FETI and BDD Preconditioners for Stokes-Mortar-Darcy Systems

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Abstract

We consider the coupling across an interface of a fluid flow and a porous media flow. The differential equations involve Stokes equations in the fluid region and Darcy equations in the porous region, and coupled through an interface with Beaver-Joseph-Saffman transmission conditions. The discretization consists of $P2/P1$ triangular Taylor-Hood finite elements in the fluid region, the lowest order triangular Raviart-Thomas finite elements in the porous region, and the mortar piecewise constant Lagrange multipliers on the interface and we allow nonmatching meshes across the interface. Due to the small values of the permeability parameter κ of the porous medium, the resulting discrete symmetric saddle point system is very ill conditioned. We design and analyze a preconditioners based on the Finite Element by Tearing and Interconnecting (FETI) and Balancing Domain Decomposition (BDD) preconditioners and derive a condition number estimate of order $C_1(1+\frac{1}{\kappa})$. In case the fluid discretization is finer than the porous side discretization, we derive a better estimate of order $C_2(\frac{\kappa+1}{\kappa+(h^p)^2})$ for the FETI preconditioner. Here h^p is the mesh size of the porous side triangulation. The constants C_1 and C_2 are independent of the permeability κ, the fluid viscosity ν, and the mesh ratio across the interface. Numerical experiments confirm the sharpness of the theoretical estimates.

Keywords. Stokes-Darcy coupling, mortar, balancing domain decomposition, FETI, saddle point problems, nonmatching grids, discontinuous coefficients, mortar elements

AMS subject classifications. 65N55, 65N30, 65N22, 35Q30, 35Q35, 76D07

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1 Introduction

We consider the coupling across an interface of a fluid flow and a porous media flow. The model consists of Stokes equations in the fluid region, Darcy equations for the filtration velocity in the porous medium, and an adequate transmission condition for coupling these equations through an interface. Such problem appears in several applications such as well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. There are works that address numerical analysis issues of this model. For inf-sup conditions and approximation results associated to the continuous and discrete formulations for Stokes-Laplacian systems we refer [14, 15], for Stokes-Darcy systems we refer [29, 37, 2], for Stokes-Mortar-Darcy systems we refer [39, 25], and for DG discretizations [11, 39]. For studies on preconditioning analysis for Stokes-Laplacian systems we refer [12, 13, 16, 17], and for Stokes-Darcy systems we refer [3]. In this paper, we are interested in Balancing Domain Decomposition (BDD) and Finite Element by Tearing and Interconnecting (FETI) preconditioned Conjugate Gradient methods for *Stokes-Mortar-Darcy* systems. For general references on BDD methods we mention [21, 33, 34, 38, 40] and for FETI methods [21, 23, 28, 35, 40] for FETI methods; see also [18, 22, 30, 31, 32, 41, 42].

In this paper we both extend some preliminary results contained in [24] and introduce and analyze new methods. We note that the BDD-I preconditioner introduced in [24] is not effective for small permeabilities (in real applications permeabilities are very small) while the preconditioner BDD-II in [24] requires constructing interface base functions which are orthogonal in the Stokes inner product (this construction is very expensive and impractical because it requires, as a precomputational step, solving many Stokes problems). Here in this paper we circunvent these issues by introducing a dual formulation and considering a FETI based methods. We propose and analyze Finite Element by Tearing and Interconnecting (FETI) methods and present numerical experiments in order to verify the theory. We note that the analysis of the FETI algorithms for Stokes-Mortar-Darcy problems is very challenging due to the following issues: 1) The mortar map from the Stokes to the Darcy side has a large kernel since the Stokes velocity space is in general richer than the Darcy velocity space on the interface; 2) The trace space of the Stokes velocity ($H^{1/2}$) is more regular than the trace space of the Darcy flux ($H^{-1/2}$), and due to a priori error estimates, see [29, 39, 25], the Stokes side must be chosen as the master side; 3) The energy associated to the Darcy region is much larger than the energy associated to the Stokes region due to the small value of the permeability. Such issues imply that the master side must be chosen on the Stokes side and where the energy is smaller and velocity space is richer. The mathematical analysis under this choice is very hard to analyze even for simpler problems such as for transmission problems with discontinuous coefficients using Mortar or DG discretizations [21, 19, 20]. For problems where both the smallest coefficient and the finest mesh are placed on the master side, as far as we know, there are no optimal preconditioners developed in the literature for transmission problems, and typically there is a

condition to rule out such choice.

The rest of the paper is organized as follows: in Section 2 we present the Stokes/Darcy coupling model. In Section 3 we describe the weak formulation of this model. In Section 4 we introduce a finite element discretization. In Section 5 we study the primal and dual formulation of the discrete problem. Section 6 is dedicated to present a complete analysis of the BDD-I preconditioner introduced in [24]. In Section 7, we design and analyze the FETI preconditioner; see Lemma 2 and Theorem 3. In particular we obtain the condition number estimate of order $C_1(1 + \frac{1}{\kappa})$ for this preconditioner and also prove Theorem 6 which gives a better estimate of order $C_2(\frac{\kappa+1}{\kappa+(h^p)^2})$ for the FETI preconditioner in case the fluid discretization is finer than the porous side discretization; the case where the Stokes mesh is not a refinement of the Darcy mesh in also discussed; see Remark 7. In Section 7 we also consider more general fluid bilinear forms by allowing the presence of a tangential interface fluid velocity energy, see Remark 9, and also translate the FETI results to analyze certain BDD methods; see Remark 8.

Here h^p is the mesh size of the porous side triangulation. The constants C_1 and C_2 are independent of the permeability κ, the fluid viscosity ν, and the mesh ratio across the interface. In Section 8 we present numerical results that confirm the theoretical estimates concerning the BDD and the FETI preconditioners.

2   Problem Setting

Let Ω^f, $\Omega^p \subset \mathbb{R}^n$ be polyhedral subdomains, define $\Omega := \text{int}(\overline{\Omega}^f \cup \overline{\Omega}^p)$ and $\Gamma := \partial\Omega^f \cap \partial\Omega^p$, with outward unit normal vectors $\boldsymbol{\eta}^i$ on $\partial\Omega^i$, $i = f, p$. The tangent vectors on Γ are denoted by $\boldsymbol{\tau}_1$ ($n = 2$), or $\boldsymbol{\tau}_l$, $l = 1, 2$ ($n = 3$). The exterior boundaries are $\Gamma^i := \partial\Omega^i \setminus \Gamma$, $i = f, p$. Fluid velocities are denoted by $\boldsymbol{u}^i : \Omega^i \to \mathbb{R}^n$, $i = f, p$, and pressures by $p^i : \Omega^i \to \mathbb{R}$, $i = f, p$.

We consider Stokes equations in the fluid region Ω^f and Darcy equations for the filtration velocity in the porous medium Ω^p. More precisely, we have the following systems of equations in each subdomain:

$$
\begin{array}{ll}
\text{Stokes equations} & \text{Darcy equations} \\
\left\{
\begin{aligned}
-\nabla \cdot T(\boldsymbol{u}^f, p^f) &= \boldsymbol{f}^f && \text{in } \Omega^f \\
\nabla \cdot \boldsymbol{u}^f &= g^f && \text{in } \Omega^f \\
\boldsymbol{u}^f &= \boldsymbol{h}^f && \text{on } \Gamma^f
\end{aligned}
\right.
&
\left\{
\begin{aligned}
\boldsymbol{u}^p &= -\frac{\kappa}{\nu}\nabla p^p && \text{in } \Omega^p \\
\nabla \cdot \boldsymbol{u}^p &= g^p && \text{in } \Omega^p \\
\boldsymbol{u}^p \cdot \boldsymbol{\eta}^p &= h^p && \text{on } \Gamma^p.
\end{aligned}
\right.
\end{array}
\quad (1)
$$

Here $T(\boldsymbol{v}, p) := -pI + 2\nu\boldsymbol{D}\boldsymbol{v}$, where ν is the fluid viscosity, $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$ is the linearized strain tensor and κ denotes the rock permeability. For simplicity on the analysis, we assume that κ is a real positive constant. We impose the following conditions:

1. *Interface matching conditions across* Γ; see [14, 15, 16, 29] and references therein.

(a) *Conservation of mass across* Γ: $\boldsymbol{u}^f \cdot \boldsymbol{\eta}^f + \boldsymbol{u}^p \cdot \boldsymbol{\eta}^p = 0$ on Γ.

(b) *Balance of normal forces across* Γ: $p^f - 2\nu\boldsymbol{\eta}^{fT}\boldsymbol{D}(\boldsymbol{u}^f)\boldsymbol{\eta}^f = p^p$ on Γ.

(c) *Beavers-Joseph-Saffman condition:* This condition is a kind of empirical law that gives an expression for the component of the Cauchy stress tensor in the tangential direction of Γ; see [4] and [27]. It is expressed by:

$$\boldsymbol{u}^f \cdot \boldsymbol{\tau}_l = -\frac{\sqrt{\kappa}}{\alpha^f}2\boldsymbol{\eta}^{fT}\boldsymbol{D}(\boldsymbol{u}^f)\boldsymbol{\tau}_l \quad l = 1, n-1; \text{ on } \Gamma.$$

2. *Compatibility condition:* The divergence and boundary data satisfy (see [25]),

$$\langle g^f, 1\rangle_{\Omega^f} + \langle g^p, 1\rangle_{\Omega^p} - \langle \boldsymbol{h}^f \cdot \boldsymbol{\eta}^f, 1\rangle_{\Gamma^f} - \langle h^p, 1\rangle_{\Gamma^p} = 0.$$

3 Weak Formulation

In this section we present the weak version of the coupled system of partial differential equations introduced above. Without loss of generality, we consider $\boldsymbol{h}^f = \boldsymbol{0}$, $g^f = 0$, $h^p = 0$ and $g^p = 0$ in (1); see [25].

The problem can be formulated as: *Find $(\boldsymbol{u}, p, \lambda) \in \boldsymbol{X} \times M_0 \times \Lambda$ such that for all $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M_0 \times \Lambda$:*

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) & = f(\boldsymbol{v}) \\ b(\boldsymbol{u}, q) & = 0 \\ b_\Gamma(\boldsymbol{u}, \mu) & = 0, \end{cases} \tag{2}$$

where $\boldsymbol{X} = \boldsymbol{X}^f \times \boldsymbol{X}^p := H_0^1(\Omega^f, \Gamma^f)^n \times \boldsymbol{H}_0(\text{div}, \Omega^p, \Gamma^p)$ and M_0 is the subset of $M := L^2(\Omega^f) \times L^2(\Omega^p) \equiv L^2(\Omega)$ of pressures with zero average value in Ω. Here $H_0^1(\Omega^f, \Gamma^f)$ denotes the subspace of $H^1(\Omega^f)$ of functions that vanish on Γ^f. The space $\boldsymbol{H}_0(\text{div}, \Omega^p, \Gamma^p)$ consists of functions in $\boldsymbol{H}(\text{div}, \Omega^p)$ with zero normal trace on Γ^p, where

$$\boldsymbol{H}(\text{div}, \Omega^p) := \left\{ \boldsymbol{v} \in L^2(\Omega^p)^n : \text{div}\,\boldsymbol{v} \in L^2(\Omega^p) \right\}.$$

For the Lagrange multiplier space we consider $\Lambda := H^{1/2}(\Gamma)$. See [25] for a discussion on the choice of the Lagrange multipliers space Λ and how to derive the weak formulation (2) and other equivalent weak formulations; see also [29].

