

# Submanifolds of codimension two attaining equality in an extrinsic inequality

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## Abstract

We provide a parametric construction in terms of minimal surfaces of the Euclidean submanifolds of codimension two and arbitrary dimension that attain equality in an inequality due to De Smet, Dillen, Verstraelen and Vrancken. The latter involves the scalar curvature, the norm of the normal curvature tensor and the length of the mean curvature vector.

Let  $f: M^n \rightarrow \mathbb{Q}_c^{n+p}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold into a space form of dimension  $n+p$  and constant sectional curvature  $c$ . Let  $s$  denote the normalized scalar curvature of  $M^n$  and let  $s_N$  be given by

$$n(n-1)s_N = \|R^\perp\|,$$

where  $R^\perp$  is the normal curvature tensor of  $f$ . Explicitly,

$$s = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \langle R(e_i, e_j)e_j, e_i \rangle$$

and

$$s_N = \frac{2}{n(n-1)} \left( \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq r < s \leq p}} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{1/2},$$

where  $R$  is the curvature tensor of  $M^n$  and  $\{e_1, \dots, e_n\}$  (resp.,  $\{\xi_1, \dots, \xi_p\}$ ) is an orthonormal basis of the tangent (resp., normal) space.

The pointwise inequality

$$s \leq c + \|H\|^2 - s_N \tag{*}$$

relates the intrinsic scalar curvature  $s$  to the extrinsic data on the right-hand-side. Here  $H$  denotes the mean curvature vector of  $f$ . It was proved for codimension  $p = 2$  by De Smet, Dillen, Verstraelen and Vrancken in [15]. Also, the pointwise structure of the

shape operators of submanifolds attaining equality was determined. It was shown that equality holds at  $x \in M^n$  if and only if there exist orthonormal bases  $\{e_1, \dots, e_n\}$  and  $\{\eta, \zeta\}$  of the tangent and normal spaces at  $x$ , respectively, such that the shape operators  $A_\eta$  and  $A_\zeta$  have the form

$$A_\eta = \begin{bmatrix} \lambda & \mu & 0 & \cdots & 0 \\ \mu & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}; \quad A_\zeta = \begin{bmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (1)$$

The inequality (\*) is also known to hold for surfaces [10] and for submanifolds with flat normal bundle [2] of any codimension as well as for various special classes of submanifolds (see [8] and the references therein). Moreover, it was conjectured in [15] to hold for any submanifold of a space form. Recently, the conjecture was proved in [3] and [14], respectively, for three dimensional submanifolds with arbitrary codimension and for submanifolds of codimension three and any codimension of space forms.

For any isometric immersion  $f: M^n \rightarrow \mathbb{Q}_c^{n+p}$ , it was shown in [8] that (\*) holds at a point  $x \in M^n$  if and only if the inequality

$$\sum_{\alpha, \beta=1}^p \|[B_\alpha, B_\beta]\|^2 \leq \left( \sum_{\alpha=1}^p \|B_\alpha\|^2 \right)^2$$

is satisfied for the traceless parts  $B_1, \dots, B_p$  of the shape operators of  $f$  with respect to any orthonormal normal frame at  $x$ .

An important consequence of this reformulation of the inequality is that it readily implies that the class of submanifolds  $f: M^n \rightarrow \mathbb{Q}_c^{n+p}$  for which equality holds is invariant under conformal transformations of the ambient space. In fact, under such a transformation the traceless parts of the shape operators only change by multiplication by a common smooth function on  $M^n$ .

By the above, the class of isometric immersions  $f: M^n \rightarrow \mathbb{R}^{n+2}$  attaining equality everywhere in (\*) contains any composition of an inversion in  $\mathbb{R}^{n+2}$  with a minimal isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+2}$  whose shape operators are as in (1) with  $\lambda = 0$ . Notice that such minimal submanifolds belong to the class of austere submanifolds of rank two, first studied in arbitrary codimension by Bryant [1] for  $n = 3$  and then by Dajczer-Florit [5] for any dimension  $n$ .

In this paper, we provide an explicit local construction of all Euclidean submanifolds  $f: M^n \rightarrow \mathbb{R}^{n+2}$  attaining equality everywhere in the inequality (\*). For  $n = 2$ , such submanifolds are precisely the surfaces in  $\mathbb{R}^4$  whose ellipses of curvature are circles at any point, and this was considered in our previous paper [6]. It turns out that several steps of the proof of the main result of that paper can be adapted to the present general case.

Our construction starts with a simply connected minimal surface  $g: M^2 \rightarrow \mathbb{R}^{n+2}$ , oriented by a global conformal diffeomorphism onto either the complex plane or the unit disk. Then we consider its conjugate minimal surface  $h: M^2 \rightarrow \mathbb{R}^{n+2}$ , each of whose components with respect to this global parameter is the harmonic conjugate of the corresponding component of  $g$ . Equivalently,  $h_* = g_* \circ J$ , where  $J$  is the complex structure on  $M^2$  compatible with its orientation. Now we decompose the position vector of  $h$  in its tangent and normal components with respect to  $g$ , i.e.,

$$h = g_* h^T + h^N.$$

Finally, on the complement of the subset of isolated points of  $M$  where  $h^N$  vanishes, let  $\Lambda_1$  be the unit bundle of the vector subbundle  $\Lambda$  of the normal bundle of  $g$  that is orthogonal to  $h^N$ . We can now state our main result.

**Theorem 1.** *Assume that  $n \geq 3$  and define a map  $\phi: \Lambda_1 \rightarrow \mathbb{R}^{n+2}$  by*

$$\phi(y, w) = g(y) + g_* J h^T(y) + \|h^N(y)\|w. \quad (2)$$

*Then, at regular points,  $\phi$  parameterizes an  $n$ -dimensional submanifold in  $\mathbb{R}^{n+2}$  attaining equality in the inequality (\*).*

*Conversely, any submanifold  $f: M^n \rightarrow \mathbb{R}^{n+2}$ ,  $n \geq 3$ , free of umbilical and minimal points and attaining equality in the inequality (\*) can be parameterized in this way.*

By combining the preceding result with the generalized Weierstrass parameterization of Euclidean minimal surfaces  $g: M^2 \rightarrow \mathbb{R}^{n+2}$  (cf. [11], [12]) we have a parametric representation of the submanifolds  $f: M^n \rightarrow \mathbb{R}^{n+2}$  attaining equality in (\*).

