Killing graphs with prescribed mean curvature and Riemannian submersions

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Abstract

It is proved the existence and uniqueness of graphs with prescribed mean curvature in Riemannian submersions fibered by flow lines of a vertical Killing vector field.

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1 Introduction

Recent papers devoted to the study of CMC surfaces in certain homogeneous three-manifolds are based in the description of these ambient spaces as Riemannian submersions over constant curvature model surfaces. For instance, this is the case of [1], [3], [7] and [8]. In particular, in [3] the authors obtained CMC graphs in the Heisenberg space regarding it as a submersion over $\mathbb{R}^2$ fibered by geodesic flow lines of a Killing vector field. The goal in these works is to extend classical results about CMC surfaces in Euclidean space as well as more recent results in nonflat space forms to a more general setting.

One of the main issues in developing a theory for CMC hypersurfaces in general Riemannian ambients is the existence of examples. Methods which rely mainly on geometric constructions may fail if the ambient space lacks appropriate symmetries or structures. However, the problem may be solvable once it is reformulated in analytical terms as the existence of CMC graphs

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for a suitable notion of graph. This is the case of Riemannian manifolds carrying a Killing vector field where the natural notion of Killing graph has been defined under additional assumptions.

The Dirichlet problem for prescribed mean curvature Killing graphs in ambient spaces endowed with a Killing field with integrable orthogonal distribution was first solved for CMC surfaces in [6]. Then, it was extended in [5] to hypersurfaces with prescribed mean curvature function. Under the integrability assumption, the ambient manifold has a warped product structure with one of the factors giving rise to a totally geodesic hypersurface foliation.

In this paper, we consider a generalization of [3] and [5] to Riemannian submersion $\pi\colon \bar{M}^{n+1} \to M^n$ whose vertical fibers are given by flow lines of a Killing field. Thus, the normal distribution to the Killing field may fail to be integrable. Our aim is to show that a natural setting of the Dirichlet problem for Killing graphs (defined in Section 2) with prescribed mean curvature function in this context is to consider these as leaves transversal to a solid cylinder of the flow lines that project on a compact domain on the base of the submersion. Using this approach, we give a unified proof of known and completely new existence results in a wide range of ambient Riemannian manifolds. Among the ambients for which this paper applies, we should mention higher-dimensional Heisenberg spaces and odd-dimensional spheres submersed in complex projective spaces.

The existence part of our result is proved using the continuity method for quasilinear elliptic PDE. In order to obtain apriori estimates essential to this method we use Killing cylinders as barriers. Given a domain Ω in M with compact closure and boundary Γ, the Killing cylinders over Γ and $\bar{\Omega}$ are, respectively, the subsets $K = \pi^{-1}(\Gamma)$ and $M_0 = \pi^{-1}(\bar{\Omega})$. We denote by H_{cyl} the inward mean curvature of K and by $\mathrm{Ric}_{\bar{M}}$ the Ricci tensor of $\bar{M}$.

With the above notations we have the following result.

Theorem 1. *Let $\Omega \subset M$ be a domain with compact closure and $C^{2,\alpha}$ boundary. Suppose that $H_{\mathrm{cyl}} > 0$ and $\inf_{\bar{M}} Ric_{\bar{M}} \geq -n \inf_\Gamma H^2_{\mathrm{cyl}}$. Let $H \in C^\alpha(\bar{\Omega})$ and $\phi \in C^{2,\alpha}(\Gamma)$ be given functions and $\iota\colon \bar{\Omega} \to M_0 \subset \bar{M}$ be a $C^{2,\alpha}$ immersion transversal to the vertical fibers such that $\pi \circ \iota = id|_{\bar{\Omega}}$. If*

$$\sup_\Omega |H| \leq \inf_\Gamma H_{\mathrm{cyl}},$$

then there exists a unique function $u \in C^{2,\alpha}(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose Killing graph Σ has mean curvature H.

The hypothesis on the existence of an immersion ι is used simultaneously to introduce a set of coordinates well suited to the problem and to define properly the notion of Killing graph. In terms of these coordinates, it may be rendered evident that the ambient metric is stationary. Moreover, $\iota(\bar{\Omega})$ is used as barrier to producing an initial minimal graph by the direct method in Calculus of Variations. In higher-dimensional Heisenberg spaces there exists a minimal leaf transverse to the flow lines of the vertical vector field. Thus, in this particular case there is no need of the hypothesis. By contrast, if we consider the example of odd-dimensional spheres submersed in the complex projective spaces, it is not guaranteed that always exist such minimal graphs with respect to the Hopf fibers.

We remark that submersions with totally geodesic fibers constitute an important example where we may construct initial Killing graphs. In fact, if we also assume that the Killing cylinder M_0 over $\bar{\Omega}$ is geodesically complete, then geodesic cones with boundary in K and vertex at the mean convex side of K may be taken, after smoothing around the vertex, as initial Killing graphs. Thus, we may rule out the hypothesis in this case.

This paper is organized as follows. In Section 2, we fix notation and made precise the notion of Killing graph. We deduce the mean curvature equation and define adapted and basic reference frames crucial in the subsequent analysis. In Section 3, we present some basic geometry of Killing cylinders. In Sections 4 and 5 we construct analytical barriers to obtain height and boundary gradient estimates. Section 6 is devoted to the proof of interior gradient estimates based in the technique of normal perturbation of the graph due to Korevaar [13]. The continuity method and the existence of the minimal initial solution are presented in the final section.

2 Killing graphs

Let $\pi\colon \bar{M}^{n+1} \to M^n$ be a Riemannian submersion such that the leaves of the vertical foliation are the trajectories of a nonsingular Killing vector field denoted by $Y \in \mathfrak{X}(\bar{M})$. Let $\Omega \subset M$ be a $C^{2,\alpha}$ domain with compact closure. We assume that the integral curves of Y in

$$M_0 := \pi^{-1}(\bar{\Omega})$$

are complete lines. Since the hypersurfaces we work with are graphs over $\bar{\Omega}$ along the integral curves, when these curves are circles we may pass to the universal cover of M_0 without loss of generality.

Let $\iota\colon \bar{\Omega} \to \bar{M}$ be an immersion satisfying $\pi \circ \iota = id_{\bar{\Omega}}$ such that the hypersurface $\Sigma_0 = \iota(\bar{\Omega})$ is transversal to the flow lines. The initial values for the flow $\Psi\colon \mathbb{R} \times \Sigma_0 \to M_0$ of Y are taken at Σ_0, i.e., Σ_0 corresponds to the level hypersurface $s = 0$ for the flow parameter s. Set $\Psi_s = \Psi(s, \cdot)$. Then, the level hypersurfaces $\Sigma_s = \Psi_s(\Sigma_0)$ constitute a foliation of M_0 by isometric hypersurfaces.

