BLOW-UP EXAMPLES FOR THE YAMABE PROBLEM

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Abstract. It has been conjectured that if solutions to the Yamabe PDE on a smooth Riemannian manifold (M^n, g) blow-up at a point $p \in M$, then all derivatives of the Weyl tensor W_g of g, of order less than or equal to $[\frac{n-6}{2}]$, vanish at $p \in M$. In this paper we will construct smooth counterexamples to the Weyl Vanishing Conjecture for any $n \geq 25$.

1. Introduction

Let (M^n, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem consists of finding metrics of constant scalar curvature in the conformal class of g. This problem reduces to a semi-linear elliptic PDE: indeed, a conformal metric of the form $u^{\frac{4}{n-2}}\, g$ has constant scalar curvature c if and only if

$$\frac{4(n-1)}{n-2}\, \Delta_g u - R_g\, u + c\, u^{\frac{n+2}{n-2}} = 0, \tag{1}$$

where Δ_g is the Laplace operator with respect to g and R_g denotes the scalar curvature of g. Every solution of (1) is a critical point of the functional

$$E_g(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}\, |du|_g^2 + R_g\, u^2\right) dvol_g}{\left(\int_M u^{\frac{2n}{n-2}}\, dvol_g\right)^{\frac{n-2}{n}}}. \tag{2}$$

The existence of a minimizing solution to the Yamabe problem is well-known and follows from the combined works of Yamabe [22], Trudinger [21], Aubin [3], and Schoen [17].

In a topics course at Stanford in 1988 Richard Schoen raised the question of compactness of the full set of solutions and proved some special cases of it. Over the past several years many authors (Schoen [19], Li-Zhu [15], Druet [8], Marques [16] and Li-Zhang [13], [14]) have studied this problem. The Compactness Conjecture is now known to be true if and only if $n \leq 24$. In [12], Khuri, Marques and Schoen have proved compactness of the full set of solutions if $n \leq 24$. The first smooth counterexamples were constructed by S. Brendle in [5] if $n \geq 52$, while in [7] Brendle and Marques have extended these counterexamples to the remaining dimensions $25 \leq n \leq 51$. See [6] for a survey of this problem.

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In this paper we address a related important question (see [20]), known as the Weyl Vanishing Conjecture. It states that if a sequence v_ν of solutions to (1) blows-up at $p \in M$, then one should have

$$\nabla_g^l W_g(p) = 0 \quad \text{for every} \quad 0 \le l \le \Big[\frac{n-6}{2}\Big]. \tag{3}$$

Here W_g denotes the Weyl tensor of the metric g.

This was in fact one of the fundamental pieces of the program proposed by Schoen in [19] to establish compactness in high dimensions. The Weyl Vanishing Conjecture has been verified for $n \le 7$ in [16], $n \le 9$ in [13], $n \le 11$ in [14] and $n \le 24$ in [12] (see Theorem 1.2 of that paper).

The goal of this paper is to construct counterexamples for any $n \ge 25$.

We will show how to use the methods of [5] and [7] to obtain blow-up examples concentrating at a point where the metric does not satisfy condition (3). We should note that the blow-up examples constructed in those papers have the property that the Weyl tensor of the underlying metric vanishes to all orders at the concentration point.

Our main theorem is:

Theorem 1.1. *Let $n, l \in \mathbb{N}$ satisfy one of the following three conditions:*

(1) $n \ge 52$ *and* $l \ge 3$;
(2) $30 \le n \le 51$ *and* $l \ge 5$;
(3) $25 \le n \le 29$ *and* $l \ge 7$.

Then there exists a Riemannian metric g on S^n (of class C^∞), a point $p \in S^n$, and a sequence of positive functions $v_\nu \in C^\infty(S^n)$ $(\nu \in \mathbb{N})$ with the following properties:

(i) v_ν *is a solution of the Yamabe PDE (1) for all* $\nu \in \mathbb{N}$
(ii) $E_g(v_\nu) < Y(S^n)$ *for all* $\nu \in \mathbb{N}$, *and* $E_g(v_\nu) \to Y(S^n)$ *as* $\nu \to \infty$
(iii) $p \in S^n$ *is a blow-up point of* v_ν
(iv) $\nabla_g^j W_g(p) = 0$ *for all* $0 \le j < l$, *but* $\nabla_g^l W_g(p) \ne 0$.

