All superconformal surfaces in $\mathbb{R}^4$ in terms of minimal surfaces.

Marcos Dajczer and Ruy Tojeiro

Abstract

We give an explicit construction of any simply-connected superconformal surface $\phi\colon M^2 \to \mathbb{R}^4$ in Euclidean space in terms of a pair of conjugate minimal surfaces $g, h\colon M^2 \to \mathbb{R}^4$. That ϕ is superconformal means that its ellipse of curvature is a circle at any point. We characterize the pairs (g, h) of conjugate minimal surfaces that give rise to images of holomorphic curves by an inversion in $\mathbb{R}^4$ and to images of superminimal surfaces in either a sphere $\mathbb{S}^4$ or a hyperbolic space $\mathbb{H}^4$ by an stereographic projection. We also determine the relation between the pairs (g, h) of conjugate minimal surfaces associated to a superconformal surface and its image by an inversion. In particular, this yields a new transformation for minimal surfaces in $\mathbb{R}^4$.

1 Introduction

For any surface $\phi\colon M^2 \to \mathbb{R}^4$ in Euclidean 4-dimensional space the pointwise inequality

$$K + |K_N| \leq \|H\|^2 \tag{1}$$

holds at every point of M^2 [15]. Here K denotes the Gaussian curvature of M^2 and K_N and H are the normal curvature and the mean curvature vector of ϕ, respectively. In fact, a similar inequality was derived by Guadalupe - Rodríguez [4] for surfaces of any codimension in space forms of sectional curvature c, namely, $K + |K_N| \leq \|H\|^2 + c$. Moreover, it was shown that equality holds at $p \in M^2$ if and only if the ellipse of curvature $E(p)$ of ϕ at p is a circle.

Recall that the *ellipse of curvature* of a surface $\phi\colon M^2 \to N^4$ in a 4-dimensional Riemannian manifold at $p \in M^2$ is the ellipse in the normal space of ϕ at p given by

$$E(p) = \{\alpha_\phi(X, X) \,:\, X \in T_pM \text{ and } \|X\| = 1\},$$

where α_ϕ is the second fundamental form of ϕ with values in the normal bundle; see [4] and references therein for several interesting facts on this concept whose study goes back almost a century to the work of Moore and Wilson [12], [13]. Observe that the property of $E(p)$ being a circle is invariant under conformal changes of the metric of N^4.

Following the terminology in [1] we call a surface $\phi\colon M^2 \to \mathbb{R}^4$ *superconformal* if at any point the ellipse of curvature is a circle. Thus, the class of superconformal surfaces is invariant under Moebius transformations of $\mathbb{R}^4$. The condition of superconformality shows up in several interesting geometric situations. For instance, for a compact oriented surface integration of (1) over M^2 yields the lower bound $\int_M \|H\|^2 \geq 2\pi \mathcal{X}(M) + |\mathcal{X}(T^\perp M)|$ due to Wintgen [15] for the Willmore functional of ϕ in terms of the Euler characteristics of the tangent and normal bundles. Moreover, we have equality if and only if K_N does not change sign and the surface is superconformal.

In this paper, we provide an explicit construction of any simply connected superconformal surface in $\mathbb{R}^4$ that is free of minimal and umbilical points. Start with a simply connected minimal surface $g\colon M^2 \to \mathbb{R}^4$, oriented by a global conformal diffeomorphism onto either the complex plane or the unit disk. Then, consider its conjugate minimal surface $h\colon M^2 \to \mathbb{R}^4$, each of whose components with respect to this global parameter is the harmonic conjugate of the corresponding component of g (see [6] or [8]). Equivalently, $h_* = g_* \circ J$, where J is the complex structure on M^2 compatible with its orientation. Notice that h is determined by g up to a vector $v \in \mathbb{R}^4$.

Now, let $\hat{J}_+$ and $\hat{J}_-$ be the two possible complex structures on $T_g^\perp M$, and consider the complex structures $\mathcal{J}_+$ and $\mathcal{J}_-$ on $g^* T\mathbb{R}^4$ given by

$$\mathcal{J}_\pm \circ g_* = g_* \circ J \quad \text{and} \quad \mathcal{J}_\pm|_{T_g^\perp M} = \hat{J}_\pm.$$

Our main result reads as follows.

Theorem 1. *Each of the maps $\phi_+\colon M^2 \to \mathbb{R}^4$ and $\phi_-\colon M^2 \to \mathbb{R}^4$ defined by*

$$\phi_\pm = g + \mathcal{J}_\pm h \tag{2}$$

parameterizes, at regular points, a superconformal surface. Moreover, ϕ_+ and ϕ_- are conformal to g and envelop a common central sphere congruence that has g as its surface of centers. Conversely, any simply connected superconformal surface that is free of minimal and umbilical points can be constructed in this way.

By combining the preceding result with the generalized Weierstrass parameterization of Euclidean minimal surfaces [6], [8] we have a parametric representation of all simply connected superconformal surfaces in $\mathbb{R}^4$.

Recall that the *central sphere congruence* (or *mean curvature sphere congruence*) of a surface in Euclidean space is the family of two-dimensional spheres that are tangent to the surface and have the same mean curvature vector as the surface at the point of tangency. Therefore, our result implies the fact already known by Rouxel [14] that superconformal surfaces $\phi\colon M^2 \to \mathbb{R}^4$ always arise in pairs (ϕ_+, ϕ_-) of *dual* surfaces that induce conformal metrics on M^2 and envelop a common sphere congruence, namely, their common central sphere congruence. Hence, the pair (ϕ_+, ϕ_-) provides a solution to the higher codimensional version studied by Ma [9] of the problem, first considered by Blaschke for surfaces in $\mathbb{R}^3$, of finding all such pairs of surfaces; see [5] for details on the latter and related facts.

We observe that superconformal minimal surfaces in $\mathbb{R}^4$ are holomorphic curves. Here and elsewhere, by a surface $f\colon M^2 \to \mathbb{R}^4$ being holomorphic we mean that f is complex with respect to a suitable complex structure of $\mathbb{R}^4$. Therefore, one obvious way to produce examples of nonminimal superconformal surfaces is to take compositions of holomorphic curves with an inversion in $\mathbb{R}^4$. Notice also that an isolated minimal point can always be removed by an inversion. Hence, locally and from the point of view of conformal geometry, assuming that the surface is free of minimal points in Theorem 1 is not essential.

Finally, we point out that there is a correspondence between holomorphic curves in $\mathbb{CP}^3$ and superconformal surfaces in $\mathbb{R}^4$. In fact, Theorem 5 in [1] states that $\phi\colon M^2 \to \mathbb{R}^4 = \mathbb{H}$ is superconformal if and only if either $[\phi, 1]$ or $[\bar{\phi}, 1]$ is the twistor projection under Penrose twistor fibration $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1 = \mathbb{S}^4$ of a holomorphic curve in $\mathbb{CP}^3$. Here $\mathbb{H}$ is the set of quaternions, $\mathbb{HP}^1$ the quaternionic projective space, $(x, y) \in \mathbb{H}^2 \setminus (0, 0) \mapsto [x, y] \in \mathbb{HP}^1$ the canonical projection and $x \in \mathbb{H} \mapsto \bar{x} \in \mathbb{H}$ the conjugation in $\mathbb{H}$.

In the next result we determine how the holomorphic representative (see [6] or [8])

$$G := g + ih\colon M^2 \to \mathbb{C}^4 \approx \mathbb{R}^4 + i\mathbb{R}^4$$

of the minimal surface $g\colon M^2 \to \mathbb{R}^4$ associated to an oriented superconformal surface ϕ is related to the holomorphic representative $\tilde{G} = \tilde{g} + i\tilde{h}$ of the minimal surface $\tilde{g}\colon M^2 \to \mathbb{R}^4$ associated to its composition $\tilde{\phi} = \mathcal{I} \circ \phi$ with an inversion $\mathcal{I}$ in $\mathbb{R}^4$ with respect to a sphere of radius R taken, for simplicity, centered at the origin.

Theorem 2. *Assume that ϕ is not the composition of a holomorphic curve with an inversion. Then $\tilde{G} = R^2 T \circ G$, where $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ is the holomorphic map*

$$T(Z) = \frac{Z}{\langle\!\langle Z, Z \rangle\!\rangle} \tag{3}$$

and $\langle\!\langle \ , \ \rangle\!\rangle\colon \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$ is the linear inner product on $\mathbb{C}^m$.

For arbitrary m, the holomorphic map $T_R := R^2 T\colon \mathbb{C}^m \to \mathbb{C}^m$ can be regarded as the *inversion* in $\mathbb{C}^m$ with respect to the quadric

$$\mathbb{Q}_k = \{Z \in \mathbb{C}^m : \langle\!\langle Z, Z \rangle\!\rangle = k\}$$

with $k = R^2$. Notice that T_R is defined on $\mathbb{C}^m \setminus \mathbb{Q}_0$ and takes any quadric $\mathbb{Q}_k$ diffeomorphically onto $\mathbb{Q}_{R^4/k}$.

As a byproduct of Theorem 2, we obtain the following remarkable property of the map T that, in particular, yields a new transformation for minimal surfaces in $\mathbb{R}^4$.

Corollary 3. *The holomorphic inversion map T preserves the class of holomorphic curves $G = g + ih\colon M^2 \to \mathbb{C}^4$ whose real and imaginary parts g and h define conjugate minimal immersions into $\mathbb{R}^4$.*

Next we characterize the minimal surfaces that give rise, by means of the construction of Theorem 1, to the superconformal surfaces that are images of holomorphic curves by an inversion in $\mathbb{R}^4$.

Theorem 4. *Let $\phi\colon M^2 \to \mathbb{R}^4$ be a superconformal surface parametrized by (2). Then, the following assertions are equivalent:*

(i) *The surface ϕ is the composition of a holomorphic curve with an inversion in $\mathbb{R}^4$.*

(ii) *The superconformal surface dual to ϕ degenerates to a constant map.*

(iii) *The minimal surface g is a holomorphic curve in $\mathbb{R}^4$.*

We point out that Rouxel [14] already observed that all spheres of the central sphere congruence of a composition of a holomorphic curve with an inversion pass through a fixed point and that, in this case, the surface of centers is a holomorphic curve. This is essentially the fact that the first assertion in Theorem 4 implies the remaining two.

