The Cauchy problem for Benney-Luke and generalized Benney-Luke equations

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Abstract. We examine the question of the minimal Sobolev regularity required to construct local solutions to the Cauchy problem for the Benney-Luke (BL) and generalized Benney-Luke (gBL) equations. As a consequence we prove that the initial value problems are globally well-posed in the energy space.

1. Introduction

An intermediate model for the evolution of weakly nonlinear, long water waves of small amplitude is given by the following equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0, \tag{1.1}$$

where $\Phi(t,\mathbf{x})$ is a real valued function, $(t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\mathbb{R}_+ = [0,\infty)$, a, b, μ, and ϵ are positive real constants and ∇ and Δ are the two-dimensional gradient and Laplacian, respectively.

In the equation (1.1), Φ is the velocity potential on the domain. After rescaling the variables, we can suppose the constants a and b are positive and such that $a - b = \alpha - \frac{1}{3} \neq 0$, where α is the Bond number, ϵ (nonlinearity coefficient) is the amplitude parameter and $\mu = (h_0/L)^2$ is the long-wave parameter (dispersion coefficient), where h_0 is the equilibrium depth and L is the length scale. This equation was first derived by Benney and Luke (see [2]) when $a = 1/6$ and $b = 1/2$ with no surface tension ($\alpha = 0$).

Pego and Quintero [7] showed that the Benney-Luke equation reduces formally to the Kadomtsev-Petviashvili (KP-I or KP-II) equation after a suitable renormalization. Indeed, putting $2\tau = \epsilon t$, $X = x - t$, $Y = \sqrt{\epsilon}y$ and $\Phi(t,x,y) = f(\tau, X, Y)$, neglecting $O(\epsilon)$ terms we find that $\eta = f_X$ satisfies

$$(\eta_\tau - (\alpha - \frac{1}{3})\eta_{XXX} + 3\eta\eta_X)_X + \eta_{YY} = 0. \tag{1.2}$$

We recall that if $\alpha > 1/3$ this equation is KP-I, if $\alpha < 1/3$ it is KP-II and, if we suppose that f does not depend on the Y variable we obtain the Korteweg de-Vries (KdV) equation. They also found traveling-wave solutions of (1.1), i.e. solutions of the form $\Phi(t,x,y) = \frac{\sqrt{\mu}}{\epsilon}v(\frac{x-ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}})$ and they showed that if the wave speed c satisfies

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$c^2 < \min(1, a/b)$ then there exists a nontrivial finite-energy solution v, where the energy associated to v is given by

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2} \{(1+c^2)v_x^2 + v_y^2 + (a+bc^2)v_{xx}^2 + (2a+bc^2)v_{xy}^2 + av_{yy}^2\}dxdy. \qquad (1.3)$$

Quintero in [10] proved that the solitary waves are orbitally stable if the wave speed c is near 0 or 1. He also showed in [9] the existence and analyticity of lump solution for generalized Benney-Luke equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta_p\Phi + 2\nabla^p\Phi\cdot\nabla\Phi_t) = 0, \qquad (1.4)$$

where ∇^p and Δ_p given by are

$$\nabla^p\Phi = ((\partial_x\Phi)^p, (\partial_y\Phi)^p) \qquad (1.5)$$

$$\Delta_p\Phi = \nabla\cdot(\nabla^p\Phi) = \partial_x(\partial_x\Phi)^p + \partial_y(\partial_y\Phi)^p. \qquad (1.6)$$

We will call equations (1.1) and (1.4) as (BL) and (gBL) equations respectively throughout this work. The family of Benney-Luke equations includes the effect of surface tension and a variety of equivalent forms of dispersion. Let us remark that the model (1.1) does not hold for $a = b$ ($\alpha = 1/3$). Paumond in [6], derived an equation that is still valid when we suppose that α is equal or close to 1/3. More precisely,

$$\Phi_{tt} - \Delta\Phi + \sqrt{\epsilon}(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(B\Delta^2\Phi_{tt} - A\Delta^3\Phi) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0 \quad (1.7)$$

where $\epsilon = \mu^2$ and the parameters A, B are linked. In [5], it was rigorously shown that the $L^2(\mathbb{R}^2)$-norm of the difference between the amplitude of the wave given by equation (1.2) and the one given by Benney-Luke (BL) equation is of order $O(\epsilon^{3/4})$ during a growing with ϵ time. Paumond in [5] also studied the Cauchy problem

$$\begin{cases} \Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0 \\ \Phi(0,\mathbf{x}) = \Phi_0(\mathbf{x}), \quad \Phi_t(0,\mathbf{x}) = \Phi_1(\mathbf{x}) \end{cases} \qquad (1.8)$$

and proved that it is globally well-posed for initial data in $H^s(\mathbb{R}^2)\times H^{s-1}(\mathbb{R}^2)$, s integer and $s \geq 2$.

In this work we study the local regularity of Benney-Luke and generalized Benney-Luke equation and the well-posedness in the energy space, $\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$. Our main purpose is to prove global well-posedness in the energy space. It is important to remember that the solitary wave of the IVP associated to (BL) and (gBL) equations lies in this space.

We obtain the well-posedness result for Benney-Luke equations using the fixed point argument and the generalized Strichartz inequalities for the wave equation. We showed a similar result for the generalized Benney-Luke equation (gBL). We also prove that the lower bound for the Sobolev exponent can be reduced from 5/2 to 2 in three space dimensions using the Strichartz estimates and the ideas of Ponce and Sideris [8]. Klainerman and Machedon [12], working in three dimensions, showed that if the nonlinearity of the inhomogeneous wave equation satisfies an "null condition", then the Sobolev exponent $s = 2$ can be achieved.

In our case if we define u such that

$$\Phi(t, \mathbf{x}) := \frac{\sqrt{\mu}}{\sqrt[p]{\epsilon}} u\left(\frac{t}{\sqrt{\mu}}, \frac{1}{\sqrt{\mu}}\mathbf{x}\right), \tag{1.9}$$

with Φ satisfying the Benney-Luke equation when $p = 1$ and if $p > 1$ Φ satisfies the generalized Benney-Luke equation, then the initial value problem (1.8) is equivalent to

$$\begin{cases} (1 - b\Delta)(u_{tt} - c^2\Delta u) = (1 - c^2)\Delta u - F_p(\partial_t u, \nabla^p u, \nabla\partial_t u) \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}) \end{cases} \tag{1.10}$$

where $c^2 = \frac{a}{b}$, $u_i(\mathbf{x}) = \frac{\sqrt[p]{\epsilon}}{\sqrt{\mu}}\Phi_i(\sqrt{\mu}\,\mathbf{x})$, $i = 0, 1$, $\mathbf{x} \in \mathbb{R}^2$ and

$$F_p(\partial_t u, \nabla^p u, \nabla\partial_t u) = p\partial_t u(u_x)^{p-1}u_{xx} + p\partial_t u(u_y)^{p-1}u_{yy} + 2\partial_t u_x(u_x)^p + 2\partial_t u_y(u_y)^p, \tag{1.11}$$

(notice that F_p is the nonlinear term of Benney-Luke equation when $p = 1$). The nonlinear term

$$(1 - b\Delta)^{-1}F_p(\partial_t u, \nabla^p u, \nabla\partial_t u)$$

does not satisfy a "null condition" but it is possible to prove that exponent $s = 2$ can be achieved in two dimensions.

We will consider the Cauchy problem (1.10) instead of the initial value problem associated to (BL) and (gBL).

This work is divided as follows: In the second section we present the main results for the BL equation in two and three space dimensions and the global result for the gBL equation in the case $p \geq 2$ integer. In the third section we will prove some lemmas and propositions that will be used in the last section for the proofs of the main results.

Notation.

