non-overlapping balancing domain decomposition methods toward simplifying their coarse problems and local solvers. They also extend restricted Schwarz methods, originally designed to overlapping domain decomposition and Dirichlet local solvers, to the case of non-overlapping domain decomposition and/or Neumann and Sommerfeld local solvers. Finally, we introduce coarse spaces based on partitions of unity and planes waves, and show how oblique projection coarse problems can be designed from restricted additive Schwarz methods. Numerical tests are presented.

Key words: Schwarz preconditioner, domain decomposition, coarse spaces, balancing, partition of unity, Sommerfeld interface condition, restricted additive Schwarz method, elliptic equations, finite elements, Helmholtz equation

1 Introduction

In this paper we introduce new two-level overlapping Schwarz preconditioners for the Helmholtz equation based on Overlapping Balancing Domain Decom-

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position (OBDD) and Restricted Additive Schwarz (RAS) algorithms. Balancing Domain Decomposition (BDD) methods [22] belong to a family of preconditioners based on non-overlapping decomposition of subregions that have been tested successfully on several challenging large scale applications [18,23,24,31,32,34]. The BDD algorithms were extended recently to the case of overlapping subregions and named Overlapping Balancing Domain Decomposition (OBDD) algorithms; see [17]. Like on the BDD methods, coarse space and weighting diagonal matrices play crucial roles in making the proposed algorithms both scalable with respect to the number of subdomains [16] as well as to make the local Neumann subproblems on the overlapping subregions consistent on each iteration of the preconditioned system. These algorithms also differ from the standard overlapping additive Schwarz (AS) methods on hybrid II form [26,31,34] since those are based on Dirichlet local problems on the overlapping subregions.

The main goal of this paper is to introduce new effective preconditioners for the Helmholtz equation based on two-level overlapping domain decomposition techniques. We generalize the OBDD methods to the Helmholtz equation by considering Sommerfeld interface condition for the local problems instead of Neumann interface condition and name them OBDD-H methods. The use of Sommerfeld boundary conditions makes the local problems solvable and hence, coarse problems are only introduced to accelerate the convergence rate of the iterative schemes. We propose coarse spaces based on combinations of partitions of unity [28,30,29] and plane waves [7,9,21,25,35]. We also introduce several OBDD-H variations based on the RAS class of preconditioners [2,3,10,34] such as WRAS, WASH and RASHO, i.e. the Weighted Restricted Additive Schwarz, the Weighted Additive Schwarz with Harmonic Extensions and the Restricted Additive Schwarz with Harmonic Extension methods. In the numerical experiment section we show that such variations improves considerably the convergence of the iterative schemes. Finally we introduce a new concept for designing very effective coarse problems. We also extend the RAS technique to define restriction and prolongation operators and coarse matrices for the coarse problem.

We remark that there are some papers that address preconditioning for the Helmholtz equation. For overlapping type methods there are the two-level overlapping RAS methods [3] and the one-level OSM-D [1,4,15]. For the non-overlapping cases we cite a few approaches such as FETI-H [7,9], the two-Lagrange multipliers methods and optimized interface conditions [5,8,11–13,19–21], and also [33,35,14] and references therein.

The paper is organized as follows: we devote Section 2 to formulate the problem and introduce its finite element discretization; in Section 3 we describe all local and global components of the OBDD-H, WASH-H, and WRAS-H, while in Section 4 we build the associated algorithms and construct the Restricted

Coarse Problems; in Section 5, we test numerically the algorithms on the Wave Guided Problem, and in Section 6 we make some conclusions.