We also characterize in terms of our construction the submanifolds that are images of austere submanifolds of rank two attaining equality in the inequality (\*) in either  $\mathbb{R}^{n+2}$ ,  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  by an inversion in the first case or a stereographic projection in the other two cases. Here,  $\mathbb{S}_d^{n+2} = \mathbb{S}^{n+2}(de_{n+3}; d)$  is the sphere in  $\mathbb{R}^{n+3}$  with radius  $d$  centered at  $de_{n+3}$  and  $\mathbb{H}_d^{n+2} = \mathbb{H}^{n+2}(-de_{n+3}; d)$  is the hyperbolic space

$$\mathbb{H}_d^{n+2} = \{X \in \mathbb{L}^{n+3} : \langle X + de_{n+3}, X + de_{n+3} \rangle = -d^2\}.$$

Moreover, we regard  $\mathbb{R}^{n+2}$  as the hyperplane through the origin and normal to the unit vector  $e_{n+3}$  in either  $\mathbb{R}^{n+3}$  or Lorentzian space  $\mathbb{L}^{n+3}$ , and by the stereographic projection of  $\mathbb{H}_d^{n+2}$  onto the open ball  $B(0; 2d) \subset \mathbb{R}^{n+2}$  we mean the map that assigns to each  $P \in \mathbb{H}_d^{n+2}$  the point of  $\mathbb{R}^{n+2}$  where the line through the points  $-2de_{n+3}$  and  $P$  intersects  $\mathbb{R}^{n+2}$ . Let

$$G = g + ih: L^2 \rightarrow \mathbb{C}^{n+2} \approx \mathbb{R}^{n+2} + i\mathbb{R}^{n+2}$$

be the holomorphic representative of the minimal surface  $g: L^2 \rightarrow \mathbb{R}^{n+2}$ . For any real number  $k$ , we denote by  $\mathcal{H}_k^{n+1} \subset \mathbb{C}^{n+2}$  the quadric

$$\mathcal{H}_k^{n+1} = \{Z \in \mathbb{C}^{n+2} : \langle\langle Z, Z \rangle\rangle = k\},$$

where  $\langle\langle \cdot, \cdot \rangle\rangle: \mathbb{C}^{n+2} \times \mathbb{C}^{n+2} \rightarrow \mathbb{C}$  denotes the linear inner product on  $\mathbb{C}^{n+2}$ .

**Theorem 2.** *The map  $\phi$  in (2) parameterizes the composition of an austere submanifold of rank two in  $\mathbb{R}^{n+2}$ ,  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  attaining equality in the inequality (\*) with an inversion in  $\mathbb{R}^{n+2}$  with respect to a hypersphere centered at the origin or a stereographic projection of  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  onto  $\mathbb{R}^{n+2}$  and  $B(0; 2d) \subset \mathbb{R}^{n+2}$ , respectively, if and only if  $G$  takes values in  $\mathcal{H}_k^{n+1}$ , with  $k = 0, 4d^2$  or  $-4d^2$ , respectively.*

As a consequence of Theorems 1 and 2, we obtain the following parameterization of all austere  $n$ -dimensional submanifolds of rank two in  $\mathbb{R}^{n+2}$  attaining equality in (\*).

**Corollary 3.** *Any austere  $n$ -dimensional submanifold of rank two in  $\mathbb{R}^{n+2}$ ,  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  attaining equality in the inequality (\*) can be parameterized by*

$$\psi = \mathcal{I} \circ \phi,$$

where  $\phi$  is given by (2) in terms of a minimal surface  $g: L^2 \rightarrow \mathbb{R}^{n+2}$  whose holomorphic representative  $g + ih$  takes values in a quadric  $\mathcal{H}_k^{n+1}$  of  $\mathbb{C}^{n+2}$  with  $k = 0, 4d^2$  or  $-4d^2$ , respectively, and  $\mathcal{I}$  is an inversion with respect to a hypersphere centered at the origin or the inverse of a stereographic projection of  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  onto  $\mathbb{R}^{n+2}$  or  $B(0; 2d) \subset \mathbb{R}^{n+2}$ , respectively.

Our next result extends Theorem 2 in [6]. Let  $G = g + ih: L^2 \rightarrow \mathbb{C}^{n+2}$  be the holomorphic representative of the minimal surface  $g: L^2 \rightarrow \mathbb{R}^{n+2}$  associated to a submanifold  $\phi: M^n \rightarrow \mathbb{R}^{n+2}$  attaining equality in the inequality (\*). Let  $\tilde{G} = \tilde{g} + i\tilde{h}$  be the holomorphic representative of the minimal surface  $\tilde{g}: L^2 \rightarrow \mathbb{R}^{n+2}$  associated to its composition  $\tilde{\phi} = \mathcal{I} \circ \phi$  with an inversion  $\mathcal{I}$  in  $\mathbb{R}^{n+2}$  with respect to a sphere of radius  $d$  taken, for simplicity, centered at the origin. Then  $G$  and  $\tilde{G}$  are related as follows.

**Theorem 4.** *If  $\phi$  is not the composition of an austere submanifold attaining equality in the inequality (\*) with an inversion, then  $\tilde{G} = \overline{T_d \circ G}$ , where  $T_d = d^2 T$  and  $T: \mathbb{C}^{n+2} \rightarrow \mathbb{C}^{n+2}$  is the holomorphic map*

$$T(Z) = \frac{Z}{\langle\langle Z, Z \rangle\rangle}.$$

As pointed out in [6], the holomorphic map  $T_d: \mathbb{C}^m \rightarrow \mathbb{C}^m$  for any  $m$  and  $T$  defined above, can be regarded as the *inversion* in  $\mathbb{C}^m$  with respect to the quadric  $\mathcal{H}_{d^2}^{m-1}$ .

The following byproduct of Theorem 4 yields, in particular, a transformation for minimal surfaces in  $\mathbb{R}^{n+2}$ .

**Corollary 5.** *The holomorphic inversion map  $T$  preserves the class of holomorphic curves  $G = g + ih: L^2 \rightarrow \mathbb{C}^{n+2}$  whose real and imaginary parts  $g$  and  $h$  define conjugate minimal immersions into  $\mathbb{R}^{n+2}$ .*

After the paper was submitted for publication, two independent proofs of the inequality (\*) for submanifolds of arbitrary codimension have appeared (see [9] and [13]). Moreover, in [9] also the pointwise structure of the second fundamental forms of the submanifolds that attain equality was determined. In particular, it was shown that the first normal spaces of such submanifolds, i.e., the subspaces of the normal spaces that are spanned by the image of the second fundamental form, have dimension either two or three. If the first case holds everywhere and the submanifold has dimension at least four and is not minimal, then it is not difficult to verify from the Codazzi equations that the first normal spaces form a parallel subbundle of the normal bundle. This can also be derived from Theorem 2 of [7]. Then, it is a standard fact that the submanifold reduces codimension to two, i.e., it is a submanifold of codimension two of a totally geodesic submanifold of the ambient space. Therefore, our main result provides a complete classification of all non-minimal submanifolds (of arbitrary codimension) of dimension at least four that attain equality in the inequality (\*) and whose first normal spaces have dimension two everywhere. It is a very interesting problem to study the remaining cases.