(Here, $Y(S^n)$ denotes the Yamabe energy of the round metric on S^n.)

The construction relies on a glueing procedure based on some local model metric. The model metrics are of the form $g(x) = \exp(h(x))$, $x \in \mathbb{R}^n$, where

$$h_{ik}(x) = f(|x|^2) \sum_{p,q} W_{ipkq} x_p x_q,$$

f is a polynomial, and $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a multi-linear form which satisfies all the algebraic properties of the Weyl tensor.

The idea is to find the blow-up solutions as critical points of the energy function defined on a finite dimensional space of approximate solutions. This energy function is well approximated at appropriate scales by an auxiliary function $F(\xi, \varepsilon)$, $\xi \in \mathbb{R}^n$, $\varepsilon \in (0, \infty)$, and we are left with the algebraic problem of finding a polynomial f such that $F(\xi, \varepsilon)$ has a strict local minimum at $(0, 1)$ with $F(0, 1) < 0$.

The bubbles concentrate at the origin along the x_1 direction, and by introducing a perturbation of higher order depending only on the $x_2, \dots, x_n$ variables, we find an example with

$$\nabla_g^{2\deg(f)+1} W_g(0) \neq 0.$$

Our examples are perturbations of the ones in [5] and [7] if $n \geq 52$ or $25 \leq n \leq 29$, respectively. If $30 \leq n \leq 51$ we improve the calculations of [7] by finding a polynomial f of one degree lower (degree 2). That is the content of Section 4.

We should note that $W_g(p) = 0$ if $n \geq 6$ and $\nabla W_g(p) = 0$ if $n \geq 8$, at a blow-up point p (see [16] and [13]). It is also possible to see that the results of [12] imply

$$\nabla^2 W_g(p) = 0,\ \nabla^3 W_g(p) = 0 \ \text{ if } \ 25 \leq n \leq 51,$$

and

$$\nabla^4 W_g(p) = 0,\ \nabla^5 W_g(p) = 0 \ \text{ if } \ 25 \leq n \leq 29.$$

We will now explain the structure of the paper. In Section 2, we recall that the problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \varepsilon)$, where ξ is a vector in $\mathbb{R}^n$ and ε is a positive real number. This idea has been used by many authors (see, e.g., [2], [4], [5], [7]). In Section 3, we show that the function $\mathcal{F}_g(\xi, \varepsilon)$ can be approximated by an auxiliary function $F(\xi, \varepsilon)$. In Section 4, we prove that if $30 \leq n \leq 51$, then there exists a polynomial f of degree 2 such that the function $F(\xi, \varepsilon)$ has a critical point, which is a strict local minimum. Finally, in Section 5, we prove Theorem 1.1 by a perturbation argument.

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2. Lyapunov-Schmidt reduction

In this section, we collect some basic results established in [5]. Let

$$\mathcal{E} = \left\{ w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |dw|^2 < \infty \right\}.$$

By Sobolev's inequality, there exists a constant K, depending only on n, such that

$$\left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq K \int_{\mathbb{R}^n} |dw|^2$$

for all $w \in \mathcal{E}$. We define a norm on $\mathcal{E}$ by $\|w\|_{\mathcal{E}}^2 = \int_{\mathbb{R}^n} |dw|^2$. It is easy to see that $\mathcal{E}$, equipped with this norm, is complete.

Given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$, we define a function $u_{(\xi,\varepsilon)} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{(\xi,\varepsilon)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2}\Big)^{\frac{n-2}{2}}.$$

The function $u_{(\xi,\varepsilon)}$ satisfies the elliptic PDE

$$\Delta u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} = 0.$$

It is well known that

$$\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} = \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}}$$

for all $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$. We next define

$$\varphi_{(\xi,\varepsilon,0)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2}\Big)^{\frac{n+2}{2}} \frac{\varepsilon^2 - |x-\xi|^2}{\varepsilon^2 + |x-\xi|^2}$$

and

$$\varphi_{(\xi,\varepsilon,k)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2}\Big)^{\frac{n+2}{2}} \frac{2\varepsilon\,(x_k - \xi_k)}{\varepsilon^2 + |x-\xi|^2}$$

for $k = 1, \ldots, n$. Finally, we define a closed subspace $\mathcal{E}_{(\xi,\varepsilon)} \subset \mathcal{E}$ by

$$\mathcal{E}_{(\xi,\varepsilon)} = \Big\{ w \in \mathcal{E} : \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, w = 0 \quad \text{for } k = 0, 1, \ldots, n \Big\}.$$