The global bilinear forms are

$$a(\boldsymbol{u}, \boldsymbol{v}) := a_{\alpha^f}^f(\boldsymbol{u}^f, \boldsymbol{v}^f) + a^p(\boldsymbol{u}^p, \boldsymbol{v}^p) \quad \text{and} \quad b(\boldsymbol{v}, p) := b^f(\boldsymbol{v}^f, p^f) + b^p(\boldsymbol{v}^p, p^p),$$

with local bilinear forms $a_{\alpha^f}^f, b^f$ and b^p defined by

$$a_{\alpha^f}^f(\boldsymbol{u}^f, \boldsymbol{v}^f) := 2\nu(\boldsymbol{Du}^f, \boldsymbol{Dv}^f)_{\Omega^f} + \sum_{\ell=1}^{n-1} \frac{\nu\alpha^f}{\sqrt{\kappa}} \langle \boldsymbol{u}^f \cdot \boldsymbol{\tau}_\ell, \boldsymbol{v}^f \cdot \boldsymbol{\tau}_\ell \rangle_\Gamma, \ \boldsymbol{u}^f, \boldsymbol{v}^f \in \boldsymbol{X}^f,$$
(3)

$$a^p(\boldsymbol{u}^p, \boldsymbol{v}^p) := (\frac{\nu}{\kappa}\boldsymbol{u}^p, \boldsymbol{v}^p)_{\Omega^p}, \quad \boldsymbol{u}^p, \boldsymbol{v}^p \in \boldsymbol{X}^p,$$
(4)

$$b^f(\boldsymbol{v}^f, q^f) := -(q^f, \nabla \cdot \boldsymbol{v}^f)_{\Omega^f}, \quad \boldsymbol{v}^f \in \boldsymbol{X}^f, \ q^f \in M^f,$$
(5)

$$b^p(\boldsymbol{v}^p, p^p) := -(p^p, \nabla \cdot \boldsymbol{v}^p)_{\Omega^p}, \quad \boldsymbol{v}^p \in \boldsymbol{X}^p, \ p^p \in M^p,$$
(6)

and with weak conservation of mass bilinear form defined by

$$b_\Gamma(\boldsymbol{v}, \mu) := \langle \boldsymbol{v}^f \cdot \boldsymbol{\eta}^f, \mu \rangle_\Gamma + \langle \boldsymbol{v}^p \cdot \boldsymbol{\eta}^p, \mu \rangle_\Gamma, \quad \boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X}, \mu \in \Lambda. \quad (7)$$

The second duality pairing of (7) is interpreted as $\langle \boldsymbol{v}^p \cdot \boldsymbol{\eta}^p, E_{\boldsymbol{\eta}^p}(\mu) \rangle_{\partial\Omega^p}$. Here $E_{\boldsymbol{\eta}^p}$ is any continuous lift-in operator from $H^{1/2}(\Gamma)$ to $H^{1/2}(\partial\Omega^p)$; recall that $\Gamma \subset \partial\Omega^p$; see [25].

The functional f in the right hand side of (2) is defined by

$$f(\boldsymbol{v}) := f^f(\boldsymbol{v}^f) + f^p(\boldsymbol{v}^p), \quad \text{for all } \boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X},$$

where $f^i(\boldsymbol{v}^i) := (\boldsymbol{f}^i, \boldsymbol{v}^i)_{L^2(\Omega^i)}$ for all $\boldsymbol{v}^i \in \boldsymbol{X}^i$, $i = f, p$.

The bilinear forms $a_{\alpha^f}^f, b^f$ are associated to Stokes equations and the bilinear forms a^p, b^p to Darcy law. The bilinear form $a_{\alpha^f}^f$ includes interface matching conditions 1.*b* and 1.*c* above. The bilinear form b_Γ is used to impose the weak version of the interface matching condition 1.*a* above. For the analysis of this weak formulation and the well-posedness of the problem; see [25, 29].

4   Discretization

¿From now on we consider only the two dimensional case. We note that the ideas developed in the following can be easily extended to case of three dimensional subdomains.

We assume that Ω^i, $i = f, p$, are *two dimensional* polygonal subdomains. Let $\mathcal{T}_{h^i}^i(\Omega^i)$ be a geometrically conforming shape regular and quasi-uniform triangulation of Ω^i with mesh size parameter h^i, $i = f, p$. We do not assume that these two triangulations match at the interface Γ. For the fluid region, let $\boldsymbol{X}_{h^f}^f$ and $M_{h^f}^f$ be $P2/P1$ triangular Taylor-Hood finite elements; see [7, 8, 10]. More precisely,

$$\boldsymbol{X}_{h^f}^f := \left\{ \boldsymbol{u} \in \boldsymbol{X}^f \ : \ \begin{array}{l} \forall K \in \mathcal{T}_{h^f}^f(\Omega_f), \ \boldsymbol{u}_K = \hat{\boldsymbol{u}}_K \circ F_K^{-1} \\ \text{and } \hat{\boldsymbol{u}}_K \in P_2(\hat{K})^2 \end{array} \right\} \cap C^0(\overline{\Omega}^f)^2, \quad (8)$$

where $\boldsymbol{u}_K := \boldsymbol{u}|_K$ and

$$M_{h^f}^f := \left\{ p \in L^2(\Omega^f) \; : \; \begin{array}{l} \forall K \in \mathcal{T}_{h^f}^f(\Omega_f), \; p_K = \hat{p}_K \circ F_K^{-1} \\ \text{and } \hat{p}_K \in P_1(\hat{K}) \end{array} \right\} \cap C^0(\overline{\Omega}^f).$$

Denote $\mathring{M}_{h^f}^f \subset M_{h^f}^f$ the discrete fluid pressures with zero average value in Ω^f. For the porous region, let $\boldsymbol{X}_{h^p}^p \subset \boldsymbol{X}^p$ and $M_{h^p}^p \subset L^2(\Omega^p)$ be the lowest order Raviart-Thomas finite elements based on triangles; see [7, 10]. Let $\mathring{M}_{h^p}^p \subset M_{h^p}^p$ be the subset of pressures in $M_{h^p}^p$ with zero average value in Ω^p.

Define $\boldsymbol{X}_h := \boldsymbol{X}_{h^f}^f \times \boldsymbol{X}_{h^p}^p \subset \boldsymbol{X}$ and $M_h := M_{h^f}^f \times M_{h^p}^p \subset L^2(\Omega^f) \times L^2(\Omega^p)$. Note that in the definition of the discrete velocities we assume that the boundary conditions are included, i.e., for $\boldsymbol{v}_{h^f}^f \in \boldsymbol{X}_{h^f}^f$ we have $\boldsymbol{v}_{h^f}^f = \boldsymbol{0}$ on Γ^f and for $\boldsymbol{v}_{h^p}^p \in \boldsymbol{X}_{h^p}^p$ we have that $\boldsymbol{v}_h^p \cdot \boldsymbol{\eta}^p = 0$ on Γ^p.

Let $\mathcal{T}_{h^p}^p(\Gamma)$ be the restriction to Γ of the porous side triangulation $\mathcal{T}_{h^p}^p(\Omega^p)$. For the Lagrange multipliers space we choose piecewise constant functions on Γ with respect to the triangulation $\mathcal{T}_{h^p}^p(\Gamma)$,

$$\Lambda_{h^p} := \left\{ \lambda \; : \; \lambda|_{e_j^p} = \lambda_{e_j^p} \; \text{ is constant in each edge } e_j^p \text{ of } \mathcal{T}_{h^p}^p(\Gamma) \right\}, \qquad (9)$$

i.e., the *master* is on the fluid region side and the *slave* is on the porous region side; see [5, 6, 21, 43]. The choice of piecewise constant Lagrange multipliers leads to a nonconforming approximation on Λ_{h^p} since piecewise constant functions do not belong to $H^{1/2}(\Gamma)$. For the analysis of this nonconforming discretization and a priori error estimates we refer to [25].

5   Primal and Dual Formulations

In order to simplify notation and since there is no danger of confusion, we will denote the finite element functions and the corresponding vector representation by the same symbol, i.e., when writing finite element functions we will drop the indices h^i. Recall that we have the pair of spaces $(\boldsymbol{X}_h, M_h)$ associated to the coupled problem and spaces associated to each subproblem: $(\boldsymbol{X}_{h^f}^f, M_{h^f}^f)$ and $(\boldsymbol{X}_{h^p}^p, M_{h^p}^p)$. We will keep the subscript h^i, $i = f, p$, in the notation for local subspaces $\boldsymbol{X}_{h^f}^f, M_{h^f}^f, \boldsymbol{X}_{h^p}^p$ and $M_{h^p}^p$.

Since we are interested in preconditioning issues we assume $\alpha^f = 0$ in the definition of the fluid side local bilinear form $a_{\alpha^f}^f$ in (3). We denote $a^f = a_0^f$. See Remark 9 for the case $\alpha^f > 0$.

With the discretization chosen in Section 4 we obtain the following symmet-

ric saddle point linear system

$$
\begin{bmatrix}
A^f & B^{fT} & 0 & 0 & C^{fT} \\
B^f & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & A^p & B^{pT} & -C^{pT} \\
0 & 0 & B^p & 0 & 0 \\
\hline
C^f & 0 & -C^p & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{u}^f \\
p^f \\
\hline
\boldsymbol{u}^p \\
p^p \\
\hline
\lambda
\end{bmatrix}
=
\begin{bmatrix}
\boldsymbol{f}^f \\
g^f \\
\hline
\boldsymbol{f}^p \\
g^p \\
\hline
0
\end{bmatrix}
\tag{10}
$$

with matrices A^i, B^i, C^i and columns vectors $\boldsymbol{f}^i, g^i$, $i = f, p$, defined by

$$
\begin{aligned}
a^i(\boldsymbol{u}^i, \boldsymbol{v}^i) &= \boldsymbol{v}^{iT} A^i \boldsymbol{u}^i, \\
b^i(\boldsymbol{u}^i, q^i) &= q^{iT} B^i \boldsymbol{u}^i, \\
(\boldsymbol{u}^i \cdot \boldsymbol{\eta}^f, \mu)_\Gamma &= \mu^T C^i \boldsymbol{u}^i, \\
f^i(\boldsymbol{v}^i) &= \boldsymbol{v}^{iT} \boldsymbol{f}^i, \\
g^i(q^i) &= q^{iT} g^i.
\end{aligned}
\tag{11}
$$

Matrix A^f corresponds to ν times the discrete version of the linearized stress tensor on Ω^f. Note that in the case $\alpha^f > 0$, the bilinear form $a^f_{\alpha^f}$ in (3) includes a boundary term; see Remark 9. The matrix A^p corresponds to ν/κ times a discrete L^2-norm on Ω^p. Matrix $-B^i$ is the discrete divergence in Ω^i, $i = f, p$, and matrices C^f and C^p correspond to the matrix form of the discrete conservation of mass on Γ. Note that ν can be viewed as a scaling factor since it appears in both matrices A^f and A^p. Therefore, it is not relevant for preconditioning issues.