Fix a local reference frame $\mathsf{v}_1, \ldots, \mathsf{v}_n$ on $\bar{\Omega}$ and set

$$\sigma_{ij} = \langle \mathsf{v}_i, \mathsf{v}_j \rangle.$$

Let $\bar{\mathsf{v}}_1, \ldots, \bar{\mathsf{v}}_n$ be the corresponding local frame on Σ_0, i.e., $\bar{\mathsf{v}}_i(p) = \iota_* \mathsf{v}_i(x)$ if $x \in \bar{\Omega}$ and $p = \iota(x)$. By means of the flux Ψ we define a local frame at $q = \Psi_s(p)$ in $\bar{M}$ by

$$\partial_s(q) = \frac{\mathrm{d}}{\mathrm{d}s}\Psi(s,p) = Y(\Psi(s,p)) = \Psi_*(s,p)\partial_s(p)$$

and

$$\bar{\mathsf{v}}_i(q) = \big(\Psi_s \circ \iota\big)_* \mathsf{v}_i(x).$$

Let $D_1, \ldots, D_n$ in $\bar{M}$ denote the basic vector fields π-related to $\mathsf{v}_1, \ldots, \mathsf{v}_n$. If $q = \Psi(s,p)$ for $p \in \Sigma_0$, then $\pi(q) = \pi \circ \Psi(s,p) = \pi(p)$. Therefore,

$$D_i(q) = \Psi_*(s,p) D_i(p)$$

since $\Psi_*(s,p) D_i(q)$ is horizontal and

$$\pi_*(q)\Psi_*(s,p)D_i(p) = (\pi \circ \Psi)_*(s,p)D_i(p) = \pi_*(p)D_i(p).$$

That π is a Riemannian submersion yields

$$\langle D_i, D_j \rangle = \langle \mathsf{v}_i, \mathsf{v}_j \rangle = \sigma_{ij}.$$

Setting

$$D_0 := f^{1/2}\,\partial_s,$$

we complete a local reference frame $D_0, D_1, \ldots, D_n$ on $\bar{M}$ where $f := 1/|Y|^2$ does not depend on s since Y is a Killing field..

We extend the frame $\bar{\mathsf{v}}_1, \ldots, \bar{\mathsf{v}}_n$ adapted to the leaves Σ_s to a frame $\bar{\nabla} s$, $\bar{\mathsf{v}}_1, \ldots, \bar{\mathsf{v}}_n$ in $\bar{M}$ by adding the gradient vector field $\bar{\nabla} s$ of the function s. Using

$$\pi_*(q)\bar{\mathsf{v}}_i = \pi_*(p)\iota_* \mathsf{v}_i(x) = \mathsf{v}_i(x) = \pi_*(q) D_i$$

and

$$1 = \partial_s s = \langle \bar{\nabla} s, \partial_s \rangle = f^{-1/2} \langle \bar{\nabla} s, D_0 \rangle,$$

we have that the two frames considered on $\bar{M}$ are related by

$$\begin{cases} \bar{\nabla} s = f^{1/2}\, D_0 + \sigma^{ji} D_j(s)\, D_i \\ \bar{\mathsf{v}}_i = \delta_i\, D_0 + D_i. \end{cases}$$

The functions δ_i are independent of s since

$$\delta_i = \langle \bar{\mathsf{v}}_i(q), D_0(q) \rangle = \langle \Psi_{s*}(p) \bar{\mathsf{v}}_i(p), \Psi_{s*}(p) D_0(p) \rangle = \langle \bar{\mathsf{v}}_i(p), D_0(p) \rangle.$$

Thus, from

$$0 = \bar{\mathsf{v}}_j(s) = \langle \bar{\nabla} s, \bar{\mathsf{v}}_j \rangle = f^{1/2} \delta_j + D_j(s)$$

we conclude that the functions $D_j(s)$ are also independent of s.

The *Killing graph* $\Sigma = \Sigma_u$ of a function $u \in C^2(\bar{\Omega})$ is the hypersurface

$$\Sigma_u = \{\Psi(u(p), p) : p \in \Sigma_0\},$$

where u is seen as a function on Σ_0 by taking $u(p) = u(x)$ when $\pi(p) = x$. Since Σ can also be considered as given by the immersion

$$\iota_u \colon\ x \in \bar{\Omega} \mapsto \Psi(u(x), \iota(x)),$$

its tangent bundle is spanned by the vector fields

$$(\iota_u)_* \mathsf{v}_i = \mathsf{v}_i(u)\, \Psi_s + (\Psi \circ \iota)_* \mathsf{v}_i = \mathsf{v}_i(u)\, \partial_s + \bar{\mathsf{v}}_i = f^{-1/2} \mathsf{v}_i(u) D_0 + \bar{\mathsf{v}}_i. \tag{1}$$

We may regard u as a function in M_0 by means of the extension

$$u(q) = u(x) \quad \text{if} \quad \pi(q) = x. \tag{2}$$

Thus $D_0(u) = f^{1/2} \partial_s u = 0$, and hence

$$D_i(u) = \bar{\mathsf{v}}_i(u) - \delta_i D_0(u) = \bar{\mathsf{v}}_i(u) = \mathsf{v}_i(u).$$

Therefore, we have using (1) that

$$(\iota_u)_* \mathsf{v}_i = (f^{-1/2} D_i(u) + \delta_i) D_0 + D_i.$$

It follows easily that a unit normal vector field to Σ pointing upwards is

$$N = \frac{1}{W}(f^{1/2}D_0 - \hat{u}^j D_j), \tag{3}$$

where

$$\hat{u}^j := \sigma^{ij} D_i(u-s) \tag{4}$$

and

$$W^2 := f + \sigma_{ij}\hat{u}^i\hat{u}^j = f + \hat{u}^i\hat{u}_i$$

for $\hat{u}_i := \sigma_{ij}\hat{u}^j$. Notice that $\hat{u}^j$ and W can also be seen as functions on M since they are independent of s.

2.1 The mean curvature equation

To compute the mean curvature of Σ assume for simplicity that the tangent frame $\mathsf{v}_1, \ldots, \mathsf{v}_n$ is orthonormal at $x \in \Omega$. Hence, the basic frame $D_0, D_1, \ldots, D_n$ is orthonormal at points of $\pi^{-1}(x)$. Thus,

$$\langle \bar{\nabla}_{D_0} N, D_0 \rangle = \frac{f^{1/2}}{W}\langle \bar{\nabla}_{D_0} D_0, D_0\rangle - \frac{\hat{u}^j}{W}\langle \bar{\nabla}_{D_0} D_j, D_0\rangle = \frac{1}{W}\langle \bar{\nabla}_{D_0} D_0, \hat{u}^j D_j\rangle,$$

where $\bar{\nabla}$ denotes the Riemannian connection on $\bar{M}$. We consider on M the vector field

$$Du := \hat{u}^j \mathsf{v}_j = \sigma^{ij} D_i(u-s)\mathsf{v}_j.$$

Since $\bar{\nabla}_{D_0} D_0$ is a horizontal vector field and $\pi_*(\hat{u}^j D_j) = \hat{u}^j \mathsf{v}_j$, we obtain

$$\langle \bar{\nabla}_{D_0} N, D_0\rangle = \frac{1}{W}\langle \pi_* \bar{\nabla}_{D_0} D_0, Du\rangle.$$

By the well-known O'Neill submersion formula [9], we have

$$\bar{\nabla}_{D_k} D_j = \big(\bar{\nabla}_{D_k} D_j\big)^h + \frac{1}{2}[D_k, D_j]^v. \tag{5}$$

Thus, we obtain for $k \geq 1$ that

$$\langle \bar{\nabla}_{D_k} N, D_k\rangle = -\frac{f^{1/2}}{W}\langle \bar{\nabla}_{D_k} D_k, D_0\rangle - \langle \bar{\nabla}_{D_k}\Big(\frac{\hat{u}^j}{W} D_j\Big), D_k\rangle = -\langle \nabla_{\mathsf{v}_k} \frac{Du}{W}, \mathsf{v}_k\rangle$$

where ∇ denotes the Riemannian connection on M. We conclude that

$$nH = -\mathrm{div}_{\bar{M}} N = \mathrm{div}_M \frac{Du}{W} - \frac{1}{W}\langle \pi_* \bar{\nabla}_{D_0} D_0, Du\rangle. \tag{6}$$