In the process of proving Theorem 4, the following interesting duality between holomorphic and anti-holomorphic curves $f\colon L^2 \to \mathbb{C}^2$ was revealed. In the next statement we denote by f^N the normal component of the position vector f in $\mathbb{R}^4 \approx \mathbb{C}^2$, and by $\mathcal{H}_+$ and $\mathcal{H}_-$ the sets of holomorphic and anti-holomorphic surfaces in $\mathbb{C}^2$, respectively.

Theorem 5. *The map between $\mathcal{H}_+$ and $\mathcal{H}_-$ given by $f \mapsto f^* = f^N/2\|f^N\|^2$ is a bijection such that $(f^*)^* = f$. Moreover, the metrics induced by f and f^* are conformal.*

Another class of superconformal surfaces in $\mathbb{R}^4$ is that of stereographic projections of superminimal surfaces in the sphere $\mathbb{S}^4$. Recall that a surface $g\colon L^2 \to \mathbb{S}^4$ is *superminimal* if it is minimal and superconformal; see [7] and the references therein for several characterizations of this very interesting class of surfaces. From a global point of view, it is worth mentioning that any minimal immersion of the sphere $\mathbb{S}^2$ into $\mathbb{S}^4$ is automatically superminimal [4], and that every compact Riemann surface admits a conformal superminimal immersion into $\mathbb{S}^4$ [2]. The latter result shows that there exist compact superconformal surfaces in $\mathbb{R}^4$ with arbitrary topology.

A further source of superconformal surfaces in $\mathbb{R}^4$ arises by taking the stereographic projections onto a ball in $\mathbb{R}^4$ of superminimal surfaces in the hyperbolic space $\mathbb{H}^4$. The latter were studied in [7] where, in particular, it was proved that there exist complete embedded simply-connected examples that are not totally geodesic.

Superminimal surfaces in $\mathbb{S}^4$ and $\mathbb{H}^4$ can be regarded as the analogues of holomorphic curves in $\mathbb{R}^4$. It is natural to ask for a similar characterization to that of Theorem 4 of the minimal surfaces that produce, by applying the procedure of Theorem 1, stereographic projections of superminimal surfaces in a sphere $\mathbb{S}^4_R = \mathbb{S}^4(Re_5; R)$ of radius R centered at Re_5, or in a hyperbolic space

$$\mathbb{H}^4_R = \mathbb{H}^4(-Re_5; R) = \{X \in \mathbb{L}^5 \,:\, \langle X + Re_5, X + Re_5 \rangle = -R^2\}.$$

Here we regard $\mathbb{R}^4$ as the hyperplane through the origin and normal to the unit vector e_5 in either $\mathbb{R}^5$ or Lorentzian space $\mathbb{L}^5$, and by the stereographic projection of $\mathbb{H}_R^4$ onto $\mathbb{R}^4$ we mean the map that assigns to each $P \in \mathbb{H}_R^4$ the point of $\mathbb{R}^4$ where the line through the points $-2Re_5$ and P intersects $\mathbb{R}^4$. Our final result is the following.

Theorem 6. *Let $\phi\colon M^2 \to \mathbb{R}^4$ be a superconformal surface parameterized by (2). Then, the following assertions are equivalent:*

(i) *Either ϕ or its dual is the composition of a superminimal surface in a sphere $\mathbb{S}_R^4$ (resp., $\mathbb{H}_R^4$) with a stereographic projection of $\mathbb{S}_R^4$ (resp., $\mathbb{H}_R^4$) onto $B(0; 2R) \subset \mathbb{R}^4$.*

(ii) *Both ϕ and its dual are compositions of superminimal surfaces in a sphere $\mathbb{S}_R^4$ (resp., $\mathbb{H}_R^4$) with a stereographic projection of $\mathbb{S}_R^4$ (resp., $\mathbb{H}_R^4$) onto $B(0; 2R) \subset \mathbb{R}^4$.*

(iii) *The holomorphic representative G of g takes values in $\mathbb{Q}_{4R^2}$ (resp., $\mathbb{Q}_{-4R^2}$).*

In order to relate the preceding result to Theorem 4, we observe that holomorphic curves in $\mathbb{R}^4$ can be characterized as the minimal surfaces whose holomorphic representatives take values in the quadric $\mathbb{Q}_0$ (see Proposition 17 below).

2 The ellipse of curvature

In this section we recall some of the basic properties of the ellipse of curvature of a surface $\phi\colon M^2 \to N^4$ in a four-dimensional Riemannian manifold.

Given an orthonormal basis $\{Y_1, Y_2\}$ of the tangent space T_pM at $p \in M^2$, denote $\alpha_{ij} = \alpha_\phi(Y_i, Y_j)$ for $1 \le i, j \le 2$. Then, we have for any $v = \cos\theta Y_1 + \sin\theta Y_2$ that

$$\alpha_\phi(v, v) = H + \cos 2\theta \frac{\alpha_{11} - \alpha_{22}}{2} + \sin 2\theta \alpha_{12}, \tag{4}$$

where $H = \frac{1}{2}(\alpha_{11} + \alpha_{22})$ is the mean curvature vector of ϕ at p. This shows that when v goes once around the unit tangent circle, the vector $\alpha_\phi(v, v)$ goes twice around an ellipse centered at H, the ellipse of curvature $E(p)$ of ϕ at p. Clearly, $E(p)$ can degenerate into a line segment or a point.

It follows from (4) that $E(p)$ is a circle if and only if for some (and hence for any) orthonormal basis of $T_{\phi(p)}M$ it holds that

$$\langle \alpha_{12}, \alpha_{11} - \alpha_{22} \rangle = 0 \quad \text{and} \quad \|\alpha_{11} - \alpha_{22}\| = 2\|\alpha_{12}\|. \tag{5}$$

Let $\{\eta, \zeta\}$ be an orthonormal basis of the normal space $T_{\phi(p)}^\perp M$ at p with $\eta = H/\|H\|$. Take $\{Y_1, Y_2\}$ as an orthonormal tangent basis of eigenvectors of the shape operator A_ζ with respect to ζ, and let μ and $-\mu$ be the corresponding eigenvalues. Denoting $\lambda_{ij} = \langle A_\eta Y_i, Y_j \rangle$ for $1 \le i, j \le 2$ we have

$$\alpha_{11} = \mu\zeta + \lambda_{11}\eta, \quad \alpha_{12} = \lambda_{12}\eta \quad \text{and} \quad \alpha_{22} = -\mu\zeta + \lambda_{22}\eta.$$

Hence, condition (5) for $E(p)$ to be a circle is that $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = \mu$. Summarizing, $E(p)$ is a circle if and only if the shape operators A_η and A_ζ have the form

$$\begin{cases} A_\eta Y_1 = \lambda Y_1 + \mu Y_2 \\ A_\eta Y_2 = \mu Y_1 + \lambda Y_2 \end{cases} \quad \text{and} \quad \begin{cases} A_\zeta Y_1 = \mu Y_1 \\ A_\zeta Y_2 = -\mu Y_2, \end{cases} \tag{6}$$

with $\lambda = \|H\|$. Notice that in this case μ is the radius of $E(p)$. In particular, $E(p)$ degenerates into a point if and only if $\mu = 0$, that is, p is an umbilical point. If N^4 is a space form, observe that the normal curvature $K_N = \langle R^\perp(Y_1, Y_2)\zeta, \eta\rangle$ with respect to the oriented orthonormal bases $\{Y_1, Y_2\}$ and $\{\eta, \zeta\}$ of $T_p M$ and $T_{\phi(p)}^\perp M$, respectively, is $K_N = 2\mu^2$.

It also follows from (5) that the property that $E(p)$ be a circle is invariant under conformal changes of the metric of N^4. Therefore, any surface with circular ellipses of curvature in a (globally) conformally flat 4-dimensional Riemannian manifold (in particular any superminimal surface in $\mathbb{S}^4$ or $\mathbb{H}^4$) gives rise to a superconformal surface in $\mathbb{R}^4$.

3   Superconformal surfaces from minimal surfaces

In the two first subsections of this section we prove Theorem 1, starting with the converse statement. In the last one we characterize dual superconformal surfaces in $\mathbb{R}^4$ that are constructed from minimal surfaces in $\mathbb{R}^3$ by the procedure of Theorem 1.

3.1   Proof of the converse of Theorem 1

Let $\phi\colon M^2 \to \mathbb{R}^4$ be a simply connected oriented surface that has nondegenerate circular ellipses of curvature everywhere and nowhere vanishing mean curvature vector H. Let $\{\eta, \zeta\}$ be an orthonormal normal frame with $\eta = H/\|H\|$. By the discussion in the previous section, there exists an orthonormal tangent frame $\{Y_1, Y_2\}$ such that the shape operators A_η and A_ζ are given everywhere by (6), with μ nowhere vanishing. Thus, we may assume that $\mu > 0$ everywhere. Hereafter, we let M^2 and $T_\phi^\perp M$ be oriented by the orthonormal frames $\{Y_1, Y_2\}$ and $\{\eta, \zeta\}$, respectively. Moreover, we always denote by J the complex structure on M^2 compatible with its orientation.

We define $g\colon M^2 \to \mathbb{R}^4$ by

$$g = \phi + r\eta, \tag{7}$$

where $r = 1/\lambda$. We write

$$\zeta = g_* Z + a\xi, \tag{8}$$

where $Z \in TM$, ξ is a unit normal vector field to g and $a \in C^\infty(M)$ is such that

$$\|Z\|^2 + a^2 = 1. \tag{9}$$

Since $\ker(A_\eta - \lambda I) \neq 0$ everywhere because $\mu \neq 0$, we have that a is nowhere vanishing. Otherwise $g_* Z$ would be somewhere normal to ϕ, which would imply, by taking tangent components for $X = Z$ in

$$\phi_* X = g_* X - \langle \nabla r, X \rangle \eta - r \eta_* X, \tag{10}$$

that $Z \in \ker(A_\eta - \lambda I)$, and this is a contradiction. Thus, we may assume that $a > 0$ everywhere. Extend ξ to an orthonormal frame $\{\xi, \delta\}$ of $T_g^\perp M$. It follows from (10) that $0 = \langle \eta, \phi_* X \rangle = \langle \eta, g_* X \rangle - \langle \nabla r, X \rangle$, hence the tangent component to g of η is $g_* \nabla r$. Thus, we may write

$$\eta = g_* \nabla r + \rho \xi + b \delta \tag{11}$$

for $\rho, b \in C^\infty(M)$ satisfying

$$\|\nabla r\|^2 + \rho^2 + b^2 = 1. \tag{12}$$

Lemma 7. *The following holds:*

   (i) $\rho = 0$, $b = \pm a$ *and* $JZ = \nabla r$.