Consider the following partition of the degrees of freedom: For $i = f, p$, let

$$
\begin{bmatrix}
\boldsymbol{u}^i_I \\
p^i_I \\
u^i_\Gamma \\
\bar{p}^i
\end{bmatrix}
\qquad
\begin{aligned}
&\text{Interior displacements + tangential velocities on } \Gamma, \\
&\text{Interior pressures with zero average in } \Omega^i, \\
&\text{Interface outward } \textit{normal velocities} \text{ on } \Gamma, \\
&\text{Constant pressure in } \Omega^i.
\end{aligned}
$$

Then, for $i = f, p$, we have the block structure:

$$
A^i = \begin{bmatrix} A^i_{II} & A^{iT}_{\Gamma I} \\ A^i_{\Gamma I} & A^i_{\Gamma\Gamma} \end{bmatrix}, \quad
B^i = \begin{bmatrix} B^i_{II} & B^{iT}_{\Gamma I} \\ 0 & \bar{B}^{iT} \end{bmatrix} \text{ and } C^i = \begin{bmatrix} 0 & 0 & \tilde{C}^i & 0 \end{bmatrix}.
$$

Note that the $(2, 1)$ entry of B^i corresponds to integrating an *interior* velocity against a constant pressure, then it vanishes due to the divergence theorem. We

have the following matrix representation of the coupled problem in (10):

$$
\left[\begin{array}{cccc|cccc|c}
A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\
B_{II}^f & 0 & B_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} & 0 & 0 & 0 & 0 & \tilde{C}^{fT} \\
0 & 0 & \bar{B}^f & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & A_{II}^p & B_{II}^{pT} & A_{\Gamma I}^{pT} & 0 & 0 \\
0 & 0 & 0 & 0 & B_{II}^p & 0 & B_{I\Gamma}^p & 0 & 0 \\
0 & 0 & 0 & 0 & A_{\Gamma I}^p & B_{I\Gamma}^{pT} & A_{\Gamma\Gamma}^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{B}^p & 0 & 0 \\
\hline
0 & 0 & \tilde{C}^f & 0 & 0 & 0 & -\tilde{C}^p & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\boldsymbol{u}_I^f \\ p_I^f \\ u_\Gamma^f \\ \bar{p}^f \\ \hline \boldsymbol{u}_I^p \\ p_I^p \\ u_\Gamma^p \\ \bar{p}^p \\ \hline \lambda
\end{array}\right]
=
\left[\begin{array}{c}
\boldsymbol{f}_I^f \\ g_I^f \\ f_\Gamma^f \\ \bar{g}^f \\ \hline \boldsymbol{f}_I^p \\ g_I^p \\ f_\Gamma^p \\ \bar{g}^p \\ \hline 0
\end{array}\right].
$$

$$(12)$$

Following [21, 38], we choose the following matrix representation in each subdomain Ω^i, $i = f, p$,

$$
\left[\begin{array}{cc|cc}
A_{II}^i & B_{II}^{iT} & A_{\Gamma I}^{iT} & 0 \\
B_{II}^i & 0 & B_{I\Gamma}^i & 0 \\
\hline
A_{\Gamma I}^i & B_{I\Gamma}^{iT} & A_{\Gamma\Gamma}^i & \bar{B}^{iT} \\
0 & 0 & \bar{B}^i & 0
\end{array}\right]
=
\left[\begin{array}{c|c}
K_{II}^i & K_{\Gamma I}^{iT} \\
\hline
K_{\Gamma I}^i & K_{\Gamma\Gamma}^i
\end{array}\right].
\tag{13}
$$

5.1 The primal formulation

From the last equation in (12) we see that the mortar condition on Γ (using the Darcy side as the slave side) can be imposed as $u_\Gamma^p = (\tilde{C}^p)^{-1} \tilde{C}^f u_\Gamma^f = \Pi u_\Gamma^f$, where Π is the $L^2(\Gamma)$ projection on the space of piecewise constant functions on each subinterval $e^p \in \mathcal{T}_{h^p}^p(\Gamma)$. We note that $\tilde{C}^p$ is a diagonal matrix for the lowest order Raviart-Thomas elements.

Now we eliminate $\boldsymbol{u}_I^i$, p_I^i, $i = f, p$, and λ, to obtain the following (saddle point) Schur complement

$$
S \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{array}\right]
=
\left[\begin{array}{c} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{array}\right].
\tag{14}
$$

Here S is given by

$$
\begin{aligned}
S : &=
\left[\begin{array}{ccc}
S_\Gamma^f & \bar{B}^{fT} & 0 \\
\bar{B}^f & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
+ \tilde{\Pi}^T
\left[\begin{array}{ccc}
S_\Gamma^p & 0 & \bar{B}^{pT} \\
0 & 0 & 0 \\
\bar{B}^p & 0 & 0
\end{array}\right]
\tilde{\Pi} \\
&= \tilde{S}^f + \tilde{S}^p \\
&=
\left[\begin{array}{c|cc}
S_\Gamma^f + \Pi^T S_\Gamma^p \Pi & \bar{B}^{fT} & \Pi^T \bar{B}^{pT} \\
\hline
\bar{B}^f & 0 & 0 \\
\bar{B}^p \Pi & 0 & 0
\end{array}\right]
=
\left[\begin{array}{cc}
S_\Gamma & \bar{B}^T \\
\bar{B} & 0
\end{array}\right],
\end{aligned}
$$

where

$$\tilde{\Pi} := \begin{bmatrix} \Pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{B}^T := [\bar{B}^{fT} \quad \Pi^T \bar{B}^{pT}]. \tag{15}$$

Here, we have denoted

$$\tilde{S}^f := \begin{bmatrix} S_\Gamma^f & \bar{B}^{fT} & 0 \\ \bar{B}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{S}^p := \tilde{\Pi}^T \begin{bmatrix} S_\Gamma^p & 0 & \bar{B}^{pT} \\ 0 & 0 & 0 \\ \bar{B}^p & 0 & 0 \end{bmatrix} \tilde{\Pi}. \tag{16}$$

The local matrices S_Γ^i and $\bar{B}^i$ and the local Schur complement S^i are given by

$$S^i = \begin{bmatrix} S_\Gamma^i & \bar{B}^{iT} \\ \bar{B}^i & 0 \end{bmatrix} := K_{\Gamma\Gamma}^i - K_{\Gamma I}^i \left(K_{\Gamma\Gamma}^i \right)^{-1} K_{\Gamma I}^{iT}, \quad i = p, f. \tag{17}$$

The right hand side of (14) is given by

$$\begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix} = \left\{ \begin{bmatrix} f_\Gamma^f \\ \bar{g}^f \\ 0 \end{bmatrix} - \begin{bmatrix} K_{\Gamma I}^f \left(K_{\Gamma\Gamma}^f \right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^f \\ g_I^f \end{bmatrix} \\ 0 \end{bmatrix} \right\} +$$
$$\left\{ \begin{bmatrix} \Pi^T f_\Gamma^p \\ 0 \\ \bar{g}^p \end{bmatrix} - \tilde{\Pi}^T \begin{bmatrix} K_{\Gamma I}^p \left(K_{\Gamma\Gamma}^p \right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^p \\ g_I^p \end{bmatrix} \\ 0 \end{bmatrix} \right\}.$$

We note that the reduced system (14), as well as the original system (12), is solvable when $\bar{b}^f + \bar{b}^p = 0$, and the solution is unique when we restrict to pressures with zero average value on Ω.

From now on we only work with functions defined on Γ and extended inside the subdomain using the discrete Stokes and Darcy problems. It is convenient to define the space

$$V_\Gamma := \left\{ v_\Gamma = (v_\Gamma^f, v_\Gamma^p) : v_\Gamma^f = \mathcal{SH}(\boldsymbol{v}^f \cdot \boldsymbol{\eta}^f|_\Gamma) \text{ and } v_\Gamma^p = \mathcal{DH}(\boldsymbol{v}^p \cdot \boldsymbol{\eta}^p|_\Gamma) \right\} \tag{18}$$

and

$$M_0 := \left\{ q \in M^h : q^i = \text{piecewise const. in } \Omega^i, i = f, p, \text{ and } \int_{\Omega^f} q^f + \int_{\Omega^p} q^p = 0 \right\}.$$

Here $\mathcal{SH}$ ($\mathcal{DH}$) is the velocity component of the discrete Stokes (Darcy) harmonic extension operator that maps discrete interface normal velocity $u_\Gamma^f \in H_{00}^{1/2}(\Gamma)$ (respectively $u_\Gamma^p \in (H^{1/2}(\Gamma))'$) to the solution of following problem: Find $\boldsymbol{u}^i \in \boldsymbol{X}_{h^i}^i$ and $p^i \in \mathring{M}_{h^i}^i$ such that for all $\boldsymbol{v}^i \in \boldsymbol{X}_{h^i}^i$ and $q^i \in \mathring{M}_{h^i}^i$, $i = f, p$, we have:

$$\begin{cases} a^f(\mathcal{SH}u^f, \boldsymbol{v}^f) + b^f(\boldsymbol{v}^f, p^f) &= 0 \\ b^f(\mathcal{SH}u^f, q^f) &= 0 \\ \mathcal{SH}u^f \cdot \boldsymbol{\eta}^f &= u_\Gamma^f \quad \text{on } \Gamma \\ \mathcal{SH}u^f &= \boldsymbol{0} \quad \text{on } \Gamma^f, \end{cases} \tag{19}$$

and

$$\begin{cases} a^p(\mathcal{DH}u^p, \boldsymbol{v}^p) + b^p(\boldsymbol{v}^p, p^p) & = 0 \\ b^p(\mathcal{DH}u^p, q^p) & = 0 \\ \mathcal{DH}u^p \cdot \boldsymbol{\eta}^p & = u_\Gamma^p \quad \text{on } \Gamma \\ \mathcal{DH}u^p \cdot \boldsymbol{\eta}^p & = 0 \quad \text{on } \Gamma^p. \end{cases} \tag{20}$$

The degrees of freedom associated with $\mathcal{SH}u^f \cdot \boldsymbol{\tau}^f$ on Γ are free. This corresponds to imposing the natural boundary condition $\boldsymbol{\tau}^T \boldsymbol{D}(\mathcal{SH}u^f)\boldsymbol{\eta}_f = 0$ on Γ.