Denote the covariant derivative in M of $Du = \hat{u}^j \mathsf{v}_j$ by

$$\nabla_{\mathsf{v}_k} Du := \hat{u}^j{}_{;k}\, \mathsf{v}_j$$

and set $\hat{u}_{k;i} := \sigma_{jk}\hat{u}^j{}_{;i}$. Computing at any point the divergence in (6) gives

$$\begin{aligned}
\mathrm{div}_M \frac{Du}{W} &= \sigma^{ik}\langle \nabla_{\mathsf{v}_i}\frac{Du}{W}, \mathsf{v}_k\rangle \\
&= \frac{\sigma^{ik}}{W^2}(W\langle \nabla_{\mathsf{v}_i} Du, \mathsf{v}_k\rangle - \mathsf{v}_i(W)\langle \hat{u}^j\,\mathsf{v}_j, \mathsf{v}_k\rangle) \\
&= \frac{\sigma^{ik}}{W^3}(W^2\langle \hat{u}^j{}_{;i}\mathsf{v}_j, \mathsf{v}_k\rangle - \frac{1}{2}\mathsf{v}_i(f + \hat{u}^l\hat{u}_l)\hat{u}^j\sigma_{jk}) \\
&= \frac{\sigma^{ik}}{W^3}(W^2\sigma_{jk}\hat{u}^j{}_{;i} - \frac{1}{2}(\mathsf{v}_i(f) + 2\hat{u}^l\hat{u}_{l;i})\hat{u}^j\sigma_{jk}) \\
&= \frac{1}{W^3}(W^2\sigma^{ik} - \hat{u}^i\hat{u}^k)\,\hat{u}_{k;i} - \frac{1}{2W^3}\,\mathsf{v}_i(f)\,\hat{u}^i.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathsf{v}_i(f) &= -f^2\mathsf{v}_i\langle Y, Y\rangle = -f^2(\delta_i D_0\langle Y, Y\rangle + D_i\langle Y, Y\rangle) \\
&= -2f^2\langle \bar{\nabla}_{D_i}Y, Y\rangle = 2f^2\langle \bar{\nabla}_Y Y, D_i\rangle = 2f\langle \bar{\nabla}_{D_0}D_0, D_i\rangle \\
&= 2f\langle \pi_*\bar{\nabla}_{D_0}D_0, \mathsf{v}_i\rangle.
\end{aligned}$$

Thus, the mean curvature equation (6) becomes

$$A^{ik}\hat{u}_{k;i} - (f + W^2)\langle \pi_*\bar{\nabla}_{D_0}D_0, Du\rangle = B, \tag{7}$$

where

$$A^{ik} := W^2\sigma^{ik} - \hat{u}^i\hat{u}^k \quad \text{and} \quad B := nH\,W^3.$$

We define the operator

$$\mathcal{Q}[u] = \frac{1}{W^3}(A^{ij}\hat{u}_{j;i} - (f + W^2)\langle \pi_*\bar{\nabla}_{D_0}D_0, Du\rangle).$$

Therefore, we have shown that Σ is a hypersurface with prescribed mean curvature function $H(x)$ and boundary condition ϕ if u is a solution to the Dirichlet problem

$$\begin{cases} \mathcal{Q}[u] = nH \\ u|_\Gamma = \phi \end{cases} \tag{8}$$

where $\Gamma = \partial\Omega$. The boundary of Σ is the Killing graph over Γ of ϕ.

2.2 A commutation formula

In this subsection, we give a commutation formula for second covariant derivatives that allow to conclude the ellipticity of the quasilinear operator defined in the proceeding one.

Since $Du = \pi_* \bar{\nabla}(u-s)$ from (4) we obtain using (5) that

$$\begin{aligned}
\nabla_{\mathsf{v}_k} Du &= \nabla_{\mathsf{v}_k} \pi_* \bar{\nabla}(u-s) \\
&= \pi_*(\bar{\nabla}_{D_k} \bar{\nabla}(u-s) - \bar{\nabla}_{D_k} \langle \bar{\nabla}(u-s), D_0 \rangle D_0) \\
&= \sigma^{il} \pi_*(\langle \bar{\nabla}_{D_k} \bar{\nabla}(u-s), D_l \rangle \, D_i - \langle \bar{\nabla}(u-s), D_0 \rangle \langle \bar{\nabla}_{D_k} D_0, D_l \rangle \, D_i) \\
&= \sigma^{il} (\bar{\nabla}^2_{D_k, D_l} (u-s) + (D_0(u) - D_0(s)) \langle \bar{\nabla}_{D_k} D_l, D_0 \rangle) \, \pi_* D_i \\
&= \sigma^{il} \big(\bar{\nabla}^2_{D_k, D_l} (u-s) - \frac{1}{2} f^{1/2} \langle [D_k, D_l], D_0 \rangle \big) \, \mathsf{v}_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{u}_{j;k} &= \sigma_{ji} \hat{u}^i{}_{;k} = \sigma_{ji} \sigma^{il} \big(\bar{\nabla}^2_{D_k, D_l} (u-s) - \frac{1}{2} f^{1/2} \langle [D_k, D_l], D_0 \rangle \big) \\
&= \bar{\nabla}^2_{D_k, D_j} (u-s) + \frac{1}{2} \gamma_{jk} \qquad (9)
\end{aligned}$$

where $\gamma_{kj} := f^{1/2} \langle [D_k, D_j], D_0 \rangle$ is skew-symmetric. Since the Hessian is symmetric we conclude that

$$\hat{u}_{j;k} - \hat{u}_{k;j} = \gamma_{jk}. \qquad (10)$$

Under the convention for u established in (2) we use the standard notation

$$u_i = D_i(u), \quad u^i = \sigma^{ij} u_j \quad \text{and} \quad u_{i;j} = \langle \bar{\nabla}_{D_i} \bar{\nabla} u, D_j \rangle.$$

Then the matrix $(\hat{u}_{i;j})$ is related with the Hessian matrices $(u_{i;j})$ and $(s_{i;j})$ by

$$\hat{u}_{i;j} = \langle \bar{\nabla}_{D_j} \bar{\nabla} u, D_i \rangle - \langle \bar{\nabla}_{D_j} \bar{\nabla} s, D_i \rangle + \frac{1}{2} \gamma_{ij} = u_{i;j} - s_{i;j} + \frac{1}{2} \gamma_{ij}. \qquad (11)$$

Hence, the principal part of the mean curvature equation (8) is given by the matrix (A^{ij}). This matrix is positive-definite. Indeed, we have that

$$f|\xi|^2 \leq A^{ij} \xi_i \xi_j \leq W^2 |\xi|^2. \qquad (12)$$

3 Killing cylinders

The *Killing cylinder* over Γ is the hypersurface $K = \pi^{-1}(\Gamma)$. Thus

$$K = \{\Psi(s, \iota(x)) : s \in \mathbb{R},\, x \in \Gamma\}$$

is ruled by the flow lines of Y through $\iota(\Gamma) \subset \Sigma_0$.

We denote by $\bar\eta$ the inward pointing unit vector field normal to K. Clearly, $\bar\eta$ is a basic vector field and $\pi_* \bar\eta = \eta$ is the unit normal vector field to Γ in M pointing inward. We work with a tangent frame satisfying that $\mathsf{v}_1 = \eta$ and $\mathsf{v}_2, \ldots, \mathsf{v}_n$ are orthogonal to v_1. In particular, their horizontal lifts D_i verify along K that $D_1 = \bar\eta$ and D_j, $2 \le j \le n$, is tangent to K. Set

$$f\langle \bar\nabla_Y Y, \bar\eta\rangle = \langle \bar\nabla_{D_0} D_0, \bar\eta\rangle = \kappa,$$

where κ can be seen as a function on $\bar\Omega$. In fact,

$$f\langle \bar\nabla_Y Y, \bar\eta\rangle = -f\langle \bar\nabla_{\bar\eta} Y, Y\rangle = \frac{1}{2f}\bar\eta(f).$$

Thus,

$$Y(\kappa) = \frac{1}{2f} Y(\bar\eta(f)) = \frac{1}{2f}[Y, \bar\eta](f) = 0$$

since $[Y, \bar\eta]^h = 0$ because $\pi_*[Y, \bar\eta]^h = [\pi_* Y, \eta] = [0, \eta]$. Hence,

$$n\, H_{\rm cyl} = \sum_{i,j} \sigma^{ij}\langle \bar\nabla_{D_i} D_j, \bar\eta\rangle + \kappa = \sum_{i,j} \sigma^{ij}\langle \nabla_{\mathsf{v}_i} \mathsf{v}_j, \eta\rangle + \kappa = (n-1)H_\Gamma + \kappa$$

where H_Γ is the mean curvature of Γ in M.