The notation to be used is mostly standard. For any $q \in [1, \infty]$, we denote by q' its conjugate, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$. Let $L^q := L^q(\mathbb{R}^n)$ be the Lebesgue space, the norm on L^q is denoted by $\|\cdot\|_q$. The homogeneous spaces and the Sobolev spaces $\dot{H}^s_q(\mathbb{R}^n)$ and $H^s_q(\mathbb{R}^n)$, respectively, are defined by $(-\Delta)^{-s/2}L^q(\mathbb{R}^n)$ and $J^{-s}L^q(\mathbb{R}^n)$ with $J := (1 - \Delta)^{1/2}$. We denote $\dot{H}^s_2(\mathbb{R}^n)$ and $H^s_2(\mathbb{R}^n)$ by $\dot{H}^s$ and H^s, respectively. The norms on $\dot{H}^s_q(\mathbb{R}^n)$ and $H^s_q(\mathbb{R}^n)$ are denoted by $\|\cdot\|_{\dot{H}^s_q}$ and $\|\cdot\|_{H^s_q}$, respectively. We will use the Sobolev spaces $L^r_t\dot{H}^\rho_q(\mathbb{R}^n)$ and $L^r_T\dot{H}^\rho_q(\mathbb{R}^n)$ endowed with the norm

$$\|u\|_{L^r_t\dot{H}^\rho_q} = \left(\int_{\mathbb{R}} \|u(t)\|^r_{\dot{H}^\rho_q} dt\right)^{1/r}, \quad \|u\|_{L^r_T\dot{H}^\rho_q} = \left(\int_0^T \|u(t)\|^r_{\dot{H}^\rho_q} dt\right)^{1/r}.$$

Throughout this note $C \geqq 0$ will stand for constants that can be changed from line to line, $a \lesssim b$ means that $a \leqslant Cb$ for some constant C greater than zero.

2. **Main results**

Our first result recover the one obtained by Paumond in [5], but by using the Strichartz inequalities for the wave equation. We also show that the local solution of the Cauchy problem associated to the BL equation (1.1) possesses certain local regularity, like for example $(\nabla u, u_t)(t) \in L^\infty(\mathbb{R}^2)$ a.e. $t \in (0, T]$.

Theorem 2.1. *Let $p = 1$. Assume that $u_0 \in H^2(\mathbb{R}^2)$ and $u_1 \in H^1(\mathbb{R}^2)$. Then there exists $T = T(\|u_0\|_{H^2(\mathbb{R}^2)}, \|u_1\|_{H^1(\mathbb{R}^2)}) > 0$ such that (1.10) has a unique solution u satisfying $u(0, \mathbf{x}) = u_0(\mathbf{x}),\ u_t(0, \mathbf{x}) = u_1(\mathbf{x})$*

$$u \in \mathcal{C}(0, T; H^2(\mathbb{R}^2)), u_t \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)),$$

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2} u(t,\cdot)\|_{L^q}^r dt\right)^{1/r} < \infty, \quad \left(\int_0^T \|(-\Delta)^{\sigma-1} u_t(t,\cdot)\|_{L^q}^r dt\right)^{1/r} < \infty,$$

$$\int_0^T \|(\nabla u, u_t)(t)\|_\infty^4 dt < \infty,$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and $2 < q < \infty$.

Moreover, for all $T' < T$ there exists a neighborhood V of $(u_0, u_1) \in H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ such that the map data solution

$$\begin{array}{ccc} V & \to & \mathcal{C}(0, T'; H^2(\mathbb{R}^2)) \cap L^r(0, T'; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{u}_0, \tilde{u}_1) & \to & \tilde{u}(t) \end{array}$$

is Lipschitz.

Our main result for the BL equation is in the energy space.

Theorem 2.2. *Let $p = 1$. Assume that $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$ and $u_1 \in H^1(\mathbb{R}^2)$. Then there exists $T = T(\|u_0\|_{\dot{H}^1(\mathbb{R}^2)}, \|u_0\|_{\dot{H}^2(\mathbb{R}^2)}, \|u_1\|_{H^1(\mathbb{R}^2)}) > 0$ such that (1.10) has a unique solution u satisfying $u(0, \mathbf{x}) = u_0(\mathbf{x}),\ u_t(0, \mathbf{x}) = u_1(\mathbf{x})$*

$$u \in \mathcal{C}(0, T; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0, T; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)),$$

$$u_t \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)) \cap L^r(0, T; \dot{H}_q^{2\sigma-2}(\mathbb{R}^2)),$$

$$\nabla u, u_t \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)) \cap L^4(0, T; L^\infty(\mathbb{R}^2)).$$

Moreover, for all $T' < T$ there exists a neighborhood V of $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ such that the map data solution

$$\begin{array}{ccc} V & \to & \mathcal{C}(0, T'; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0, T'; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{u}_0, \tilde{u}_1) & \to & \tilde{u}(t) \end{array}$$

is Lipschitz.

Remark 2.3. *It is important to observe that the flow of (1.10) preserves the Hamiltonian*

$$H(u)(t) = \|\partial_t u(t)\|_2^2 + \mu\ b\|\partial_t u(t)\|_{\dot{H}^1}^2 + \|u(t)\|_{\dot{H}^1}^2 + \mu\ a\|u(t)\|_{\dot{H}^2}^2 = H(u)(0), \quad (2.12)$$

for all $p \geq 1$, integer. See [5] *for the proof of (2.12).*

Using previous Remark, it is possible to establish an *a priori* estimate to prove the following global result for BL in the energy space.

Corollary 2.4. *Let $p = 1$. For any $T > 0$, $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$ and $u_1 \in H^1(\mathbb{R}^2)$ there exists a unique solution u of (1.10) such that*

$$\nabla u \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)),\ \partial_t u \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)).$$

Using the Strichartz estimates and mixed norms we prove that the (gBL) equation is locally and globally well-posed in the energy space.

Theorem 2.5. *Assume that $p \geq 2$ integer and $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$. Then there exist $T = T(\|u_0\|_{\dot{H}^1(\mathbb{R}^2)}, \|u_0\|_{\dot{H}^2(\mathbb{R}^2)}, \|u_1\|_{H^1(\mathbb{R}^2)}) > 0$ and a unique solution u of (1.10) such that*

$$u \in \mathcal{C}(0, T; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0, T; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)),$$
$$\partial_t u \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)) \cap L^r(0, T; \dot{H}_q^{2\sigma-2}(\mathbb{R}^2)),$$

and

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2} u(t, \cdot)\|_{L^q}^r dt\right)^{1/r} < \infty,$$
$$\left(\int_0^T \|(-\Delta)^{\sigma-1} u_t(t, \cdot)\|_{L^q}^r dt\right)^{1/r} < \infty,$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and $2 < q < q(p)$, where

$$q(p) = \begin{cases} \infty, & p = 2, 3, 4, \\ \dfrac{2p}{p-4}, & p > 4. \end{cases} \quad (2.13)$$

Moreover, for all $T' < T$ there exists a neighborhood V of $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ such that the map data solution

$$\begin{array}{ccc} V & \rightarrow & \mathcal{C}(0, T'; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0, T'; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{u}_0, \tilde{u}_1) & \rightarrow & \tilde{u}(t) \end{array}$$

is Lipschitz.

Corollary 2.6. *Let $p \geq 2$ integer and $T > 0$. Then for all the functions u_0, u_1 such that $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$, exists a unique solution u of (1.10) such that*

$$\nabla u \in \mathcal{C}(0, T; H^1(\mathbb{R}^2)),\ \partial_t u \in \mathcal{C}(0, T, H^1(\mathbb{R}^2)),$$

and

$$\int_0^T \|(\nabla u, \partial_t u)(\cdot,t)\|_{L^\infty}^4 dt < \infty.$$

We also consider the Cauchy problem of the Benney-Luke equation (1.8) in three spatial dimensions. In this case we have the following.