Notice that minimal submanifolds that attain equality in (\*) have necessarily first normal spaces of dimension two but may have arbitrary codimension. These submanifolds were considered in [5]. In particular, it was shown that complete examples must be Riemannian products  $L^3 \times \mathbb{R}^{n-3}$ . Moreover, when the manifold is Kaehler a complete classification in terms of a Weierstrass-type representation was given. By composing such submanifolds with an inversion in Euclidean space one obtains non-minimal submanifolds that attain equality in (\*) whose first normal spaces have dimension three but do not reduce codimension.

## 1 The proofs

We first prove the converse of Theorem 1. Let  $f: M^n \rightarrow \mathbb{R}^{n+2}$  be an isometric immersion attaining equality everywhere in the inequality (\*) and free of minimal and umbilical points. We must prove that there exists a minimal surface  $g: L^2 \rightarrow \mathbb{R}^{n+2}$  and a diffeomorphism  $\psi: \Lambda_1 \rightarrow M^n$ , with  $\Lambda_1$  defined as in the statement, such that  $\phi := f \circ \psi$  is given by (2).

By the result in [15], there exist orthonormal tangent and normal frames  $\{e_1, \dots, e_n\}$  and  $\{\eta, \zeta\}$ , respectively, with respect to which the shape operators  $A_\eta$  and  $A_\zeta$  have the form (1). Our assumption that  $f$  is free of minimal and umbilical points is equivalent to  $\lambda$  and  $\mu$  being nowhere vanishing, respectively. By Lemma 5.2 in [15], we also have that  $e_k(\lambda) = 0$  and  $\nabla_{e_k}^\perp \eta = 0$  for  $k \geq 3$ . Therefore  $\lambda\eta$  is a *Dupin principal normal* of multiplicity  $n - 2$ . This means that the subspaces

$$E_\eta(x) = \{T \in T_x M: \alpha_f(T, X) = \lambda \langle T, X \rangle \eta, \text{ for all } X \in T_x M\},$$

where  $\alpha_f$  is the second fundamental form of  $f$  with values in the normal bundle, define

a smooth distribution  $E_\eta$  of rank  $n - 2$  satisfying

$$T(\lambda) = 0 \text{ and } \nabla_T^\perp \eta = 0 \text{ for any } T \in E_\eta. \quad (3)$$

In addition, since  $\mu \neq 0$  we have that  $\lambda\eta$  is *generic*, in the sense that  $E_\eta = \ker(A_\eta - \lambda I)$ . It is well-known that  $E_\eta$  is an involutive distribution whose leaves are (mapped by  $f$  into) round  $(n - 2)$ -dimensional spheres in  $\mathbb{R}^{n+2}$ .

By the first equation in (3), the function  $r = 1/\lambda$  gives rise to a smooth function on the quotient space  $L^2$  of leaves of  $E_\eta$ , which is also denoted by  $r$ . Let  $g: M^n \rightarrow \mathbb{R}^{n+2}$  be given by  $g = f + r\eta$ . From (3) we have

$$g_*T = f_*T - rf_*A_\eta T = 0,$$

hence  $g$  also factors through a map on  $L^2$ , still denoted by  $g$ . By Proposition 1 in [4] there exist a smooth unit vector field  $\xi$  normal to  $g$  and a diffeomorphism  $\psi: \Lambda_1 \rightarrow M^n$ , where  $\Lambda_1$  is the unit bundle of the vector subbundle  $\Lambda$  of the normal bundle of  $g$  that is orthogonal to  $\xi$ , such that

$$\eta(y, w) = (\eta \circ \psi)(y, w) = g_*\nabla r(y) + \rho(y)\xi(y) + \Omega(y)w$$

for  $\rho, \Omega \in C^\infty(L)$  satisfying

$$\|\nabla r\|^2 + \rho^2 + \Omega^2 = 1. \quad (4)$$

Moreover, we have

$$\phi(y, w) = (f \circ \psi)(y, w) = g(y) - r(y)\eta(y, w). \quad (5)$$

We identify the tangent space  $T_{(y,w)}\Lambda_1$  with the direct sum  $T_yL \oplus \{w\}^\perp$ , where  $\{w\}^\perp$  denotes the orthogonal complement of  $\text{span}\{w\}$  in  $\Lambda(y)$ , and write vectors  $Y \in T_{(y,w)}\Lambda_1$  as  $Y = (X, V)$  according to this decomposition. We also denote by  $\mathcal{V}$  the corresponding *vertical* subbundle of  $T\Lambda_1$ , whose fiber  $\mathcal{V}(y, w)$  at  $(y, w) \in \Lambda_1$  is  $\{w\}^\perp$ . Clearly, we have that  $\psi_*\mathcal{V} = E_\eta$ .

Since  $\lambda\eta$  is a Dupin principal normal of  $f$ , the orthogonal complement  $\eta^\perp$  of  $\text{span}\{\eta\}$  in  $T_f^\perp M$  is constant in  $\mathbb{R}^{n+2}$  along  $E_\eta$ . Therefore, if  $\zeta$  is a unit vector field spanning  $\eta^\perp$ , then the map  $\zeta \circ \psi$ , which we also denote simply by  $\zeta$ , is constant along  $\mathcal{V}$ . Thus we may write

$$\zeta = g_*Z + a\xi \quad (6)$$

for a smooth vector field  $Z$  and  $a \in C^\infty(L)$  satisfying

$$\|Z\|^2 + a^2 = 1. \quad (7)$$

Since  $(\ker(A_\eta - \lambda I))^\perp$  has rank two everywhere, the function  $a$  is nowhere vanishing. Otherwise  $g_*Z$  would be somewhere normal to  $\phi$ , which would imply, by taking tangent components for  $X = Z$  in

$$\phi_*(X, 0) = g_*X - \langle \nabla r, X \rangle \eta - r\eta_*(X, 0),$$

that  $\psi_*(Z, 0) \in \ker(A_\eta - \lambda I)$ , a contradiction.

From now on we follow closely the proof of Theorem 1 of [6].

**Lemma 6.** *The following facts hold:*

- (i)  $\rho = 0$  and  $L^2$  is orientable with a complex structure  $J$  such that  $JZ = \nabla r$ ,
- (ii)  $h = -r\zeta$  satisfies that  $h_* = g_* \circ J$ .