In the sequel, we deduce some useful properties of the distance function $d = \mathrm{dist}(\,\cdot\,, K)$ from K. We denote by Γ_ϵ and K_ϵ the level sets $d = \epsilon$ in M and $\bar M$, respectively. Thus, Γ_ϵ and K_ϵ are equidistant from Γ and K, respectively. It is immediate that K_ϵ is a Killing cylinder over Γ_ϵ. Since Γ is assumed to be $C^{2,\alpha}$, the function d is also $C^{2,\alpha}$ at points of $\Psi(\mathbb{R} \times \Omega_\epsilon)$, where $\Omega_\epsilon \subset \Omega$ is a tubular ϵ-neighborhood of Γ in M for small $\epsilon > 0$.

Given $q \in \Psi(\mathbb{R} \times \Omega_\epsilon)$ we write $q = \exp_p d\,\eta$ for some $p \in K$. Hence,

$$|D_1| = |\bar\nabla d| = 1.$$

It follows that

$$0 = \frac{1}{2} D_i \langle \bar\nabla d, \bar\nabla d\rangle = \langle \bar\nabla_{D_i} \bar\nabla d, \sigma^{jk} D_j(d) D_k\rangle = d^k d_{i;k}.$$

We also have

$$\langle \bar\nabla_{D_0}\bar\nabla d, D_0\rangle = -\langle \bar\nabla_{D_0} D_0, \bar\nabla d\rangle := -\kappa_\epsilon$$

and

$$\langle \bar\nabla_{D_1}\bar\nabla d, D_1\rangle = \frac{1}{2} D_1|\bar\nabla d|^2 = 0.$$

Therefore,

$$\Delta d|_{d=\epsilon} = -\kappa_\epsilon \sigma^{ij}\langle \bar\nabla_{D_i}\bar\nabla d, D_j\rangle = -\kappa_\epsilon - \sigma^{ij}\, b^\epsilon_{ij} = -nH^\epsilon_{\mathrm{cyl}}, \tag{13}$$

where b^ϵ_{ij} are the components of the Weingarten operator A_ϵ and $H^\epsilon_{\mathrm{cyl}}$ the mean curvature of K_ϵ.

Fact 2. All of the above calculations on the distance function d remain valid if we replace Ω_ϵ by the larger subset Ω_0 in Ω consisting of the points which can be joined to Γ by a *unique* minimizing geodesic. It was shown in [12] that in this set d has the same regularity as Γ.

In this paper the ambient Ricci tensor in direction v is defined by

$$\mathrm{Ric}_{\bar M}(v) = \sum_{i=1}^{n} \langle \bar R(e_i, v)v, e_i\rangle,$$

where $\bar R$ is the curvature tensor in $\bar M$ and $e_1, \ldots, e_n, v$ is an orthonormal basis. We follow [11] or [15] and use the result in Fact 2 for the proof of the following result.

Lemma 3. *Assume that the Ricci curvature satisfies* $\mathrm{Ric}_{\bar M} \geq -n \inf_\Gamma H^2_{\mathrm{cyl}}$. *Let* $y_0 \in \Gamma$ *be the closest point to a given point* $x_0 \in \Gamma_\epsilon \subset \Omega_0$. *If* $H_{\mathrm{cyl}} > 0$, *then, we have*

$$H_{\mathrm{cyl}}(\epsilon)|_{x_0} \geq H_{\mathrm{cyl}}|_{y_0}.$$

Proof: At $d = \epsilon$ and since D_1 is the unit speed of a geodesic, on one hand we have that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle A_\epsilon D_i, D_j\rangle &= D_1\langle \bar\nabla_{D_i} D_1, D_j\rangle = \langle \bar\nabla_{D_1}\bar\nabla_{D_i} D_1, D_j\rangle + \langle \bar\nabla_{D_i} D_1, \bar\nabla_{D_1} D_j\rangle \\ &= -\bar R_\epsilon(D_i, D_j) - \langle \nabla_{[D_i, D_1]} D_1, D_j\rangle + \langle \bar\nabla_{D_i} D_1, \bar\nabla_{D_1} D_j\rangle \\ &= -\bar R_\epsilon(D_i, D_j) - \langle \nabla_{D_j} D_1, [D_i, D_1]\rangle + \langle \bar\nabla_{D_i} D_1, \bar\nabla_{D_1} D_j\rangle \end{aligned}$$

where $\bar{R}_\epsilon = \langle R(\cdot, D_1)D_1, \cdot\rangle|_{d=\epsilon}$. On the other hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle A_\epsilon D_i, D_j\rangle &= \langle \bar{\nabla}_{D_1} A_\epsilon D_i, D_j\rangle + \langle A_\epsilon D_i, \bar{\nabla}_{D_1} D_j\rangle \\ &= \langle (\bar{\nabla}_{D_1} A_\epsilon) D_i, D_j\rangle + \langle A_\epsilon D_j, \bar{\nabla}_{D_1} D_i\rangle - \langle \bar{\nabla}_{D_i} D_1, \bar{\nabla}_{D_1} D_j\rangle \\ &= \langle A'_\epsilon D_i, D_j\rangle - \langle \bar{\nabla}_{D_i} D_1, \bar{\nabla}_{D_1} D_j\rangle - \langle \bar{\nabla}_{D_j} D_1, \bar{\nabla}_{D_1} D_i\rangle.\end{aligned}$$

Adding the above equations we have the Ricatti equation

$$A'_\epsilon - A_\epsilon^2 - \bar{R}_\epsilon = 0,$$

Taking traces, we obtain

$$n\frac{\mathrm{d}}{\mathrm{d}\epsilon} H^\epsilon_{\mathrm{cyl}} = D_1(\mathrm{tr}\, A_\epsilon) = \mathrm{tr}\bar{\nabla}_{D_1} A_\epsilon = \mathrm{tr}\big(A_\epsilon^2 + \bar{R}_\epsilon\big) \geq n(H^\epsilon_{\mathrm{cyl}})^2 + \mathrm{Ric}_{\bar{M}}(D_1).$$

From our hypothesis on $\mathrm{Ric}_{\bar{M}}$ we have that $z(d) = H_{\mathrm{cyl}}(d) - H_{\mathrm{cyl}}(y_0)$ satisfies

$$z'(d) \geq H^2_{\mathrm{cyl}}(d) - \inf_\Gamma H^2_{\mathrm{cyl}} \geq H^2_{\mathrm{cyl}}(d) - H^2_{\mathrm{cyl}}(y_0) = (H_{\mathrm{cyl}}(d) + H_{\mathrm{cyl}}(y_0))z(d).$$

Since $H_{\mathrm{cyl}} > 0$, it follows that $z'(d) \geq c\, z(d)$ in some interval $d \in [0, d_0 > 0]$ for a constant $c > 0$. We obtain that $H^\epsilon_{\mathrm{cyl}}$ does not decrease with increasing d. This concludes the proof of the lemma.

4 The C^0 estimate

In this section, we obtain apriori C^0 estimates for solutions of the Dirichlet problem (8).