2   The Finite Element Formulation

Consider the Helmholtz problem:

$$
\begin{aligned}
-\Delta u^* - k^2 u^* &= f &&\text{in}\quad \Omega \\
u^* &= g_D &&\text{on}\quad \partial\Omega_D \\
\frac{\partial u^*}{\partial n} &= g_N &&\text{on}\quad \partial\Omega_N \\
\frac{\partial u^*}{\partial n} + iku^* &= g_S &&\text{on}\quad \partial\Omega_S
\end{aligned}
\tag{1}
$$

where Ω is a bounded polygonal region in $\Re^n$, and $\partial\Omega_D$, $\partial\Omega_N$, and $\partial\Omega_S$ are the subsets of $\partial\Omega$ where the Dirichlet, Neumann, and Sommerfeld boundary conditions are imposed, respectively. From a Green's formula and conjugation of the test functions, we can reduce (1) into the following problem on the variational form: find $u^* - u_D^* \in H_D^1(\Omega)$ such that,

$$
\begin{aligned}
a(u^*, v) &= \int_\Omega (\nabla u^* \cdot \nabla \bar{v} - k^2 u^* \bar{v})\, dx + ik \int_{\partial\Omega_S} u^* \bar{v}\, ds \\
&= \int_\Omega f\bar{v}\, dx + \int_{\partial\Omega_N} g_N \bar{v}\, ds + \int_{\partial\Omega_S} g_S \bar{v} = F(v), \ \ \forall v \in H_D^1(\Omega),
\end{aligned}
\tag{2}
$$

where u_D^* is an extension of g_D to $H^1(\Omega)$, and $H_D^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions vanishing on $\partial\Omega_D$.

Let $V \subset H_D^1(\Omega)$ be the finite element space of continuous piecewise linear functions vanishing on $\partial\Omega_D$ and associated with a standard triangulation $\mathcal{T}_h(\Omega)$. We also assume that u_D on $\partial\Omega_D$ is a piecewise linear continuous function on $\mathcal{T}^h(\partial\Omega_D)$ and we have eliminated u_D by a trivial extension by zero on all the remaining nodes of $\mathcal{T}_h(\Omega)$. We then obtain a discrete problem on the following form: find $u \in V$ such that

$$
a(u, v) = f(v), \ \ \forall\, v \in V.
\tag{3}
$$

Using the standard basis functions, (3) can be rewritten as a linear system of equations of the form

$$
Au = f.
\tag{4}
$$

3 Overlapping Schwarz Methods

3.1 Partitioning and Subdomains

Given the triangulation $\mathcal{T}^h(\Omega)$, we assume that a domain partition by elements has been applied and resulted in N non-overlapping connected subdomains $\Omega_i, i = 1, \ldots N$, such that

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for} \quad j \neq i.$$

We define the overlapping subdomains Ω_i^δ as follows: let Ω_i^1 be the one-overlap element extension of Ω_i, where $\Omega_i^1 \supset \Omega_i$ is obtained by including all the immediate neighboring elements $\tau_h \in \mathcal{T}^h(\Omega)$ of Ω_i such that $\overline{\tau}_h \cap \overline{\Omega}_i \neq \emptyset$. Using the idea recursively, we can define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i = \Omega_i^0 \subset \Omega_i^1 \subset \cdots \subset \Omega_i^\delta.$$

Here the integer $\delta \geq 0$ indicates the level of element extension and δh the approximate length of the extension.

3.2 Partitions of the Unity

In this section we construct a partition of unity (PU) on $\overline{\Omega}$. We first construct the piecewise linear and continuous function $\hat{\vartheta}_i^e$ and then we define the partition of unity ϑ_i^e. Here the integer $e \geq 1$ has similar meaning as δ except that we associate the extension $e = c$ to define the partition of unity functions that will be used as coarse basis functions for the coarse problem, and $e = w$ to define the weighting diagonal matrices used on the restriction and extension local operators. We associate δ to define the extended subdomains where the local solvers are defined. As we will see, the support of ϑ_i^e is going to be $\overline{\Omega}_i^e$, i.e. as if we had extended $\overline{\Omega}_i$ recursively e layers. The $\hat{\vartheta}_i^e$ is built as follows: let $\hat{\vartheta}_i^e(x) = 1$ on nodes x of $\overline{\Omega}_i$. For the first layer of neighboring nodes x of $\overline{\Omega}_i$ we set $\hat{\vartheta}_i^e(x) = 1 - 1/e$, and recursively until $m = e$, we set $\hat{\vartheta}_i^e(x) = 1 - m/e$ for the $(m)st$ layer of neighboring nodes x of $\overline{\Omega}_i$. For the remaining nodes, i.e. for nodes x on $\overline{\Omega} \backslash \overline{\Omega}_i^e$ we define $\hat{\vartheta}_i^e(x) = 0$. Here and later in this paper, we implicitly assume that a piecewise linear and continuous function is constructed from the nodal values. The partition of unity ϑ_i^e is defined as