Before proving Lemma 6, let us see how it yields the converse statement of the theorem. It follows from Lemma 6-(ii) that  $g$  and  $h$  are conjugate minimal surfaces. Moreover, from (6) we obtain  $-ra\xi = h^N$  and  $-rg_*\nabla r = g_*Jh^T$ , where  $J$  is the complex structure on  $L^2$  given by Lemma 6-(i). Since  $\rho = 0$  and  $\|Z\| = \|\nabla r\|$  by Lemma 6-(i), it follows that  $a^2 = \Omega^2$  by (4) and (7). Thus  $|r\Omega| = \|h^N\|$ , and then (5) reduces to (2).

The proof of Lemma 6 will be given in several steps. We start with the following preliminary facts.

**Sublemma 7.** *We have*

$$\langle B_w Z, X \rangle = a \langle \nabla_X^\perp w, \xi \rangle \quad \text{for all } X \in TM \text{ and } w \in \Lambda \quad (8)$$

and

$$\text{Hess}r(Z) - \frac{1}{r}Z + B_\xi(a\nabla r - \rho Z) + a\nabla\rho = 0, \quad (9)$$

where  $B_w$  and  $B_\xi$  denote the shape operators of  $g$  in the normal directions  $w$  and  $\xi$ , respectively.

*Proof:* From the fact that  $\mathcal{V} = \ker A_\zeta$  we obtain

$$0 = \langle \zeta_*(y, w)(X, 0), \phi_*(y, w)(0, V) \rangle \quad \text{for any } y \in L^2, w \in \Lambda_1(y) \text{ and } V \in w^\perp.$$

Then (8) follows by differentiating (6) and using that  $\phi_*(0, V) = -r\Omega V$ . Thus

$$\zeta_*(X, 0) = g_*DX + \langle K, X \rangle \xi, \quad (10)$$

where

$$DX = \nabla_X Z - aB_\xi X \quad \text{and} \quad K = \nabla a + B_\xi Z. \quad (11)$$

The orthogonality between  $\eta$  and  $\zeta$  yields

$$\langle Z, \nabla r \rangle + a\rho = 0. \quad (12)$$

Hence,

$$\begin{aligned} \langle \nabla_X Z, \nabla r \rangle &= XZ(r) - \langle Z, \text{Hess}r(X) \rangle \\ &= -X(a)\rho - aX(\rho) - \langle Z, \text{Hess}r(X) \rangle \\ &= -\langle \rho\nabla a + a\nabla\rho + \text{Hess}r(Z), X \rangle. \end{aligned} \quad (13)$$

It follows from (10), (11) and (13) that

$$\langle \zeta_*(X, 0), \eta \rangle = \langle DX, \nabla r \rangle + \rho \langle K, X \rangle = -\langle \text{Hess } r(Z) + B_\xi(a\nabla r - \rho Z) + a\nabla \rho, X \rangle. \quad (14)$$

On the other hand,

$$\langle \zeta, \phi_*(X, 0) \rangle = X \langle \zeta, g \rangle - \langle \zeta_*(X, 0), g - r\eta \rangle = \langle Z, X \rangle + r \langle \zeta_*(X, 0), \eta \rangle, \quad (15)$$

thus (9) follows from (14), (15) and the fact that  $\zeta$  is normal to  $\phi$ . ■

We now express in terms of  $g$  the condition that the shape operators of  $f$  are given by (1). It is convenient to use the orthonormal frame  $Y_1, Y_2, Y_j = e_j$  for  $3 \leq j \leq n$ , with

$$Y_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad Y_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$$

With respect to this frame, the matrices in (1) become

$$A_\eta = \begin{bmatrix} \lambda + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}; \quad A_\zeta = \begin{bmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Therefore,

$$\begin{cases} \eta_* Y_1 = -(\lambda + \mu)\phi_* Y_1 - \omega(Y_1)\zeta \\ \eta_* Y_2 = -(\lambda - \mu)\phi_* Y_2 - \omega(Y_2)\zeta \\ \zeta_* Y_i = -\mu\phi_* Y_j + \omega(Y_i)\eta, \quad 1 \leq i \neq j \leq 2, \end{cases} \quad (16)$$

where  $\omega(Y) = \langle \nabla_Y^\perp \zeta, \eta \rangle$  and

$$\langle \phi_* Y_i(y, w), \phi_*(0, V) \rangle = 0, \quad 1 \leq i \leq 2, \quad \text{for any } (y, w) \in \Lambda_1 \text{ and } V \in \{w\}^\perp. \quad (17)$$

Write  $Y_1 = (X_1, V_1)$  and  $Y_2 = (X_2, V_2)$  according to the splitting  $T_{(y,w)}\Lambda_1 = T_y L \oplus \{w\}^\perp$ . Using that

$$\eta_*(0, V) = \Omega V, \quad \phi_*(0, V) = -r\Omega V \quad \text{and} \quad \zeta_*(0, V) = 0,$$

we obtain from (17) that

$$(\eta_*(X_i, 0))_{w^\perp} = -\Omega V_i, \quad 1 \leq i \leq 2. \quad (18)$$

Taking components in  $H = g_* TL \oplus \text{span}\{\xi\} \oplus \text{span}\{w\}$  in (16) gives

$$\begin{cases} r^2 \mu (\eta_*(X_1, 0))_H = \theta_1 (g_* X_1 - r_1 \eta) + r \omega((X_1, 0)) \zeta \\ r^2 \mu (\eta_*(X_2, 0))_H = \theta_2 (-g_* X_2 + r_2 \eta) - r \omega((X_2, 0)) \zeta \\ \zeta_*(X_i, 0) = -\mu g_* X_j + \mu (r (\eta_*(X_j, 0))_H + r_j \eta) + \omega((X_i, 0)) \eta, \quad 1 \leq i \neq j \leq 2, \end{cases} \quad (19)$$

where  $\theta_1 = 1 + r\mu$ ,  $\theta_2 = 1 - r\mu$  and  $r_i = \langle \nabla r, X_i \rangle$  for  $1 \leq i \leq 2$ .