For $i = f, p$, define the normal trace component of $\boldsymbol{X}_{h^i}^i$ by

$$Z_{h^i}^i = \left\{ \boldsymbol{v}^i \cdot \boldsymbol{\eta}^i |_\Gamma \;:\; \boldsymbol{v}^i \in \boldsymbol{X}_{h^i}^i \right\}. \tag{21}$$

Associated with the coupled problem (12) we introduce the *balanced subspace*

$$V_{\Gamma, \bar{B}} := \left\{ v_\Gamma^f \in Z_{h^f}^f \;:\; (v_\Gamma^f, \Pi v_\Gamma^f) \in V_\Gamma, \text{ and } \int_\Gamma v_\Gamma^f \cdot \boldsymbol{\eta}_f = 0 \right\}, \tag{22}$$

with V_Γ defined in (18); see [38]. Observe that $V_{\Gamma, \bar{B}} = \operatorname{Ker}\bar{B}$, where $\bar{B}$ is defined in (15) and (17). Then for $v_\Gamma^f \in V_{\Gamma, \bar{B}}$ we have $\bar{B}v_\Gamma^f = 0$. We will refer to functions $v_\Gamma^f \in V_{\Gamma, \bar{B}}$ as *balanced functions*. If $v_\Gamma^p = \Pi v_\Gamma^f$ and v_Γ^f is a balanced function then we also say that v_Γ^p is a balanced function or the pair $(v_\Gamma^f, \Pi v_\Gamma^f)$ is balanced.

5.2   Dual formulation

In the system (12), we first eliminate the unknowns $\boldsymbol{u}_I^f, p_I^f$ and $\boldsymbol{u}_I^p, p_I^p$. We obtain

$$\left[\begin{array}{cc|cc|c} S_\Gamma^f & \bar{B}^{fT} & 0 & 0 & \tilde{C}^{fT} \\ \bar{B}^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & S_\Gamma^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\ 0 & 0 & \bar{B}^p & 0 & 0 \\ \hline \tilde{C}^f & 0 & -\tilde{C}^p & 0 & 0 \end{array} \right] \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \hline u_\Gamma^p \\ \bar{p}^p \\ \hline \lambda \end{array} \right] = \left[\begin{array}{c} \tilde{b}^f \\ \hline \tilde{b}^p \\ \hline 0 \end{array} \right], \tag{23}$$

where S_Γ^i, K_{II}^i and $K_{I\Gamma}^i$, $i = f, p$, are defined in (17) and (13). The right hand side of (23) is given by

$$\left[\begin{array}{c} \tilde{b}^f \\ \hline \tilde{b}^p \\ \hline 0 \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} f_\Gamma^f \\ g^f \end{array} \right] - K_{\Gamma I}^f \left(K_{\Gamma\Gamma}^f \right)^{-1} \left[\begin{array}{c} \boldsymbol{f}_I^f \\ g_I^f \end{array} \right] \\ \hline \left[\begin{array}{c} f_\Gamma^p \\ \bar{g}^p \end{array} \right] - K_{\Gamma I}^p \left(K_{\Gamma\Gamma}^p \right)^{-1} \left[\begin{array}{c} \boldsymbol{f}_I^p \\ g_I^p \end{array} \right] \\ \hline 0 \end{array} \right].$$

Let $N_i := \begin{bmatrix} \tilde{C}^i & 0 \end{bmatrix}$ and consider S^i, $i = f, p$, defined in (17). Then the matrix in the left hand side of (23) can be rewritten as

$$
\left[
\begin{array}{c|c|c}
S^f & 0 & N^{fT} \\
\hline
0 & S^p & -N^{pT} \\
\hline
N^f & -N^p & 0
\end{array}
\right] .
$$

Now we eliminate the unknowns $u_\Gamma^f, \bar{p}^f$ and $u_\Gamma^p, \bar{p}^p$. We end up with the reduced system

$$ F\lambda = c, \tag{24} $$

where the operator F is defined by

$$ F := N^f (S^f)^{-1} N^{fT} + N^p (S^p)^{-1} N^{pT}, \tag{25} $$

and the right hand side c is given by

$$
c = N^f (S^f)^{-1} \left\{ \begin{bmatrix} f_\Gamma^f \\ \bar{g}^f \end{bmatrix} - K_{\Gamma I}^f \left(K_{\Gamma\Gamma}^f \right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^f \\ g_I^f \end{bmatrix} \right\} -
$$
$$
N^p (S^p)^{-1} \left\{ \begin{bmatrix} f_\Gamma^p \\ \bar{g}^p \end{bmatrix} - K_{\Gamma I}^p \left(K_{\Gamma\Gamma}^p \right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^f \\ g_I^p \end{bmatrix} \right\} .
$$

Note that F is positive semidefinite and since a discrete Lagrange multiplier in Λ_{h^p} does not have necessarily zero mean average value on Γ, then, the operator F has one simple zero eigenvalue corresponding to a constant Lagrange multiplier. The above linear system, as well as the original linear system (12), is solvable for zero mean right hand side, i.e., for $c^T \cdot (1, \ldots, 1) = 0$.

6   BDD Preconditioner

In this section we design and analyze a BDD type preconditioner for the Schur complement system (14); see [9, 21, 40] and also [1, 20, 33, 38, 41]. For the sake of simplicity on the analysis we assume that $\Gamma = \{1\} \times (0, 1)$, $\Omega^f = (1, 2) \times (0, 1)$ and $\Omega^p = (0, 1) \times (0, 1)$. We introduce the velocity coarse space on Γ as the span of the normal velocity $v_0 = y(1 - y)$ (with v_0 also denoting its vector representation). Define:

$$ R_0 := \begin{bmatrix} v_0^T & 0 \\ 0 & I_{2\times 2} \end{bmatrix}, \quad S_0 := R_0 S R_0^T \quad \text{and} \quad Q_0 := R_0^T S_0^\dagger R_0. \tag{26} $$

The system (14) is solvable when the right hand side satisfy $\bar{b}^f + \bar{b}^p = 0$ with uniqueness of the solution in the space of vectors with pressure component having zero average value on Ω. Then, we have that S_0 is invertible restricted to vectors with pressure component in M_0. The low dimensionality of the coarse space (which is spanned by ϕ_0^f and a constant pressure per subdomain Ω^i, $i = f, p$,) and the fact that the functions ϕ_0^f is independent of the triangulation

parameters imply stable discrete inf-sup condition for the coarse problem.

Denote $\tilde{S}_0 := v_0^T S_\Gamma v_0$ and $\tilde{S} := \bar{B} v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T$. We can write (see (15) and (26))

$$S_0 = \left[\begin{array}{cc} \tilde{S}_0 & (\bar{B} v_0)^T \\ \bar{B} v_0 & 0 \end{array} \right].$$

A simple calculation using the formula for the inverse of a saddle point matrix gives

$$Q_0 = \left[\begin{array}{cc} v_0 \tilde{S}_0^{-1} v_0^T - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T & v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \\ \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T & \tilde{S}^{-1} \end{array} \right],$$

and using (15) we obtain

$$Q_0 S = \left[\begin{array}{cc} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma + v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} & 0 \\ \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - \tilde{S}^{-1} \bar{B} & I \end{array} \right],$$

or $Q_0 S = \left[\begin{array}{cc} \mathcal{P} & 0 \\ \mathcal{G} & I \end{array} \right]$. Here, we have defined

$$\mathcal{P} := \left(v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma \right) + v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B}$$

$$\mathcal{G} := \tilde{S}^{-1} \bar{B} - \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma.$$

With this notation we have that $I - Q_0 S = \left[\begin{array}{cc} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{array} \right]$. Elementary calculations show that $\mathcal{P}^2 = \mathcal{P}$ and $\bar{B}(I - \mathcal{P}) = 0$, hence $I - \mathcal{P}$ is a projection and its image is contained on the balanced subspace defined in (22); see also [38].

Given a residual $r = \left[\begin{array}{cc} f_\Gamma^T & \bar{g}^T \end{array} \right]^T$, the coarse problem $Q_0 r$, with Q_0 defined in (26), is the solution of the coupled problem (12) with one velocity degree of freedom (v_0), and a constant pressure per subdomain Ω^i, $i = f, p$, with mean zero in $\Omega = \text{int}(\overline{\Omega^f} \cup \overline{\Omega^p})$. Note that the matrix S_0 defined in (26) can be computed easily and in order to ensure zero mean pressure on Ω we can use a Lagrange multiplier.

For balanced functions v_Γ^f and u_Γ^f, the S_Γ-inner product is defined by (see (15)):

$$\langle u_\Gamma^f, v_\Gamma^f \rangle_{S_\Gamma} := \langle S_\Gamma u_\Gamma^f, v_\Gamma^f \rangle = u_\Gamma^{fT} S_\Gamma v_\Gamma^f.$$

Recall that $\bar{B} u_\Gamma^f = 0$ when u_Γ^f is balanced. Then, on this subspace of balanced functions, the S_Γ inner product coincides with the S-inner product defined by

$$\left\langle \left[\begin{array}{c} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{array} \right], \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{array} \right] \right\rangle_S := \left[\begin{array}{c} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{array} \right]^T S \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{array} \right] = \left[\begin{array}{c} v_\Gamma^f \\ \bar{q} \end{array} \right]^T \left[\begin{array}{cc} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{array} \right] \left[\begin{array}{c} u_\Gamma^f \\ \bar{p} \end{array} \right],$$

where $\bar{p}^T = [\ \bar{p}^p \quad \bar{p}^p \ ]^T$. Consider the BDD preconditioner operator given by

$$S_N^{-1} := Q_0 + (I - Q_0 S)\,(\tilde{S}^f)^\dagger\,(I - SQ_0)\,, \tag{27}$$

where $\tilde{S}^f$ is defined in (16); see [21, 38]. The notation $(\tilde{S}^f)^\dagger$ stands for the pseudo-inverse of $\tilde{S}^f$, i.e.,

$$(\tilde{S}^f)^\dagger = \left[\begin{array}{cc} (S^f)^{-1} & 0 \\ 0 & 0 \end{array} \right],$$

with S^f defined in (17). The preconditioned operator is given by

$$
\begin{aligned}
S_N^{-1} S &= Q_0 S + (I - Q_0 S)\,(\tilde{S}^f)^\dagger S\,(I - Q_0 S) \\
&= \left[\begin{array}{cc} \mathcal{P} & 0 \\ \mathcal{G} & I \end{array} \right] + \left[\begin{array}{cc} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{array} \right] (\tilde{S}^f)^\dagger \left[\begin{array}{cc} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{array} \right] \left[\begin{array}{cc} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{array} \right]
\end{aligned} \tag{28}
$$

Note that applying $(S^f)^{-1}$ to a vector $\left[\begin{array}{c} u_\Gamma^f \\ \bar{p} \end{array} \right]$ is equivalent to solving the linear system

$$
\left[\begin{array}{cccc} A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 \\ B_{II}^f & 0 & B_{I\Gamma}^f & 0 \\ A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} \\ 0 & 0 & \bar{B}^f & 0 \end{array} \right] \left[\begin{array}{c} w_I^f \\ s_I^f \\ w_\Gamma^f \\ \bar{s}^f \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ u_\Gamma^f \\ \bar{p}^f \end{array} \right].
$$

If u_Γ^f is balanced, so is balanced the velocity component of $(S^f)^{-1} \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \end{array} \right]$. Then using elementary calculations with the matrices in (28) we obtain that

$$\langle S_N^{-1} S \left[\begin{array}{c} u_\Gamma \\ \bar{p} \end{array} \right], \left[\begin{array}{c} v_\Gamma \\ \bar{q} \end{array} \right] \rangle_S = \langle (S_\Gamma^f)^{-1} S_\Gamma u_\Gamma, v_\Gamma \rangle_{S_\Gamma},$$

for $u_\Gamma, v_\Gamma \in \text{Range}(I - \mathcal{P})$. In order to bound the condition number of the preconditioned operator $S_N^{-1} S$, we only need to analyze the condition of the operator $(S_\Gamma^f)^{-1} S_\Gamma$. Note that

$$c\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \le \langle \left(S^f \right)^{-1} S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \le C \langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma}$$

is equivalent to

$$c\langle S^f u_\Gamma^f, u_\Gamma^f \rangle \le \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \le C \langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \tag{29}$$

The next theorem shows that the condition number estimate for the BDD method introduced in (27) is of order $O(1 + \frac{1}{\kappa})$, where κ is the permeability of the porous medium; see (1).