$$\vartheta_i^e = I_h\left(\frac{\hat{\vartheta}_i^e}{\sum_{j=1}^N \hat{\vartheta}_j^e}\right), \quad i = 1, \cdots, N,$$

where I_h is the nodal piecewise linear interpolant on $\mathcal{T}^h(\Omega)$. It is easy to verify that $\sum_{i=1}^N \vartheta_i^e(x) = 1, \quad 0 \leq \vartheta_i^e(x) \leq 1$, and $|\nabla \vartheta_i^e(x)| \leq C/(eh)$, when $x \in \overline{\Omega}$,

and the support of ϑ_i^e is $\overline{\Omega}_i^e$. On Figure 1 we illustrate a case in which the function ϑ_i^3 is associated to a subdomain Ω_i touching the boundary $\partial\Omega$.
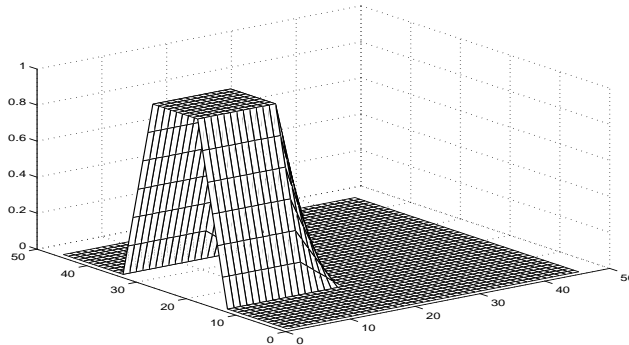

Fig. 1. An illustration of a function ϑ_i^3

We note that the ϑ_i^e does not vanish on the Dirichlet boundary $\partial\Omega_D$ and therefore, they do not belong to V. Hence, we modify ϑ_i^e by defining $\theta_i^e(x) = \vartheta_i^e(x)$ on the nodes x in $\overline{\Omega}\backslash\partial\Omega_D$, and $\theta_i^e(x) = 0$ on the nodes x on $\partial\Omega_D$. We obtain $\theta_i^e \in V$. We remark that we could have defined smoother $\theta_i^e \in V$ by forcing a controlled energy decrease near $\partial\Omega_D$ (see[28]), however the numerical tests indicate that such a strategy does not bring any gain for the Helmholtz problem.

3.3 Coarse Spaces and Coarse Problems

Let c be a positive integer. For the Poisson problem case, the coarse space $V_0^{c,p} \subset V$ is defined as the space spanned by the discrete functions $I_h(\theta_i^c Q_j), i = 1,\ldots,N$, and $j = 1,\cdots,p$. For example, if we take $p = 1$ we let $Q_1(x) = 1$, and if we take $p = 3$ we include also the linear functions, $Q_2(x) = x_1$ and $Q_3(x) = x_2$ i.e. the first and second coordinate of x. Or we can consider Q_i as the i eigenfunction associated to the i lowest eigenvalue of the discrete Poisson problem on Ω_i^δ with Neumann data; see [30]. For the Helmholtz equation, the coarse basis functions are built from the θ_i^c and planar waves, where the coarse space $V_0^{c,p} \subset V$ is defined as the space spanned by the discrete functions $I_h(\theta_i^c Q_j), i = 1,\ldots,N$, and $j = 1,\cdots,p$, where $Q_j(x) = e^{ik\Theta_j^T x}$, $\Theta_j^T = (\cos(\theta_j),\sin(\theta_j))$, and $\theta_j = (j-1) \times \frac{2\pi}{p}, j = 1,\cdots,p$. The use of planar waves for coarse space functions or for finite element discretizations are widely used nowadays and can be found for instance in [9,21,25,35]. For an illustration for the case $p = 8$, see Fig 2. We define the coarse space projection $P_0^{c,p} : V \rightarrow V_0^{c,p}$ as