We have

$$\eta_*(X, 0) = g_*Q_w X + \langle T_w, X \rangle \xi + \langle P_w, X \rangle w + (\eta_*(X, 0))_{w^\perp},$$

where

$$\begin{cases} Q_w = \text{Hess } r - \rho B_\xi - \Omega B_w \\ T_w = \nabla \rho + B_\xi \nabla r + \frac{\Omega}{a} B_w Z \\ P_w = \nabla \Omega + B_w \nabla r - \frac{\rho}{a} B_w Z. \end{cases}$$

It follows immediately from (9) and (14) that

$$\omega(X, 0) = -\frac{1}{r} \langle Z, X \rangle,$$

in terms of the metric induced by  $g$ . Using this, the  $w$ -component of (19) gives

$$\begin{cases} r^2 \mu \langle P_w, X_1 \rangle = -\theta_1 \Omega r_1 \\ r^2 \mu \langle P_w, X_2 \rangle = \theta_2 \Omega r_2 \\ r^2 \mu \langle P_w, X_i \rangle = -r \mu \Omega r_i + \Omega \langle Z, X_j \rangle, \quad 1 \leq i \neq j \leq 2. \end{cases} \quad (20)$$

Replacing the first two equations into the last two yields

$$r_1 = -\langle Z, X_2 \rangle \quad \text{and} \quad r_2 = \langle Z, X_1 \rangle. \quad (21)$$

Taking the tangent component to  $g$  of (19) and using (21) we obtain

$$\begin{cases} r^2 \mu Q_w X_1 - \theta_1 S X_1 + r_2 Z = 0 \\ r^2 \mu Q_w X_2 + \theta_2 S X_2 + r_1 Z = 0 \\ r D X_1 + r \mu S X_2 - r^2 \mu Q_w X_2 + r_2 \nabla r = 0 \\ r D X_2 + r \mu S X_1 - r^2 \mu Q_w X_1 - r_1 \nabla r = 0, \end{cases} \quad (22)$$

where we denoted

$$S = I - \langle \nabla r, * \rangle \nabla r. \quad (23)$$

Finally, computing the  $\xi$ -component of (19) gives

$$\begin{cases} r^2 \mu \langle T_w, X_1 \rangle = -\theta_1 \rho r_1 - a \langle Z, X_1 \rangle \\ r^2 \mu \langle T_w, X_2 \rangle = \theta_2 \rho r_2 + a \langle Z, X_2 \rangle \\ r \langle K, X_i \rangle = r^2 \mu \langle T_w, X_j \rangle + r \mu \rho r_j - \rho \langle Z, X_i \rangle, \quad 1 \leq i \neq j \leq 2. \end{cases} \quad (24)$$

We now use that

$$\delta_{ij} = \langle \phi_* Y_i, \phi_* Y_j \rangle, \quad 1 \leq i, j \leq 2. \quad (25)$$

From (18) we get

$$\langle \phi_*(0, V_1), \phi_*(X_2, 0) \rangle = \langle -r\Omega V_1, -r\eta_*(X_2, 0) \rangle = -r^2\Omega^2 \langle V_1, V_2 \rangle = \langle \phi_*(0, V_2), \phi_*(X_1, 0) \rangle.$$

On the other hand,

$$\langle \phi_*(0, V_1), \phi_*(0, V_2) \rangle = r^2\Omega^2 \langle V_1, V_2 \rangle = \langle (\phi_*(X_1, 0))_{w^\perp}, (\phi_*(X_2, 0))_{w^\perp} \rangle.$$

Hence (25) reduces to  $\delta_{ij} = \langle (\phi_*(X_1, 0))_H, (\phi_*(X_2, 0))_H \rangle$ , which gives

$$\begin{aligned} \delta_{ij} = & \langle X_i, X_j \rangle - r_i r_j - 2r \langle Q_w X_i, X_j \rangle + r^2 (\langle Q_w X_i, Q_w X_j \rangle \\ & + \langle T_w, X_i \rangle \langle T_w, X_j \rangle + \langle P_w, X_i \rangle \langle P_w, X_j \rangle). \end{aligned} \quad (26)$$

Then, we argue exactly as in the proof of Sublemma 7 in [6] to prove that

$$\|X_1\|^2 = r^2\mu^2 + r_1^2 + r_2^2 = \|X_2\|^2. \quad (27)$$

First, taking inner products of the first and second equations in (22) by  $X_2$  and  $-X_1$ , respectively, and adding them up, bearing in mind (21), yields

$$\langle X_1, X_2 \rangle = 0.$$

On the other hand, we compute from the first two equations in (22) that

$$\begin{cases} r^2\mu \langle Q_w X_1, X_1 \rangle = \theta_1 (\|X_1\|^2 - r_1^2) - r_2^2 \\ r^2\mu \langle Q_w X_2, X_2 \rangle = -\theta_2 (\|X_2\|^2 - r_2^2) + r_1^2 \\ r \langle Q_w X_1, X_2 \rangle = -r_1 r_2. \end{cases} \quad (28)$$

Using (4), (7), (12) and the first two equations in (22) we have

$$\begin{cases} r^4\mu^2 \|Q_w X_1\|^2 = \theta_1^2 (\|X_1\|^2 - (1 + \rho^2 + \Omega^2)r_1^2) - 2\theta_1 (r_2^2 + a\rho r_1 r_2) + (1 - a^2)r_2^2 \\ r^4\mu^2 \|Q_w X_2\|^2 = \theta_2^2 (\|X_2\|^2 - (1 + \rho^2 + \Omega^2)r_2^2) - 2\theta_2 (r_1^2 - a\rho r_1 r_2) + (1 - a^2)r_1^2 \\ r^4\mu^2 \langle Q_w X_1, Q_w X_2 \rangle = (\theta_1\theta_2(1 + \rho^2 + \Omega^2) - \theta_1 - \theta_2 - a^2 + 1)r_1 r_2 - a\rho(\theta_1 r_1^2 - \theta_2 r_2^2). \end{cases} \quad (29)$$

From (21) and (24) we obtain

$$r^2\mu \langle T_w, X_1 \rangle = -\theta_1 \rho r_1 - a r_2 \quad \text{and} \quad r^2\mu \langle T_w, X_2 \rangle = \theta_2 \rho r_2 - a r_1.$$

Thus,

$$\begin{cases} r^4\mu^2 \langle T_w, X_1 \rangle^2 = \theta_1^2 \rho^2 r_1^2 + a^2 r_2^2 + 2\theta_1 a \rho r_1 r_2 \\ r^4\mu^2 \langle T_w, X_1 \rangle \langle T_w, X_2 \rangle = (a^2 - \theta_1\theta_2\rho^2)r_1 r_2 + \theta_1 a \rho r_1^2 - \theta_2 a \rho r_2^2 \\ r^4\mu^2 \langle T_w, X_2 \rangle^2 = \theta_2^2 \rho^2 r_2^2 + a^2 r_1^2 - 2\theta_2 a \rho r_1 r_2. \end{cases} \quad (30)$$

From the first two equations in (20) we get

$$\begin{cases} r^4 \mu^2 \langle P_w, X_1 \rangle^2 = \theta_1^2 \Omega^2 r_1^2 \\ r^4 \mu^2 \langle P_w, X_2 \rangle^2 = \theta_2^2 \Omega^2 r_2^2 \\ r^4 \mu^2 \langle P_w, X_1 \rangle \langle P_w, X_2 \rangle = -\theta_1 \theta_2 \Omega^2 r_1 r_2. \end{cases} \quad (31)$$

Replacing (28), (29), (30) and (31) into (26) we end up with (27).