Clearly, $u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 1. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. There exists a positive constant $\alpha_0 \leq 1$, depending only on n, with the following significance: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$, then, given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$ and any function $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, there exists a unique function $w = G_{(\xi,\varepsilon)}(f) \in \mathcal{E}_{(\xi,\varepsilon)}$ such that*

$$\int_{\mathbb{R}^n} \Big(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, w\, \psi - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w\, \psi \Big) = \int_{\mathbb{R}^n} f\, \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Moreover, we have $\|w\|_{\mathcal{E}} \leq C\, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, where C is a constant that depends only on n.

Proposition 2. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. Moreover, let $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$. There exists a positive constant $\alpha_1 \leq \alpha_0$, depending only on n, with the following significance: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}^n$, then there exists a function $v_{(\xi,\varepsilon)} \in \mathcal{E}$ such that $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ and*

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, \psi - n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$*. Moreover, we have the estimate*

$$\begin{aligned}&\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{\mathcal{E}} \\ &\leq C \, \Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},\end{aligned}$$

where C *is a constant that depends only on* n*.*

We next define a function $\mathcal{F}_g : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}_g(\xi,\varepsilon) &= \int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, v_{(\xi,\varepsilon)}^2 - (n-2)^2 \, |v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} \Big) \\ &\quad - 2(n-2) \, \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}.\end{aligned}$$

If we choose α_1 small enough, then we obtain the following result:

Proposition 3. *The function* $\mathcal{F}_g$ *is continuously differentiable. Moreover, if* $(\bar{\xi},\bar{\varepsilon})$ *is a critical point of the function* $\mathcal{F}_g$*, then the function* $v_{(\bar{\xi},\bar{\varepsilon})}$ *is a non-negative weak solution of the equation*

$$\Delta_g v_{(\bar{\xi},\bar{\varepsilon})} - \frac{n-2}{4(n-1)} \, R_g \, v_{(\bar{\xi},\bar{\varepsilon})} + n(n-2) \, v_{(\bar{\xi},\bar{\varepsilon})}^{\frac{n+2}{n-2}} = 0.$$

3. An estimate for the energy of a "bubble"

Throughout this paper, we fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. We assume that W_{ijkl} satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of W are non-zero, so that

$$\sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 > 0.$$

For abbreviation, we put

$$H_{ik}(x) = \sum_{p,q=1}^n W_{ipkq} \, x_p \, x_q$$

and

$$\overline{H}_{ik}(x) = f(|x|^2) \, H_{ik}(x),$$

where $f(s)$ is a polynomial of degree m. We have that $H_{ik}(x)$ is trace-free, $\sum_{i=1}^n x_i \, H_{ik}(x) = 0$, and $\sum_{i=1}^n \partial_i H_{ik}(x) = 0$ for all $|x| \leq 1$.

Let $T_{ik}(x)$ be a symmetric matrix of smooth functions so that $T_{ik}(x)$ is trace-free, $\sum_{i=1}^n x_i \, T_{ik}(x) = 0$, and $\sum_{i=1}^n \partial_i T_{ik}(x) = 0$ for all $x \in \mathbb{R}^n$. We will also assume

$$\sum_{i,k=1}^n \Big(|T_{ik}(x)| + |x||\partial T_{ik}(x)| + |x|^2 |\partial^2 T_{ik}(x)| \Big) \leq \beta \, |x|^t$$

for some integer t, $0 \leq \beta \leq 1$, and all $|x| \leq 1$.

Throughout the rest of the paper we will suppose

$$2m + 2 < \min\{t, \frac{n-2}{2}\}. \tag{4}$$

We consider a Riemannian metric of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$$

for all $x \in \mathbb{R}^n$, and

$$h_{ik}(x) = \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x) \ + \ T_{ik}(x)$$

for $|x| \leq \rho$. We assume that the parameters λ, μ, and ρ are chosen such that $\mu \leq 1$ and $\lambda \leq \rho \leq 1$. Note that $\sum_{i=1}^n x_i\, h_{ik}(x) = 0$ and $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, there exists a unique function $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ such that

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)}\, \psi \Big) \\ &= -\int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi\end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Moreover, by Proposition 2, there exists a unique function $v_{(\xi,\varepsilon)}$ such that $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ and

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, \psi - n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$.