Theorem 1 *If u_Γ^f is a balanced function then*

$$\langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle \le \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \prec \left(1 + \frac{1}{\kappa} \right) \langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle.$$

Proof. The lower bound follows trivially from $\tilde{S}_\Gamma^f$ and $\tilde{S}_\Gamma^p$ being positive on the subspace of balanced functions. Next we concentrate on the upper bound.

Let v_Γ^f be a balanced function and $v_\Gamma^p = \Pi v_\Gamma^f$. Define $\boldsymbol{v}^p = \mathcal{DH} v_\Gamma^p$; see (20). Using properties of the discrete operator $\mathcal{DH}$, see [36], we obtain

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle = a^p(\boldsymbol{v}^p, \boldsymbol{v}^p) \asymp \frac{\nu}{\kappa} \|v_\Gamma^p\|_{(H^{1/2})'(\Gamma)}^2.$$

Using the L_2-stability property of mortar projection Π, we have

$$\|v_\Gamma^p\|_{(H^{1/2})'(\Gamma)}^2 \prec \|v_\Gamma^p\|_{L^2(\Gamma)}^2 = \|v_\Gamma^f\|_{L^2(\Gamma)}^2 \prec \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2.$$

With $\mathcal{SH}$ defined in (19), define $\boldsymbol{v}^f = \mathcal{SH} v_\Gamma^f$. Using properties of $\mathcal{SH}$, see [38], we have

$$\nu \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2 \asymp a^f(\boldsymbol{v}^f, \boldsymbol{v}^f)$$

and then

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle \prec \frac{1}{\kappa} \langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \tag{30}$$

This gives the upper bound and finishes the proof. ∎

---

1. Initialize

$$x^{(0)} = Q_0 b + w$$
$$d^{(0)} = b - S x^{(0)}$$

with $w \in \text{Range}(I - Q_0 S)$. Recall that all vectors have three components, for instance, $x = \begin{bmatrix} x_\Gamma \\ \bar{x}^f \\ \bar{x}^p \end{bmatrix}$ and $b = \begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix}$.

2. Iterate $k = 1, 2, \ldots$ until convergence

$$\text{Precondition:} \quad z^{(k-1)} = (\tilde{S}^f)^\dagger d^{(k-1)}$$

$$\text{Project:} \quad y^{(k-1)} = (I - Q_0 S) z^{(k-1)}$$

$$\begin{aligned} \beta^k &= \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle y^{(k-2)}, d^{(k-1)} \rangle \quad [\beta^{(1)} = 0] \\ r^{(k)} &= y^{(k-1)} + \beta^{(k)} r^{(k)} \quad [r^{(1)} = y^{(0)}] \\ \alpha^{(k)} &= \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle d^{(k)}, S r^{(k)} \rangle \\ x^{(k)} &= x^{(k-1)} + \alpha^{(k)} r^{(k)} \\ d^{(k)} &= d^{(k-1)} - \alpha^{(k)} S r^{(k)} \end{aligned}$$
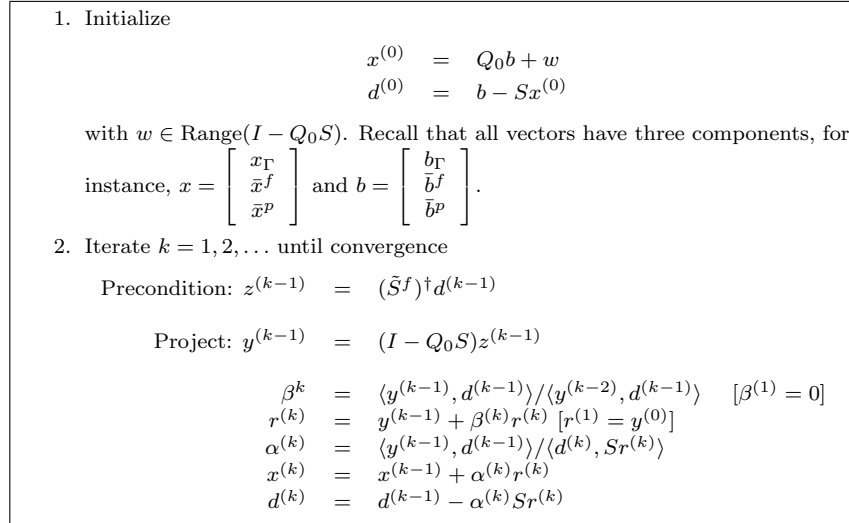
Figure 1: Implementation of the projected preconditioned conjugate gradient algorithm for the system (14) involving the BDD preconditioner (27).

Recall that we consider the preconditioned projected conjugate gradient method applied to the Schur complement problem (14). We have written the algorithm in Figure 1.

7 FETI Preconditioner

In this section we analyze a FETI preconditioner for the reduced linear system (24); see [9, 21, 40] and also [23, 28, 35]. Recall the definition of F in (25). We propose the following preconditioner

$$(N^p)^\dagger (S^p)(N^p)^{\dagger T}, \tag{31}$$

where $(N^p)^\dagger$ is the pseudo-inverse $(N^p)^\dagger = \begin{bmatrix} (\tilde{C}^p)^{-1} & 0 \end{bmatrix}$.

Note that after computing the action of $(S^f)^{-1}$ and $(S^p)^{-1}$, in the application of F to a zero average Lagrange multiplier, we end up with balanced functions. Therefore, in order to apply the preconditioned operator $(N^p)^\dagger (S^p)(N^p)^{\dagger T} F$ to a zero mean Lagrange multiplier, we do not need to solve a coarse problem at the beginning of the CG, neither inside of the CG iteration.

The FETI preconditioner in (31) can be considered as the dual preconditioner of the BDD preconditioner defined in (27); see the proof of Lemma 2 below.

Recall the definition of S^i, $i = f, p$, in (17) and the definition of space of balanced functions $V_\Gamma = V_\Gamma^f \times V_\Gamma^p$ in (22) and (21). We prove the following result.

Lemma 2 *Let $\lambda \in \Lambda_{h^p} \cap L_0^2(\Gamma)$ be a zero mean Lagrange multiplier. Then*

$$\langle N^f (S^f)^{-1} N^{fT} \lambda, \lambda \rangle \prec \frac{1}{\kappa} \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle.$$

Proof. Consider a zero mean Lagrange multiplier λ. Define $t = (S^p)^{-\frac{1}{2}} N^{pT} \lambda$ and $w^f = N^{fT} \lambda$. Then it is enough to prove that

$$\|(S^f)^{-\frac{1}{2}} w^f\|^2 \prec \|t\|^2.$$

Since w^f is balanced, i.e., $w^f \in V_\Gamma^f$, we have that

$$
\begin{aligned}
\|(S^f)^{-\frac{1}{2}} w^f\|^2 &= \sup_{z^f \in Z_{h_f}^f} \frac{\langle (S^f)^{-\frac{1}{2}} w^f, z^f \rangle^2}{\|z^f\|^2} \\
&= \sup_{v^f \text{ balanced}} \frac{\langle w^f, v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\
&= \sup_{v^f \text{ balanced}} \frac{\langle \lambda, N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\
&= \sup_{v^f \text{ balanced}} \frac{\langle (S^p)^{-\frac{1}{2}} N^p \lambda, (S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2}.
\end{aligned}
$$

Then using the Cauchy-Schwarz inequality and (30) in the proof of Theorem 1, we have

$$
\begin{aligned}
\|(S^f)^{-\frac{1}{2}} w^f\|^2 &= \sup_{v^f \text{ balanced}} \frac{\langle t, (S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\
&\leq \|t\|^2 \sup_{v^f \text{ balanced}} \frac{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f\|^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\
&\prec \frac{1}{\kappa} \|t\|^2.
\end{aligned}
$$

■

Using Lemma 2 we can derive the following estimate for the condition number of the FETI preconditioner defined in (31).

Theorem 3 *Let λ be a zero mean Lagrange multiplier. Then*

$$
\langle N^p (S^p)^{-1} N_p^T \lambda, \lambda \rangle \prec \langle F\lambda, \lambda \rangle \prec \left(1 + \frac{1}{\kappa} \right) \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle.
$$

The condition number estimate $O(\frac{\kappa+1}{\kappa})$ can be improved in the case where the fluid side triangulation is finer than the porous side triangulation. This case has some advantages when κ is small. In order to fix ideas and simplify notation we analyze in detail the case where the triangulation of the fluid side is a *refinement* of the porous side triangulation. In particular, in Theorem 6, we will prove that the condition of the FETI preconditioned operator is of order $O(\frac{\kappa+1}{\kappa+(h^p)^2})$ in this simpler situation. The analysis that we will present to prove Theorem 6 can be extended easily for the case where the fluid side triangulation is finer than (and not necessarily a refinement of) the porous side triangulation; see Remark 7.

We assume that the fluid side discretization on Γ, $\mathcal{T}_{h^f}^f(\Omega^f)|_\Gamma$, is a refinement of the corresponding porous side discretization, $\mathcal{T}_{h^p}^p(\Omega^p)|_\Gamma$. That is, assume that $h^p = r h^f$ for some positive integer r. We will refer to this assumption as the *nested refinement assumption*. For $j = 1, \ldots, m^p$, we introduce the normal fluid velocity ϕ_j^f as the $P2$ bubble function defined on $\mathcal{T}_{h^p}^p(\Omega^p)|_\Gamma$ and with support on the interval $e_j^p = \{0\} \times [(j-1)h^p, jh^p]$. Recall that we are using $P2/P1$ Taylor-Hood discretization on the fluid side. Under the nested refinement assumption we have that $\phi_j^f \in Z_{h^f}^f$ with $Z_{h^f}^f$ defined in (21). Denote $Z_{h^f,b}^f$ as the subspace of $Z_{h^f}^f$ spanned by all ϕ_j^f, $j = 1, \ldots, m^p$, and set $Z_{h^f,0}^f$ as the subspace of $Z_{h^f}^f$ spanned by functions with zero average on all edges e_j^p, $j = 1, \ldots, m^p$. Note that $Z_{h^f,b}^f$ and $Z_{h^f,0}^f$ form a direct sum for $Z_{h^f}^f$ and the image $\Pi Z_{h^f,0}^f$ is the zero vector.

Before deriving the condition number estimate of the FETI preconditioner under the nested refinement assumption we first prove a preliminary lemma.