It follows from (21) and (27) that  $\nabla r$  and  $Z$  are orthogonal vector fields on  $L^2$  with the same norm, thus there exists a complex structure  $J$  on  $L^2$  such that  $JZ = \nabla r$ . We conclude from (12) that  $\rho = 0$ , and the proof of (i) is completed.

We now prove (ii). Replacing the first two equations of (22) into the last two gives

$$\begin{cases} rDX_1 + SX_2 + r_1Z + r_2\nabla r = 0 \\ rDX_2 - SX_1 + r_2Z - r_1\nabla r = 0, \end{cases}$$

which can be written as

$$rDX = -JX - \langle \nabla r, X \rangle Z. \quad (32)$$

On the other hand, replacing the first two equations of (24) into the last two yields

$$\begin{cases} r\langle K, X_1 \rangle = a\langle Z, X_2 \rangle \\ r\langle K, X_2 \rangle = -a\langle Z, X_1 \rangle. \end{cases} \quad (33)$$

Taking (21) into account, the preceding equations reduce to

$$rB_\xi Z + \nabla(ar) = 0. \quad (34)$$

From (8) we have

$$\tilde{\nabla}_X \xi = -g_* B_\xi X + \nabla_X^\perp \xi = -g_* B_\xi X - \frac{1}{a}(\alpha_g(Z, X) - \langle B_\xi Z, X \rangle \xi),$$

where  $\alpha_g$  denotes the second fundamental form of  $g$ . Hence,

$$-ar\tilde{\nabla}_X \xi + r\langle B_\xi Z, X \rangle \xi = arg_* B_\xi X + r\alpha_g(Z, X).$$

In view of (34) the left-hand-side is  $\tilde{\nabla}_X(-ar\xi)$ . For the right-hand-side we have

$$\begin{aligned} arg_* B_\xi X + r\alpha_g(Z, X) &= arg_* B_\xi X + r(\tilde{\nabla}_X g_* Z - g_* \nabla_X Z) \\ &= g_*(arB_\xi X - r\nabla_X Z - X(r)Z) + \tilde{\nabla}_X(rg_* Z). \end{aligned}$$

Therefore, we obtain using (32) that

$$h_* X = g_*(arB_\xi X - r\nabla_X Z - X(r)Z) = g_*(-rDX - X(r)Z) = g_* JX.$$

We now prove the direct statement of Theorem 1. We need the following fact from [6].

**Proposition 8.** *Let  $g: M^2 \rightarrow \mathbb{R}^{n+2}$  be a simply connected oriented minimal surface with complex structure  $J$  compatible with the orientation and let  $h: M^2 \rightarrow \mathbb{R}^{n+2}$  be a conjugate minimal surface such that  $h_* = g_* \circ J$ . Then  $r = \|h\|$  satisfies  $\|\nabla r\| \leq 1$  everywhere. Moreover, on the complement of the subset of isolated points of  $M^2$  where  $a = \sqrt{1 - \|\nabla r\|^2}$  vanishes, there exists a smooth unit normal vector field  $\xi$  to  $g$  such that*

$$h = r(g_*J\nabla r - a\xi).$$

Furthermore,

$$\langle B_\delta J\nabla r, X \rangle + a\langle \nabla_X^\perp \delta, \xi \rangle = 0 \quad \text{for all } \delta \in \text{span}\{\xi\}^\perp \quad (35)$$

and

$$B_\xi = \frac{1}{ar}(r\text{Hess}r - S) \circ J, \quad (36)$$

where  $S$  is given by (23).

Setting  $\eta(y, w) = g_*\nabla r(y) - aw$ , we have by Proposition 8 that  $\phi = g - r\eta$ . From

$$\phi_*(X, 0) = g_*X - \langle \nabla r, X \rangle \eta - r\eta_*(X, 0) \quad \text{and} \quad \phi_*(0, V) = -arV$$

it follows that  $\eta$  is a unit normal vector field to  $\phi$ . Moreover, since  $\phi + r\eta = g$  does not depend on  $w$ , we have that  $A_\eta|_{\mathcal{V}} = r^{-1}I$ .

Let  $\zeta$  be defined by (6) with  $Z = -J\nabla r$ . Then  $\zeta$  has unit length and is orthogonal to  $\eta$ . We obtain from (35) that (10) holds, hence we have (14) with  $\rho = 0$ , and also (15). From (36) we get

$$r\text{Hess}r(Z) - Z + arB_\xi\nabla r = 0,$$

which implies, using (14) (with  $\rho = 0$ ) and (15), that  $\zeta$  is normal to  $\phi$ . It also follows from (35) that  $A_\zeta|_{\mathcal{V}} = 0$ .

Therefore, to complete the proof it suffices to show that there exists an orthonormal frame  $\{Y_1, Y_2\}$  on the open subset of regular points of  $\phi$  (with respect to the metric induced by  $\phi$ ) satisfying (16) and (17).

For each  $w \in \Lambda_1$ , since  $B_w$  and  $B_\xi$  are traceless symmetric  $2 \times 2$  matrices, we have

$$(B_w + B_\xi J)^2 = \alpha^2 I \quad (37)$$

for some  $\alpha \in \mathbb{R}$ . We need only prove the existence of the orthonormal frame  $\{Y_1, Y_2\}$  on the complement of the subset with empty interior of points of  $\Lambda_1$  where  $\alpha$  is nonzero. At such a point, set  $\mu = -a/r^2\alpha$ . Since  $B_w + B_\xi J = \alpha R_w$  for some reflection  $R_w$  by (37), it follows using (36) that

$$B_w = \frac{1}{a}\left(\text{Hess}r - \frac{1}{r}S\right) - \frac{a}{r^2\mu}R_w. \quad (38)$$

For each  $w \in \Lambda_1$ , let  $\{\bar{X}_1, \bar{X}_2\}$  be the orthonormal basis of  $TL$  (with respect to the metric induced by  $g$ ) formed by eigenvectors of  $R_w$ , with  $\bar{X}_1$  corresponding to the eigenvalue  $+1$  and  $\bar{X}_2 = J\bar{X}_1$ . Define

$$X_i = \frac{r\mu}{a}\bar{X}_i, \quad 1 \leq i \leq 2,$$

and set

$$V_i = -\frac{1}{a}(\eta_*(X_i, 0))_{w^\perp}, \quad 1 \leq i \leq 2.$$

We claim that  $\{Y_1, Y_2\}$  given by  $Y_i = (X_i, V_i)$  is the desired orthonormal frame.