$$a(P_0^{c,p}u, v) = a(u, v), \quad \forall v \in V_0^{c,p}.$$

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We next express $P_0^{c,p}$ in matrix notation. Defining the extension matrix $(R_0^{c,p})^T$: $Z^{N*p} \to V$ consisting of columns $I_h(\theta_i^c Q_j)$, the projection $P_0^{c,p}$ can be written as

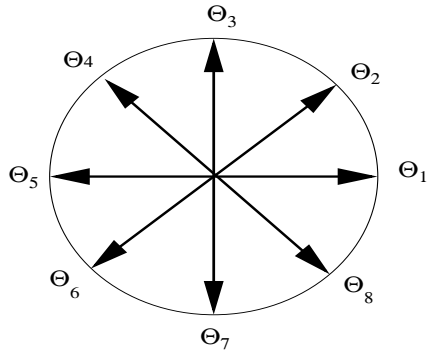$$P_0^{c,p} = (R_0^{c,p})^T [R_0^{c,p} A (R_0^{c,p})^T]^{-1} R_0^{c,p}. \tag{5}$$

Fig. 2. An illustration of planar waves with $p = 8$

3.4 Local Problems

We next review some of the known local problems based on overlapping Schwarz methods [1–3,10,17,31,33] and then we propose new ones.

3.4.1 Additive Schwarz

Let Ω_i^δ be the extended subdomain associated to Ω_i with $1 \leq \delta$. Let us denote by $V_i^\delta \subset V$, $i = 1, \cdots, N$, the local space of functions in $H^1(\Omega_i^\delta)$ which are continuous and piecewise linear on the elements of $\mathcal{T}^h(\Omega_i^\delta)$ and vanishing on $\partial\Omega_i^\delta \backslash \{\partial\Omega_N \cup \partial\Omega_S\}$. For each subdomain Ω_i^δ, we define the corresponding restriction operator $R_i^\delta : V \to V_i^\delta$, $i = 1, \cdots, N$, by $v_i^\delta = R_i^\delta v$, where $v_i^\delta(x) = v(x)$ for any node $x \in \Omega_i^\delta \cup \{\partial\Omega_N \cup \partial\Omega_S\}$ and $v_i^\delta(x) = 0$ for any node $x \in \partial\Omega_i^\delta \backslash \partial\Omega_D$. We note that $(R_i^\delta)^T V_i^\delta \subset V$ and $V = \sum_{i=1}^N (R_i^\delta)^T V_i^\delta$.

We define the local space projection operators $\tilde{T}_{i,AS} : V \to V_i^\delta$ as

$$a_i(\tilde{T}_{i,AS} u, v) = a(u, (R_i^\delta)^T v), \quad \forall v \in V_i^\delta,$$

where

$$a_i(u, v) = a((R_i^\delta)^T u, (R_i^\delta)^T v), \forall u, v \in V_i^\delta,$$

and define $T_{i,AS} = (R_i^\delta)^T \tilde{T}_{i,AS}$. In matrix notation,

$$T_{i,AS} = (R_i^\delta)^T [R_i^\delta A (R_i^\delta)^T]^{-1} R_i^\delta A.$$

Except for sufficient small subdomains, we cannot guarantee that local Dirichlet problems are invertible for the Helmholtz problem; see [1,4,25].