Lemma 4 *Assume that $h^p = rh^f$, where r is a positive integer. If $v_{\Gamma,b}^f \in Z_{h^f,b}^f$ and $v_{\Gamma,b}^f$ is a balanced function then*

$$\langle S_\Gamma^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \prec \frac{\kappa}{(h^p)^2} \langle S_\Gamma^p \Pi v_{\Gamma,b}^f, \Pi v_{\Gamma,b}^f \rangle.$$

Proof. Let $v_{\Gamma,b}^f = \sum_{j=1}^{m^p} \beta_j \phi_j^f \in Z_{h^f,b}^f \subset Z_{h^f}^f$ and note that since the basis functions ϕ_j^f, $j = 1, \ldots, m^p$, do not overlap each other on Γ, they are orthogonal in $L^2(\Gamma)$ and also in $H_0^1(\Gamma)$. Then

$$\|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2 = \sum_{j=1}^{m^p} \beta_j^2 \|\phi_j^f\|_{L^2(\Gamma)}^2 \asymp h^p \sum_{j=1}^{m^p} \beta_j^2, \tag{32}$$

and

$$|v_{\Gamma,b}^f|_{H^1(\Gamma)}^2 = \sum_{j=1}^{m^p} \beta_j^2 |\phi_j^f|_{H_0^1(e_j^p)}^2 \asymp \frac{1}{h^p} \sum_{j=1}^{m^p} \beta_j^2. \tag{33}$$

Using (32), (33) and a interpolation estimate we see that

$$\|v_{\Gamma,b}^f\|_{H_{00}^{1/2}(\Gamma)}^2 \asymp \sum_{j=1}^{m^p} \beta_j^2 \asymp \frac{1}{h^p} \|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2.$$

Note also that

$$\langle S^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \leq a^f(\mathcal{SH} v_{\Gamma,b}^f, \mathcal{SH} v_{\Gamma,b}^f) \asymp \nu \|v_{\Gamma,b}^f\|_{H_{00}^{1/2}(\Gamma)}^2.$$

Denote by $z_{\Gamma,b}^p = \sum_{j=1}^{m^p} \rho_j \chi_{e_j^p}$ the unique piecewise constant function such that $\Pi v_{\Gamma,b}^f = z_{\Gamma,b}^p$. Observe that $|\rho_j| \asymp |\beta_j|$, $j = 1, \ldots, m^p$. We obtain

$$\langle S_\Gamma^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \quad \prec \quad \frac{\nu}{h^p} \|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2 \asymp \frac{\nu}{h^p} \|z_{\Gamma,b}^p\|_{L^2(\Gamma)}^2 \tag{34}$$

$$\prec \quad \frac{\nu}{(h^p)^2} \|z_{\Gamma,b}^p\|_{(H^{1/2})'(\Gamma)}^2 \asymp \frac{\kappa}{(h^p)^2} \langle S_\Gamma^p z_{\Gamma,b}^p, z_{\Gamma,b}^p \rangle, \tag{35}$$

where we have used an inverse inequality for piecewise constant functions. ∎

We now translate Lemma 4 in a result concerning to our dual preconditioner.

Lemma 5 *Assume that $h^p = rh^f$, where r is a positive integer and let λ be a zero mean Lagrange multiplier. Then*

$$\frac{(h^p)^2}{\kappa} \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle \prec \langle N^f (S^f)^{-1} N^{fT} \lambda, \lambda \rangle.$$

Proof. We proceed as before. Let $t = (S^f)^{-\frac{1}{2}} N^{fT} \lambda$ and $w = N^p \lambda$. Then

$$
\begin{aligned}
\|(S^p)^{-\frac{1}{2}} w\|^2 &= \sup_{z^f \in Z_{h^f}^f} \frac{\langle (S^p)^{-\frac{1}{2}} w, z^f \rangle^2}{\|z^p\|^2} \qquad\qquad (36) \\
&= \sup_{v^p \text{ balanced}} \frac{\langle w, v^f \rangle^2}{\|(S^p)^{\frac{1}{2}} v^p\|^2} \\
&= \sup_{v^p \text{ balanced}} \frac{\langle \lambda, N^p v^p \rangle^2}{\|(S^p)^{\frac{1}{2}} v^p\|^2} \\
&= \sup_{v_b^f \text{ balanced}} \frac{\langle \lambda, N^f v_b^f \rangle^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\
&= \sup_{v_b^f \text{ balanced}} \frac{\langle (S^f)^{-\frac{1}{2}} N^{fT} \lambda, (S^f)^{\frac{1}{2}} v_b^f \rangle^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\
&\leq \|t\|^2 \sup_{v_b^f \text{ balanced}} \frac{\|(S^f)^{\frac{1}{2}} v_b^f\|^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\
&\prec \frac{\kappa}{(h^p)^2} \|t\|^2,
\end{aligned}
$$

where the last step follows from Lemma 4. ∎

¿From Lemmas 2 and 5, the next theorem follows.

Theorem 6 *Assume that $h^p = r h^f$, where r is a positive integer. Let λ be a zero mean Lagrange multiplier, then*

$$
\left(1 + \frac{(h^p)^2}{\kappa} \right) \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle \prec \langle F \lambda, \lambda \rangle \prec \left(1 + \frac{1}{\kappa} \right) \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle.
$$

Remark 7 *Theorem 6 can be extended for the case where $h^f \leq 2h^p$. We only need to extend the argument given in the proof of Lemma 4. The basic idea in the proof of Lemma 4 is to associate a bubble function $\phi_j^f \in Z_{h^f}^f$ to each porous side element e_j^p, $j = 1, \ldots, m^p$, in such a way that we can construct a one to one and continuous map $v_{\Gamma,b}^f \mapsto z_{\Gamma,b}^p$. The bubble functions ϕ_j^f, $j = 1, \ldots, m^p$, can be chosen orthogonal in $L^2(\Gamma)$ and in $H_0^1(\Gamma)$. This can also be done when $h^f \leq h^p$. The smaller the h^f, the closer is the size of the support of the bubble ϕ_j^f to the size of the element e_j^p since more and more elements e^f can be associated to only one element e^p. This construction can also be carried out in the case $h_p < h^f \leq 2h^p$ where non-orthogonal Taylor-Hood basis functions must be used. This last situation leads to the appearance of an additional constant that depends on the non-orthogonality; see Section 8.*

Remark 8 *We note that Lemma 4 can be used directly to obtain a bound for the balancing domain decomposition preconditioner similar to the one presented in Section 6 but with $\tilde{S}^p$ instead of $\tilde{S}^f$ in (27); see Proposition 2 of [24]. In this case an additional variable elimination is needed. We have to eliminate the component of the normal fluid velocity in the space $Z_{h^f,0}^f$ and work with the Schur complement with respect to the space $Z_{h^f,b}^f$. This is rather difficult to implement (we can use Lagrange multipliers in this case). Then passing to the dual preconditioner permit us to take advantage of the case where the fluid side discretization on Γ is a refinement of the corresponding porous side discretization.*

Remark 9 *Theorems 1, 3 and 6 are also valid for the case $\alpha^f > 0$ in (3). To see this we need to compare, for different values of α^f, the energy of discrete extensions for a given normal velocity defined on Γ. Given the outward normal velocity v_Γ^f on Γ, let $\mathcal{SH}_{\alpha^f} v_\Gamma^f$ denote the discrete harmonic extension in the sense of $(a_{\alpha^f}^f, b^f)$, that is, the solution of problem (19) with a^f replaced by $a_{\alpha^f}^f$. Recall that $a^f = a_0^f$, where $a_0^f = a_{\alpha^f}$ when $\alpha^f = 0$, and therefore, $\mathcal{SH} v_\Gamma^f = \mathcal{SH}_0 v_\Gamma^f$. Note that in (19) we have imposed the natural boundary condition $\boldsymbol{\tau}^T \boldsymbol{D}(\mathcal{SH} u^f)\boldsymbol{\eta}_f = 0$ on Γ. Now we define another extension denoted by $\widehat{\mathcal{SH}} v_\Gamma^f$. Given the outward normal velocity v_Γ^f on Γ, let $\widehat{\mathcal{SH}} v_\Gamma^f$ be the (a^f, b^f)-discrete harmonic extension given by the solution of (19) with the boundary condition $\widehat{\mathcal{SH}} v_\Gamma^f \cdot \boldsymbol{\tau} = 0$. For both $\mathcal{SH}$ and $\widehat{\mathcal{SH}}$ are imposed essential boundary condition v_Γ^f for the normal component on Γ. The difference between them is in how the boundary condition is imposed for the tangential component on Γ. For the $\mathcal{SH}$ is imposed homogeneous natural boundary condition, while for $\widehat{\mathcal{SH}}$ is imposed homogeneous essential boundary condition. Note that both extensions, $\mathcal{SH}_{\alpha^f}$ and $\widehat{\mathcal{SH}}$, satisfy the zero discrete divergence condition, i.e., the second equation of (19).*
Using the minimization property of the $(a_{\alpha^f}^f, b^f)$-discrete harmonic extension $\mathcal{SH}_{\alpha^f}$ and the (a^f, b^f)-discrete harmonic extension $\widehat{\mathcal{SH}}$ we get

$$
\begin{aligned}
a^f(\mathcal{SH} v_\Gamma^f, \mathcal{SH} v_\Gamma^f) &= a_0^f(\mathcal{SH} v_\Gamma^f, \mathcal{SH} v_\Gamma^f) \quad \text{(by definition)} \\
&\leq a_0^f(\mathcal{SH}_{\alpha^f} v_\Gamma^f, \mathcal{SH}_{\alpha^f} v_\Gamma^f) \quad \text{(by the minimization property of } \mathcal{SH}) \\
&\leq a_{\alpha^f}^f(\mathcal{SH}_{\alpha^f} v_\Gamma^f, \mathcal{SH}_{\alpha^f} v_\Gamma^f) \quad (\alpha^f > 0) \\
&\leq a_{\alpha^f}^f(\widehat{\mathcal{SH}} v_\Gamma^f, \widehat{\mathcal{SH}} v_\Gamma^f) \quad \text{(by the minimization property of } \mathcal{SH}_{\alpha^f}) \\
&= a_0^f(\widehat{\mathcal{SH}}_0 v_\Gamma^f, \widehat{\mathcal{SH}}_0 v_\Gamma^f) \quad \text{(because } \widehat{\mathcal{SH}} u^f \cdot \boldsymbol{\tau}^f = 0 \text{ on } \Gamma) \\
&\asymp \nu \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2 \\
&\asymp a^f(\mathcal{SH}_0 v_\Gamma^f, \mathcal{SH} v_\Gamma^f).
\end{aligned}
$$