We construct barriers for u in (8) on Ω_0 (see Fact 2) by

$$\varphi(x) = \sup_\Gamma \phi + h(d(x))$$

where $d = \mathrm{dist}(\,\cdot\,, \Gamma)$ is regarded as the distance from Γ on M and the function h will be chosen later. We work with the frame $\mathsf{v}_1 := \nabla d, \mathsf{v}_2, \ldots, \mathsf{v}_n$ and the corresponding frame $D_0, D_1, \ldots, D_n$. Thus,

$$D_i(d) = \langle \bar{\nabla} d, D_i\rangle = \langle D_1, D_i\rangle = \langle \mathsf{v}_1, \mathsf{v}_i\rangle = \langle \nabla d, \mathsf{v}_i\rangle = \mathsf{v}_i(d).$$

We have,

$$\varphi_i = h' d_i \quad \text{and} \quad \varphi_{i;j} = h'' d_i d_j + h' d_{i;j}.$$

We obtain from (4) and (11) that

$$\hat{\varphi}^j = \sigma^{ij}(\varphi_i - s_i) = \sigma^{ij}(h' d_i - s_i) = h' d^i - s^i$$

and

$$\hat{\varphi}_{i;j} = \varphi_{i;j} - s_{i;j} + \frac{1}{2}\,\gamma_{ij} = h'' d_i d_j + h' d_{i;j} - s_{i;j} + \frac{1}{2}\,\gamma_{ij}.$$

Since γ_{ij} is skew-symmetric, we have from (3) and (3) that

$$\begin{aligned} A^{ij}\hat{\varphi}_{j;i} &= W^2\,\hat{\varphi}^j_{;i} - \hat{\varphi}^i\hat{\varphi}^j\hat{\varphi}_{j;i} \\ &= W^2(h'' + h' d^i_{;i} - s^i_{;i}) - (h' d^i - s^i)\big(h' d^j - s^j)(h'' d_i d_j + h' d_{j;i} - s_{j;i}) \\ &= (W^2 - h'^2 + 2h'\langle\bar{\nabla} d, \bar{\nabla} s\rangle - \langle\bar{\nabla} d, \bar{\nabla} s\rangle^2)\,h'' + W^2 h' d^i_{;i} + R \end{aligned}$$

where

$$R := -h' s^i s^j d_{i;j} - W^2 s^i{}_{;i} + (h'^2 d^i d^j - h'(d^i s^j + d^j s^i) + s^i s^j) s_{i;j}. \tag{14}$$

Using that

$$W^2 = f + \hat{\varphi}^k\hat{\varphi}_k = f + h'^2 - 2h'\langle\bar{\nabla} d, \bar{\nabla} s\rangle + |\bar{\nabla} s|^2, \tag{15}$$

we conclude that

$$A^{ij}\hat{\varphi}_{j;i} = (f + |\bar{\nabla} s|^2 - \langle\bar{\nabla} d, \bar{\nabla} s\rangle^2)\,h'' + W^2 h' d^i_{;i} + R \tag{16}$$

where R is a polynomial of second degree in h' and its coefficients are just functions on M.

We have from (13) that

$$d^i_{;i} = \sigma^{ij} d_{i;j} = \sigma^{ij}\langle\bar{\nabla}_{D_i}\bar{\nabla} d, D_j\rangle = -\sigma^{ij}\, b^{\epsilon}_{ij} = \kappa_\epsilon - nH^{\epsilon}_{\mathrm{cyl}}.$$

Since $D\varphi = \pi_*(h'\,\bar{\nabla} d - \bar{\nabla} s)$ from (4), we also have

$$\langle\pi_*\bar{\nabla}_{D_0} D_0, D\varphi\rangle = h'\kappa_\epsilon - \langle\bar{\nabla}_{D_0} D_0, \bar{\nabla} s\rangle. \tag{17}$$

Thus, we obtain

$$W^3\mathcal{Q}[\varphi] = (f + |\bar{\nabla} s|^2 - \langle\bar{\nabla} d, \bar{\nabla} s\rangle^2)h'' - (f\kappa_\epsilon + nW^2 H^{\epsilon}_{\mathrm{cyl}})h' + R^*$$

where

$$R^* := R + (f + W^2)\langle\bar{\nabla}_{D_0} D_0, \bar{\nabla} s\rangle.$$

We choose for (4) the test function

$$h = \frac{e^{CA}}{C}\left(1 - e^{-Cd}\right)$$

where $A > \operatorname{diam}(\bar{\Omega})$ and $C > 0$ is a constant to be chosen later. Then,

$$h' = e^{C(A-d)} \quad \text{and} \quad h'' = -Ch'.$$

Hence,

$$\mathcal{Q}[\varphi] \leq -(C + \kappa_\epsilon)\frac{fh'}{W^3} - \frac{h'}{W} nH^\epsilon_{\mathrm{cyl}} + \frac{R^*}{W^3}.$$

Observe that $f/W^2 \leq 1$. Moreover, as $C \to \infty$ we have that $1/W \to 0$ and

$$\frac{h'}{W} = \frac{h'}{(h'^2 - 2h'\langle \bar{\nabla} d, \bar{\nabla} s\rangle + |\bar{\nabla} s|^2 + f)^{1/2}} \to 1.$$

In particular, we have

$$\frac{R^*}{W^3} \to 0 \quad \text{as} \quad C \to \infty.$$

Choose $C \gg 0$ such that, in particular, $C + \kappa_\epsilon > 0$. Using $\sup_\Omega |H| \leq \inf_\Gamma H_{\mathrm{cyl}}$ and Lemma 3, we obtain

$$\mathcal{Q}[\varphi] < -n|H| \leq nH.$$

Thus, one has at points of Ω_0 that

$$\mathcal{Q}[\varphi] < \mathcal{Q}[u] = nH, \qquad \varphi|_\Gamma \geq u|_\Gamma.$$

We now prove that $\varphi \geq u$ on $\bar{\Omega}$. By contradiction, assume that there exist points for which the continuous function $u^* := u - \varphi$ satisfies $u^* > 0$. Hence $m := u^*(y) > 0$ at a maximum point $y \in \bar{\Omega}$ of u^*. Choose a minimizing geodesic γ joining y to Γ for which the distance $d = d(y, \Gamma)$ is attained. Thus, $\gamma(t) = \exp_{y_0} t\eta$, $0 \leq t \leq d$, starts from a point $y_0 \in \Gamma$ with unit speed η. Since γ is minimizing, we have $d(\gamma(t), \Gamma) = t$ and the function φ restricted to γ is differentiable with $\varphi'(\gamma(t)) = e^{C(A-t)}$. Since the maximum of u^* restricted to γ occurs at $t = d$, i.e., at the point y, one has that

$$u'(\gamma(d)) - \varphi'(\gamma(d)) = (u^*)'(\gamma(d)) \geq 0.$$

This implies that

$$\langle \nabla u(y), \gamma'(d)\rangle \geq \varphi'(\gamma(d)) = e^{C(A-d)} > 0.$$