3.4.2 Restricted Additive Schwarz

Let $1 \leq w \leq \delta$, and Ω_i^δ, V_i^δ and R_i^δ be given as before. We introduce the weighting diagonal matrix $D_i^w : V \to V$ by defining $v_i^w = D_i^w v$, where $v_i^w(x) = \theta_i^w(x)v(x)$ for any node $x \in \overline{\Omega}$. Note that $v_i^w \in (R_i^\delta)^T V_i^\delta$ whenever $1 \leq w \leq \delta$. We introduce two versions of RAS local solvers: the Weighted Restricted Additive Schwarz (WRAS) and the Weighted Additive Schwarz with Harmonic Extension (WASH); see [3,10]. Algebraically these two local solvers are defined as follows:

$$T_{i,WRAS}^w = (R_i^\delta D_i^w)^T [R_i^\delta A (R_i^\delta)^T]^{-1} R_i^\delta A,$$

or

$$T_{i,WASH}^w = (R_i^\delta)^T [R_i^\delta A (R_i^\delta)^T]^{-1} R_i^\delta D_i^w A.$$

Again, except for sufficient small subdomains, we cannot guarantee that the local problems are invertible for the Helmholtz problem.

3.4.3 Overlapping Balancing Domain Decomposition for Helmholtz

Let Ω_i^δ and D_i^w be given as above, and here we assume $1 \leq w \leq \delta + 1$. The fundamental difference between the local problems introduced previously and the ones to be introduced here is that previously we had used zero Dirichlet boundary condition on the interior interfaces for the local problems while here we use Sommerfeld boundary conditions. As a consequence we can allow non-overlapping ($\delta = 0$) and the more general case $w = \delta + 1$, and so permitting the weighting matrices D_i^w not to vanish on the boundary nodes of the subdomains.

Let us denote by $\tilde{V}_i^\delta$, $i = 1, \cdots, N$, the local space of functions in $H^1(\Omega_i^\delta)$ which are continuous and piecewise linear on $\mathcal{T}^h(\Omega_i^\delta)$ and vanishing on $\partial\Omega_i^\delta \cap \partial\Omega_D$. For each subdomain Ω_i^δ, we define the corresponding restriction operator $\tilde{R}_i^\delta : V \to \tilde{V}_i^\delta$, $i = 1, \cdots, N$, by $v_i^\delta = \tilde{R}_i^\delta v$, where $v_i^\delta(x) = v(x)$ for any $x \in \overline{\Omega}_i^\delta$. For the local solvers, we respect the original boundary condition and impose Sommerfeld boundary condition on $\partial\Omega_i^\delta \backslash \partial\Omega$. The corresponding local bilinear forms on $\tilde{V}_i^\delta$ are defined as

$$\tilde{a}_i(u_i, v_i) = \int_{\Omega_i^\delta} (\nabla u_i \cdot \nabla \overline{v}_i - k^2 u_i \overline{v}_i)\, dx + ik \int_{\partial\Omega_i^\delta \backslash (\partial\Omega_D \cup \partial\Omega_N)} u_i \overline{v}_i\, ds.$$

We note that having Sommerfeld boundary condition on a positive measure set guarantees that local Helmholtz problems are well-posed on the continuous partial differential equation sense; see [25]. Such approach, i.e. considering Sommerfeld boundary conditions on the interior boundaries, has been widely

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used on methods based on non-overlapping subdomains; see [1,5,9]. The associated local problem operators for the OBDD-H methods are then defined as $\tilde{T}^w_{i,OBDD-H} : V \rightarrow \tilde{V}^\delta_i$

$$\tilde{a}_i(\tilde{T}^w_{i,OBDD-H}u, v) = a(u, (\tilde{R}^\delta_i D^w_i)^T v), \quad \forall v \in \tilde{V}^\delta_i, \ i = 1, \cdots, N, \qquad (6)$$

and then setting $T^w_{i,OBDD-H} = (\tilde{R}^\delta_i D^w_i)^T \tilde{T}^w_{i,OBDD-H}$. The matrix form of the local problems (6) are given by

$$T^w_{i,OBDD-H} = (\tilde{R}^\delta_i D^w_i)^T (\tilde{A}^\delta_i)^{-1} \tilde{R}^\delta_i D^w_i A \qquad (7)$$

where the matrix $\tilde{A}^\delta_i$ is obtained from the $\tilde{a}_i(\cdot, \cdot)$ by using the basis functions of $\tilde{V}^\delta_i$.