The last two equivalences follow from properties of the (a^f, b)-discrete harmonic extensions $\mathcal{SH}$ and $\widehat{\mathcal{SH}}$ (which coincides with the discrete Stokes harmonic ex-
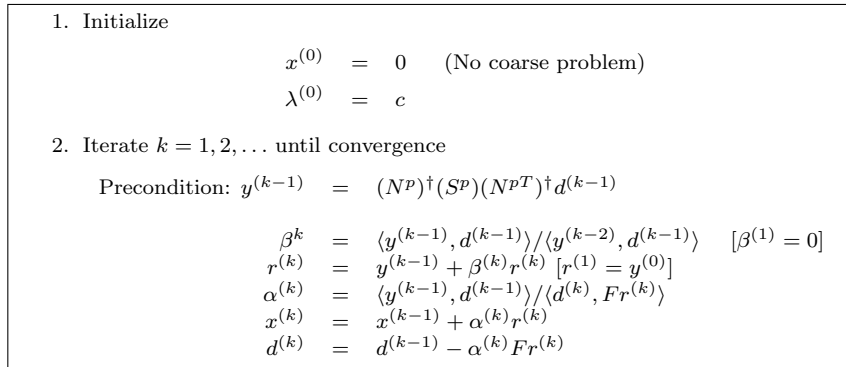
1. Initialize

$$x^{(0)} = 0 \quad \text{(No coarse problem)}$$
$$\lambda^{(0)} = c$$

2. Iterate $k = 1, 2, \ldots$ until convergence

Precondition: $y^{(k-1)} = (N^p)^\dagger (S^p)(N^{pT})^\dagger d^{(k-1)}$

$$\beta^k = \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle y^{(k-2)}, d^{(k-1)} \rangle \quad [\beta^{(1)} = 0]$$
$$r^{(k)} = y^{(k-1)} + \beta^{(k)} r^{(k)} \, [r^{(1)} = y^{(0)}]$$
$$\alpha^{(k)} = \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle d^{(k)}, Fr^{(k)} \rangle$$
$$x^{(k)} = x^{(k-1)} + \alpha^{(k)} r^{(k)}$$
$$d^{(k)} = d^{(k-1)} - \alpha^{(k)} Fr^{(k)}$$

Figure 2: Implementation of the preconditioned conjugate gradient algorithm for the system (24) involving the FETI preconditioner (31).

tension); see [26, 38]. The two equivalences appearing above are independent of the permeability, fluid viscosity and mesh sizes. Then, the energy of the $(a^f_{\alpha^f}, b)$-discrete harmonic extensions is equivalent to the energy of the (a^f, b)-discrete harmonic extension, i.e., the discrete Stokes harmonic extension. This equivalence guarantees the extensions of Theorems 1, 3 and 6 to the case $\alpha^f > 0$.

We solve the system (24) using preconditioned conjugate gradient. We have written the algorithm in Figure 2.

8   Numerical Results

In this section we present numerical tests in order to verify the estimates in Theorems 1, 3 and 6. We consider $\Omega^f = (1, 2) \times (0, 1)$ and $\Omega^p = (0, 1) \times (0, 1)$. See [11] and [25] for examples of exact solutions and compatible divergence and boundary data. Note that the reduced systems (14) and (24) involve only degrees of freedom on the interface Γ. In our test problems we compute the eigenvalues of the preconditioned operators.

To solve both reduced systems (14) and (24) we can use the PCG algorithms described in Figures 1 and 2. Recall that the original system (10) is a "three times" saddle point problem. Note that since the finite element basis of $M^f_{h^f} \times M^p_{h^p}$ and Λ^{h^p} have no zero mean, the finite element matrix in (12) has the kernel composed by constant pressures in $\Omega = \text{int}(\overline{\Omega^f} \cup \overline{\Omega^p})$ and constant Lagrange multipliers on Γ. The corresponding system is solved up to a constant pressure and a constant Lagrange multiplier. These constants can be recovered when imposing the zero average pressure constraint; see [25].

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{0}$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{0}$	1, 1.0189	1, 1.0198	1, 1.0194	1, 1.0193	1, 1.0193
$2^{-1} * 2^{-1}$	1, 1.0209	1, 1.0200	1, 1.0197	1, 1.0196	1, 1.0196
$2^{-1} * 2^{-2}$	1, 1.0217	1, 1.0205	1, 1.0202	1, 1.0201	1, 1.0201
$2^{-1} * 2^{-3}$	1, 1.0220	1, 1.0208	1, 1.0204	1, 1.0203	1, 1.0203
$2^{-1} * 2^{-4}$	1, 1.0221	1, 1.0209	1, 1.0205	1, 1.0204	1, 1.0204

Table 1: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here $\kappa = 1$ and $\alpha^f = 0$.

8.1 BDD preconditioner

In the case of the BDD preconditioner (27) for (14), we solve a coarse problem before reducing the system to ensure balanced velocities at the beginning of the CG iterations.

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{0}$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{0}$	1, 27.7647	1, 21.0147	1, 20.6035	1, 20.3686	1, 20.2893
$2^{-1} * 2^{-1}$	1, 28.1390	1, 21.3303	1, 20.8549	1, 20.6550	1, 20.5836
$2^{-1} * 2^{-2}$	1, 28.8104	1, 22.0017	1, 21.3392	1, 21.1424	1, 21.0735
$2^{-1} * 2^{-3}$	1, 29.0687	1, 22.2367	1, 21.6045	1, 21.3626	1, 21.2955
$2^{-1} * 2^{-4}$	1, 29.1810	1, 22.3479	1, 21.7006	1, 21.4666	1, 21.3929

Table 2: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here $\kappa = 10^{-3}$ and $\alpha^f = 0$.

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{0}$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{0}$	1, 1891.43	1, 1977.08	1, 1945.05	1, 1932.10	1, 1928.32
$2^{-1} * 2^{-1}$	1, 2095.91	1, 1997.27	1, 1972.77	1, 1961.34	1, 1957.88
$2^{-1} * 2^{-2}$	1, 2168.17	1, 2053.57	1, 2021.03	1, 2010.27	1, 2006.90
$2^{-1} * 2^{-3}$	1, 2201.81	1, 2079.68	1, 2044.05	1, 2032.42	1, 2029.13
$2^{-1} * 2^{-4}$	1, 2215.58	1, 2090.10	1, 2054.33	1, 2042.26	1, 2038.90

Table 3: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$.

We consider $\alpha^f = 0$ and $\nu = 1$, and different values of h^f and h^p with non-matching grids across the interface Γ; see Table 1 for the results when $\kappa = 1$, Table 2 for $\kappa = 10^{-3}$ and Table 3 for the case $\kappa = 10^{-5}$. These three tables reveal growth of order $O(1 + \frac{1}{\kappa})$ in κ and hence, verify the sharpness of the estimate in Theorem 1.

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-5}$
$2^{-1} * 2^{-0}$	1.0000, 1.0208	1.0000, 1.0194	1.0000, 1.0193	1.0000, 1.0193
$2^{-1} * 2^{-1}$	1.0017, 1.0200	1.0000, 1.0197	1.0000, 1.0196	1.0000, 1.0196
$2^{-1} * 2^{-2}$	1.0026, 1.0205	1.0004, 1.0202	1.0000, 1.0200	1.0000, 1.0201
$2^{-1} * 2^{-3}$	1.0027, 1.0208	1.0007, 1.0204	1.0001, 1.0203	1.0000, 1.0203
$2^{-1} * 2^{-4}$	1.0028, 1.0209	1.0007, 1.0205	1.0002, 1.0204	1.0000, 1.0204
$2^{-1} * 2^{-5}$	1.0028, 1.0209	1.0007, 1.0206	1.0002, 1.0205	1.0000, 1.0204

Table 4: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here $\kappa = 1$ and $\alpha^f = 0$.

$h_f \downarrow h_p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.000, 20.7608	1.000, 20.4405	1.000, 20.3110	1.000, 20.2732
$2^{-1} * 2^{-1}$	2.707, 20.9627	1.000, 20.7177	1.000, 20.6034	1.000, 20.5688
$2^{-1} * 2^{-2}$	3.634, 21.5257	1.425, 21.2003	1.000, 21.0927	1.000, 21.0590
$2^{-1} * 2^{-3}$	3.714, 21.7868	1.651, 21.4305	1.106, 21.3142	1.000, 21.2813
$2^{-1} * 2^{-4}$	3.760, 21.891	1.663, 21.5333	1.162, 21.4126	1.026, 21.3790
$2^{-1} * 2^{-5}$	3.771, 21.937	1.673, 21.5768	1.164, 21.4561	1.040, 21.4220

Table 5: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here $\kappa = 10^{-3}$ and $\alpha^f = 0$.

8.2   FETI preconditioner

In the case of the FETI preconditioner (31), we solve the reduced system (24) up to a constant Lagrange multiplier and a constant pressure. These constants are recovered after enforcing zero mean pressure on $\Omega = \text{int}\,(\overline{\Omega}^f \cup \overline{\Omega}^p)$; see [25]. We note that the FETI method can be viewed as the dual preconditioner counterpart of the BDD preconditioner. We repeat the same experiments mentioned above for this preconditioner.

We consider $\alpha^f = 0$, $\nu = 1$ and different values of h^f and h^p with nonmatching grids across the interface Γ; see Table 4 for the results when $\kappa = 1$, Table 5 for $\kappa = 10^{-3}$ and Table 6 for the case $\kappa = 10^{-5}$. Note that in Tables 4, 5 and 6 the minimum eigenvalues strictly greater than one when $h^f \leq 2h^p$, and the value of the minimum eigenvalues seem to stabilize very quick for smaller

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 1977.08	1.00, 1945.05	1.00, 1932.10	1.00, 1928.32
$2^{-1} * 2^{-1}$	171.72, 1997.27	1.00, 1972.77	1.00, 1961.34	1.00, 1957.88
$2^{-1} * 2^{-2}$	264.44, 2053.57	43.45, 2021.03	1.00, 2010.27	1.00, 2006.90
$2^{-1} * 2^{-3}$	272.35, 2079.68	66.10, 2044.05	11.58, 2032.42	1.00, 2029.13
$2^{-1} * 2^{-4}$	276.95, 2090.10	67.29, 2054.33	17.20, 2042.26	3.64, 2038.90
$2^{-1} * 2^{-5}$	278.09, 2094.70	68.32, 2058.68	17.42, 2046.61	5.04, 2043.20

Table 6: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$.

h^f with fixed h^p. This confirms the extension of Theorem 6 for the case where $h^f \leq 2h^p$; see Remark 7. In Table 7 we present the numerical results where one of the meshes on the interface is a refinement of the other side triangulation on the interface. We observe a behavior similar to the behavior of Table 6 with a bigger value for the minimum eigenvalue when $h_f \leq h_p$. This verifies the estimates of Theorem 6. This shows that the FETI preconditioner is scalable for the parameters faced in practice, i.e., the fluid side mesh finer than the porous side mesh and a small permeability κ. We conclude that the numerical experiments concerning the FETI preconditioner reveal the sharpness of the results obtained in Theorems 3 and 6 and Remark 7.

$h^f \downarrow h^p \rightarrow$	$2^{-1} * 2^{-1}$	$2^{-1} * 2^{-2}$	$2^{-1} * 2^{-3}$	$2^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 2002.47	1.00, 1961.35	1.00, 1937.86	1.00, 1929.93
$2^{-1} * 2^{-1}$	690.43, 2034.03	1.00, 1986.49	1.00, 1966.50	1.00, 1959.36
$2^{-1} * 2^{-2}$	627.36, 2101.17	176.56, 2034.92	1.00, 2015.24	1.00, 2008.35
$2^{-1} * 2^{-3}$	639.68, 2124.67	151.62, 2061.45	44.91, 2037.26	1.00, 2030.55
$2^{-1} * 2^{-4}$	642.44, 2135.79	154.45, 2071.06	38.04, 2047.66	11.98, 2040.29
$2^{-1} * 2^{-5}$	643.47, 2140.73	154.86, 2075.43	38.73, 2051.91	10.20, 2044.66

Table 7: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$. The refinement condition of Theorem 6 is satisfied under the diagonal.