In particular $\nabla u(y) \neq 0$, and hence the level hypersurface

$$S = \{x \in \Omega \cap B_r(y) : u(x) = u(y)\}$$

is regular for small radius r. Along S we have

$$u^*(x) + \varphi(x) = u^*(y) + \varphi(y) \geq u^*(x) + \varphi(y),$$

and since φ is an increasing function of $d(\,\cdot\,, \Gamma)$ we have $d(x, \Gamma) \geq d(y, \Gamma) = d$. From this we conclude that the points in S are at a distance at least d from Γ. Since S is C^2 it satisfies the interior sphere condition: there exists a small ball $B_\varepsilon(z)$ touching S at y contained in the side to which $\nabla u(y)$ and $\gamma'(d)$ points. Thus, the points of $B_\varepsilon(z)$ satisfy $u(x) \geq u(y)$, and hence

$$\varphi(x) + m \geq u(x) \geq u(y) = \varphi(y) + m, \quad x \in B_\varepsilon(z),$$

where in the first inequality we used the definition of m. Again because φ is an increasing function of d, we have $d(x, \Gamma) \geq d$ on $B_\varepsilon(z)$ and therefore this ball is contained in the interior of Ω far away from Γ. This allows us to extend the geodesic γ through $B_\varepsilon(z)$. We claim that the center z of the ball is contained in this extension. Otherwise, the broken line consisting of γ and of the radius in $B_\varepsilon(z)$ from z to y has length smaller than a minimizing geodesic joining z to $y_0 \in \Gamma$ (for a suitable small ε such a geodesic must cross the level hypersurface S at a point $x \neq y$ at distance to Γ greater than d). Thus, if there exists at least two distinct minimizing geodesics joining y to Γ, then the point z is contained in the extension of both geodesics after its intersection at y. Choosing ε sufficiently small, we see that this configuration is not possible (the construction we made above applies to both geodesics). This contradiction implies that the maximum point y belongs to Ω_0. However, in this case, $u^*(y) \leq 0$, a contradiction. We conclude that $u \leq \varphi$ throughout $\bar{\Omega}$ and therefore φ is a continuous super-solution for the Dirichlet problem (8).

In a similar way, we may construct lower barriers for u, that is, continuous sub-solutions for (8). It is a standard fact that the existence of these barriers implies the desired C^0 apriori estimates. Thus, we have proved the following result.

Lemma 4. *Under the assumptions of Theorem 1 there exists a constant* $C = C(\Omega, H)$ *such that*

$$|u|_0 \leq C + |\phi|_0$$

if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies* $\mathcal{Q}[u] = nH$ *and* $u|_\Gamma = \phi$.

5 Boundary gradient estimates

In this section our task is to produce apriori gradient estimates along Γ for the Dirichlet problem (8). This is accomplished by constructing local lower and upper barriers for Σ in a tubular neighborhood of Γ.

We construct barriers of the form $w + \phi$ along a tubular neighborhood Ω_ϵ of Γ as defined in Section 3. Here, $w = \psi(d(x))$ for some real function ψ to be chosen and $d = \operatorname{dist}(\,\cdot\,, \Gamma)$. Moreover, the boundary data ϕ is extended to a function in Ω_ϵ along the normal geodesics in a way we make precise later.

We denote

$$\tilde{\mathcal{Q}}[u] = \mathcal{Q}[u] - nH.$$

A simple estimate using (12) and then (17) gives

$$\begin{aligned} \tilde{\mathcal{Q}}[w+\phi] &= a^{ij}(x, \nabla w + \nabla\phi)(\hat{w}_{i;j} + \hat{\phi}_{i;j}) + b(x, \nabla w + \nabla \phi) - nH \\ &\leq a^{ij}\hat{w}_{i;j} + \frac{1}{W}|\phi|_{2,\alpha} + b - nH, \end{aligned} \tag{18}$$

where

$$a^{ij} := \frac{A^{ij}}{W^3} = \frac{1}{W}\sigma^{ij} - \frac{1}{W^3}(\hat{w}^i + \hat{\phi}^i)(\hat{w}^j + \hat{\phi}^j)\hat{w}_{i;j} \tag{19}$$

and

$$b = -\frac{f + W^2}{W^3}(\psi'\kappa_\epsilon + \langle \pi_* \bar{\nabla}_{D_0} D_0, D\phi\rangle - \langle \bar{\nabla}_{D_0} D_0, \bar{\nabla} s\rangle)$$

since $\kappa_\epsilon = \langle \bar{\nabla}_{D_0} D_0, \bar{\nabla} d\rangle$ and

$$D(w+\phi) = \sigma^{ij}(\psi' d_j + \phi_j - s_j)\mathsf{v}_j = D\phi + \pi_*(\psi'\,\bar{\nabla} d - \bar{\nabla} s).$$

From now on R_j, $j \geq 1$, denotes a polynomial of at most second degree in ψ' whose coefficients are functions in M. As in (15) and (16) we first obtain,

$$W^2 = f + \psi'^2 - 2\psi'\langle \bar{\nabla} d, \bar{\nabla} s - \bar{\nabla}\phi\rangle + |\bar{\nabla} s - \bar{\nabla}\phi|^2,$$

and then

$$W^2\hat{w}^i{}_{;i} - \hat{w}^i\hat{w}^j\hat{w}_{i;j} = (f + |\bar{\nabla}s - \bar{\nabla}\phi|^2 - \langle\bar{\nabla}d, \bar{\nabla}s - \bar{\nabla}\phi\rangle^2)\psi'' + W^2\psi' d^i_{;i} + R_1.$$

Moreover,

$$\begin{aligned}\hat{w}^i\hat{\phi}^j\hat{w}_{i;j} &= (\psi' d^i - s^i)\hat{\phi}^j(\psi'' d_i d_j + \psi' d_{i;j} - s_{i;j}) \\ &= -\psi'\psi''\langle\bar{\nabla}d, \bar{\nabla}s - \bar{\nabla}\phi\rangle + \psi''\langle\bar{\nabla}d, \bar{\nabla}s\rangle\langle\bar{\nabla}d, \bar{\nabla}s - \bar{\nabla}\phi\rangle + R_2\end{aligned}$$

and

$$\hat{\phi}^i\hat{\phi}^j\hat{w}_{i;j} = \hat{\phi}^i\hat{\phi}^j(\psi'' d_i d_j + \psi' d_{i;j} - s_{i;j}) + R_3.$$

Now define

$$\psi(d) = \mu\ln(1 + Kd)$$

for constants $\mu > 0$ and $K > 0$ to be chosen later. We have

$$\psi' = \frac{\mu K}{1 + Kd} \quad \text{and} \quad \psi'' = -\frac{1}{\mu}\psi'^2.$$

Then using $d^i_{;i} = -nH^\epsilon_{\text{cyl}} + \kappa_\epsilon$ we obtain

$$W^2\hat{w}^i{}_{;i} - \hat{w}^i\hat{w}^j\hat{w}_{i;j} = -\psi'(nH^\epsilon_{\text{cyl}} - \kappa_\epsilon)W^2 + R_4,$$

$$\hat{w}^i\hat{\phi}^j\hat{w}_{i;j} = -\psi'\psi''\langle\bar{\nabla}d, \bar{\nabla}s - \bar{\nabla}\phi\rangle + R_5$$

and

$$\hat{\phi}^i\hat{\phi}^j\hat{w}_{i;j} = R_6.$$

Since (19) gives

$$W^3 a^{ij}\hat{w}_{i;j} = W^2\hat{w}^i{}_{;i} - \hat{w}^i\hat{w}^j\hat{w}_{i;j} - (\hat{w}^i\hat{\phi}^j + \hat{w}^j\hat{\phi}^i)\hat{w}_{i;j} - \hat{\phi}^i\hat{\phi}^j\hat{w}_{i;j},$$

we now conclude from (18) that

$$W^3\tilde{\mathcal{Q}}[w+\phi] \le -\psi'(nH^\epsilon_{\text{cyl}} - \kappa_\epsilon)W^2 - \frac{2}{\mu}\psi'^3\langle\bar{\nabla}d, \bar{\nabla}s - \bar{\nabla}\phi\rangle + (b - nH)W^3 + R_7.$$

From the expressions above for b and W^2 it follows that

$$bW^3 + \psi'\kappa_\epsilon W^2 = R_8.$$

Hence, we obtain

$$W^3\tilde{\mathcal{Q}}[w+\phi] \le -(n(H+H^{\epsilon}_{\rm cyl}) + \frac{2}{\mu}\langle\bar\nabla d, \bar\nabla s - \bar\nabla\phi\rangle)\psi'^3 + R_9.$$