3.4.4 *Restricted Additive Schwarz for Helmholtz*

We next extend the RAS technique to the OBDD-H local problems. The Weighted Restricted Additive Schwarz (WRAS-H) and the Weighted Additive Schwarz with Harmonic Extension (WASH-H) local solvers for the Helmholtz problems are defined as

$$T^w_{i,WRAS-H} = (\tilde{R}^\delta_i D^w_i)^T (\tilde{A}^\delta_i)^{-1} \tilde{R}^\delta_i A, \qquad (8)$$

and

$$T^w_{i,WASH-H} = (\tilde{R}^\delta_i)^T (\tilde{A}^\delta_i)^{-1} \tilde{R}^\delta_i D^w_i A. \qquad (9)$$

4 Two-level Hybrid Preconditioners

In this section we build several preconditioners on a hybrid II format; see [31]. We describe implementation issues in more detail for the Overlapping Sommerfeld-Sommerfeld preconditioner and then setup the stage to present the other algorithms.

OBDD-H Algorithm:

$$T^{w,c,p}_{OBDD-H} = P^{c,p}_0 + (I - P^{c,p}_0)(\sum_{i=1}^{N} T^w_{i,OBDD-H})(I - P^{c,p}_0). \qquad (10)$$

Let us define an initial guess $z \in V^\delta_0$ by $z = P^{c,p}_0 u$. The function z can be calculated without knowledge of the exact solution u because

$$P^{c,p}_0 u = (R^{c,p}_0)^T [R^{c,p}_0 A (R^{c,p}_0)^T]^{-1} R^{c,p}_0 f.$$

Let the error e be defined by $e = u - z$. Using the definition of $T_{OBDD-H}^{w,c,p}$, and the fact that $P_0^{c,p}$ is an orthogonal projection, we obtain

$$T_{OBDD-H}^{w,c,p}e = g, \tag{11}$$

where

$$g = (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,OBDD-H}^{w})(I - P_0^{c,p})u.$$

The function g can be calculated without knowledge of the exact solution u because

$$(\sum_{i=1}^{N} T_{i,OBDD-H}^{w})(I - P_0^{c,p})u = \sum_{i=1}^{N} (\tilde{R}_i^w)^T (\tilde{A}_i^\delta)^{-1} \tilde{R}_i^w (f - Az).$$

Instead of solving $Au = f$, we solve (11) by PGMRES. If a PGMRES is used, the Krylov space is built by forming powers $(T_{i,OBDD-H}^w)^m g$. Hence, using that $(I - P_0^{c,p})^2 = I - P_0^{c,p}$, we need to solve only one coarse problem per iteration of the preconditioned system.

To resume the section, we write down the other operators based on the local problems previously defined.

AS Algorithm:

$$T_{AS}^{w,c,p} = P_0^{c,p} + (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,AS}^{w})(I - P_0^{c,p}), \tag{12}$$

WRAS Algorithm:

$$T_{WRAS}^{w,c,p} = P_0^{c,p} + (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,WRAS}^{w})(I - P_0^{c,p}), \tag{13}$$

WASH Algorithm:

$$T_{WASH}^{w,c,p} = P_0^{c,p} + (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,WASH}^{w})(I - P_0^{c,p}). \tag{14}$$

WRAS-H Algorithm:

$$T_{WRAS-H}^{w,c,p} = P_0^{c,p} + (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,WRAS-H}^{w})(I - P_0^{c,p}), \tag{15}$$

WASH-H Algorithm:

$$T_{WASH-H}^{w,c,p} = P_0^{c,p} + (I - P_0^{c,p})(\sum_{i=1}^{N} T_{i,WASH-H}^w)(I - P_0^{c,p}). \qquad (16)$$

When using the right preconditioned GMRES, our numerical experiments show that the **WRAS-H Algorithm** outperforms the other algorithms; see Tables 1-3. We next show how can we improve even further the convergence rate of this algorithm by introducing new coarse problems based on RAS.