It follows from  $(\eta_*(X_i, 0))_{w^\perp} = -aV_i$  that (17) is satisfied. In particular, in order to check that  $\{Y_1, Y_2\}$  is an orthonormal frame it suffices to verify (26). It also follows from  $(\eta_*(X_i, 0))_{w^\perp} = -aV_i$  that the  $w^\perp$ -components of both sides of all equations in (16) coincide. Therefore, it suffices to prove that (19), or equivalently, (20), (22) and (24), holds for  $X_1$  and  $X_2$ .

Since we have (21), because  $JX_1 = X_2$  and  $JZ = \nabla r$ , system (20) reduces to its first two equations. These are in turn equivalent to

$$rB_w \nabla r + \frac{a}{r\mu}R_w \nabla r + \nabla(ar) = 0,$$

which follows from (38).

Now, (38) also implies that

$$rQ_w = S + \frac{a^2}{r\mu}R_w.$$

Moreover, from (36) we get (32), hence (22) is satisfied.

From (36) we obtain (34), and hence (33). Moreover, (36) and (38) imply that

$$B_\xi \nabla r + B_w Z + \frac{a}{r^2\mu}R_w Z = 0,$$

thus (24) is satisfied.

Finally, we now have (28), (29), (30) and (31), hence (26) follows by using that  $\langle X_1, X_2 \rangle = 0$  and  $\|X_i\| = r\mu/a$  for  $1 \leq i \leq 2$ . ■

**Remark 9.** It was shown in [4] that an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+p}$ ,  $n \geq 4$ , that carries a generic Dupin principal normal  $\eta$  of multiplicity  $n - 2$  is a rotational submanifold over a surface whenever  $\text{trace } A_\eta \neq n\|\eta\|$  and  $\text{trace } A_\eta$  is constant along the leaves of the corresponding eigendistribution. Our result shows that the assumption that  $\text{trace } A_\eta \neq n\|\eta\|$  can not be removed.

For the proof of Theorem 2, we first recall that, given an isometric immersion  $f: M^n \rightarrow \mathbb{R}^N$  and an inversion with respect to a sphere of radius  $d$  centered at  $P_0 \in \mathbb{R}^N$ , the map

$$\mathcal{P}\xi = \xi - 2 \frac{\langle f - P_0, \xi \rangle}{\langle f - P_0, f - P_0 \rangle} (f - P_0) \quad (39)$$

is a vector bundle isometry between the normal bundles  $T_f^\perp M$  and  $T_{\mathcal{I} \circ f}^\perp M$ . Moreover, the shape operators  $A_\xi$  and  $\tilde{A}_{\mathcal{P}\xi}$  of  $f$  and  $\mathcal{I} \circ f$  with respect to  $\xi$  and  $\mathcal{P}\xi$ , respectively, are related by

$$\tilde{A}_{\mathcal{P}\xi} = \frac{1}{d^2} (\langle f - P_0, f - P_0 \rangle A_\xi + 2 \langle f - P_0, \xi \rangle I). \quad (40)$$

Similar results hold for an ‘‘inversion’’

$$\mathcal{I}(P) = P_0 - \frac{d^2}{\langle P - P_0, P - P_0 \rangle} (P - P_0), \quad P \neq P_0,$$

in Lorentzian space  $\mathbb{L}^N$  with respect to a hyperbolic space

$$\mathbb{H}^{N-1}(P_0; d) := \{P \in \mathbb{L}^N : \langle P - P_0, P - P_0 \rangle = -d^2\}$$

(see Lemma 15 of [6]), with  $d^2$  replaced by  $-d^2$  in formula (40). We also observe that a stereographic projection of  $\mathbb{S}_d^N$  onto  $\mathbb{R}^N$  can be regarded as the restriction to  $\mathbb{S}_d^N$  of an inversion in  $\mathbb{R}^{N+1}$  with respect to the sphere of radius  $2d$  centered at  $2de_{N+1}$ . Similarly, a stereographic projection of  $\mathbb{H}_d^N$  onto  $B(0; 2d) \subset \mathbb{R}^N$  can be viewed as the restriction to  $\mathbb{H}_d^N$  of an inversion in  $\mathbb{L}^{N+1}$  with respect to  $\mathbb{H}^N(-2de_{N+1}; 2d)$ . In both cases, we regard  $\mathbb{R}^N$  as the hyperplane through the origin orthogonal to  $e_{N+1}$  in either  $\mathbb{R}^{N+1}$  or  $\mathbb{L}^{N+1}$ , respectively.

Let us denote by  $\mathbb{Q}_\epsilon^{n+2}$  either  $\mathbb{R}^{n+2}$ ,  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$ , according as  $\epsilon = 0$ ,  $\epsilon = 1$  or  $\epsilon = -1$ , respectively. Given an austere isometric immersion  $f: M^n \rightarrow \mathbb{Q}_\epsilon^{n+2}$  attaining equality in the inequality (\*), let  $\hat{J}$  be the complex structure on  $T_f^\perp M$  determined by the opposite orientation to that induced by the vector bundle isometry  $\mathcal{P}: T_f^\perp M \rightarrow T_{\mathcal{I} \circ f}^\perp M$  given by (39) from the orientation on  $T_{\mathcal{I} \circ f}^\perp M$  defined by the orthonormal frame  $\{\eta, \zeta\}$  as in (1). Here  $\mathcal{I}$  is an inversion in  $\mathbb{R}^{n+2}$  with respect to a hypersphere of radius  $2d$  centered at the origin or a stereographic projection of  $\mathbb{S}_d^{n+2}$  or  $\mathbb{H}_d^{n+2}$  onto  $\mathbb{R}^{n+2}$  or  $B(0; 2d) \subset \mathbb{R}^{n+2}$ , according as  $\epsilon = 0$ ,  $\epsilon = 1$  or  $\epsilon = -1$ , respectively. Set also  $\bar{\epsilon} = 1$  if  $\epsilon = 1$  or  $0$  and  $\bar{\epsilon} = -1$  if  $\epsilon = -1$ .