Theorem 2.7. *Assume that* $(u_0,u_1) \in H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ *and* $2 < s \leqslant 5/2$. *Then there exists* $T>0$ *such that the Cauchy problem:*

$$\begin{cases}(1-bc^{-2}\Delta)(u_{tt}-\Delta u) = c^{-2}(1-c^2)\Delta u - c^{-2}\left(u_t\Delta u + 2\nabla u\cdot\nabla u_t\right), \\ u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0,\mathbf{x}) = u_1(\mathbf{x}),\end{cases} \tag{2.14}$$

where u *is such that*

$$\Phi(t,\mathbf{x}) := \frac{\sqrt{\mu}}{\epsilon}u(\frac{t}{\sqrt{\mu}}, \frac{c}{\sqrt{\mu}}\ \mathbf{x}),$$

with Φ *satisfying the Benney-Luke equation,* $c^2 = a/b$, $(t,\mathbf{x})\in\mathbb{R}_+\times\mathbb{R}^3$, $\mathbb{R}_+ = [0,\infty)$, a *and* b *are positive real constants and* ∇ *and* Δ *are the three-dimensional gradient and Laplacian, respectively, has a unique solution* u *satisfying*

$$u\in\mathcal{C}(0,T;H^s(\mathbb{R}^3)), \partial_t u\in\mathcal{C}(0,T;H^{s-1}(\mathbb{R}^3)),$$

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t,\cdot)\|_{L^q}^r dt\right)^{1/r} < \infty, \qquad \left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t,\cdot)\|_{L^q}^r dt\right)^{1/r} < \infty$$

and

$$\int_0^T \|(\nabla u, u_t)(t)\|_\infty^2 dt < \infty$$

with $r = \dfrac{2q}{q-2}$, $\sigma = \dfrac{s}{2}+\dfrac{1}{q}$ *and* $(s-2)^{-1} < q < \infty$.

3. Linear and Nonlinear Estimates

To show our results we will use some estimates for solutions of linear problem as well as the commutators of Kato-Ponce type [4].

3.1. Linear Estimates.

The linear problem associated to (1.10) is

$$\begin{cases}u_{tt}-\Delta u + a\Delta^2 u - b\Delta u_{tt} = 0 \\ u(0,x) = f(x), \qquad u_t(0,x) = g(x).\end{cases} \tag{3.15}$$

Let $h(\xi) = \left(\dfrac{1+a|\xi|^2}{1+b|\xi|^2}\right)^{1/2}$,

$$(\widehat{W(t)g})(\xi) = (|\xi|h(\xi))^{-1}\sin\left(|\xi|h(\xi)t\right)\hat{g}(\xi)$$

and

$$(\widehat{\dot{W}(t)f})(\xi) = \cos\left(|\xi|h(\xi)t\right)\hat{f}(\xi).$$

Then a solution of

$$\begin{cases} u_{tt} - \Delta u + a\Delta^2 u - b\Delta u_{tt} = G(u) \\ u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{cases} \tag{3.16}$$

when $a \neq b$ and f, g are smooth, is given by

$$u(t,x) = \dot{W}(t)f + W(t)g + \int_0^t W(t-t')G(u)(t')dt'. \tag{3.17}$$

If $a = b$ and f, g are smooth, the solution of (3.16) is given by

$$u(t,x) = \dot{\mathbf{K}}(t)f + \mathbf{K}(t)g + \int_0^t \mathbf{K}(t-t')(1-b\Delta)^{-1}G(u)(t')dt', \tag{3.18}$$

where $\{\mathbf{K}(t)\}_t$ is the classical wave semigroup,

$$(\widehat{\mathbf{K}(t)g})(\xi) = |\xi|^{-1}\sin(|\xi|t)\hat{g}(\xi)$$

with

$$(\widehat{\dot{\mathbf{K}}(t)f})(\xi) = \cos(|\xi|t)\hat{f}(\xi),$$

and $(1-b\Delta)^{-1}G(u)$ is defined via the Fourier transform as

$$((1-b\Delta)^{-1}G(u))\hat{}\,(\xi) = (1+b|\xi|^2)^{-1}\widehat{G(u)}(\xi).$$

It is clear that $W(t)$ is bounded in $L^2(\mathbb{R}^n)$, for all $a, b > 0$, since

$$\begin{aligned} \|W(t)g\|_2 &= \|(\widehat{W(t)g})(\cdot)\|_2 \\ &\leq \|\,(|\cdot|h(\cdot))^{-1}\sin(|\cdot|h(\cdot)t)\,\|_\infty \|\hat{g}(\cdot)\|_2 \end{aligned}$$

and $\|h\|_\infty \leq \max\{1, \sqrt{a/b}\}$.
Then

$$\|W(t)g\|_2 \lesssim |t|\|g\|_2. \tag{3.19}$$

Moreover, for all $s \geq 0$

$$\begin{aligned} &\|W(t)g\|_{\dot{H}^s} \leq \max\{1, \sqrt{a/b}\}\|g\|_{\dot{H}^{s-1}}, \\ &\|\dot{W}(t)f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^s}. \end{aligned} \tag{3.20}$$

Remark 3.1. *If we write equation (1.10) as*

$$\begin{cases} (1-bc^{-2}\Delta)(u_{tt}-\Delta u) = c^{-2}(1-c^2)\Delta u - c^{-2}\left(u_t\Delta u + 2\nabla u\cdot\nabla u_t\right) \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0,\mathbf{x}) = u_1(\mathbf{x}) \end{cases} \tag{3.21}$$

we will see that it is sufficient to have the estimates (3.19) and (3.20) for $\mathbf{K}(t)$.

3.2. **Strichartz Estimates for $\mathbf{K}(t)$ and $\dot{\mathbf{K}}(t)$.** Let

$$v(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1,$$
$$w(t) = \int_0^t \mathbf{K}(t-t')f(t')dt'.$$

Proposition 3.2. *If $2 \le r_1, r_2 \le \infty$, $2 \le q_1, q_2 < \infty$, $\rho_1, \rho_2, \mu \in \mathbb{R}$ satisfy*

$$0 \le \frac{2}{r_i} \le \min\left\{1, (n-1)\left(\frac{1}{2}-\frac{1}{q_i}\right)\right\} \quad i = 1, 2, \tag{3.22}$$

$$\left(\frac{2}{r_i}, (n-1)\left(\frac{1}{2}-\frac{1}{q_i}\right)\right) \neq (1,1) \quad i = 1, 2 \tag{3.23}$$

$$\rho_1 + n\left(\frac{1}{2}-\frac{1}{q_1}\right) - \frac{1}{r_1} = \mu \tag{3.24}$$

$$\rho_1 + n\left(\frac{1}{2}-\frac{1}{q_1}\right) - \frac{1}{r_1} = 1 - \left(\rho_2 + n\left(\frac{1}{2}-\frac{1}{q_2}\right) - \frac{1}{r_2}\right), \tag{3.25}$$

then the following generalized Strichartz estimates for $\mathbf{K}(t)$ and $\dot{\mathbf{K}}(t)$ hold :

$$\|v\|_{L_t^{r_1}\dot{H}_{q_1}^{\rho_1}} + \|\partial_t v\|_{L_t^{r_1}\dot{H}_{q_1}^{\rho_1-1}} \le C\left(\|u_0\|_{\dot{H}^{\mu}} + \|u_1\|_{\dot{H}^{\mu-1}}\right), \tag{3.26}$$

$$\|w\|_{L_T^{r_1}\dot{H}_{q_1}^{\rho_1}} + \|\partial_t w\|_{L_T^{r_1}\dot{H}_{q_1}^{\rho_1-1}} \le C\|f\|_{L_T^{r_2'}\dot{H}_{q_2'}^{-\rho_2}}. \tag{3.27}$$