For abbreviation, let

$$\Omega = \Big\{ (\xi, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < 1, \, \frac{1}{2} < \varepsilon < 2 \Big\}.$$

If $(\xi, \varepsilon) \in \lambda\, \Omega$, then the function $w_{(\xi,\varepsilon)}$ satisfies the estimates

$$\begin{aligned}|w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{2m+4-n} \\ |\partial w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{2m+3-n} \\ |\partial^2 w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{2m+2-n}\end{aligned}$$

for all $x \in \mathbb{R}^n$ (see [5]).

The following result is proved in the Appendix A of [5]. A similar formula is derived in [2]. We use repeated indices to indicate summation.

Proposition 4. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying*

$|h(x)| \leq 1$ for all $x \in \mathbb{R}^n$. Let R_g be the scalar curvature of g. There exists a constant C, depending only on n, such that

$$\Big|R_g - \partial_i\partial_k h_{ik} + \partial_i(h_{il}\,\partial_k h_{kl}) - \frac{1}{2}\,\partial_i h_{il}\,\partial_k h_{kl} + \frac{1}{4}\,\partial_l h_{ik}\,\partial_l h_{ik}\Big| \\ \leq C\,|h|^2\,|\partial^2 h| + C\,|h|\,|\partial h|^2.$$

In what follows $\theta_k = 1$ if $k = \frac{n-2}{2}$, and $\theta_k = 0$ otherwise.

Proposition 5. *Assume that $(\xi,\varepsilon) \in \lambda\,\Omega$. Then we have*

$$\Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ \leq C\,\lambda^{2m+2}\,\mu + C\,\beta\,\lambda^t\,\Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

and

$$\Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \\ + \sum_{i,k=1}^n \mu\,\lambda^{2m}\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ \leq C\,\lambda^{\frac{(2m+2)(n+2)}{n-2}}\,\mu^2 + C\,\beta\,\lambda^t\,\Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}.$$

Proof. Note that $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$. Hence, it follows from Proposition 4 that

$$\begin{aligned} |R_g(x)| &\leq C\,|h(x)|^2\,|\partial^2 h(x)| + C\,|\partial h(x)|^2 \\ &\leq C\,\mu^2\,(\lambda + |x|)^{4m+2} + C\,\beta^2\,(\lambda + |x|)^{2t-2} \end{aligned}$$

for $|x| \leq \rho$. This implies

$$\Big|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big| \\ = \Big|\sum_{i,k=1}^n \partial_i\big[(g^{ik} - \delta_{ik})\,\partial_k u_{(\xi,\varepsilon)}\big] - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)}\Big| \\ \leq C\,\lambda^{\frac{n-2}{2}}\,\mu\,(\lambda + |x|)^{2m+2-n} + C\,\beta\,\lambda^{\frac{n-2}{2}}\,(\lambda + |x|)^{t-n}$$

and

$$\Big|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} + \sum_{i,k=1}^n (h_{ik} \,-\, T_{ik})\, \partial_i\partial_k u_{(\xi,\varepsilon)}\Big|$$
$$= \Big|\sum_{i,k=1}^n \partial_i\big[(g^{ik} - \delta_{ik} + h_{ik})\, \partial_k u_{(\xi,\varepsilon)}\big] - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)}\Big|$$
$$+ \Big|\sum_{i,k=1}^n T_{ik}\, \partial_i\partial_k u_{(\xi,\varepsilon)}\Big|$$
$$\leq C\, \lambda^{\frac{n-2}{2}}\, \mu^2\, (\lambda + |x|)^{4m+4-n} + C\, \beta\, \lambda^{\frac{n-2}{2}}\, (\lambda + |x|)^{t-n}$$

for $|x| \leq \rho$. From this the assertion follows.

Corollary 6. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\, \lambda^{2m+2}\, \mu + C\, \beta\, \lambda^t\, \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C\, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

whenever $(\xi, \varepsilon) \in \lambda\,\Omega$.