As we will see in the numerical experiments, the larger the size of the support of the coarse basis functions is, the more effective the preconditioner is. We note that after solving the local problems the fine residual is more concentrated near the boundary of the subdomains. The idea of introducing the restricted coarse problem is to force the coarse basis functions to act on the residuals near their center of gravity and hence, the coarse problem is going to be better conditioned. The restricted coarse matrix extension $(\tilde{R}_0^{w,p})^T : Z^{N*p} \rightarrow V$ consists of the columns $(R_i^w)^T R_i^w I_h Q_j$. In matrix notation the coarse problem can be written as

$$P_{0,RC}^{c,w,p} = (R_0^{c,p})^T [\tilde{R}_0^{w,p} A(R_0^{c,p})^T]^{-1} \tilde{R}_0^{w,p},$$

and the **WRAS-H Algorithm** with restricted coarse problems as

WRAS-H-RC Algorithm:

$$T_{WRAS-H-RC}^{w,c,p} = P_{0,RC}^{c,w,p} + (I - P_{0,RC}^{c,w,p})(\sum_{i=1}^{N} T_{i,WRAS-H}^w)(I - P_{0,RC}^{c,w,p}). \qquad (17)$$

5   Numerical Results

As a test problem we consider the **Wave Guided Problem** for solving the Helmholtz's equation on the unit square with all three boundary conditions: homogeneous Neumann on the horizontal sides, homogeneous Sommerfeld on the right vertical side and a constant Dirichlet condition identical to 1 on the left vertical side; see [7,9]. The constant k denotes the wave number associated with the original problem. For each subdomain, on a preconditioning coarse level, we use a superposition of p local planar waves smoothed by a partition of unity. The parameter δ refers to the size of the overlap; $\delta = 0$ means a non-overlapping domain decomposition method. We note that δ is an important factor for the cost of the computation because it decides the size of local problems, hence, the small overlap cases [6] are stressed on the tests. The parameter w refers to the support of the weighted restriction operator: if we

impose Dirichlet boundary condition on inner boundaries, then we require $1 \leq w \leq \delta$, or if we impose Sommerfeld condition instead, we require $1 \leq w \leq \delta+1$. The parameter c refers to the size of the support of the coarse basis functions. We also check the numerical behavior of the preconditioners with respect to different meshes measured by n, the number of nodes in each direction. The integer $nsub$ represents the number of subdomains in each direction. In all experiments we run right preconditioned GMRES [27] until the l_2 initial residual is reduced by a factor of 10^{-6}.

From Tables 1-3 we test a variety of combinations of overlap (δ, w, c), mesh sizes and number of subdomains. The restricted version of Overlapping Sommerfeld-Sommerfeld Balancing Domain Decomposition Methods outperforms the other versions. In addition, we see that the new restricted coarse problem has an impressive performance for making the Helmholtz preconditioners very effective even in the cases where the overlap is small or zero.

We now focus on the **WRAS-H-RC Algorithm** since it outperforms the other algorithms considered in this paper. We also focus on the case $w = \delta+1$ since it gives the best combination between w and δ for best performance; see Table 1 and a comparison between Tables 2 and 3. This combination is equivalent to choosing a w as large as possible so that the partition of unity $(\tilde{R}_i^\delta)^T \tilde{R}_i^\delta D_i^w$ is preserved. The Table 4 indicates that the coarse functions should be as smooth as possible, i.e. the parameter c should be as larger as possible. Also, the more we increase the overlap size δ of the subdomains the more effective the preconditioner is going to be in terms of iterations. On Table 5 we can see that the preconditioner is more than scalable when we increase the number of subdomains and keep the size of the local problems fixed. This effect is well-known since the discrete problem becomes very ill-conditioned for the Helmholtz problem when the mesh size resolution gets coarse compared to the wave length $O(1/k)$; see also Tables 6 and 7. Comparing Tables 5, 8 and 9 we see that the number of plane waves p to build the coarse problem is also an important factor to improve the performance of the preconditioners when k gets larger.