Proof. See [3]. □

3.3. **Nonlinear Estimates.**

Proposition 3.3. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, $s \in \mathbb{Z}^+$ then exists $C = C_{n,s} > 0$ such that*

$$\begin{aligned}\sum_{|\alpha|=s} \|[\partial_x^\alpha, f]g\|_2 &= \sum_{|\alpha|=s} \|\partial_x^\alpha(fg) - f\partial_x^\alpha g\|_2 \\ &\le C(\|\nabla f\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta g\|_2 + \|g\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta f\|_2).\end{aligned} \tag{3.28}$$

Proof. It follows by Leibniz rule and Gagliardo-Nirenberg Inequality. □

Proposition 3.4. Commutators of Kato-Ponce type. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, $s \ge 1$ then there exists $C = C_{n,s} > 0$ such that*

$$\|[J^s; f]g\|_2 \le C(\|\nabla f\|_\infty \|J^{s-1}g\|_2 + \|g\|_\infty \|J^s f\|_2). \tag{3.29}$$

Proof. See [4]. □

Lemma 3.5. *If $f \in \mathcal{S}(\mathbb{R}^2)$, $2 < q < \infty$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and $0 < s_0 = 1 - \dfrac{2}{q} < 1$, then*

$$\|f\|_\infty \le C\left(\|f\|_{\dot{H}^{s_0}} + \|f\|_{\dot{H}_q^{2\sigma-2}}\right). \tag{3.30}$$

Proof. By Sobolev's inequality we have

$$\begin{aligned}\|f\|_\infty \leq & C\|(1-\Delta)^{1/q+}f\|_q \\ \leq & C\|f\|_q + C\|(-\Delta)^{1/q+}f\|_q \\ \leq & C\|(-\Delta)^{s_0/2}f\|_2 + C\|(-\Delta)^{\sigma-1}f\|_q.\end{aligned} \tag{3.31}$$

Since our assumptions imply that $1/q < \sigma - 1$ we have the result. □

Lemma 3.6. *If $f, g \in \mathcal{S}(\mathbb{R}^2)$, $2 < q < \infty$, $s_0 = 1 - \frac{2}{q}$ and $\sigma = \frac{9}{8} + \frac{3}{4q}$ then*

$$\begin{aligned}\|[\Delta, f]g\|_2 \leq & C\{\|\nabla f\|_{\dot{H}^{s_0}} + \|(-\Delta)^{\sigma-1/2}f\|_q\}\|g\|_{\dot{H}^1} \\ & + C\{\|g\|_{\dot{H}^{s_0}} + \|(-\Delta)^{\sigma-1}g\|_q\}\|f\|_{\dot{H}^2}.\end{aligned} \tag{3.32}$$

Proof. By Proposition 3.3 with $s = 2$, we have

$$\|[\Delta, f]g\|_2 \leq C\left(\|\nabla f\|_\infty \|g\|_{\dot{H}^1} + \|g\|_\infty \|f\|_{\dot{H}^2}\right). \tag{3.33}$$

An application of Lemma 3.5 yields the result. □

Remark 3.7. *We notice that for $0 < s < 1$ we have the following interpolated inequality*

$$\|w\|_{\dot{H}^{s+1}} \lesssim \|w\|_{\dot{H}^1}^{1-s} \|w\|_{\dot{H}^2}^{s}. \tag{3.34}$$

It follows by Hölder's inequality with $p = \frac{1}{s}$ and $q = \frac{1}{1-s}$ (see [1]).

4. Proof of the main results

4.1. Proof of Theorem 2.1.

We begin by rewriting the equation (1.10) when $p = 1$ in the equivalent form

$$(1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}(1-c^2)\Delta u - c^{-2}[\Delta, u]u_t. \tag{4.35}$$

We will use the notation $G(u) = G_1(u) + G_2(u)$ where

$$G_1(u) = c^{-2}(1-c^2)\Delta(1 - bc^{-2}\Delta)^{-1}u \tag{4.36}$$

and

$$G_2(u) = -c^{-2}(1 - bc^{-2}\Delta)^{-1}[\Delta, u]u_t. \tag{4.37}$$

Then we can write the solution of the IVP associated to (4.35) as

$$u(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')(G_1(u) + G_2(u))(t')dt'. \tag{4.38}$$

We first prove the local well-posedness for the IVP associated to (4.35) in $H^2 \times H^1$. To do so we will use the fixed point argument as we mentioned in the introduction.

For $M, T > 0$ and $2 < q < \infty$, define the complete metric space

$$X_T^M = \{u \in \mathcal{C}([0, T]; H^2(\mathbb{R}^2)) : |||u||| \leq M\},$$

where

$$|||u||| = \|u\|_{L^\infty_T H^2} + \|u_t\|_{L^\infty_T H^1} + \|u\|_{L^r_T H^{2\sigma-1}_q} + \|u_t\|_{L^r_T H^{2\sigma-2}_q} \tag{4.39}$$

with $r = \dfrac{4q}{q-2}$ and $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.
We shall prove that for an appropriate choice of T and M the operator

$$\mathbb{F}_{(u_0,u_1)}(u)(t) = \mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')dt' \tag{4.40}$$

is a contraction on X_T^M.

We estimate $\|\mathbb{F}(u)\|_{H^2}$ and $\|\partial_t \mathbb{F}(u)\|_{H^1}$ using the linear estimate (3.19), as follows,

$$\begin{aligned} &\|\mathbb{F}(u)(t)\|_{H^2} + \|\partial_t\mathbb{F}(u)(t)\|_{H^1} \lesssim \|u_0\|_{H^2} + (1+t)\|u_1\|_{H^1} \\ &+ \int_0^t (t-t')\|G(u)(t')\|_2 dt' + \int_0^t \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_2 dt'. \end{aligned} \tag{4.41}$$

Note that

$$\|(-\Delta)^{(s-1)/2} G_2(u)(t)\|_2 \lesssim \|[\Delta, u]u_t(t)\|_2 \tag{4.42}$$

if $1 \le s \le 3$, $n \ge 1$.

Using (4.42), Lemma 3.6 and Hölder's inequality we have

$$\begin{aligned} I_1 :=& \int_0^t (t-t')\|G(u)(t')\|_2 dt' \\ \le& C\int_0^t (t-t')\|u_t\|_{H^1}\left(\|u\|_{\dot{H}^{s_0+1}} + \|(-\Delta)^{\sigma-\frac{1}{2}}u\|_q\right)dt' \\ &+ C\int_0^t (t-t')\|u\|_{\dot{H}^2}\left(|1-c^2| + \|u_t\|_{H^1} + \|(-\Delta)^{\sigma-1}u_t\|_q\right)dt' \\ \lesssim& 2T^2\ \|u\|_{L^\infty_T H^2}\|u_t\|_{L^\infty_T H^1} + |1-c^2|\ T^2\ \|u\|_{L^\infty_T H^2} \\ &+ \|u_t\|_{L^\infty_T H^1}(\int_0^t (t-t')^{r'}dt')^{\frac{1}{r'}}(\int_0^t \|(-\Delta)^{\sigma-\frac{1}{2}}u\|_q^r\,dt')^{1/r} \\ &+ \|u\|_{L^\infty_T H^2}(\int_0^t (t-t')^{r'}dt')^{\frac{1}{r'}}(\int_0^t \|(-\Delta)^{\sigma-1}u_t\|_q^r\,dt')^{1/r}. \end{aligned} \tag{4.43}$$

Then

$$\begin{aligned} I_1 \lesssim& |1-c^2|\ T^2\ \|u\|_{L^\infty_T H^2} +\ T^2\ \|u\|_{L^\infty_T H^2}\|u_t\|_{L^\infty_T H^1} + \\ &T^\beta\{\|u_t\|_{L^\infty_T H^1}\ \|u\|_{L^r_T \dot{H}^{2\sigma-1}_q} + \|u\|_{L^\infty_T H^2}\ \|u_t\|_{L^r_T \dot{H}^{2\sigma-2}_q}\}, \end{aligned} \tag{4.44}$$

where $\beta = 1 + \dfrac{1}{r'} > \dfrac{7}{4}$.