Proof. It follows from Proposition 2 that

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}$$
$$\leq C\, \Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},$$

where C is a constant that depends only on n. Hence, the assertion follows from Proposition 5.

Corollary 7. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}$$
$$\leq C\, \lambda^{\frac{(2m+2)(n+2)}{n-2}}\, \mu^{\frac{n+2}{n-2}} + C\, \beta\, \lambda^t \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C\, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

whenever $(\xi, \varepsilon) \in \lambda\,\Omega$.

Proof. Consider the functions

$$B_1 = \sum_{i,k=1}^n \partial_i\big[(g^{ik} - \delta_{ik})\, \partial_k w_{(\xi,\varepsilon)}\big] - \frac{n-2}{4(n-1)}\, R_g\, w_{(\xi,\varepsilon)}$$

and

$$B_2 = \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i\partial_k u_{(\xi,\varepsilon)}.$$

By definition of $w_{(\xi,\varepsilon)}$, we have

$$\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} \, R_g \, w_{(\xi,\varepsilon)} \, \psi - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)} \, \psi \Big) = -\int_{\mathbb{R}^n} (B_1 + B_2) \, \psi$$

for all functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Since $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$, it follows that

$$w_{(\xi,\varepsilon)} = -G_{(\xi,\varepsilon)}(B_1 + B_2).$$

Moreover, we have

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_3 + n(n-2) \, B_4\big),$$

where

$$B_3 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$B_4 = |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} \, v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}).$$

Thus, we conclude that

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_1 + B_2 + B_3 + n(n-2) \, B_4\big).$$

Note that

$$\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \, \lambda^{\frac{(2m+2)(n+2)}{n-2}} \, \mu^2 + C \, \lambda^{2m+2+t} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n} \theta_{2m+2+t}} \, \mu \, \beta + C \, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

and

$$\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \, \lambda^{\frac{(2m+2)(n+2)}{n-2}} \, \mu^2 + C \, \beta \, \lambda^t \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n} \theta_t} + C \, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

by Proposition 5. Moreover, we have

$$\begin{aligned} \|B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} &\leq C \, \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{n+2}{n-2}} \\ &\leq C \, \lambda^{\frac{(2m+2)(n+2)}{n-2}} \, \mu^{\frac{n+2}{n-2}} + C \, \lambda^{\frac{t(n+2)}{n-2}} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{(n+2)^2}{2n(n-2)} \theta_t} \beta^{\frac{n+2}{n-2}} \\ &\quad + C \, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n+2}{2}} \end{aligned}$$

by Corollary 6. Hence, it follows from Proposition 1 that

$$\begin{aligned} &\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C \, \big\|B_1 + B_2 + B_3 + n(n-2) \, B_4\big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\leq C \, \lambda^{\frac{(2m+2)(n+2)}{n-2}} \, \mu^{\frac{n+2}{n-2}} + C \, \lambda^t \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n} \theta_t} \, \beta + C \, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}. \end{aligned}$$

This completes the proof.

Proposition 8. *We have*

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, (v_{(\xi,\varepsilon)}^2 - u_{(\xi,\varepsilon)}^2) \Big) \\
&\qquad + \int_{\mathbb{R}^n} n(n-2)\, (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} \\
&\qquad - \int_{\mathbb{R}^n} n(n-2)\, (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \\
&\qquad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, w_{(\xi,\varepsilon)} \Big| \\
&\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^{2m+2}\, \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\, \beta^2\, \lambda^{2t} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{n}\theta_t} \\
&+ C\, \beta\, \lambda^{t+2m+2} \mu \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C \left(\frac{\lambda}{\rho}\right)^{n-2}.
\end{aligned}$$

whenever $(\xi, \varepsilon) \in \lambda\, \Omega$.

Proof. By definition of $v_{(\xi,\varepsilon)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 - \langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \Big) \\
&\qquad - \int_{\mathbb{R}^n} n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) = 0.
\end{aligned}$$

Using Proposition 5 and Corollary 6, we obtain

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \Big(\langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \Big) \\
&\quad - \int_{\mathbb{R}^n} n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\
&\quad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \bigg| \\
&\leq \Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \\
&\quad + \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&\quad \cdot \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^3 + C\, \lambda^{2m+2}\, \mu\, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\, \beta^2\, \lambda^{2t} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{n}\theta_t} \\
&\quad + C \lambda^{t+2m+2} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t}\, \mu\, \beta + C\, \Big(\frac{\lambda}{\rho}\Big)^{n-2}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}) \bigg| \\
&\leq C\, (\lambda^{2m+2}\, \mu)\, \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^{2m+2}\, \mu\, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} \\
&\qquad + C\mu\, \beta\, \lambda^{t+2m+2} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t}
\end{aligned}$$

by Corollary 7. Putting these facts together, the assertion follows.