Recall that we have assumed $\alpha^f = 0$. Now consider $\alpha^f > 0$. Numerical experiment were performed with $\alpha^f > 0$ reveling results similar to the ones presented above for the case $\alpha^f = 0$. We only include Table 8 which shows the extreme eigenvalues of the FETI preconditioned operator for the case $\alpha^f = 1$, $\nu = 1$ and $\kappa = 10^{-5}$. This table presents a similar behavior to the one with $\alpha^f = 0$ in Table 6 and hence, confirms Remark 9 which says that the parameter α^f does not play much role for preconditioning.

$h_f \downarrow h_p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 1705.47	1.00, 1678.07	1.00, 1666.84	1.00, 1663.55
$2^{-1} * 2^{-1}$	162.74, 1814.26	1.00, 1787.53	1.00, 1776.50	1.00, 1773.22
$2^{-1} * 2^{-2}$	251.56, 1843.50	41.65, 1812.69	1.00, 1801.61	1.00, 1798.29
$2^{-1} * 2^{-3}$	267.47, 1849.46	63.63, 1816.43	11.24, 1804.66	1.00, 1801.34
$2^{-1} * 2^{-4}$	272.29, 1850.65	66.82, 1817.38	16.75, 1805.30	3.58, 1801.91
$2^{-1} * 2^{-5}$	273.34, 1851.08	67.99, 1817.68	17.37, 1805.57	4.97, 1802.14

Table 8: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 1$.

9 Conclusions and Final Comments

We consider the problem of coupling fluid flows with porous media flows with Beavers-Joseph-Saffman condition on the interface. We choose a discretization consisting of Taylor-Hood finite elements of order two on the free fluid side and the lowest order Raviart-Thomas finite element on the porous fluid side. The meshes are allowed to be nonmatching across the interface.

We design and analyze two preconditioners for the resulting symmetric linear system. We note that the original linear system is symmetric indefinite and involves three Lagrange multipliers: one for each subdomain pressure and a third one to impose the weak conservation of mass across the interface Γ; see Section 1.

One preconditioner is based on BDD methods and the other one is based on FETI methods. In the case of the BDD preconditioner, the energy is controlled by the Stokes side, while in the FETI preconditioner, the energy is controlled by the Darcy system; see Theorems 1 and 3. In both cases a bound $C_1(\frac{\kappa+1}{\kappa})$ is derived. Furthermore, under the assumption that the fluid side mesh on the interface is finer than the corresponding porous side mesh, we derive the better bound $C_2(\frac{\kappa+1}{\kappa+(h^p)^2})$ for the FETI preconditioner; see Theorem 6 and Remark 7. This better bound also shows that the FETI preconditioner is more scalable for parameters faced in practice, e.g., problems with small permeability κ and where the fluid side mesh is finer than the porous side mesh. The constants C_1 and C_2 above are independent of the fluid viscosity ν, the mesh ratio across the interface, and the permeability κ.

References

[1] YVES ACHDOU, YVON MADAY, AND OLOF B. WIDLUND, *Iterative substructuring preconditioners for mortar element methods in two dimensions*, SIAM J. Numer. Anal., 36 (1999), pp. 551–580.

[2] TODD ARBOGAST AND DANA S. BRUNSON, *A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium*, Comput. Geosci., 11 (2007), pp. 207–218.

[3] TODD ARBOGAST AND GOMEZ M .S. M., *A discretization and multigrid solvers for a Darcy-Stokes system of three dimensional vuggy porous media*, Comput. Geosci. To appear.

[4] GORDON S. BEAVERS AND DANIEL D. JOSEPH, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech., 30 (1967), pp. 197–207.

[5] FAKER BEN BELGACEM AND YVON MADAY, *The mortar element method for three-dimensional finite elements*, RAIRO Modél. Math. Anal. Numér., 31 (1997), pp. 289–302.

[6] CHRISTINE BERNARDI, YVON MADAY, AND ANTHONY T. PATERA, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), vol. 299 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1994, pp. 13–51.

[7] DIETRICH BRAESS, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, Cambridge, 2001. Second Edition.

[8] SUSANNE C. BRENNER AND L. RIDGWAY SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, 1994.

[9] SUSANNE C. BRENNER AND LI-YENG SUNG, *BDDC and FETI-DP without matrices or vectors*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1429–1435.

[10] FRANCO BREZZI AND MICHEL FORTIN, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[11] ERIK BURMAN AND PETER HANSBO, *A unified stabilized method for Stokes' and Darcy's equations*, J. Comput. Appl. Math., 198 (2007), pp. 35–51.

[12] MARCO DISCACCIATI, *Domain decomposition methods for the coupling of surface and groundwater flows*, PhD thesis, Ecole Polytechnique Fédérale, Lausanne (Switzerland), 2004. Thèse n. 3117.

[13] MARCO DISCACCIATI, *Iterative methods for Stokes/Darcy coupling*, in Domain decomposition methods in science and engineering, vol. 40 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2005, pp. 563–570.

[14] MARCO DISCACCIATI, EDIE MIGLIO, AND ALFIO QUARTERONI, *Mathematical and numerical models for coupling surface and groundwater flows*, Appl. Numer. Math., 43 (2002), pp. 57–74. 19th Dundee Biennial Conference on Numerical Analysis (2001).

[15] MARCO DISCACCIATI AND ALFIO QUARTERONI, *Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations*, in ENUMATH 2001, F. Brezzi, A. Buffa, S. Corsaro, and A Murli, eds., Numerical Mathematics and Advanced Applications, Springer-Verlag, 2003, pp. 3–20.

[16] ——, *Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations*, Comput. Vis. Sci., 6 (2004), pp. 93–103.

[17] MARCO DISCACCIATI, ALFIO QUARTERONI, AND ALBERTO VALLI, *Robin-Robin domain decomposition methods for the Stokes-Darcy coupling*, SIAM J. Numer. Anal., 45 (2007), pp. 1246–1268 (electronic).

[18] CLARK R. DOHRMANN, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci. Comput., 25 (2003), pp. 246–258 (electronic).

[19] MAKSYMILIAN DRYJA, JUAN GALVIS, AND MARCUS SARKIS, *BDDC methods for discontinuous Galerkin discretization of elliptic problems*, J. Complexity, 23 (2007), pp. 715–739.

[20] MAKSYMILIAN DRYJA, HYEA HYUN KIM, AND OLOF B. WIDLUND, *A BDDC algorithm for problems with mortar discretization*, Tech. Report TR2005-873, Courant Institute of Mathematical Sciences, Computer Science Department, September 2005. http://cs.nyu.edu/web/Research/TechReports/.

[21] MAKSYMILIAN DRYJA AND W. PROSKUROWSKI, *On preconditioners for mortar discretization of elliptic problems*, Numer. Linear Algebra Appl., 10 (2003), pp. 65–82. Dedicated to the 60th birthday of Raytcho Lazarov.

[22] CHARBEL FARHAT, MICHAEL LESOINNE, AND KENDALL PIERSON, *A scalable dual-primal domain decomposition method*, Numer. Linear Algebra Appl., 7 (2000), pp. 687–714. Preconditioning techniques for large sparse matrix problems in industrial applications (Minneapolis, MN, 1999).

[23] CHARBEL FARHAT AND FRANCOIS-XAVIER ROUX, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., 32 (1991), pp. 1205 – 1227.

[24] JUAN GALVIS AND MARCUS SARKIS, *Balancing domain decomposition methods for mortar coupling Stokes-Darcy systems*, in Domain Decomposition Methods in Science and Engineering XVI, David Keyes and Olof B. Widlund, eds., vol. 55 of Lecture Notes in Computational Science and Engineering, Springer, 2006, pp. 373–380.

[25] JUAN GALVIS AND MARCUS SARKIS, *Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations*, Electron. Trans. Numer. Anal., 26 (2007), pp. 350–384.

[26] VIVETTE GIRAULT AND PIERRE-ARNAUD RAVIART, *Finite element methods for Navier-Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[27] WILLI JÄGER AND ANDRO MIKELIĆ, *On the interface boundary condition of Beavers, Joseph, and Saffman*, SIAM J. Appl. Math., 60 (2000), pp. 1111–1127.

[28] Axel Klawonn and Olof B. Widlund, *FETI and Neumann-Neumann iterative substructuring methods: connections and new results*, Comm. Pure Appl. Math., 54 (2001), pp. 57–90.

[29] William J. Layton, Friedhelm Schieweck, and Ivan Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 40 (2002), pp. 2195–2218 (2003).

[30] Jing Li, *A dual-primal FETI method for incompressible Stokes equations*, Numer. Math., 102 (2005), pp. 257–275.

[31] Jing Li and Olof Widlund, *BDDC algorithms for incompressible Stokes equations*, SIAM J. Numer. Anal., 44 (2006), pp. 2432–2455 (electronic).

[32] ——, *A BDDC preconditioner for saddle point problems*, in Domain decomposition methods in science and engineering XVI, vol. 55 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2007, pp. 413–420.

[33] Jan Mandel, *Balancing domain decomposition*, Comm. Numer. Methods Engrg., 9 (1993), pp. 233–241.

[34] Jan Mandel and Marian Brezina, *Balancing domain decomposition for problems with large jumps in coefficients*, Math. Comp., 65 (1996), pp. 1387–1401.

[35] Jan Mandel and Radek Tezaur, *Convergence of a substructuring method with Lagrange multipliers*, Numer. Math., 73 (1996), pp. 473–487.

[36] Tarek P. Mathew, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretizations of Elliptic Problems*, PhD thesis, Courant Institute of Mathematical Sciences, September 1989. Tech. Rep. 463, Department of Computer Science, Courant Institute.

[37] Mo Mu and Jinchao Xu, *A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 45 (2007), pp. 1801–1813 (electronic).

[38] Luca F. Pavarino and Olof B. Widlund, *Balancing Neumann-Neumann methods for incompressible Stokes equations*, Comm. Pure Appl. Math., 55 (2002), pp. 302–335.

[39] Béatrice Rivière and Ivan Yotov, *Locally conservative coupling of Stokes and Darcy flows*, SIAM J. Numer. Anal., 42 (2005), pp. 1959–1977.

[40] Andrea Toselli and Olof Widlund, *Domain decomposition methods—algorithms and theory*, vol. 34 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2005.

[41] Xuemin Tu, *A BDDC algorithm for a mixed formulation of flow in porous media*, Electron. Trans. Numer. Anal., 20 (2005), pp. 164–179.

[42] ——, *A BDDC algorithm for flow in porous media with a hybrid finite element discretization*, Electron. Trans. Numer. Anal., 26 (2007), pp. 146–160.

[43] Barbara I. Wohlmuth, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal., 38 (2000), pp. 989–1012 (electronic).