  (ii) $h = -r\zeta$ *satisfies* $h_* = g_* \circ J$.

Before proving Lemma 7, let us see how it yields the converse statement of the theorem. It follows from part (ii) that g and h are conjugate minimal surfaces. By (7), (11) and part (i), we have

$$\phi = g - r\eta = g - r g_* \nabla r \pm a r \delta. \tag{13}$$

Now,

$$-r g_* \nabla r = -r g_* JZ = \mathcal{J}_\pm(-r g_* Z) = \mathcal{J}_\pm h^T, \tag{14}$$

where h^T denotes the tangent component to g of the position vector h. On the other hand, if $\hat{J}_+$ and $\hat{J}_-$ are the complex structures on $T_g^\perp M$ defined by $\hat{J}_\pm \xi = \mp \delta$, then

$$\pm a r \delta = \hat{J}_\pm(-a r \xi) = \hat{J}_\pm h^N = \mathcal{J}_\pm h^N, \tag{15}$$

where h^N is the normal component to g of the position vector h. We obtain from (13), (14) and (15) that ϕ is given by (2).

Proof of Lemma 7: The proof of Lemma 7 will be given in several steps. We start with the following preliminary facts, where B_ν denotes the shape operator of g for $\nu \in T_g^\perp M$.

Sublemma 8. *We have*

$$\langle B_\delta Z, X \rangle = a \langle \nabla_X^\perp \delta, \xi \rangle \quad \text{for any } X \in TM \tag{16}$$

and

$$\operatorname{Hess} r(Z) - \frac{1}{r} Z + B_\xi(a \nabla r - \rho Z) + a \nabla \rho = 0. \tag{17}$$

The Codazzi equations for ϕ yields

$$\begin{cases} Y_1(\mu) = -\lambda\psi(Y_1) + \mu\psi(Y_2) + 2\mu\Gamma_{22}^1 \\ Y_2(\mu) = \lambda\psi(Y_2) - \mu\psi(Y_1) + 2\mu\Gamma_{11}^2 \end{cases} \quad \text{and} \quad \begin{cases} Y_1(\mu) = Y_2(\lambda) + \mu\psi(Y_2) + 2\mu\Gamma_{22}^1 \\ Y_2(\mu) = Y_1(\lambda) - \mu\psi(Y_1) + 2\mu\Gamma_{11}^2, \end{cases}$$

where $\Gamma_{ii}^j = \langle \nabla_{Y_i} Y_i, Y_j \rangle$ and ψ is the normal connection form of ϕ given by $\psi(X) = \langle \nabla_X^\perp \zeta, \eta \rangle$. Subtracting each equation of the first system from the corresponding equation of the second yields

$$\begin{cases} Y_2(\lambda) + \lambda\psi(Y_1) = 0 \\ Y_1(\lambda) - \lambda\psi(Y_2) = 0. \end{cases} \tag{18}$$

Differentiating $g = \phi + \dfrac{1}{\lambda}\eta$ and using (6) and (18) gives

$$\begin{cases} g_* Y_1 = -\dfrac{1}{\lambda}\left(\psi(Y_2)\eta + \mu\phi_* Y_2 + \psi(Y_1)\zeta\right) \\ g_* Y_2 = -\dfrac{1}{\lambda}\left(-\psi(Y_1)\eta + \mu\phi_* Y_1 + \psi(Y_2)\zeta\right). \end{cases}$$

Therefore, the vector field $\psi(Y_1)\phi_* Y_1 - \psi(Y_2)\phi_* Y_2 + \mu\eta$ is normal to g, and since it is orthogonal to ξ, it is in the direction of δ. Thus, from

$$\begin{cases} \zeta_* Y_1 = -\mu\phi_* Y_1 + \psi(Y_1)\eta \\ \zeta_* Y_2 = \mu\phi_* Y_2 + \psi(Y_2)\eta, \end{cases}$$

we obtain that $\langle \zeta_* Y_j, \delta \rangle = 0$, which is easily seen to be equivalent to (16). Then,

$$\zeta_* X = g_* DX + \langle K, X \rangle \xi, \tag{19}$$

where

$$DX = \nabla_X Z - aB_\xi X \quad \text{and} \quad K = \nabla a + B_\xi Z. \tag{20}$$

The orthogonality between η and ζ yields

$$\langle Z, \nabla r \rangle + a\rho = 0. \tag{21}$$

Hence,

$$\begin{aligned} \langle \nabla_X Z, \nabla r \rangle &= X\langle Z, \nabla r \rangle - \langle Z, \operatorname{Hess} r(X) \rangle \\ &= -X(a)\rho - aX(\rho) - \langle Z, \operatorname{Hess} r(X) \rangle \\ &= -\langle \rho\nabla a + a\nabla\rho + \operatorname{Hess} r(Z), X \rangle. \end{aligned} \tag{22}$$

It follows from (19), (20) and (22) that

$$\langle \zeta_* X, \eta \rangle = \langle DX, \nabla r \rangle + \rho\langle K, X \rangle = -\langle \operatorname{Hess} r(Z) + B_\xi(a\nabla r - \rho Z) + a\nabla\rho, X \rangle. \tag{23}$$

8

On the other hand,

$$\langle \zeta, \phi_* X \rangle = X \langle \zeta, g \rangle - \langle \zeta_* X, g - r\eta \rangle = \langle Z, X \rangle + r \langle \zeta_* X, \eta \rangle, \tag{24}$$

and thus (17) follows from (23), (24) and the fact that ζ is normal to ϕ.

The next step is to express (6) in terms of g. It is convenient to use the orthonormal frame

$$X_1 = \frac{1}{\sqrt{2}}(Y_1 + Y_2), \qquad X_2 = \frac{1}{\sqrt{2}}(Y_1 - Y_2),$$

with respect to which (6) becomes

$$\begin{cases} A_\eta X_1 = (\lambda + \mu) X_1 \\ A_\eta X_2 = (\lambda - \mu) X_2 \end{cases} \quad \text{and} \quad \begin{cases} A_\zeta X_1 = \mu X_2 \\ A_\zeta X_2 = \mu X_1. \end{cases}$$

Hence,

$$\begin{cases} \eta_* X_1 = -(\lambda + \mu)\phi_* X_1 - \psi(X_1)\zeta \\ \eta_* X_2 = -(\lambda - \mu)\phi_* X_2 - \psi(X_2)\zeta \\ \zeta_* X_i = -\mu\phi_* X_j + \psi(X_i)\eta, \quad 1 \le i \ne j \le 2, \end{cases} \tag{25}$$

In view of (10), this is equivalent to

$$\begin{cases} r^2\mu\eta_* X_1 = \theta_1(g_* X_1 - r_1\eta) + r\psi(X_1)\zeta \\ r^2\mu\eta_* X_2 = \theta_2(-g_* X_2 + r_2\eta) - r\psi(X_2)\zeta \\ \zeta_* X_i = -\mu g_* X_j + \mu(r\eta_* X_j + r_j\eta) + \psi(X_i)\eta, \quad 1 \le i \ne j \le 2, \end{cases} \tag{26}$$

where $\theta_1 = (1 + r\mu)$, $\theta_2 = (1 - r\mu)$ and $r_i = \langle \nabla r, X_i \rangle$ for $1 \le i \le 2$.

We have

$$\eta_* X = g_* QX + \langle T, X \rangle \xi + \langle P, X \rangle \delta \tag{27}$$

where

$$\begin{cases} Q = \operatorname{Hess} r - \rho B_\xi - b B_\delta \\ T = \nabla\rho + B_\xi\nabla r + \dfrac{b}{a} B_\delta Z \\ P = \nabla b + B_\delta\nabla r - \dfrac{\rho}{a} B_\delta Z. \end{cases}$$

To proceed we use that the normal connection form ψ of ϕ can be written as

$$\psi(X) = -\frac{1}{r}\langle Z, X \rangle \quad \text{for any } X \in TM \tag{28}$$

in terms of data related to g. This follows immediately from (17) and (23).