Proposition 9. *We have*

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \bigg| \\
&\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^{\frac{2nt}{n-2}} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{n-2}\theta_t}\, \beta^{\frac{2n}{n-2}} + C\, \Big(\frac{\lambda}{\rho}\Big)^n
\end{aligned}$$

whenever $(\xi, \varepsilon) \in \lambda\, \Omega$.

Proof. We have the pointwise estimate

$$\Big|(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n}\, (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big|$$
$$\leq C\, |v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}},$$

where C is a constant that depends only on n. This implies

$$\Big|\int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big|$$
$$\leq C\, \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{2n}{n-2}}$$
$$\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^{\frac{2nt}{n-2}} \Big(1 + \log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{n-2}\theta_t}\, \beta^{\frac{2n}{n-2}} + C\, \Big(\frac{\lambda}{\rho}\Big)^n.$$

Proposition 10. *We have*

$$\Big|\int_{\mathbb{R}^n} \Big(|du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)}^2 - n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}\Big)$$
$$- \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)}$$
$$+ \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2\, u_{(\xi,\varepsilon)}^2\Big|$$
$$\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^3 + C\, \lambda^{\frac{2nt}{n-2}}\, \beta^3 + C\, \Big(\frac{\lambda}{\rho}\Big)^{n-2}$$

whenever $(\xi,\varepsilon) \in \lambda\, \Omega$.

Proof. Note that

$$\Big|g^{ik}(x) - \delta_{ik} + h_{ik}(x) - \frac{1}{2} \sum_{l=1}^n h_{il}(x)\, h_{kl}(x)\Big|$$
$$\leq C\, |h(x)|^3 \leq C\, \mu^3\, (\lambda + |x|)^{3(2m+2)} + C\, \beta^3\, (\lambda + |x|)^{3t}$$

for $|x| \leq \rho$. This implies

$$\Big|\int_{\mathbb{R}^n} (|du_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|^2) + \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)}$$
$$- \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)}\Big|$$
$$\leq C\, \lambda^{\frac{(2m+2)2n}{n-2}}\, \mu^3 + C\, \lambda^{\frac{2nt}{n-2}}\, \beta^3 + C\, \Big(\frac{\lambda}{\rho}\Big)^{n-2}.$$

At this point, we use the formula

$$\begin{aligned}&\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,\partial_i\partial_k(u_{(\xi,\varepsilon)}^2)\\ &= \frac{1}{n}\Big(|du_{(\xi,\varepsilon)}|^2 - \frac{n-2}{4(n-1)}\,\Delta(u_{(\xi,\varepsilon)}^2)\Big)\,\delta_{ik}.\end{aligned}$$

Since h_{ik} is trace-free, we obtain

$$\sum_{i,k=1}^n h_{ik}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} = \frac{n-2}{4(n-1)}\sum_{i,k=1}^n h_{ik}\,\partial_i\partial_k(u_{(\xi,\varepsilon)}^2),$$

hence

$$\int_{\mathbb{R}^n}\sum_{i,k=1}^n h_{ik}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} = \int_{\mathbb{R}^n}\frac{n-2}{4(n-1)}\sum_{i,k=1}^n \partial_i\partial_k h_{ik}\,u_{(\xi,\varepsilon)}^2.$$

Since $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$, it follows that

$$\Big|\int_{\mathbb{R}^n}\sum_{i,k=1}^n h_{ik}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)}\Big| \leq C\,\rho^2\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}.$$