Again using (4.42), Lemma 3.6, (3.34) and Hölder's inequality we have

$$\begin{aligned} I_2 :=& \int_0^t \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_2 dt' \\ \lesssim& T\ \|u\|_{L_T^\infty \dot{H}^2} \|u_t\|_{L_T^\infty H^1} + T|1-c^2|\ \|u\|_{L_T^\infty \dot{H}^1} \\ &+ T\|u_t\|_{L_T^\infty \dot{H}^1} \|u\|^{1-s_0}_{L_T^\infty \dot{H}^1} \|u\|^{s_0}_{L_T^\infty \dot{H}^2} \\ &+ T^{\beta-1}\{\|u_t\|_{L_T^\infty H^1}\ \|u\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|u\|_{L_T^\infty \dot{H}^2}\ \|u_t\|_{L_T^r \dot{H}_q^{2\sigma-2}}\}. \end{aligned} \tag{4.45}$$

Then using (4.41), (4.44) and (4.45) it follows that

$$\begin{aligned} \|\mathbb{F}(u)(t)\|_{L_T^\infty H^2} + \|\partial_t \mathbb{F}(u)\|_{L_T^\infty H^1} \leq& C\|u_0\|_{H^2} + C(1+T)\|u_1\|_{H^1} \\ &+ C\left(|1-c^2|T^2 + P(T)|||u|||\right) |||u|||. \end{aligned} \tag{4.46}$$

where $P(T) = T^\beta + T^2 + T^{\beta-1} + T$.

Now we want to estimate the mixed norms. First, we recall that $\sigma - 1 = \dfrac{3}{4q} + \dfrac{1}{8}$, $r = \dfrac{4q}{q-2}$, $2 < q < \infty$ and $n = 2$. If $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1$, $\mu = 2$ and $-\rho_2 = 1$ then $r_i,\ q_i,\ \rho_i$ and μ, $i = 1, 2$, satisfy (3.22), (3.23), (3.24) and (3.25) and using the Strichartz estimates (3.26) and (3.27) we have

$$\begin{aligned} I_3 =& \|\mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|\partial_t \mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-2}} \\ \leq& C\left(\|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_2 dt'\right). \end{aligned} \tag{4.47}$$

Using the same technique used to estimate I_2, we get

$$I_3 \leq C\left(\|u_0\|_{H^2} + (1+T)\|u_1\|_{H^1} + T|1-c^2|\ \|u\|_{L_T^\infty \dot{H}^1} + P(T)|||u|||^2\right). \tag{4.48}$$

Putting together the estimates (4.46) and (4.48) it follows that

$$|||\mathbb{F}(u)||| \leq C\left(\ \|u_0\|_{H^2} + (1+T)\|u_1\|_{H^1}\right) + C\left(|1-c^2|(T+T^2)\ + P(T)|||u|||\right) |||u|||. \tag{4.49}$$

Let $\delta = \|u_0\|_{H^2} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|1-c^2|(T+T^2) + C(T^2 + T^\beta + T + T^{\beta-1})M \leq 1/2 \tag{4.50}$$

then we have that $\mathbb{F}(X_T^M) \subset X_T^M$.

Noticing that

$$\begin{aligned} |||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim& \int_0^T (T-t')\|G(u)(t') - G(\tilde{u})(t')\|_2 dt' \\ &+ \int_0^T \|(-\Delta)^{1/2}\left(G(u)(t') - G(\tilde{u})(t')\right)\|_2 dt', \end{aligned} \tag{4.51}$$

and using the fact that

$$G_2(u) - G_2(\tilde{u}) = -c^{-2}(1 - bc^{-2}\Delta)^{-1}([\Delta, (u - \tilde{u})]\partial_t u + [\Delta, \tilde{u}]\partial_t(u - \tilde{u})) \tag{4.52}$$

it follows from Lemma 3.6 that if $u, \tilde{u} \in X_M^T$ then

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim \left((T + T^2)|1 - c^2| + (T + T^{1/r'})M\right) |||u - \tilde{u}|||. \tag{4.53}$$

Thus, there exists a unique fixed point of $\mathbb{F}$ which is a solution of the integral equation (4.40) if

$$C\left((T + T^2)|1 - c^2| + (T + T^{1/r'})M\right) < 1. \tag{4.54}$$

Thus, we have proved the existence and uniqueness in an apropiate class of the solution of equation (4.38). For standards arguments it is possible to prove the uniqueness of the solution in the space $H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

To prove the continuous dependence of $\mathbb{F}(u)(t) = \mathbb{F}_{(u_0,u_1)}(u)(t)$ with respect to (u_0, u_1), note that if u, v are the corresponding solutions of (4.40) with initial data (u_0, u_1), (v_0, v_1), respectively, then

$$u(t) - v(t) = \dot{\mathbf{K}}(t)(u_0 - v_0) + \mathbf{K}(t)(u_1 - v_1) + \int_0^t \mathbf{K}(t - t')(G(u) - G(v))(t')dt'. \tag{4.55}$$

Therefore, the same argument used in (4.41) and (4.47) implies

$$\begin{aligned} |||u - v||| \leq & C\left(\|u_0 - v_0\|_{H^2} + (1 + T)\|u_1 - v_1\|_{H^1}\right) \\ & + C\int_0^T (T - t')\|(G(u) - G(v))(t')\|_2 dt' \\ & + C\int_0^T \|(-\Delta)^{1/2}(G(u) - G(v))(t')\|_2 dt'. \end{aligned} \tag{4.56}$$

Using the fact that $G(u) = G_1(u) + G_2(u)$ and (4.52) it follows from Lemma 3.6 that if $u, v \in X_M^T$ then

$$\begin{aligned} & C\int_0^T (T - t')\|(G(u) - G(v))(t')\|_2 dt' + C\int_0^T \|(-\Delta)^{1/2}(G(u) - G(v))(t')\|_2 dt' \\ & \leq \left((T + T^2)|1 - c^2| + (T + T^{1/r'})M\right) |||u - v|||. \end{aligned} \tag{4.57}$$

As a consequence, if $T > 0$ satisfies (4.50) and (4.54), then

$$|||u - v||| \leq C(\|u_0 - v_0\|_{H^2} + \|u_1 - v_1\|_{H^1}), \tag{4.58}$$

which completes the proof. □

4.2. **Proof of Theorem 2.2.** Fix q, $2 < q < \infty$. Let $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2), u_1 \in H^1(\mathbb{R}^2)$ and

$$\mathbb{F}_{(u_0,u_1)}(u)(t) = \mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')dt'. \tag{4.59}$$

We define the complete metric space

$$Y_T^M = \{u \in \mathcal{C}([0,T]; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) : |||u|||_Y \leq M\}$$

with

$$\begin{aligned}|||u|||_Y &= \|u\|_{L_T^\infty \dot{H}^2} + \|u\|_{L_T^\infty \dot{H}^1} + \|u_t\|_{L_T^\infty H^1} \\ &+ \left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t)\|_{L^q}^r dt\right)^{1/r} + \left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t)\|_{L^q}^r dt\right)^{1/r},\end{aligned} \tag{4.60}$$

$r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.

We shall prove that for an appropriate choice of T and M the operator given by (4.59) is a contraction on Y_T^M.