Using (28), the δ-component of (26) gives

$$\begin{cases} r^2\mu\langle P, X_1\rangle = -\theta_1 br_1 \\ r^2\mu\langle P, X_2\rangle = \theta_2 br_2 \\ r^2\mu\langle P, X_i\rangle = -r\mu br_i + b\langle Z, X_j\rangle, \quad 1 \le i \ne j \le 2. \end{cases} \tag{29}$$

Replacing the first two equations into the last two yields

$$r_1 = -\langle Z, X_2\rangle \quad \text{and} \quad r_2 = \langle Z, X_1\rangle. \tag{30}$$

Taking the tangent component to g of (26) and using (30) we obtain

$$\begin{cases} r^2\mu QX_1 - \theta_1 SX_1 + r_2 Z = 0 \\ r^2\mu QX_2 + \theta_2 SX_2 + r_1 Z = 0 \\ rDX_1 + r\mu SX_2 - r^2\mu QX_2 + r_2\nabla r = 0 \\ rDX_2 + r\mu SX_1 - r^2\mu QX_1 - r_1\nabla r = 0, \end{cases} \tag{31}$$

where

$$S = I - \langle\nabla r, *\rangle\nabla r. \tag{32}$$

Finally, computing the ξ-component of (26) yields

$$\begin{cases} r^2\mu\langle T, X_1\rangle = -\theta_1\rho r_1 - a\langle Z, X_1\rangle \\ r^2\mu\langle T, X_2\rangle = \theta_2\rho r_2 + a\langle Z, X_2\rangle \\ r\langle K, X_i\rangle = r^2\mu\langle T, X_j\rangle + r\mu\rho r_j - \rho\langle Z, X_i\rangle, \quad 1 \le i \ne j \le 2. \end{cases} \tag{33}$$

We now prove:

Sublemma 9. *The metrics induced by g and ϕ are conformal. Namely,*

$$\langle g_* X, g_* Y\rangle = \frac{r^2\mu^2}{a^2}\langle\phi_* X, \phi_* Y\rangle. \tag{34}$$

From (10) and (27) we have

$$\begin{aligned} \delta_{ij} = \langle\phi_* X_i, \phi_* X_j\rangle = \langle X_i, X_j\rangle - r_i r_j - 2r\langle QX_i, X_j\rangle \\ + r^2(\langle QX_i, QX_j\rangle + \langle T, X_i\rangle\langle T, X_j\rangle + \langle P, X_i\rangle\langle P, X_j\rangle). \end{aligned} \tag{35}$$

Taking inner products of the first and second equations in (31) by X_2 and $-X_1$, respectively, and adding them up taking (30) into account, yields

$$\langle X_1, X_2\rangle = 0.$$

10

We compute from the first two equations in (31), bearing in mind (30), that

$$\begin{cases} r^2\mu\langle QX_1, X_1\rangle = \theta_1(\|X_1\|^2 - r_1^2) - r_2^2 \\ r^2\mu\langle QX_2, X_2\rangle = -\theta_2(\|X_2\|^2 - r_2^2) + r_1^2 \\ r\langle QX_1, X_2\rangle = -r_1 r_2. \end{cases} \tag{36}$$

Using (9) and (21), we have

$$\begin{cases} r^4\mu^2\|QX_1\|^2 = \theta_1^2(\|X_1\|^2 + (\|\nabla r\|^2 - 2)r_1^2) - 2\theta_1(r_2^2 + a\rho r_1 r_2) + (1-a^2)r_2^2 \\ r^4\mu^2\|QX_2\|^2 = \theta_2^2(\|X_2\|^2 + (\|\nabla r\|^2 - 2)r_2^2) - 2\theta_2(r_1^2 - a\rho r_1 r_2) + (1-a^2)r_1^2 \\ r^4\mu^2\langle QX_1, QX_2\rangle = (\theta_1\theta_2(1 + \rho^2 + b^2) - \theta_1 - \theta_2 - a^2 + 1)r_1 r_2 - a\rho(\theta_1 r_1^2 - \theta_2 r_2^2). \end{cases} \tag{37}$$

From (30) and (33) we obtain

$$r^2\mu\langle T, X_1\rangle = -\theta_1\rho r_1 - ar_2 \quad \text{and} \quad r^2\mu\langle T, X_2\rangle = \theta_2\rho r_2 - ar_1.$$

Thus,

$$\begin{cases} r^4\mu^2\langle T, X_1\rangle^2 = \theta_1^2\rho^2 r_1^2 + a^2 r_2^2 + 2\theta_1 a\rho r_1 r_2 \\ r^4\mu^2\langle T, X_1\rangle\langle T, X_2\rangle = (a^2 - \theta_1\theta_2\rho^2)r_1 r_2 + \theta_1 a\rho r_1^2 - \theta_2 a\rho r_2^2 \\ r^4\mu^2\langle T, X_2\rangle^2 = \theta_2^2\rho^2 r_2^2 + a^2 r_1^2 - 2\theta_2 a\rho r_1 r_2. \end{cases} \tag{38}$$

From the first two equations in (29) we get

$$\begin{cases} r^4\mu^2\langle P, X_1\rangle^2 = \theta_1^2 b^2 r_1^2 \\ r^4\mu^2\langle P, X_2\rangle^2 = \theta_2^2 b^2 r_2^2 \\ r^4\mu^2\langle P, X_1\rangle\langle P, X_2\rangle = -\theta_1\theta_2 b^2 r_1 r_2. \end{cases} \tag{39}$$

Replacing (36), (37), (38) and (39) into (35) we end up with

$$\|X_1\|^2 = r^2\mu^2 + r_1^2 + r_2^2 = \|X_2\|^2,$$

and (34) follows easily.

It follows from Sublemma 9 and (30) that $JZ = \nabla r$. We conclude from (21) that $\rho = 0$, hence $b = \pm a$ by (9) and (12), and the proof of (i) is completed.

We now prove (ii). Replacing the first two equations of (31) into the last two gives

$$\begin{cases} rDX_1 + SX_2 + r_1 Z + r_2\nabla r = 0 \\ rDX_2 - SX_1 + r_2 Z - r_1\nabla r = 0, \end{cases} \tag{40}$$

that can be written as

$$rDX = -JX - \langle\nabla r, X\rangle Z. \tag{41}$$

11

On the other hand, replacing the first two equations of (33) into the last two yields

$$\begin{cases} r\langle K, X_1 \rangle = a\langle Z, X_2 \rangle \\ r\langle K, X_2 \rangle = -a\langle Z, X_1 \rangle. \end{cases} \tag{42}$$

Taking (30) into account, the preceding equations reduce to

$$r B_\xi Z + \nabla(ar) = 0. \tag{43}$$

From (16) we have

$$\tilde{\nabla}_X \xi = -g_* B_\xi X + \nabla_X^\perp \xi = -g_* B_\xi X - \frac{1}{a}(\alpha_g(Z, X) - \langle B_\xi Z, X \rangle \xi),$$

where α_g denotes the second fundamental form of g. Hence,

$$-ar\tilde{\nabla}_X \xi + r\langle B_\xi Z, X \rangle \xi = arg_* B_\xi X + r\alpha_g(Z, X).$$

In view of (43) the left-hand-side is $\tilde{\nabla}_X(-ar\xi)$. For the right-hand-side we have

$$\begin{aligned} arg_* B_\xi X + r\alpha_g(Z, X) &= arg_* B_\xi X + r(\tilde{\nabla}_X g_* Z - g_* \nabla_X Z) \\ &= g_*(ar B_\xi X - r\nabla_X Z - X(r)Z) + \tilde{\nabla}_X(rg_* Z). \end{aligned}$$

Therefore, we obtain using (41) that

$$h_* X = g_*(ar B_\xi X - r\nabla_X Z - X(r)Z) = g_*(-rDX - X(r)Z) = g_* JX. \quad \blacksquare$$

3.2   Proof of the direct statement of Theorem 1

For the proof of the direct statement we need the following general fact.

Proposition 10. *Let* $g\colon M^2 \to \mathbb{R}^{n+2}$ *be a simply connected oriented minimal surface with complex structure J compatible with the orientation and let $h\colon M^2 \to \mathbb{R}^{n+2}$ be a conjugate minimal surface such that*

$$h_* = g_* \circ J. \tag{44}$$

Then $r = \|h\|$ satisfies $\|\nabla r\| \le 1$ everywhere. Moreover, on the complement of the subset of isolated points of M^2 where $a = \sqrt{1 - \|\nabla r\|^2}$ vanishes, there exists a smooth unit normal vector field ξ to g such that

$$h = -r(g_* \nabla r + a\xi).$$

Furthermore,

$$\langle B_\delta J\nabla r, X \rangle + a\langle \nabla_X^\perp \delta, \xi \rangle = 0 \quad \text{for all } \delta \in span\{\xi\}^\perp \tag{45}$$

and

$$B_\xi = \frac{1}{ar}(r\,Hess\,r - S) \circ J, \tag{46}$$

where S is given by (32).

Proof: Decompose $h = g_*T + h^N$ into tangent and normal components to g. From (44) we obtain

$$\begin{cases} \nabla_X T - B_{h^N} X = JX \\ \alpha_g(X, T) + \nabla_X^\perp h^N = 0. \end{cases} \tag{47}$$

It also follows from (44) that the tangent components of the position vector h with respect to g and h coincide. Since the latter is $h_*(r\nabla r)$, we get

$$g_*T = h_*(r\nabla r) = g_*(rJ\nabla r),$$

hence $T = rJ\nabla r$. From $\|h\|^2 = \|T\|^2 + \|h^N\|^2$ we get $r^2 = r^2\|\nabla r\|^2 + \|h^N\|^2$, which implies that $\|\nabla r\| \leq 1$ holds everywhere. By the real analyticity of g and h the points where the function a vanishes are isolated. On the complement of the subset of such points we have $\|h^N\| = ar$. Thus, we can write $h^N = -ar\xi$ for a unit normal vector field ξ. Then, using that $J \circ B_\xi = -B_\xi \circ J$, for B_ξ is traceless, the first equation in (47) reduces to (46), whereas the span$\{\xi\}^\perp$-component of the second yields (45). ∎

Setting $\delta_\pm = \hat{J}_\pm \xi$ and $\eta_\pm = g_*\nabla r + a\delta_\pm$, we have from Proposition 10 that

$$\phi_\pm = g - r\eta_\pm.$$

It follows from (10) that $\eta_\pm$ is a unit normal vector field to $\phi_\pm$. Let ζ be defined by (8) with $Z = -J\nabla r$. Then ζ has unit length and is orthogonal to $\eta_\pm$. We obtain from (45) that (19) holds, hence we have (23) with $\rho = 0$, and also (24). ¿From (46) we get

$$r\,\mathrm{Hess}\,r(Z) - Z + arB_\xi\nabla r = 0, \tag{48}$$

which implies, using (23) (with $\rho = 0$) and (24), that ζ is normal to $\phi_\pm$.