By Proposition 4, the scalar curvature of g satisfies the estimate

$$\begin{aligned}&\Big|R_g(x) + \frac{1}{4}\sum_{i,k,l=1}^n (\partial_l h_{ik}(x))^2\Big|\\ &\leq C\,|h(x)|^2\,|\partial^2 h(x)| + C\,|h(x)|\,|\partial h(x)|^2\\ &\leq C\,\mu^3\,(\lambda + |x|)^{6m+4} + C\,\beta^3\,(\lambda + |x|)^{3t-2}\end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} R_g\,u_{(\xi,\varepsilon)}^2 + \int_{\mathbb{R}^n}\frac{1}{4}\sum_{i,k,l=1}^n (\partial_l h_{ik})^2\,u_{(\xi,\varepsilon)}^2\Big|\\ &\leq C\,\lambda^{\frac{(2m+2)2n}{n-2}}\,\mu^3 + C\,\lambda^{\frac{2nt}{n-2}}\,\beta^3 + C\,\rho^2\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}.\end{aligned}$$

Putting these facts together, the assertion follows.

Therefore we have proved

Corollary 11. *The function $\mathcal{F}_g(\xi,\varepsilon)$ satisfies the estimate*

$$\begin{aligned}
&\Big|\mathcal{F}_g(\xi,\varepsilon) - \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \\
&\qquad + \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2\, u_{(\xi,\varepsilon)}^2 \\
&\qquad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^{2m}\, f(\lambda^{-2}\,|x|^2)\, H_{ik}(x)\; \partial_i\partial_k u_{(\xi,\varepsilon)}\, w_{(\xi,\varepsilon)}\Big| \\
&\leq C\,\lambda^{\frac{(2m+2)2n}{n-2}}\,\mu^{\frac{2n}{n-2}} + C\,\lambda^{2m+2}\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\,\Big(1+\log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{n}\theta_t} \lambda^{2t}\,\beta^2 \\
&+ C\lambda^{t+2m+2}\mu\beta\Big(1+\log(\frac{\rho}{\lambda})\Big)^{\frac{n+2}{2n}\theta_t} + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}.
\end{aligned}$$

whenever $(\xi,\varepsilon) \in \lambda\,\Omega$.

We define a function $F : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ as follows: given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$, we define

$$\begin{aligned}
F(\xi,\varepsilon) &= \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, \partial_i u_{(\xi,\varepsilon)}(x)\, \partial_k u_{(\xi,\varepsilon)}(x) \\
&\quad - \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, u_{(\xi,\varepsilon)}(x)^2 \\
&\quad + \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}(x)\, \partial_i\partial_k u_{(\xi,\varepsilon)}(x)\, z_{(\xi,\varepsilon)}(x),
\end{aligned}$$

where $z_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ satisfies the relation

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\langle dz_{(\xi,\varepsilon)}, d\psi\rangle - n(n+2)\, u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}}\, z_{(\xi,\varepsilon)}\, \psi\Big) \\
&= -\int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}\, \partial_i\partial_k u_{(\xi,\varepsilon)}\, \psi
\end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$.

The next Proposition follows from Corollary 11.

Proposition 12. *The function $\mathcal{F}_g(\xi,\varepsilon)$ satisfies the estimate*

$$\begin{aligned}&\Big|\mathcal{F}_g(\lambda\,\xi,\lambda\,\varepsilon)-\mu^2\,\lambda^{2(2m+2)}\,F(\xi,\varepsilon)\Big|\\&\leq C\,\mu^{\frac{2n}{n-2}}\,\lambda^{\frac{(2m+2)2n}{n-2}}+C\,\mu\,\lambda^{2m+2}\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}\\&+C\,\mu\,\beta\,\lambda^{t+2m+2}\Big(1+\log(\frac{\rho}{\lambda})\Big)+C\beta^2\lambda^{2t}\Big(1+\log(\frac{\rho}{\lambda})\Big)^2+C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}\end{aligned}$$

whenever $(\xi,\varepsilon)\in\Omega$.

4. Finding a critical point of an auxiliary function

In this section we will prove that for all $30\leq n\leq 51$ there exists a polynomial f of degree 2 such that the corresponding $F(\xi,\varepsilon)$ has a strict local minimum at $(0,1)$ with $F(0,1)<0$. It has been proved that in dimensions $n\geq 52$ (see [5]) one can choose a polynomial of degree 1, while in dimensions $25\leq n\leq 51$ (see [7]) there exists a polynomial of degree 3.