Let $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1,\ \mu = 2$ and $-\rho_2 = 1$ then $r_i,\ q_i,\ \rho_i$ and μ, $i = 1, 2$, satisfy (3.22), (3.23), (3.24), (3.25). Now, using the linear estimate (3.19), the Strichartz estimates (3.26) and (3.27) with $r_i,\ q_i,\ \rho_i$ and μ, $i = 1, 2$ we get

$$\begin{aligned}&\|\mathbb{F}(u)\|_{L_T^\infty \dot{H}^2} + \|\mathbb{F}(u)\|_{L_T^\infty \dot{H}^1} + \|\partial_t\mathbb{F}(u)\|_{L_T^\infty H^1} + \|\mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|\partial_t\mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-2}} \\ &\lesssim \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + (1+T)\|u_1\|_{H^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}}G(u)(t')\|_{L^2}dt'.\end{aligned} \tag{4.61}$$

From (4.45) we have

$$\begin{aligned}|||\mathbb{F}(u)|||_Y \leq &C\left(\|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + (1+T)\|u_1\|_{H^1}\right) \\ &+ C\left(|1-c^2|T\ + (T^{\beta-1}+T)|||u|||_Y\right)|||u|||_Y,\end{aligned} \tag{4.62}$$

where $\beta = 1 + \dfrac{1}{r'} > \dfrac{7}{4}$.

Let $\delta = \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|1-c^2|T + C(T + T^{\beta-1})M \leq 1/2 \tag{4.63}$$

then we have that $\mathbb{F}(Y_T^M) \subset Y_T^M$.

Since

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})|||_Y \lesssim \int_0^T \|(-\Delta)^{1/2}\left(G(u)(t') - G(\tilde{u})(t')\right)\|_2 dt'. \tag{4.64}$$

Using (4.52) and Lemma 3.6 we get

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})|||_Y \leq C\left(|1-c^2|T\ + (T^{\beta-1}+T)M\right)|||u-\tilde{u}|||_Y \tag{4.65}$$

whenever $u, \tilde{u} \in Y_T^M$.

Then, there exists a unique fixed point of $\mathbb{F}$ if

$$C(|1-c^2|T \;+(T^{\beta-1}+T)M)<1.$$

Therefore, the existence and uniqueness of the solution of the problem (4.38) have been proved in the metric space Y_T^M. The uniqueness of the solution in the space $\dot{H}^1(\mathbb{R}^2)\cap\dot{H}^2(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ is obtained by standard arguments.

Using similar arguments to the those applied in the continuous dependence proof in Theorem 2.1 one can show that the map data solution is locally Lipschitz. □

4.3. **Proof of Corollary 2.4.** Now we will show that the local solution obtained in Theorem 2.2 can be extended to $[0,T]$, for any $T>0$, time interval. It suffices to prove the existence of a uniform bound for $\|u(t)\|^2_{\dot{H}^1}$, $\|u(t)\|^2_{\dot{H}^2}$, $\|\partial_t u(t)\|^2_2$ and $\|\partial_t u(t)\|^2_{\dot{H}^1}$. This allows us to establish an *a priori* estimate and then make use of the local theory to extend the solution. To do so we use the following conserved quantity

$$0\leq H(u)(t)=\|\partial_t u(t)\|_2^2+\mu\, b\|\partial_t u(t)\|^2_{\dot{H}^1}+\|u(t)\|^2_{\dot{H}^1}+\mu\, a\|u(t)\|^2_{\dot{H}^2}=H(u)(0), \tag{4.66}$$

satisfied by the flow of (1.10) for $p\geq 1$ integer.

4.4. **Proof of Theorem 2.5.** The basic tools of the proof are the same one: Fixed point theorem, generalized Strichartz estimates for the wave equation and Lemma 3.5.

Using the scale change $\tilde{\mathbf{x}}=c\,\mathbf{x}$ and denoting the new function with the same variable we get the following equivalent equation for (gBL)

$$u_{tt}-\Delta u=B^{-1}G(u) \tag{4.67}$$

with

$$\begin{aligned}
&G(u)=G_0(u)+G_p(u),\\
&G_0(u)=(1-c^2)c^{-2}\Delta u=m_2\Delta u,\\
&G_p(u)=-c^{-(p+1)}F_p(\partial_t u,\nabla^p u,\nabla\partial_t u)=k_pF_p(\partial_t u,\nabla^p u,\nabla\partial_t u),\\
&B\phi=(1-m_1^2\Delta)\phi,\quad m_1^2=\frac{b^2}{a}.
\end{aligned} \tag{4.68}$$

Fix $p\geq 2$ integer, $2<q<q(p)$ where $q(p)$ is giving by (2.13) and let $r=\dfrac{4q}{q-2}$, $\sigma=\dfrac{9}{8}+\dfrac{3}{4q}$ and for $T,M>0$ define the complete metric space

$$X_T^M=\{u\in\mathcal{C}([0,T];\dot{H}^1(\mathbb{R}^2)\cap\dot{H}^2(\mathbb{R}^2)):|||u|||_X\leq M\},$$

where

$$|||u|||_X=\|u\|_{L^\infty_T\dot{H}^1}+\|u\|_{L^\infty_T\dot{H}^2}+\|\partial_t u\|_{L^\infty_T H^1}+\|u\|_{L^r_T\dot{H}^{2\sigma-1}}+\|\partial_t u\|_{L^r_T\dot{H}^{2\sigma-2}}, \tag{4.69}$$

and let

$$\mathbb{F}_{(u_0,u_1)}(u)(t)=\mathbb{F}(u)(t)=\dot{\mathbf{K}}(t)u_0+\mathbf{K}(t)u_1+\int_0^t\mathbf{K}(t-t')B^{-1}G(u)(t')dt', \tag{4.70}$$

where G and B are given by (4.68). Using the linear estimates for $\mathbf{K}(t)$ and $\dot{\mathbf{K}}(t)$ we have

$$\begin{aligned}&\|\mathbb{F}(u)(t)\|_{\dot H^1}+\|\mathbb{F}(u)(t)\|_{\dot H^2}+\|\partial_t\mathbb{F}(u)(t)\|_{H^1}\\ &\lesssim \|u_0\|_{\dot H^1}+\|u_0\|_{\dot H^2}+(1+t)\|u_1\|_{H^1}+\int_0^t\|(-\Delta)^{\frac12}B^{-1}G(u)(t')\|_2dt'.\end{aligned}\tag{4.71}$$

Now the Strichartz estimates imply

$$\begin{aligned}&\|\mathbb{F}(u)\|_{L^r_T\dot H^{2\sigma-1}}+\|\partial_t\mathbb{F}(u)\|_{L^r_T\dot H^{2\sigma-2}}\\ &\lesssim \|u_0\|_{\dot H^2}+\|u_1\|_{H^1}+\int_0^T\|(-\Delta)^{\frac12}B^{-1}G(u)(t')\|_2dt'.\end{aligned}\tag{4.72}$$

Therefore,

$$|||\mathbb{F}(u)|||_X\lesssim(1+T)(\|u_0\|_{\dot H^1}+\|u_0\|_{\dot H^2}+\|u_1\|_{H^1})+\int_0^T\|(-\Delta)^{\frac12}B^{-1}G(u)(t')\|_2dt'.\tag{4.73}$$

Remark 4.1. *Note that, from Plancherel's identity and the definition of B (see (4.68)) we can get*

$$\begin{aligned}\|(-\Delta)^{\frac12}B^{-1}G(u)(t')\|_2 =& \ \|(-\Delta)^{\frac12}B^{-1}(m_2\Delta u+k_pF_p(\partial_tu,\nabla^pu,\nabla\partial_tu)(t'))\|_2\\ \lesssim& \ |m_2|\|u(t')\|_{\dot H^1}+|k_p|\|F_p(\partial_tu,\nabla^pu,\nabla\partial_tu)(t'))\|_2.\end{aligned}$$

This remark and (4.73) imply that

$$|||\mathbb{F}(u)|||_X\lesssim(1+T)(\|u_0\|_{\dot H^1}+\|u_0\|_{\dot H^2}+\|u_1\|_{H^1})+T|m_2|\|u\|_{L^\infty_T\dot H^1}+|k_p|\int_0^T\|F_p(u)\|_2dt'.\tag{4.74}$$