Therefore, to complete the proof it suffices to show that there exists an orthonormal tangent frame $\{X_1, X_2\}$ (with respect to the metric induced by $\phi_\pm$) satisfying (25). Since $B_{\delta_\pm}$ and B_ξ are traceless symmetric 2×2 matrices, we have

$$(B_{\delta_\pm} + B_\xi J)^2 = \alpha_\pm^2 I \tag{49}$$

for some smooth functions $\alpha_\pm$. By analyticity, either $\alpha_\pm$ vanishes identically or it vanishes only at isolated points. In the first case, a standard argument shows that the complex structure $\mathcal{J} = J \oplus \hat{J}_\pm$ on $g^*T\mathbb{R}^4$ is a parallel tensor, hence defines a complex structure on $\mathbb{R}^4$ with respect to which g is holomorphic. Then, in this case the conclusion follows from Theorem 4. Therefore, we may assume in the sequel that $\alpha_\pm$ is nowhere vanishing, hence there exists $\mu_\pm \in C^\infty(M)$ such that $\alpha_\pm = -a/r^2\mu_\pm$. Since $B_{\delta_\pm} + B_\xi J = \alpha_\pm R_\pm$ for some reflection $R_\pm$ by (49), it follows using (46) that

$$B_{\delta_\pm} = \frac{1}{a}(\mathrm{Hess}\,r - \frac{1}{r}S) - \frac{a}{r^2\mu_\pm}R_\pm. \tag{50}$$

13

Let $\{\bar{X}_1^\pm, \bar{X}_2^\pm\}$ be the orthonormal basis of eigenvectors of $R_\pm$ (with respect to the metric induced by g), with $\bar{X}_1^\pm$ corresponding to the eigenvalue $+1$ and $\bar{X}_2^\pm = J\bar{X}_1^\pm$. Define

$$X_j^\pm = \frac{r\mu_\pm}{a} \bar{X}_j^\pm, \quad 1 \le j \le 2.$$

We claim that $\{X_1^\pm, X_2^\pm\}$ is the desired orthonormal frame. In order to prove (25), it suffices to show that (26), or equivalently, (29), (31) and (33), holds for $X_1^\pm$ and $X_2^\pm$.

Since we have (30), because $JX_1^\pm = X_2^\pm$ and $JZ = \nabla r$, system (29) reduces to its first two equations. These are in turn equivalent to

$$rB_{\delta_\pm}\nabla r + \frac{a}{r\mu_\pm}R_\pm\nabla r + \nabla(ar) = 0,$$

which follows from (50). Now, (50) also implies that

$$rQ = S + \frac{a^2}{r\mu_\pm}R_\pm.$$

Moreover, from (46) we get (41), hence (31) is satisfied.

From (46) we also obtain (43), and hence (42). Moreover, (46) and (50) imply that

$$B_\xi\nabla r + B_{\delta_\pm}Z + \frac{a}{r^2\mu_\pm}R_\pm Z = 0,$$

thus (33) is satisfied. Since we now have (36), (37), (38) and (39), then

$$\langle \phi_{\pm*}X_i^\pm, \phi_{\pm*}X_j^\pm \rangle = \delta_{ij}$$

follows by using that $\langle X_1^\pm, X_2^\pm \rangle = 0$ and $\|X_j^\pm\|^2 = r^2\mu_\pm^2/a^2$ for $1 \le j \le 2$.

Finally, that ϕ_+ and ϕ_- envelop a common central sphere congruence, with g as the surface of centers, follows from the facts that for each $p \in M^2$ we have

$$\phi_+(p) + H_+(p)/\|H_+(p)\|^2 = g(p) = \phi_-(p) + H_-(p)/\|H_-(p)\|^2$$

and

$$(\phi_+)_*T_pM \oplus \mathrm{span}\{H_+(p)\} = (\phi_-)_*T_pM \oplus \mathrm{span}\{H_-(p)\},$$

for $\zeta(p)$ is orthogonal to both subspaces. ∎

Remarks 11. (1) It follows from the proof that $\phi_\pm$ may fail to be regular only at points where the function a in Lemma 10 vanishes, that is, at points where the position vector of h is tangent to g, and at points where the shape operator of g with respect to any normal direction β satisfies $B_{\hat{j}_{\pm}\beta} = -B_\beta \circ J$. The latter can be seen as the "holomorphic" points of g, that is, points where the ellipse of curvature of g is a circle.

(2) If we change the conjugate minimal surface h by $h + v$ for any $v \in \mathbb{R}^4$, then the corresponding surfaces $\phi_\pm$ are changed by addition of $\mathcal{J}_\pm v$. One can check that the latter

is just a parameterization of an open subset of the two-dimensional equator in $\mathbb{S}^3(\|v\|)$ orthogonal to v. Moreover, the parameterization is conformal to g and singularities occur at points where the ellipse of curvature of g is a circle.

(3) It follows from Sublemma 9 that the metrics $\langle\,,\,\rangle_+$ and $\langle\,,\,\rangle_-$ induced on M^2 by ϕ_+ and ϕ_-, respectively, are related by $\mu_+^2\langle\,,\,\rangle_+ = \mu_-^2\langle\,,\,\rangle_-$.

(4) For any element of the associated family $g_\theta = \cos\theta g + \sin\theta h$ of the minimal surface g, we have a pair $(\phi_+^\theta, \phi_-^\theta)$ of dual superconformal surfaces in $\mathbb{R}^4$ determined by the pair (g_θ, h_θ) of conjugate minimal surfaces g_θ and $h_\theta = -\sin\theta g + \cos\theta h$, which also satisfy $h_{\theta*} = g_{\theta*} \circ J$. Namely,

$$\phi_\pm^\theta = g_\theta + \mathcal{J}_\pm h_\theta = \mathcal{J}_\pm^\theta g + \tilde{\mathcal{J}}_\pm^\theta h,$$

where

$$\mathcal{J}_\pm^\theta \circ g_* = g_*(cos\theta I - \sin\theta J), \qquad \tilde{\mathcal{J}}_\pm^\theta \circ g_* = g_*(\sin\theta I + \cos\theta J)$$

and $\mathcal{J}_\pm^\theta|_{T_g^\perp M}$ and $\tilde{\mathcal{J}}_\pm^\theta|_{T_g^\perp M}$ are given, respectively, by

$$\mathcal{J}_\pm^\theta|_{T_g^\perp M} = cos\theta I - \sin\theta\mathcal{J}_\pm, \qquad \tilde{\mathcal{J}}_\pm^\theta|_{T_g^\perp M} = \sin\theta I + \cos\theta\mathcal{J}_\pm.$$

It is an interesting question whether $\{\phi_\pm^\theta\}$ coincides with the associated family of $\phi_\pm$ in the sense of [10], Corollary 2.7.

3.3 Superconformal surfaces from minimal surfaces in $\mathbb{R}^3$

In this subsection we consider the problem of determining when a pair of dual superconformal surfaces ϕ_+ and ϕ_- differ by an inversion in $\mathbb{R}^4$.

Proposition 12. *Let (ϕ_+, ϕ_-) be a pair of dual superconformal surfaces constructed from a minimal surface $g\colon L^2 \to \mathbb{R}^3 \subset \mathbb{R}^4$ by the procedure of Theorem 1. Then, the following holds:*

(i) The maps ϕ_+ and ϕ_- differ by a reflection with respect to $\mathbb{R}^3$.

(ii) For any inversion $\mathcal{I}$ with respect to a sphere centered at a point $P_0 \notin \mathbb{R}^3$, the maps $\mathcal{I}\circ\phi_+$ and $\mathcal{I}\circ\phi_-$ define dual superconformal surfaces that differ by an inversion with respect to the sphere $\mathcal{I}(\mathbb{R}^3)$.

Conversely, any pair of dual superconformal surfaces that differ by an inversion in $\mathbb{R}^4$ arises as in (ii).

Proof: If $g\colon L^2 \to \mathbb{R}^4$ is a minimal surface such that $g(L^2) \subset \mathbb{R}^3$, then

$$\phi_\pm = g + Jh^T \pm \langle h, N\rangle e_4,$$

where N is a unit normal vector field to g in $\mathbb{R}^3$. This proves the first assertion. The second follows from a well known property of inversions. For the converse, if a pair (ϕ_+, ϕ_-) of

dual superconformal surfaces differ by an inversion in $\mathbb{R}^4$ with respect to a hypersphere S, then each element of their common central sphere congruence is orthogonal to S, since it passes through a pair of inverse points. Then the image of S by an inversion $\mathcal{I}$ with respect to a point contained in it is a hyperplane $\mathbb{R}^3$ and any element of the common central sphere congruence of $\mathcal{I} \circ \phi_+$ and $\mathcal{I} \circ \phi_-$, being orthogonal to $\mathbb{R}^3$, has its center therein. Therefore $\mathcal{I} \circ \phi_+$ and $\mathcal{I} \circ \phi_-$ are constructed as in Theorem 1 from a minimal surface $g \colon L^2 \to \mathbb{R}^3$. ∎

The following example shows that minimal surfaces in $\mathbb{R}^3$ give rise to nontrivial examples of superconformal surfaces in $\mathbb{R}^4$ by means of the construction in Theorem 1.

Example 13. Consider the catenoid and the helicoid in $\mathbb{R}^3$ parameterized, respectively, by $g(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$ and $h(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u)$. Then the corresponding dual superconformal surfaces given by Theorem 1 are

$$\phi_\pm = \frac{1}{\cosh v}(\cos u - u \sin u, \sin u + u \cos u, v \cosh v - \sinh v, \pm u \sinh v).$$

4 Proof of Theorem 2

For the proof of Theorem 2 we need the following well-known fact.