We choose μ in such a way that $\mu \to 0$ as $K \to \infty$. Namely,

$$\mu = \frac{C}{\ln(1+K)}$$

for some constant $C > 0$ to be chosen later. As $K \to \infty$ we have that

$$\psi'(0) = \frac{CK}{\ln(1+K)} \to +\infty.$$

It also holds that $\psi'/W \sim 1$ as $K \to \infty$. Thus, at points of Γ the last inequality becomes

$$W^3\tilde{\mathcal{Q}}[w+\phi] \le -(n(H+H_{\rm cyl}) + \frac{2}{\mu}\langle\bar\nabla s - \bar\nabla\phi, \eta\rangle)\psi'^3 + R_9.$$

We choose the extension of ϕ in such a way that at points of Γ it holds

$$\langle\bar\nabla\phi, \eta\rangle < \langle\bar\nabla s, \eta\rangle.$$

Therefore, assuming that $H_{\rm cyl} + H \ge 0$ and choosing K large enough, we assure that $\tilde{\mathcal{Q}}[w+\phi] < 0$ on a small tubular neighborhood Ω_ϵ of Γ. Notice that $(w+\phi)|_\Gamma = \phi|_\Gamma$. Choosing C and K large enough we also have that $w + \phi \ge u|_{\Gamma_\epsilon} + \phi$. Therefore, $w+\phi$ is a locally defined upper barrier for the Dirichlet problem (8). A lower barrier may be constructed in a similar way. Thus, we have proved the following fact.

Lemma 5. *Assume that $u \in C^2(\Omega)\cap C^1(\bar\Omega)$ satisfies $\mathcal{Q}[u] = nH$ and $u|_\Gamma = \phi$. If $|u|$ is bounded in $\bar\Omega$, then*

$$\sup_\Gamma |\nabla u| \le C$$

by a constant that depends on $|u|_0$.

6 Interior gradient estimates

6.1 The prescribed mean curvature case

In this general case, we adopt ideas from the classical estimate of Korevaar [13]. Suppose that the maximum of $|Du|$ is attained at an interior point, say $x_0 \in \Omega$, where we may assume that $|Du| \neq 0$ without loss of generality. Consider a geodesic ball $B = B(x_0, \rho) \subset \Omega$ centered at x_0 with small radius $\rho \leq 1$ so that $|Du| \geq C$ at points of $\bar{B}$ for some positive constant C. Without loss of generality, we may assume after a translation along the flow lines of Y, if necessary, that $u < 0$ at points of the solid cylinder $\pi^{-1}(\bar{B})$.

Let $\eta(x, s) \geq 0$ be a continuous function defined in $\bar{B} \times \mathbb{R}^-$ that vanishes on $\partial B \times \mathbb{R}^-$ and is smooth wherever it is positive. Then, let $\bar{\Sigma}$ be the normal geodesic graph over Σ defined by

$$q = \exp_p \epsilon \eta(p) N(p)$$

where $p \in \Sigma$ is parametrized by $(x, u(x))$. Recall that N given in (3) was fixed to be upwards.

For small $\epsilon > 0$, we may describe $\bar{\Sigma}$ as a Killing graph of some function $\bar{u}$ defined in $\bar{\Omega}$. We denote by y the point in Ω that maximizes $\bar{u} - u$. It is clear that $y \in B$ and that $D_i \bar{u} = D_i u$ at this point. From (3) the tangent planes to both graphs have the same slope with respect to the flow line $\pi^{-1}(y)$ of Y.

We claim that

$$H_{\bar{u}}(y) \leq H_u(y) \tag{20}$$

where H_u and $H_{\bar{u}}$ denote the mean curvature of Σ and $\bar{\Sigma}$, respectively. In fact, moving Σ upward along the flow lines until the points $(y, u(y)) \in \Sigma$ and $(y, \bar{u}(y)) \in \bar{\Sigma}$ coincide, we obtain a tangency point for both graphs. Moreover, by the choice of y it is clear that the translated copy of Σ is above $\bar{\Sigma}$ locally around the point. Thus, the inequality (20) is consequence of the comparison principle for the mean curvature PDE.

In analytical terms, the above geometrical reasoning is justified in the following way: one has $H_u(x) = \mathcal{Q}[u](x)$ and $H_{\bar{u}}(x) = \mathcal{Q}[\bar{u}](x)$ since both hypersurfaces are described as Killing graphs. By construction, $u = \bar{u}$ at ∂B and $u \leq \bar{u}$ in $\bar{B}$ (this time, we are not considering the translation of the geometric proof). If $\mathcal{Q}[\bar{u}] \geq \mathcal{Q}[u]$ in B, then the analytical comparison principle (cf. Thm. 10.1 em [11]) assures that $\bar{u} \leq u$ in B. Thus, this contradiction shows that (20) holds.

It is a well-known fact that since the variation of Σ we consider is along the normal direction, then the mean curvature may be expanded as

$$nH_{\bar{u}}(\bar{x}) = nH_u(x) + \epsilon J\eta + O(\epsilon^2), \tag{21}$$

where $(x, u(x))$ and $(\bar{x}, \bar{u}(\bar{x}))$ parametrize correspondent points in Σ and $\bar{\Sigma}$ along the same normal geodesic and

$$J = \Delta_\Sigma + |A|^2 + \mathrm{Ric}_{\bar{M}}(N, N)$$

is the Jacobi operator produced by the linearization of the mean curvature equation. Here, Δ_Σ is the Laplace-Beltrami operator on Σ and $|A|$ denotes the norm of its second fundamental form.

Let $\bar{x} = y$ for some x. It follows from (20) and (21) that

$$\epsilon J\eta + O(\epsilon^2) = n(H_{\bar{u}}(y) - H_u(x)) \leq n(H_u(y) - H_u(x)).$$

On the other hand, Taylor's expansion of H_u gives

$$H_u(y) = H_u(x) + \epsilon\eta H_i T^i + O(\epsilon^2),$$

where T^i are the components of the horizontal projection of the normal vector field N. Thus, we get at y that

$$\Delta_\Sigma\eta + |A|^2\eta + \mathrm{Ric}(N, N)\eta \leq n\eta H_i T^i + O(\epsilon).$$

Therefore,

$$\Delta_\Sigma\eta - M\eta \leq O(\epsilon) \tag{22}$$

for some constant $M > 0$ which does not depend on η.

Next we proceed as in [13] choosing $\eta = g(\theta(x, s))$ for some real function g to be chosen and a function θ defined so that $\Delta_\Sigma\eta$ is large for sufficiently large $|Du(x)|$. Since ϵ is chosen small, then (22) will give a contradiction. Observe that C being large implies that the tangent hyperplanes to Σ near $(y, u(y))$ are very sloppy.