We have from (1.11)

$$\begin{aligned}I_4:=\int_0^T\|F_p(u)\|_2dt'\lesssim p&\int_0^T\|\partial_tu(t')\|_\infty\|\nabla u(t')\|_\infty^{p-1}\|u(t')\|_{\dot H^2}dt'\\ &+2\int_0^T\|\nabla u(t')\|_\infty^p\|\partial_tu(t')\|_{\dot H^1}dt'.\end{aligned}$$

So from Lemma 3.5

$$\begin{aligned}I_4\lesssim p&\int_0^T\left\{\|\partial_tu(t')\|_{\dot H^{s_0}}+\|\partial_tu(t')\|_{\dot H_q^{2\sigma-2}}\right\}\\ &\times\left\{\|u(t')\|_{\dot H^{s_0+1}}+\|u(t')\|_{\dot H_q^{2\sigma-1}}\right\}^{p-1}\|u(t')\|_{\dot H^2}dt'\\ &+2\|\partial_tu\|_{L^\infty_T\dot H^1}\int_0^T\left(\|u(t')\|_{\dot H^{s_0+1}}+\|u(t')\|_{\dot H_q^{2\sigma-1}}\right)^pdt'.\end{aligned}$$

Therefore

$$\begin{aligned} I_4 \lesssim & p\|\partial_t u\|_{L^\infty_T \dot H^{s_0}} \|u\|_{L^\infty_T \dot H^2} \int_0^T \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H^{2\sigma-1}_q}\right)^{p-1} dt' \\ & + p\|u\|_{L^\infty_T \dot H^2} \int_0^T \|\partial_t u(t')\|_{\dot H^{2\sigma-2}_q} \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H^{2\sigma-1}_q}\right)^{p-1} dt' \\ & + 2\|\partial_t u\|_{L^\infty_T \dot H^1} \int_0^T \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H^{2\sigma-1}_q}\right)^{p} dt'. \end{aligned} \tag{4.75}$$

Hence

$$\begin{aligned} I_4 \lesssim & pT\|\partial_t u\|_{L^\infty_T \dot H^{s_0}} \|u\|_{L^\infty_T \dot H^2} \|u\|^{p-1}_{L^\infty_T \dot H^{s_0+1}} + p\|\partial_t u\|_{L^\infty_T \dot H^{s_0}} \|u\|_{L^\infty_T \dot H^2} \int_0^T \|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q} dt' \\ & + p\|u\|_{L^\infty_T \dot H^2} \|u\|^{p-1}_{L^\infty_T \dot H^{s_0+1}} \int_0^T \|\partial_t u(t')\|_{\dot H^{2\sigma-2}_q} dt' \\ & + p\|u\|_{L^\infty_T \dot H^2} \int_0^T \|\partial_t u(t')\|_{\dot H^{2\sigma-2}_q} \|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q} dt' \\ & + 2T\|\partial_t u\|_{L^\infty_T \dot H^1} \|u\|^p_{L^\infty_T \dot H^{s_0+1}} + 2\|\partial_t u\|_{L^\infty_T \dot H^1} \int_0^T \|u(t')\|^p_{\dot H^{2\sigma-1}_q} dt'. \end{aligned} \tag{4.76}$$

Using the Hölder inequality we have

$$\begin{aligned} & \int_0^T \|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q} dt' \le T^{1+(1-p)/r} \|u\|^{p-1}_{L^r_T \dot H^{2\sigma-1}_q}, \\ & \int_0^T \|\partial_t u(t')\|_{\dot H^{2\sigma-2}_q} dt' \le T^{1-1/r} \|\partial_t u\|_{L^r_T \dot H^{2\sigma-2}_q}, \\ & \int_0^T \|\partial_t u(t')\|_{\dot H^{2\sigma-2}_q} \|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q} dt' \le T^{1-p/r} \|\partial_t u\|_{L^r_T \dot H^{2\sigma-2}_q} \|u\|^{p-1}_{L^r_T \dot H^{2\sigma-1}_q}, \\ & \int_0^T \|u(t')\|^p_{\dot H^{2\sigma-1}_q} dt' \le T^{1-p/r} \|u\|^p_{L^r_T \dot H^{2\sigma-1}_q}. \end{aligned} \tag{4.77}$$

From (4.74), (4.76), (4.77) we have

$$|||\mathbb{F}(u)|||_X \le C(1+T)(\|u_0\|_{\dot H^1} + \|u_0\|_{\dot H^2} + \|u_1\|_{H^1}) + CT|m_2|\, M + C|k_p| P(T) M^{p+1} \tag{4.78}$$

with $P(T) = T + T^{1-p/r} + T^{1+(1-p)/r} + T^{1-1/r}$.

Let $\delta = \|u_0\|_{\dot H^1} + \|u_0\|_{\dot H^2} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|m_2|T + C|k_p|P(T)M^p \le 1/2 \tag{4.79}$$

then we have that $\mathbb{F}(X^M_T) \subset X^M_T$.

Now we proceed to show that, with suitable choice of T and M, $\mathbb{F}$ is a contraction. We have

$$\mathbb{F}(u)(t) - \mathbb{F}(\tilde u)(t) = \int_0^t \mathbf{K}(t-t') B^{-1}(G(u) - G(\tilde u))(t') dt',$$

from (4.71) and (4.72)

$$|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})|||_X \leq C\int_0^T \|(-\Delta)^{\frac{1}{2}}B^{-1}(G(u)-G(\tilde{u}))(t')\|_2 dt'.$$

The Remark 4.1 implies

$$|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})|||_X \leq CT|m_2|\|u-\tilde{u}\|_{L_T^\infty \dot{H}^1} + |k_p|\int_0^T \|(F_p(u)-F_p(\tilde{u}))(t')\|_2 dt'.$$

To estimate the last term we notice that

$$\|\nabla^p u - \nabla^p \tilde{u}\|_\infty \lesssim \|\nabla(u-\tilde{u})\|_\infty \left\{ \|\sum_{k=0}^{p-1}(\partial_x u)^{p-1-k}(\partial_x \tilde{u})^k\|_\infty + \|\sum_{k=0}^{p-1}(\partial_y u)^{p-1-k}(\partial_y \tilde{u})^k\|_\infty \right\}. \tag{4.80}$$

Using

$$\begin{aligned}
\|\Delta_p u - \Delta_p \tilde{u}\|_2 &\lesssim p\|(\partial_x u)^{p-1}-(\partial_x\tilde{u})^{p-1}\|_\infty \|\partial_x^2 u\|_2 + p\|\partial_x \tilde{u}\|_\infty^{p-1}\|\partial_x^2(u-\tilde{u})\|_2 \\
&\quad + p\|(\partial_y u)^{p-1}-(\partial_y\tilde{u})^{p-1}\|_\infty \|\partial_y^2 u\|_2 + p\|\partial_y \tilde{u}\|_\infty^{p-1}\|\partial_y^2(u-\tilde{u})\|_2 \\
&\lesssim p\|\nabla(u-\tilde{u})\|_\infty \|\sum_{k=0}^{p-2}(\partial_x u)^{p-2-k}(\partial_x\tilde{u})^k\|_\infty \|u\|_{\dot{H}^2} \qquad (4.81)\\
&\quad + p\|\nabla(u-\tilde{u})\|_\infty \|\sum_{k=0}^{p-2}(\partial_y u)^{p-2-k}(\partial_y\tilde{u})^k\|_\infty \|u\|_{\dot{H}^2} \\
&\quad + p\|\nabla\tilde{u}\|_\infty^{p-1}\|u-\tilde{u}\|_{\dot{H}^2}.
\end{aligned}$$