Lemma 14. *Let $f \colon M^n \to \mathbb{R}^N$ be an isometric immersion and let $\mathcal{I}$ be an inversion with respect to a sphere with radius R centered at $P_0 \in \mathbb{R}^N$. Then,*

$$\mathcal{P}\xi = \xi - 2\frac{\langle f - P_0, \xi\rangle}{\langle f - P_0, f - P_0\rangle}(f - P_0) \tag{51}$$

is a vector bundle isometry between the normal bundles $T_f^\perp M$ and $T_{\mathcal{I}\circ f}^\perp M$. Moreover, the shape operators A_ξ and $\tilde{A}_{\mathcal{P}\xi}$ are related by

$$\tilde{A}_{\mathcal{P}\xi} = \frac{1}{R^2}\left(\langle f - P_0, f - P_0\rangle A_\xi + 2\langle f - P_0, \xi\rangle I\right). \tag{52}$$

In Section 6 we will need the following analogue of Lemma 14 for isometric immersions $f \colon M^n \to \mathbb{L}^N$ into Lorentzian space and the "inversion"

$$\mathcal{I}(P) = P_0 - \frac{R^2}{\langle P - P_0, P - P_0\rangle}(P - P_0), \quad P \neq P_0.$$

with respect to the hyperbolic space $\mathbb{H}_R^{N-1} := \mathbb{H}^{N-1}(P_0; R)$ of radius R "centered" at $P_0 \in \mathbb{L}^N$, i.e.,

$$\mathbb{H}_R^{N-1} = \{P \in \mathbb{L}^N : \langle P - P_0, P - P_0\rangle = -R^2\}.$$

Lemma 15. *Let $f \colon M^n \to \mathbb{L}^N$ be an isometric immersion and let $\mathcal{I}$ be an inversion with respect to $\mathbb{H}_R^{N-1}$. Then (51) and (52) hold true if we replace R^2 by $-R^2$ in the latter.*

16

Proof: We provide a proof for the sake of completeness, which also applies for Lemma 14. An easy computation shows that

$$\tilde{f}_* = -\frac{R^2}{\langle f - P_0, f - P_0 \rangle} \mathcal{P} \circ f_*, \tag{53}$$

where $\mathcal{P} \colon f^* T \mathbb{L}^N \to f^* T \mathbb{L}^N$ is given by

$$\mathcal{P} Z = Z - 2 \langle f - P_0, Z \rangle \langle f - P_0, f - P_0 \rangle^{-2} (f - P_0).$$

Since $\mathcal{P}$ is easily seen to be a vector bundle isometry, it follows from (53) that it restricts to a vector bundle isometry of $T_f^\perp M$ onto itself. Denoting by $\bar{\nabla}$ the derivative in $\mathbb{L}^N$, equation (52) follows by taking tangent components in

$$-\tilde{f}_* \tilde{A}_{\mathcal{P}\xi} X + \tilde{\nabla}^\perp \mathcal{P} \xi = \bar{\nabla}_X \tilde{f}_* Y = \bar{\nabla}_X (\xi - 2 \frac{\langle f - P_0, \xi \rangle}{\langle f - P_0, f - P_0 \rangle} (f - P_0))$$

$$= -\mathcal{P} f_* (A_\xi + 2 \frac{R^2}{\langle f - P_0, f - P_0 \rangle} \langle f - P_0, \xi \rangle I) + \mathcal{P} \nabla_X^\perp \xi. \quad \blacksquare$$

Proof of Theorem 2: Define

$$\tilde{\zeta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2} (\bar{\nu} \mathcal{P} \eta - \bar{\lambda} \mathcal{P} \zeta) \quad \text{and} \quad \tilde{\eta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2} (\bar{\lambda} \mathcal{P} \eta + \bar{\nu} \mathcal{P} \zeta)$$

where $R^2 \bar{\lambda} = \lambda \|\phi\|^2 + 2 \langle \phi, \eta \rangle$, $R^2 \bar{\nu} = 2 \langle \phi, \zeta \rangle$ and $\mathcal{P}$ is given by (51). Using (52), we obtain that the shape operators $\tilde{A}_{\tilde{\eta}}$ and $\tilde{A}_{\tilde{\zeta}}$ of $\tilde{\phi} = \mathcal{I} \circ \phi = \frac{R^2}{\|\phi\|^2} \phi$ are given as in (6) with λ and μ replaced, respectively, by

$$\tilde{\lambda} = (\bar{\lambda}^2 + \bar{\nu}^2)^{1/2} \quad \text{and} \quad \tilde{\mu} = \frac{1}{R^2} \|\phi\|^2 \mu.$$

The pair $(\tilde{g}, \tilde{h})$ of conjugate minimal surfaces associated to $\tilde{\phi}$ is

$$\tilde{g} = \tilde{\phi} + \tilde{r} \tilde{\eta} = \tilde{\phi} + \frac{\bar{\lambda} \mathcal{P} \eta + \bar{\nu} \mathcal{P} \zeta}{\bar{\lambda}^2 + \bar{\nu}^2} \quad \text{and} \quad \tilde{h} = -\tilde{r} \tilde{\zeta} = -\frac{\bar{\nu} \mathcal{P} \eta - \bar{\lambda} \mathcal{P} \zeta}{\bar{\lambda}^2 + \bar{\nu}^2}. \tag{54}$$

We have
$$R^4 (\bar{\nu}^2 + \bar{\lambda}^2) = 4 (\langle \phi, \zeta \rangle^2 + \langle \phi, \eta \rangle^2 + \lambda \langle \phi, \eta \rangle \|\phi\|^2) + \lambda^2 \|\phi\|^4. \tag{55}$$

On the other hand, from

$$\mathcal{P} \eta = \eta - 2 \frac{\langle \phi, \eta \rangle}{\|\phi\|^2} \phi \quad \text{and} \quad \mathcal{P} \zeta = \zeta - 2 \frac{\langle \phi, \zeta \rangle}{\|\phi\|^2} \phi,$$

we obtain

$$R^2 (\bar{\lambda} \mathcal{P} \eta + \bar{\nu} \mathcal{P} \zeta) = 2 \langle \phi, \zeta \rangle \zeta + 2 \langle \phi, \eta \rangle (\eta - \lambda \phi) + \lambda \|\phi\|^2 \eta - \frac{4 (\langle \phi, \eta \rangle^2 + \langle \phi, \zeta \rangle^2)}{\|\phi\|^2} \phi \tag{56}$$

17

and

$$R^2(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta) = 2\langle\phi,\zeta\rangle(\eta + \lambda\phi) - (2\langle\phi,\eta\rangle + \lambda\|\phi\|^2)\zeta. \tag{57}$$

Using that $\lambda = 1/r$, $\phi = g - r\eta$ and $h = -r\zeta$, we have

$$\langle\phi,\zeta\rangle = -\frac{1}{r}\langle g,h\rangle, \quad \langle\phi,\eta\rangle = \langle g,\eta\rangle - r \quad \text{and} \quad \|\phi\|^2 = \|g\|^2 - 2r\langle g,\eta\rangle + r^2.$$

Thus,

$$\bar{\lambda}^2 + \bar{\nu}^2 = \frac{1}{R^4\|h\|^2}(4\langle g,h\rangle^2 + (\|g\|^2 - \|h\|^2)^2).$$

Hence, from (54) we get

$$\frac{1}{R^2}\tilde{g} = \frac{\phi}{\|\phi\|^2} + \frac{\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta}{R^2(\bar{\nu}^2 + \bar{\lambda}^2)} = \frac{(\|g\|^2 - \|h\|^2)g + 2\langle g,h\rangle h}{4\langle g,h\rangle^2 + (\|g\|^2 - \|h\|^2)^2}$$

and

$$\frac{1}{R^2}\tilde{h} = \frac{2\langle g,h\rangle g - (\|g\|^2 - \|h\|^2)h}{4\langle g,h\rangle^2 + (\|g\|^2 - \|h\|^2)^2}.$$

Therefore,

$$(\tilde{g},\tilde{h}) = F \circ (g,h),$$

where $F\colon \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ is given by

$$\frac{1}{R^2}F(U,V) = \frac{((\|U\|^2 - \|V\|^2)U + 2\langle U,V\rangle V, 2\langle U,V\rangle U - (\|U\|^2 - \|V\|^2)V)}{4\langle U,V\rangle^2 + (\|U\|^2 - \|V\|^2)^2}.$$

Then, as a map $F\colon \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$, we can write F as

$$\frac{1}{R^2}F(Z) = \frac{\langle\!\langle Z,Z\rangle\!\rangle}{|\langle\!\langle Z,Z\rangle\!\rangle|^2}\bar{Z} = \frac{\bar{Z}}{\overline{\langle\!\langle Z,Z\rangle\!\rangle}} = \overline{T(Z)}. \quad \blacksquare$$

Remark 16. For an inversion $\mathcal{I}$ with respect to a sphere centered at an arbitrary point $P_0 \in \mathbb{R}^4$, the formula in Theorem 2 becomes $\tilde{g} - P_0 - i\tilde{h} = T_R \circ (g - P_0 + ih)$.

5 Superconformal surfaces and holomorphic curves

The main goal of this section is to prove Theorems 4 and 5. Along the way we establish some additional facts on holomorphic curves and their compositions with an inversion in $\mathbb{R}^4$. We also look at the Whitney sphere in the light of our results.

5.1 A characterization of holomorphic curves

In this subsection we prove the following characterization of holomorphic curves that is interesting on its own right.

Proposition 17. *Let $g\colon M^2 \to \mathbb{R}^4$ be a simply connected oriented minimal surface whose holomorphic representative $g + ih\colon M^2 \to \mathbb{C}^4$ takes values in the quadric $\mathbb{Q}_0 \subset \mathbb{C}^4$. Then, g is holomorphic with respect to some complex structure $\mathcal{J}$ on $\mathbb{R}^4$ and its conjugate minimal surface is $h = \mathcal{J}g$.*

Proof: By the assumption we have

$$\langle g, h \rangle = 0 \quad \text{and} \quad \|g\|^2 = \|h\|^2. \tag{58}$$

Differentiating the first of equations (58) and using that $h_* = g_* \circ J$, we obtain that the tangent components g^T and h^T of the position vectors of g and h, respectively, are related by $h^T = Jg^T$. Then, by (58) there exists a complex structure $\hat{J}$ on $T_g^\perp M$ such that $\hat{J}g^N = h^N$.

Differentiating (58) twice gives

$$\langle \alpha_h(X, Y), g^N \rangle = -\langle \alpha_g(X, Y), h^N \rangle = -\langle \alpha_g(X, Y), \hat{J}g^N \rangle = \langle \hat{J}\alpha_g(X, Y), g^N \rangle$$

and

$$\langle \alpha_h(X, Y), h^N \rangle = \langle \alpha_g(X, Y), g^N \rangle = -\langle \alpha_g(X, Y), \hat{J}h^N \rangle = \langle \hat{J}\alpha_g(X, Y), h^N \rangle.$$

Hence, $\alpha_h = \hat{J} \circ \alpha_g$. Since conjugate minimal surfaces satisfy $\alpha_h(X, Y) = \alpha_g(X, JY)$, it follows that

$$\alpha_g(X, JY) = \hat{J}\alpha_g(X, Y).$$

Thus, the complex structure $\mathcal{J} = J \oplus \hat{J}$ on $g^*T\mathbb{R}^4$ extends to a complex structure on $\mathbb{R}^4$ with respect to which g is holomorphic, and we have that $h = \mathcal{J}g$. ∎

5.2 Minimal surfaces and inversions of holomorphic curves

We now determine the holomorphic representative of the minimal surface associated to the composition of an inversion with a holomorphic curve. Given an oriented holomorphic curve $f\colon M^2 \to \mathbb{R}^4$, we denote by $\hat{J}$ the complex structure on $T_f^\perp M$ determined by the opposite orientation to that induced by the vector bundle isometry $\mathcal{P}\colon T_f^\perp M \to T_{\mathcal{I} \circ f}^\perp M$ from the orientation on $T_{\mathcal{I} \circ f}^\perp M$ defined as in the proof of Theorem 1.