6   Conclusions

We present new two-level overlapping Schwarz preconditioners for the Helmholtz equation. These methods are based on Overlapping Balancing Domain Decomposition and Restricted Additive Schwarz methods. The preconditioners are of algebraic nature and are easy to implement on unstructured meshes. We also introduce the Restricted Coarse Problem technique to improve the convergence rate. The numerical experiments on the Wave Guided Problem show that among all the algorithms considered in this paper the **WRAS-H-**

RC Algorithm is the most robust and scalable with respect to overlapping sizes, mesh sizes and number of subdomains. The results also show that the new Restricted Coarse Problem is a very effective preconditioner tool for the Helmholtz equation.

Table 1
Conditions: $n = 257$, $nsub = 8$, $p = 4$, $k = 20$.

$(\delta, w, c) =$	(1,1,7)	(2,2,7)	(0,1,7)	(1,2,7)	(2,3,7)
$OBDD - H$	153	84	158	85	43
$WASH - H$	33	30	166	84	42
$WRAS - H$	26	21	150	74	36
$WRAS - H - RC$	23	17	34	19	13

Table 2
Conditions: $p = 4$, $k = 20$, and extension sizes $(\delta = 1, w = 1, c = 7)$

n $(nsub) =$	65 (2)	129 (4)	257 (8)	513 (16)
$OBDD - H$	66	158	153	41
$WRAS - H$	33	65	26	31
$WRAS - H - RC$	32	62	23	11

Table 3
Conditions: $p = 4$, $k = 20$, and extension sizes $(\delta = 1, w = 2, c = 7)$

n $(nsub) =$	65 (2)	129 (4)	257 (8)	513 (16)
$OBDD - H$	51	106	85	20
$WRAS - H$	43	87	74	17
$WRAS - H - RC$	40	71	19	6

Table 4
WRAS-H-RC Conditions: $n = 257$, $nsub = 8$, $p = 4$, $k = 20$

	$w = \delta + 1$							
$c=$	1	2	3	4	5	6	7	8
$\delta = 0$	78	67	54	46	40	37	34	32
$\delta = 1$		36	31	25	22	21	19	18
$\delta = 2$			19	18	16	14	13	12

Table 5
WRAS-H-RC Conditions: $p = 4$, $k = 20$

	$\delta = 1, w = 2$					
n (nsub), $c=$	2	3	4	5	6	7
65 (2)	41	40	40	40	40	40
129 (4)	76	77	71	69	68	71
257 (8)	36	31	25	22	21	19
513 (16)	10	10	8	7	7	6

Table 6
WRAS-H-RC Conditions: $p = 4$, $n = 257$

(δ, w, c)	(0, 1, 7)			
$nsub \setminus k=$	5	10	20	40
2	17	28	74	103
4	10	32	104	229
8	4	13	34	257

Table 7
WRAS-H-RC Conditions: $p = 4$, $n = 513$

(δ, w, c)	(0, 1, 7)		
$nsub \setminus k=$	10	20	40
4	38	107	238
8	12	49	285
16	4	12	42

Table 8
WRAS-H-RC Conditions: $p = 8$, $k = 20$

	$\delta = 1, w = 2$				
n (nsub), $c=$	2	3	4	5	6
65 (2)	37	36	35	34	34
129 (4)	13	14	14	13	14
257 (8)	4	5	5	4	4
513 (16)	2	2	2	2	2

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Table 9
WRAS-H-RC Conditions: $k = 20$, c=6

n ($nsub$), p=	$\delta = 1, w = 2$							
	1	2	3	4	5	6	7	8
129 (4)	153	98	112	68	61	31	33	14
257 (8)	299	109	64	21	17	7	13	4

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