Since

$$\begin{aligned}
F_p(u)-F_p(\tilde{u}) = &2\nabla(u-\tilde{u})_t\cdot\nabla^p u + (u-\tilde{u})_t\Delta_p u \\
&+2\nabla\tilde{u}_t\cdot(\nabla^p u - \nabla^p\tilde{u}) + \tilde{u}_t(\Delta_p u - \Delta_p\tilde{u})
\end{aligned}$$

and using the definitions of $\nabla^p u$ and $\Delta_p u$ (see (1.5) and (1.6)) we have

$$\begin{aligned}
&\int_0^T \|(F_p(u)-F_p(\tilde{u}))(t')\|_2\, dt' \\
&\lesssim \int_0^T \|\nabla^p u\|_\infty \|(u-\tilde{u})_t\|_{\dot{H}^1}\, dt' + p\int_0^T \|(u-\tilde{u})_t\|_\infty \|\nabla u\|_\infty^{p-1}\|u\|_{\dot{H}^2}\, dt' \qquad (4.82)\\
&\quad + \int_0^T \|\nabla^p u - \nabla^p\tilde{u}\|_\infty \|\tilde{u}_t\|_{\dot{H}^1}\, dt' + \int_0^T \|\tilde{u}_t\|_\infty \|\Delta_p u - \Delta_p\tilde{u}\|_2\, dt'.
\end{aligned}$$

Remark 4.2. *We notice that using Lemma 3.5 and interpolation result (3.34) we have*

$$\|\nabla w\|_\infty \lesssim \|w\|_{\dot{H}^{s_0+1}} + \|w\|_{\dot{H}_q^{2\sigma-1}} \lesssim \|w\|_{\dot{H}^1}^{1-s_0}\|w\|_{\dot{H}^2}^{s_0} + \|w\|_{\dot{H}_q^{2\sigma-1}}$$

also

$$\int_0^T \|\nabla w(t',\cdot)\|_\infty \, dt' \lesssim T\|w\|_{L^\infty_T \dot{H}^{s_0+1}} + T^{1/r'}\|w\|_{L^r_T \dot{H}^{2\sigma-1}} \lesssim (T + T^{1/r'})|||w|||_X.$$

Therefore we only consider the following terms

$$\int_0^T \|\tilde{u}_t\|_{\dot{H}^1}\|u-\tilde{u}\|_{\dot{H}^{2\sigma-1}_q}\|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q}\|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \tag{4.83}$$

and

$$\int_0^T \|\tilde{u}_t\|_{\dot{H}^1}\|u-\tilde{u}\|_{\dot{H}^{s_0+1}}\|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q}\|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \tag{4.84}$$

to illustrate the proof of

$$\begin{aligned} &|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})|||_X \\ &\qquad \lesssim (T|m_2| + |k_p|(T + T^{1-p/r} + T^{1+(1-p)/r} + T^{1-1/r})M^p)|||u-\tilde{u}|||_X. \end{aligned} \tag{4.85}$$

Using Hölder inequality we have

$$\begin{aligned} &\int_0^T \|\tilde{u}_t\|_{\dot{H}^1}\|u-\tilde{u}\|_{\dot{H}^{2\sigma-1}_q}\|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q}\|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim T^{1-p/r}|||u|||_X^{p-1-k}|||\tilde{u}|||_X^{k+1}|||u-\tilde{u}|||_X \end{aligned} \tag{4.86}$$

and

$$\begin{aligned} &\int_0^T \|\tilde{u}_t\|_{\dot{H}^1}\|u-\tilde{u}\|_{\dot{H}^{s_0+1}}\|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q}\|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim \|\tilde{u}_t\|_{L^\infty_T H^1}\|u-\tilde{u}\|^{1-s_0}_{L^\infty_T \dot{H}^1}\|u-\tilde{u}\|^{s_0}_{L^\infty_T \dot{H}^2}\int_0^T \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q}\|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim T^{1+(1-p)/r}|||u|||_X^{p-1-k}|||\tilde{u}|||_X^{k+1}|||u-\tilde{u}|||_X. \end{aligned} \tag{4.87}$$

Hence by standard arguments we can guarantee the existence and uniqueness of a solution of the Cauchy problem associated to the equation (1.4).

We will show that the map data solution is locally Lipschitz using the similar argument to the one used in the continuous dependence proof in Theorem 2.1.

Let u and v be the corresponding solutions of (4.70) with initial data (u_0, u_1), (v_0, v_1), respectively, then

$$u(t) - v(t) = \dot{\mathbf{K}}(t)(u_0 - v_0) + \mathbf{K}(t)(u_1 - v_1) + \int_0^t \mathbf{K}(t-t')B^-1(G(u) - G(v))(t')dt'. \tag{4.88}$$

The same arguments used in (4.71) and (4.72) imply

$$\begin{aligned} |||u-v|||_X \leq & C\left(\|u_0-v_0\|_{\dot{H}^2} + \|u_0-v_0\|_{\dot{H}^1} + (1+T)\|u_1-v_1\|_{H^1}\right) + \\ & C\int_0^T \|(-\Delta)^{1/2}B^{-1}(G(u)-G(v))(t')\|_2 dt'. \end{aligned} \tag{4.89}$$

By Remark 4.1 we have

$$\int_0^T \|(-\Delta)^{1/2}B^{-1}(G(u)-G(v))(t')\|_2 dt' \le CT|m_2|\|u-v\|_{L^\infty_T \dot{H}^1} + |k_p|\int_0^T \|(F_p(u)-F_p(v))(t')\|_2 dt'. \tag{4.90}$$

The arguments used for estimate (4.82) imply

$$\int_0^T \|(F_p(u)-F_p(v))(t')\|_2 dt' \le C(T+T^{1-p/r}+T^{1+(1-p)/r}+T^{1-1/r})M^p)|||u-v|||_X, \tag{4.91}$$

if $u, v \in X_T^M$. Using (4.79) we have

$$|||u-v|||_X \le \tilde{C}\left(\|u_0-v_0\|_{\dot{H}^2}+\|u_0-v_0\|_{\dot{H}^1}+\|u_1-v_1\|_{H^1}\right), \tag{4.92}$$

which completes the proof. □

4.5. **Proof of Corollary 2.6.** To prove that the local solution obtained in Theorem 2.5 can be extended to $[0,T]$, for any $T>0$, we use, as we mentioned in Remark 2.3, that the flow of (1.10) satisfies the conserved quantity

$$H(u)(t) = \|\partial_t u(t)\|_2^2 + \mu\, b\|\partial_t u(t)\|_{\dot{H}^1}^2 + \|u(t)\|_{\dot{H}^1}^2 + \mu\, a\|u(t)\|_{\dot{H}^2}^2 = H(u)(0). \tag{4.93}$$

According to the above, we can establish an *a priori* estimate for $\|u(t)\|_{\dot{H}^1}^2$, $\|u(t)\|_{\dot{H}^2}^2$, $\|\partial_t u(t)\|_2^2$ and $\|\partial_t u(t)\|_{\dot{H}^1}^2$ and then apply the local theory. □

4.6. **Proof of Theorem 2.7.** Fix s, $2 < s \le 5/2$, and take $q \in (1/(s-2), \infty)$. For $T, M > 0$ define the complete metric space

$$X_T^M = \{u \in \mathcal{C}(0,T;H^s(\mathbb{R}^3)) : |||u||| \le M\}$$

where

$$|||u||| = \|u\|_{L^\infty_T H^s} + \|u_t\|_{L^\infty_T H^{s-1}} + (\int_0^T \|(-\Delta)^{\sigma-1/2}u(t)\|_{L^q}^r dt)^{1/r} + (\int_0^T \|(-\Delta)^{\sigma-1}u_t(t)\|_{L^q}^r dt)^{1/r}, \tag{4.94}$$

with $r = \dfrac{2q}{q-2}$, $\sigma = \dfrac{s}{2}+\dfrac{1}{q}$. It is not difficult to prove that $\mathbb{F}$ is a contraction in X_T^M using the same arguments as in [8], (4.42) and Proposition 3.3.□

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