Proposition 18. *Let $f\colon M^2 \to \mathbb{R}^4$ be a holomorphic curve and let $\mathcal{I}$ be the inversion in $\mathbb{R}^4$ with respect to the sphere of radius R centered at the origin. Then, the holomorphic curve in $\mathbb{C}^4$ associated to $\tilde{f} = \mathcal{I} \circ f$ is*

$$\tilde{g} + i\tilde{h} = \frac{R^2}{2\|f^N\|^2}(f^N + i\hat{J}f^N). \tag{59}$$

Proof: Following the proof of Theorem 2 with $\lambda = 0$, we obtain from (55) that

$$R^4(\bar{\nu}^2 + \bar{\lambda}^2) = 4\|f^N\|^2.$$

On the other hand, (56) and (57) now reduce, respectively, to

$$R^2(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta) = 2f^N - 4\frac{\|f^N\|^2}{\|f\|^2}f \tag{60}$$

and

$$R^2(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta)) = 2\langle f, \zeta\rangle\eta - 2\langle f, \eta\rangle\zeta = -2\hat{J}f^N. \tag{61}$$

Then (59) follows from (54), (60) and (61). ∎

Remark 19. For an inversion $\mathcal{I}$ with respect to a sphere centered at an arbitrary point $P_0 \in \mathbb{R}^4$, formula (59) becomes

$$\tilde{g} + i\tilde{h} = P_0 + \frac{R^2}{2\|(f - P_0)^N\|^2}((f - P_0)^N + i\hat{J}(f - P_0)^N),$$

where $(f - P_0)^N$ is the normal component of the position vector of f with respect to P_0.

The first assertion in Theorem 5 is a consequence of the following general fact.

Proposition 20. *Let $f\colon M^n \to \mathbb{C}^{n+p}$ be a holomorphic isometric immersion of a Kaehler manifold of real dimension $2n$. Then $F = f^N/\|f^N\|^2$ is anti-holomorphic with respect to the complex structure J of $\mathbb{C}^{n+p}$ if and only if $f^T \in \ker \alpha_f^\perp$, where $\alpha_f^\perp$ is the $\mathrm{span}\{f^N, Jf^N\}^\perp$ component of α_f. In particular, this is always the case if the complex codimension is $p = 1$.*

Proof: Denote for simplicity $g = f^N$ and $h = f^T$. We have

$$g_*Z = Z - \nabla_Z h - \alpha_f(Z, h)$$

and

$$g_*Z = -A_g^f Z + \nabla_Z^\perp g.$$

We conclude that

$$g_*Z = -A_g^f Z - \alpha_f(Z, h).$$

Then

$$F_*Z = \frac{1}{\|g\|^4}(\|g\|^2 g_*Z - 2\langle g_*Z, g\rangle g),$$

and hence

$$\|g\|^4 F_*Z = -\|g\|^2(A_g^f Z + \alpha_f(Z, h)) + 2\langle \alpha_f(Z, h), g\rangle g.$$

Equivalently,

$$\|g\|^2 F_*Z = -A_g^f Z + \frac{1}{\|g\|^2}(\langle \alpha_f(Z, h), g\rangle g - \langle \alpha_f(Z, h), Jg\rangle Jg) - \alpha_f^\perp(Z, h).$$

It is now easy to see that

$$F_* \circ J = -J \circ F_* \quad \text{if and only if} \quad h \in \ker \alpha_f^\perp. \quad \blacksquare$$

20

Remark 21. Now that we know that for a holomorphic curve $f \colon M^2 \to \mathbb{R}^4$ with respect to a complex structure J on $\mathbb{R}^4$ the map $f^N/\|f^N\|^2$ defines an anti-holomorphic curve with respect to J, it follows that the complex structure $\hat{J}$ on $T_f^\perp M$ is the restriction of $-J$ to $T_f^\perp M$.

5.3 Proof of Theorems 4 and 5

Proof of Theorem 4: If either ϕ or its dual parameterizes the composition of an inversion with respect to a sphere of radius R centered, say, at the origin, with a holomorphic curve with respect to a complex structure J, then the associated minimal surface is $g = \frac{R^2 f^N}{2\|f^N\|^2}$, which is anti-holomorphic with respect to J by the assertion in Theorem 5 already proved. If either ϕ or its dual only takes the value 0, then the holomorphic representative $g+ih$ of the associated minimal surface g takes values in $\mathbb{Q}_0$, and hence (iii) follows from Proposition 17. Conversely, if g is holomorphic with respect to some complex structure $\mathcal{J}$ on $\mathbb{R}^4$, $h = \mathcal{J}g$ and $\hat{J}_+$ is the restriction of $-\mathcal{J}$ to $T_g^\perp M$, then ϕ_- reduces to the constant map $\phi_- = 0$ and its dual is $\phi_+ = 2g^N$. Thus, the assertion in Theorem 5 already proved implies that $\phi = \phi_+ = 2g^N$ is as in (i). ∎

Proof of Theorem 5: We already know that $f \mapsto f^* = f^N/2\|f^N\|^2$ maps $\mathcal{H}_+$ into $\mathcal{H}_-$. Since f^* is the minimal surface associated to $f/\|f\|^2$ by Proposition 18, that f and f^* induce conformal surfaces on L^2 follows from Theorem 1. Finally, since f^* is anti-holomorphic with respect to J, the fact that f^* is the minimal surface associated to $f/\|f\|^2$ implies that $f/\|f\|^2 = 2(f^*)^N$. Hence,

$$
f = \frac{2(f^*)^N}{\|2(f^*)^N\|^2} = \frac{(f^*)^N}{2\|(f^*)^N\|^2} = (f^*)^*.
$$

In particular, this implies that the map $f \mapsto f^*$ is a bijection between $\mathcal{H}_+$ and $\mathcal{H}_-$. ∎

5.4 The Whitney sphere

The *Whitney sphere* is the immersion

$$
(x, y, z) \in \mathbb{S}^2 \mapsto \frac{1}{1 + z^2}(x(1 + iz), y(1 + iz)) \in \mathbb{C}^2
$$

of the unit sphere $\mathbb{S}^2(1)$ into the complex Euclidean plane. Among several interesting characterizations, the one that concerns this paper is as the only Lagrangean superconformal surface in $\mathbb{C}^2$ due to Castro [3].

The Whitney sphere is just the composition $\mathcal{I} \circ f$ of the holomorphic curve $f \colon \mathbb{C}^* \to \mathbb{C}^2$ given by

$$
f(z) = (z, 1/z)
$$

with the inversion $\mathcal{I}$ with respect to the sphere of unit radius centered at the origin.

By Proposition 18, if we orient $\mathbb{S}^2$ and $T^{\perp}_{\mathcal{I}\circ f}\mathbb{S}^2$ as described in the paragraph preceding it, then the pair (g, h) of conjugate minimal surfaces associated to $\mathcal{I} \circ f$ is given by

$$g = \frac{1}{2}\mathcal{I} \circ f^N \quad \text{and} \quad h = \frac{i}{2}\mathcal{I} \circ f^N.$$

An easy computation shows that

$$g(z) = \frac{1}{4}(1/\bar{z}, \bar{z}) \quad \text{and} \quad h(z) = \frac{i}{4}(1/\bar{z}, \bar{z}).$$

In fact, the construction of Theorem 1 applied to g gives

$$g + \mathcal{J}_- h = 2g^N = \frac{1}{2}\mathcal{I} \circ \frac{1}{2}f = \mathcal{I} \circ f.$$

6 Superconformal and superminimal surfaces

In this section we prove Theorem 6 and illustrate our result with the Veronese surface.

6.1 Proof of Theorem 6

First we consider the case of $\mathbb{S}^4_R = \mathbb{S}^4(Re_5; R)$. Regard the stereographic projection of $\mathbb{S}^4_R$ onto $\mathbb{R}^4$ as the restriction to $\mathbb{S}^4_R$ of the inversion $\mathcal{I}$ in $\mathbb{R}^5$ with respect to the sphere of radius $2R$ centered at $2Re_5$. Now, given a superminimal surface $f \colon M^2 \to \mathbb{S}^4_R$, we obtain as in Proposition 18 (see also Remark 19) that the pair (g, h) of conjugate minimal surfaces associated to $\mathcal{I} \circ f$ is

$$g = 2Re_5 + \frac{2R^2(f - 2Re_5)^N}{\|(f - 2Re_5)^N\|^2} \quad \text{and} \quad h = \frac{2R^2\hat{J}(f - 2Re_5)^N}{\|(f - 2Re_5)^N\|^2},$$

where $(f - 2Re_5)^N$ is the normal component of f with respect to $2Re_5$. Thus,

$$\langle g, h \rangle = 0 \quad \text{and} \quad \|g - 2Re_5\| = \|h\|, \tag{62}$$

and hence $g + ih$ takes values in $\mathbb{Q}_{4R^2}$.

Conversely, assume that the pair (g, h) of conjugate minimal surfaces that gives rise to ϕ_+ and ϕ_- satisfies (62). We claim that $\mathcal{I} \circ \phi_\pm$ is a superminimal surface in $\mathbb{S}^4_R$. The arguments for ϕ_+ and ϕ_- being the same, in the sequel we omit the subscript $\pm$ for simplicity of notation.