**Proposition 10.** *Let  $f: M^n \rightarrow \mathbb{Q}_\epsilon^{n+2}$  be an austere isometric immersion that attains equality in the inequality (\*). Then the holomorphic representative  $G$  of the minimal surface associated to  $\mathcal{I} \circ f$  is given by*

$$G = \epsilon 2de_{n+3} + 2\bar{\epsilon}d^2 \frac{(f - \epsilon 2de_{n+3})^N + i\hat{J}(f - \epsilon 2de_{n+3})^N}{\langle (f - \epsilon 2de_{n+3})^N, (f - \epsilon 2de_{n+3})^N \rangle}, \quad (41)$$

where  $(f - \epsilon 2de_{n+3})^N$  denotes the normal component (in  $\mathbb{Q}_\epsilon^{n+2}$ ) of the position vector  $f - \epsilon 2de_{n+3}$  in either  $\mathbb{R}^{n+2}$ ,  $\mathbb{R}^{n+3}$  or  $\mathbb{L}^{n+3}$ , according as  $\epsilon = 0$ ,  $\epsilon = 1$  or  $\epsilon = -1$ , respectively.

*Proof:* Set  $\bar{d} = 2d$  and  $P_0 = \epsilon \bar{d}e_{n+3}$ . Define

$$\tilde{\zeta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2}(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta) \quad \text{and} \quad \tilde{\eta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2}(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta), \quad (42)$$

where  $\mathcal{P}$  is given by (39),

$$\bar{\epsilon}\bar{d}^2\bar{\lambda} = 2\langle f - P_0, \eta \rangle \quad \text{and} \quad \bar{\epsilon}\bar{d}^2\bar{\nu} = 2\langle f - P_0, \zeta \rangle.$$

Using (40), we obtain that the shape operators  $\tilde{A}_{\tilde{\eta}}$  and  $\tilde{A}_{\tilde{\zeta}}$  of  $\tilde{f} = \mathcal{I} \circ f$  are given as in (1) with  $\lambda$  and  $\mu$  replaced, respectively, by

$$\tilde{\lambda} = (\bar{\lambda}^2 + \bar{\nu}^2)^{1/2} \quad \text{and} \quad \tilde{\mu} = \frac{\langle f - P_0, f - P_0 \rangle}{\bar{\epsilon}\bar{d}^2}\mu. \quad (43)$$

The holomorphic curve  $G = g + ih$  associated to  $\tilde{f}$  is given by

$$g = \tilde{f} + \tilde{r}\tilde{\eta} = \tilde{f} + \frac{\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta}{\bar{\lambda}^2 + \bar{\nu}^2} \quad \text{and} \quad h = -\tilde{r}\tilde{\zeta} = -\frac{\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta}{\bar{\lambda}^2 + \bar{\nu}^2}, \quad (44)$$

where  $\tilde{r} = 1/\tilde{\lambda}$ . We have

$$\bar{d}^4(\bar{\lambda}^2 + \bar{\nu}^2) = 4(\langle f - P_0, \zeta \rangle^2 + \langle f - P_0, \eta \rangle^2) = 4\langle (f - P_0)^N, (f - P_0)^N \rangle. \quad (45)$$

On the other hand, from

$$\mathcal{P}\eta = \eta - 2\frac{\langle f - P_0, \eta \rangle}{\langle f - P_0, f - P_0 \rangle}(f - P_0) \quad \text{and} \quad \mathcal{P}\zeta = \zeta - 2\frac{\langle f - P_0, \zeta \rangle}{\langle f - P_0, f - P_0 \rangle}(f - P_0)$$

we obtain

$$\bar{\epsilon}\bar{d}^2(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta) = 2(f - P_0)^N - 4\frac{\langle (f - P_0)^N, (f - P_0)^N \rangle}{\langle f - P_0, f - P_0 \rangle}(f - P_0) \quad (46)$$

and

$$\bar{\epsilon}\bar{d}^2(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta) = 2\langle f - P_0, \zeta \rangle\eta - 2\langle f - P_0, \eta \rangle\zeta = -2\hat{J}(f - P_0)^N. \quad (47)$$

Then (41) follows from (44), (45), (46) and (47). ■

*Proof of Theorem 2:* It follows from Proposition 10 that the holomorphic curve  $G = g + ih$  associated to  $\mathcal{I} \circ f$  satisfies

$$\langle g, h \rangle = 0 \quad \text{and} \quad \langle g - P_0, g - P_0 \rangle = \langle h, h \rangle, \quad (48)$$

with  $P_0 = \epsilon 2de_{n+3}$ . Hence  $G$  takes values in  $\mathcal{H}_{\epsilon 4d^2}$ .

Conversely, assume that the holomorphic curve  $G = g + ih$  associated to  $f$  satisfies (48). We claim that  $\mathcal{I} \circ f$  is an austere isometric immersion into  $\mathbb{Q}_\epsilon^{n+2}$ .

Define  $\tilde{\zeta}$  and  $\tilde{\eta}$  as in (42), where  $\mathcal{P}$  is the vector bundle isometry between  $T_f^\perp M$  and  $T_{\mathcal{I} \circ f}^\perp M$  given by (39) and  $\bar{\nu}, \bar{\lambda}$  are now given by

$$\bar{\epsilon} d^2 \bar{\nu} = 2\langle f - P_0, \zeta \rangle \quad \text{and} \quad \bar{\epsilon} d^2 \bar{\lambda} = \lambda \langle f - P_0, f - P_0 \rangle + 2\langle f - P_0, \eta \rangle.$$

As before, we obtain using (40) that the shape operators  $\tilde{A}_{\tilde{\eta}}$  and  $\tilde{A}_{\tilde{\zeta}}$  of  $\mathcal{I} \circ f$  are given as in (1) with  $\lambda$  and  $\mu$  replaced, respectively, by  $\tilde{\lambda}$  and  $\tilde{\mu}$  given by (43). Since (48) holds, using that  $h = -\frac{1}{\lambda}\zeta$  and  $g - P_0 = f - P_0 + \frac{1}{\lambda}\eta$  we obtain

$$\langle f - P_0, \zeta \rangle = \langle g - P_0 - \frac{1}{\lambda}\eta, \zeta \rangle = 0$$

and

$$\begin{aligned} \frac{-2}{\lambda} \langle f - P_0, \eta \rangle &= \langle f - P_0, f - P_0 \rangle + \frac{1}{\lambda^2} - \langle g - P_0, g - P_0 \rangle \\ &= \langle f - P_0, f - P_0 \rangle + \frac{1}{\lambda^2} - \langle h, h \rangle = \langle f - P_0, f - P_0 \rangle. \end{aligned}$$

Thus,  $\bar{\nu} = 0 = \bar{\lambda}$ , and hence  $\tilde{\lambda} = 0$ . Therefore  $\mathcal{I} \circ f$  is austere. ■

The proof of Theorem 4 is similar to that of Theorem 2 of [6] and will be omitted.

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