That a tangent hyperplane to Σ is almost vertical means the tangential component $\nabla_\Sigma\theta$ of the gradient of θ is approximately θ_s. Then, we define

$$\theta(x, s) = (Ks + (\rho^2 - r^2))^+$$

for some small constant $K > 0$, where $r(x) = \mathrm{dist}_M(x_0, x)$ is the geodesic distance measured from the center x_0 of B and $(\,\cdot\,)^+$ means positive part. We have that $0 \leq \theta \leq \rho$. Since we are assuming height estimates for Σ, we may choose K sufficiently small in such a way that $\theta > 0$ in a neighborhood of $(y, u(y))$ in $B \times \mathbb{R}^-$. We restrict ourselves to points where θ is differentiable. There,

$$\theta_s = K > 0.$$

Since

$$\Delta_\Sigma \eta = g'' |\nabla_\Sigma \theta|^2 + g' \Delta_\Sigma \theta, \tag{23}$$

we have from (22) and (23) that

$$g'' |\nabla_\Sigma \theta|^2 + g' \Delta_\Sigma \theta - Mg \leq O(\epsilon). \tag{24}$$

By hypothesis, the tangent plane of Σ at $(y, u(y))$ is not horizontal. Otherwise, we obtain from (3) that $Du(y) = 0$. Let e be the unit vector that gives the steepest ascent direction in the tangent plane of Σ at $(y, u(y))$, namely,

$$e = \frac{1}{W|Du|}(|Du|^2 D_0 + f^{1/2} \hat{u}^j D_j).$$

Denoting by $\bar{\nabla}\theta$ the ambient gradient of θ and using that $\rho \leq 1$, we have

$$\langle \nabla_\Sigma \theta, e \rangle = \langle \bar{\nabla}\theta, e \rangle = \frac{f^{1/2}}{W}\left(K|Du| + \frac{\hat{u}^j D_j(\theta)}{|Du|} \right) \geq \frac{f^{1/2}}{W}(K|Du| - \hat{C}K - 2),$$

where $\hat{C} > 0$ is a constant independent of u given by the following estimate:

$$\frac{\hat{u}^j}{|Du|} D_j(\theta) = \frac{\hat{u}^j}{|Du|}(K D_j(s) - 2r\mathsf{v}_j(r)) \geq -2 - \hat{C}K.$$

Since K and $\hat{C}$ are independent of u and the parameter s, we may assume that $|Du| > 2/K + \hat{C}$, and conclude that

$$|\nabla_\Sigma \theta| > 0.$$

Finally, for $C_1 > 0$ large we choose

$$g(\theta) = e^{C_1 \theta} - 1.$$

It is easily seen that this choice leads to a contradiction with (24). We conclude that $|Du|$ and therefore $|\nabla u|$ is bounded by some constant which does not depend on u.

Lemma 6. *Assume that $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies $\mathcal{Q}[u] = nH$ and $u|_\Gamma = \phi$. If u is bounded in Ω and $|\nabla u|$ is bounded in Γ, then $|\nabla u|$ is bounded in Ω by a constant that depends only on $|u|_0$ and $\sup_\Gamma |\nabla u|$.*

The usual elliptic regularity results guarantee that the above estimate is also true for a $C^{2,\alpha}$ function (see [11]).

6.2 The constant mean curvature case

In this case a standard argument works. In fact, consider the positive function

$$\Theta := \langle N, Y\rangle = \frac{f}{\sqrt{f + |Du|^2}}$$

since a lower estimate of Θ clearly yields an upper estimate for $|\nabla u|$. Under the assumption that H is constant and being Y a Killing field, it is well-known (cf. [2]) that Θ is a Jacobi field, namely, $J\Theta = 0$. By assumption the Ricci tensor is bounded from below. Thus, since Σ is compact there is a constant $c \geq 0$ such that $|A|^2 + \mathrm{Ric}_{\bar{M}}(N, N) \geq -c$. Thus Θ is a supersolution to the elliptic operator $\Delta_\Sigma - c$. Hence, the classical minimum principle states that

$$\min_\Sigma \Theta \geq \min_{\partial\Sigma} \Theta.$$

This assures that $|\nabla u|$ is uniformly bounded from above by a constant involving the boundary estimates for $|\nabla u|$.

7 The proof of the theorem

In view of the Continuity Method, one must seek for an initial minimal surface with boundary given by Γ. This may be accomplished by defining the sets

$$\mathcal{C} = \{u \in C^{0,1}(\Omega) : u|_\Gamma = 0\}$$

and, given $k > 0$,

$$\mathcal{C}_k = \{u \in \mathcal{C} : |u|_{0,1} \leq k\}.$$

The hypothesis on the existence of an immersion $\iota\colon \bar{\Omega} \to \bar{M}$ assures that the set $\mathcal{C}$ is non-empty since we may consider the hypersurface $\iota(\bar{\Omega})$ as the graph Σ_0 of the function $u = 0$. For the case $\kappa = 0$, if we assume M_0

is geodesically complete, the immersion ι may be obtained as follows: we construct a geodesic cone by joining points of a Killing graph in K over Γ to a vertex p_0 inside M_0. This cone is contained inside M_0 since the Killing cylinder K is mean convex and M_0 is geodesically complete. Moreover, it may be smoothed out near the vertex. The resulting hypersurface may be given as a Killing graph since the geodesic cone is always transversal to the geodesic vertical fibers.

We then formulate the issue of the existence of a minimal graph spanning $\iota(\Gamma)$ as the minimization of the functional

$$\mathcal{I}(u) = \int_\Omega W(x, \nabla u(x))\,\sqrt{\sigma}\mathrm{d}x, \quad u \in \mathcal{C},$$

where

$$W = \sqrt{f + \hat{u}_i \hat{u}^i}$$

and the first derivatives of u are taken in a weak sense. Notice that f and $\hat{u}_i = u_i - f^{1/2}\delta_i$ do not depend on u. It is clear that u is a critical point of $\mathcal{I}$ if and only if is a weak solution of the mean curvature equation in divergence form. Since the principal part of the mean curvature equation is positive-definite and the coefficients of this equation (including H) do not depend on the function, it follows from Theorem 11.10 and Theorem 11.11 in [11] that $\mathcal{I}$ has a extremum in $\mathcal{C}$. In fact, these theorems require upper bounds in the Lipschitz norm of the candidates $u \in \mathcal{C}$ which may be obtained from the apriori C^1 estimates we derived earlier.

The $C^{2,\alpha}$ regularity of the minimizer function u_0 follows from very general results found in [14]. This function defines a minimal graph over Ω with boundary Γ.

For the proof of the existence part we apply the well-known continuity method to the family of Dirichlet problems

$$\mathcal{Q}_\sigma[u] = n\sigma H, \quad u|_\Gamma = \sigma\phi,$$

where $\sigma \in [0, 1]$. The subset I of $[0, 1]$ consisting of values of σ for which there is a solution is non-empty since we have an initial minimal graph spanning the boundary data ϕ. The openness of I is a direct consequence of a standard application of the implicit function theorem since the derivative of Q_σ is a linear homeomorphism. The closedness of I follows from the apriori estimates we had proved and linear elliptic PDE theory. Thus, the continuity method assures that $1 \in I$.

In order to prove the uniqueness statement, we deduce a kind of flux formula. We suppose that there exists a hypersurface Σ' in M_0 with $\partial\Sigma' = \Gamma$ and whose mean curvature is the same as Σ at corresponding points in flow lines. This means that if $x = \pi(p)$ for $p \in \Sigma'$ then the mean curvature of Σ' at p is $H(x)$. Translating Σ' we may suppose that Σ, Σ' and a part of the cylinder K form an oriented cycle which bounds a domain U in M_0. Since Y is tangent to the part of the boundary of U contained in K, we conclude from divergence theorem applied to the field HY in U that

$$\int_{\Sigma'} \langle HY, N' \rangle = \int_{\Sigma} \langle HY, N \rangle$$

where N and N' define respectively the orientations in Σ and Σ'. Applying now the divergence theorem to the hypersurfaces Σ and Σ' we obtain that

$$\int_{\Gamma} \langle Y, \nu \rangle = \int_{\Gamma'} \langle Y, \nu' \rangle$$

where ν and ν' are respectively the outward unit co-normals to Γ with respect to Σ and Σ'. This implies that there exists a point p in Γ where Σ and Σ' are tangent, that is, where $\nu|_p = \nu'|_p$. Thus, since Σ' is locally a graph near p, we conclude from the the boundary maximum principle that $\Sigma = \Sigma'$. This concludes the proof of the theorem.

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