Set $\bar{R} = 2R$, $P_0 = 2Re_5$ and define

$$\tilde{\zeta} = \frac{\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta}{(\bar{\lambda}^2 + \bar{\nu}^2)^{1/2}} \quad \text{and} \quad \tilde{\eta} = \frac{\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta}{(\bar{\lambda}^2 + \bar{\nu}^2)^{1/2}},$$

where $\mathcal{P}$ is the vector bundle isometry between $T_f^\perp M$ and $T_{\mathcal{I} \circ f}^\perp M$ given by (51),

$$\bar{R}^2 \bar{\nu} = 2\langle \phi - P_0, \zeta \rangle \quad \text{and} \quad \bar{R}^2 \bar{\lambda} = \lambda \|\phi - P_0\|^2 + 2\langle \phi - P_0, \eta \rangle.$$

Using Lemma 14, we obtain that the shape operators $\tilde{A}_{\tilde{\eta}}$ and $\tilde{A}_{\tilde{\zeta}}$ of $\mathcal{I} \circ \phi$ are given as in (6) with λ and μ replaced, respectively, by

$$\tilde{\lambda} = (\bar{\lambda}^2 + \bar{\nu}^2)^{1/2} \quad \text{and} \quad \tilde{\mu} = \frac{\|\phi - P_0\|^2}{\bar{R}^2} \mu.$$

Since (62) holds, using that $h = -\dfrac{1}{\lambda}\zeta$ and $g - P_0 = \phi - P_0 + \dfrac{1}{\lambda}\eta$ we obtain

$$\langle \phi - P_0, \zeta \rangle = \langle g - P_0 - \frac{1}{\lambda}\eta, \zeta \rangle = 0$$

and

$$\frac{-2}{\lambda}\langle \phi - P_0, \eta \rangle = \|\phi - P_0\|^2 + \frac{1}{\lambda^2} - \|g - P_0\|^2 = \|\phi - P_0\|^2 + \frac{1}{\lambda^2} - \|h\|^2 = \|\phi - P_0\|^2.$$

Thus, $\bar{\nu} = 0 = \bar{\lambda}$, and hence $\tilde{\lambda} = 0$, which is equivalent to $\mathcal{I} \circ \phi$ being superminimal.

In the case of $\mathbb{H}_R^4 = \mathbb{H}^4(-Re_5; R)$, let $\mathcal{I}: \mathbb{H}_R^4 \to \mathbb{R}^4$ denote the stereographic projection defined in the introduction. Notice that the image of $\mathbb{H}_R^4$ by $\mathcal{I}$ is the open ball $B(0; 2R) \subset \mathbb{R}^4$, the induced metric on $B(0; 2R)$ being the Poincaré hyperbolic metric of constant sectional curvature $-1/R^2$. Observe also that $\mathcal{I}$ can be regarded as the restriction to $\mathbb{H}_R^4$ of the "inversion" $\mathcal{I}$ on $\mathbb{L}^5$ with respect to $\mathbb{H}^4(-2Re_5; 2R)$ (see Lemma 15). Taking Lemma 15 into account, the remaining of the proof is entirely similar to that of the spherical case, the conclusion now being that (g, h) is the pair of conjugate minimal surfaces associated to $\mathcal{I} \circ f$, where $f: M^2 \to \mathbb{H}_R^4$ is a superminimal surface, if and only if

$$\langle g, h \rangle = 0 \quad \text{and} \quad \langle g + 2Re_5, g + 2Re_5 \rangle = \langle h, h \rangle, \tag{63}$$

that is, if and only if $g + ih$ takes values in $\mathbb{Q}_{-4R^2}$. $\blacksquare$

6.2 The Veronese surface

The Veronese surface $f: \mathbb{S}_{1/3}^2 \to \mathbb{S}^4 \subset \mathbb{R}^5$ given by

$$f(x, y, z) = \frac{1}{2\sqrt{3}}(2xy, 2xz, 2yz, x^2 - y^2, \frac{1}{\sqrt{3}}(x^2 + y^2 - 2z^2))$$

is a superminimal surface with constant normal curvature in the sphere. In spherical coordinates $(x, y, z) = \sqrt{3}(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, we can write f as

$$\frac{2}{\sqrt{3}}f(\varphi, \theta) = \sin^2\varphi X_1 + \sin 2\varphi X_2 + \frac{1}{\sqrt{3}}(1 - 3\cos^2\varphi)e_5,$$

23

where $(\varphi, \theta) \in (0, \pi) \times (0, 2\pi)$ and

$$X_1 = \sin 2\theta e_1 + \cos 2\theta e_4, \qquad X_2 = \cos \theta e_2 + \sin \theta e_3.$$

We have

$$\frac{2}{\sqrt{3}} \frac{\partial f}{\partial \varphi} = \sin 2\varphi X_1 + 2 \cos 2\varphi X_2 + \sqrt{3} \sin 2\varphi e_5$$

and

$$\frac{2}{\sqrt{3}} \frac{\partial f}{\partial \theta} = 2 \sin^2 \varphi X_3 + \sin 2\varphi X_4,$$

where

$$X_3 = \cos 2\theta e_1 - \sin 2\theta e_4, \qquad X_4 = - \sin \theta e_2 + \cos \theta e_3.$$

Thus, an orthonormal frame $\{\eta, \zeta\}$ of $T_f^\perp \mathbb{S}_{1/3}^2$ is given by

$$\eta = - \cos \varphi X_3 + \sin \varphi X_4, \qquad 2\zeta = (1 + \cos^2 \varphi) X_1 - \sin 2\varphi X_2 - \sqrt{3} \sin^2 \varphi e_5.$$

Identifying $\mathbb{R}^4$ with the tangent space of $\mathbb{S}(e_5, 1) \subset \mathbb{R}^5$ at the origin and viewing f as a map into $\mathbb{S}(e_5, 1)$, the pair (g, h) of conjugate minimal surfaces associated to the stereographic projection of f onto $\mathbb{R}^4$ is

$$g = 2e_5 + 2 \frac{(f - 2e_5)^N}{\|(f - 2e_5)^N\|^2} \qquad \text{and} \qquad h = 2 \frac{\hat{J}(f - 2e_5)^N}{\|(f - 2e_5)^N\|^2},$$

where $(f - 2e_5)^N$ denotes the component of the position vector $f - 2e_5$ in the normal bundle of f (as a map into $\mathbb{S}(e_5, 1)$). Using that $(f - e_5)^N = 0$, we obtain

$$\frac{1}{2} g = e_5 - \frac{\langle \eta, e_5 \rangle \eta + \langle \zeta, e_5 \rangle \zeta}{\langle \eta, e_5 \rangle^2 + \langle \zeta, e_5 \rangle^2} = e_5 - \frac{\zeta}{\langle \zeta, e_5 \rangle} \qquad \text{and} \qquad \frac{1}{2} h = - \frac{\eta}{\langle \zeta, e_5 \rangle}.$$

Therefore, we have the conjugate minimal surfaces

$$g = \frac{2}{\sqrt{3} \sin^2 \varphi} \left((1 + \cos^2 \varphi) X_1 - 2 \sin \varphi \cos \varphi X_2 \right) \quad \text{and} \quad h = \frac{4}{\sqrt{3} \sin^2 \varphi} \left(\cos \varphi X_3 - \sin \varphi X_4 \right),$$

which induce the complete metric

$$ds^2 = \frac{4(1 + 3 \cos^2 \varphi)}{\sin^6 \varphi} (\sin^2 \varphi d^2\theta + d^2\varphi).$$

Amazingly enough, in this case we have a compact superconformal surface generated as in (2) by a complete unbounded minimal surface.

References

[1] F. Burstall, D. Ferus, K. Leschke, F. Pedit, U. Pinkall, Conformal Geometry of Surfaces in the 4-Sphere and Quaternions, Lecture Notes in Mathematics vol 1772, Springer-Verlag, 2002.

[2] R. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere.* J. Diff. Geom. **17** (1982), 455–473.

[3] I. Castro, *Lagrangian surfaces with circular ellipse of curvature in complex space forms.* Math. Proc. Cambridge Phil. Soc. **136** (2004), 239–245.

[4] I. Guadalupe and L. Rodríguez, *Normal curvature of surfaces in space forms.* Pacific J. Math. **106** (1983), 95–103.

[5] U. Hertrich-Jeromin, *Introduction to Möbius differential geometry*, London Math. Lect. Notes Series, vol. 300, Cambridge Univ. Press, Cambridge, 2003.

[6] D. Hoffman and R. Osserman, *The geometry of the generalized Gauss map.* Mem. Amer. Math. Soc. **28** (1980), no. 236.

[7] T. Friedrich, *On superminimal surfaces.* Arch. Math. **33** (1997), 41–56.

[8] B. Lawson Jr, Lectures on minimal submanifolds. Mathematics Lecture Series, vol. 9, Publish or Perish Inc., Wilmington, Del., 1980.

[9] X. Ma, *Isothermic and S-Willmore surfaces as solutions to a problem of Blaschke.* Results Math. **48** (2005), 301–309.

[10] X. Ma, *Adjoint Transform of Willmore Surfaces in n-sphere.* Manuscripta Math. **120** (2006), 163–179.

[11] X. Ma, *Willmore surfaces of constant Moebius curvature.* Preprint (available at arXiv math.DG/0609057).

[12] C. Moore and E. Wilson, *A general theory of surfaces.* J. Nat. Acad Proc. **2** (1916), 273–278.

[13] C. Moore and E. Wilson, *Differential geometry of two-dimensional surfaces in hyperspaces.* Proc. of the Academy of Arts and Sciences, **52** (1916), 267–368.

[14] B. Rouxel, *Harmonic spheres of a submanifold in Euclidean space.* Proc. of the 3rd Congress of Geometry, Thessaloniki (1991), 357–364 .

[15] P. Wintgen, *Sur l'inegalité de Chen-Willmore.* C. R. Acad. Sc. Paris T., **288** (1979), Serie A, 993–995.

IMPA – Estrada Dona Castorina, 110
22460-320 – Rio de Janeiro – Brazil
E-mail: marcos@impa.br

Universidade Federal de São Carlos
13565-905 – São Carlos – Brazil
E-mail: tojeiro@dm.ufscar.br