Proposition 13. *The function $F(\xi,\varepsilon)$ satisfies $F(\xi,\varepsilon)=F(-\xi,\varepsilon)$ for all $(\xi,\varepsilon)\in\mathbb{R}^n\times(0,\infty)$. Consequently $\frac{\partial}{\partial\xi_p}F(0,\varepsilon)=0$ and $\frac{\partial^2}{\partial\varepsilon\,\partial\xi_p}F(0,\varepsilon)=0$ for all $\varepsilon>0$ and $p=1,\dots,n$.*

Proof. This follows immediately from the relation $\overline{H}_{ik}(-x)=\overline{H}_{ik}(x)$.

The following Proposition was proved in [7].

Proposition 14. *We have*

$$\begin{aligned}F(0,\varepsilon)=-&\frac{n-2}{16n(n-1)(n+2)}\,|S^{n-1}|\sum_{i,j,k,l=1}^{n}(W_{ijkl}+W_{ilkj})^2\\&\cdot\int_0^\infty\varepsilon^{n-2}\,(\varepsilon^2+r^2)^{2-n}\,r^{n+1}\\&\qquad\cdot\Big[(n+2)\,f(r^2)^2+4\,r^2\,f(r^2)\,f'(r^2)+2\,r^4\,f'(r^2)^2\Big]\,dr.\end{aligned}$$

For the rest of this section we will choose

$$f(s)=\tau+5s-\frac{3}{4}s^2,$$

where τ is a real parameter.

Proposition 15. *The function $F(0,\varepsilon)$ can be written in the form*

$$\begin{aligned}F(0,\varepsilon)=-&\frac{n-2}{16n(n-1)(n+2)}\,|S^{n-1}|\sum_{i,j,k,l=1}^{n}(W_{ijkl}+W_{ilkj})^2\\&\cdot I(\varepsilon^2)\int_0^\infty(1+r^2)^{2-n}\,r^{n+7}\,dr,\end{aligned}$$

where

$$\begin{aligned}I(s) &= \frac{n-12}{n+6}\,\frac{n-10}{n+4}\,(n-8)\,\tau^2\,s^2 + 10\,\frac{n-12}{n+6}\,(n-10)\,\tau\,s^3 \\ &+ \Big(25(n+8)\frac{(n-12)}{n+6} - \frac{3(n-12)}{2}\tau\Big)\,s^4 - \frac{15(n+12)}{2}\,s^5 \\ &+ \frac{n+8}{n-14}\,\frac{9(n+18)}{16}\,s^6.\end{aligned}$$

Proof. It is straightforward to check that

$$\begin{aligned}&(n+2)\,f(s)^2 + 4s\,f(s)\,f'(s) + 2s^2\,f'(s)^2 \\ &= (n+2)\tau^2 + 10(n+4)\tau\,s + \Big(25(n+8) - \frac{3(n+6)}{2}\tau\Big)\,s^2 \\ &- \frac{15(n+12)}{2}\,s^3 + \frac{9(n+18)}{16}\,s^4.\end{aligned}$$

This implies

$$\begin{aligned}&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{2-n}\,r^{n+1} \\ &\qquad\cdot\Big[(n+2)\,f(r^2)^2 + 4\,r^2\,f(r^2)\,f'(r^2) + 2r^4\,f'(r^2)^2\Big]\,dr \\ &= (n+2)\tau^2\,\varepsilon^4 \int_0^\infty (1+r^2)^{2-n}\,r^{n+1}\,dr \\ &+ 10(n+4)\tau\,\varepsilon^6 \int_0^\infty (1+r^2)^{2-n}\,r^{n+3}\,dr \\ &+ \Big(25(n+8) - \frac{3(n+6)}{2}\tau\Big)\,\varepsilon^8 \int_0^\infty (1+r^2)^{2-n}\,r^{n+5}\,dr \\ &- \frac{15(n+12)}{2}\,\varepsilon^{10} \int_0^\infty (1+r^2)^{2-n}\,r^{n+7}\,dr \\ &+ \frac{9(n+18)}{16}\,\varepsilon^{12} \int_0^\infty (1+r^2)^{2-n}\,r^{n+9}\,dr.\end{aligned}$$

Using the identity

$$\int_0^\infty (1+r^2)^{2-n}\,r^{\beta+2}\,dr = \frac{\beta+1}{2n-\beta-7}\int_0^\infty (1+r^2)^{2-n}\,r^{\beta}\,dr,$$

we obtain

$$\begin{aligned}&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{2-n}\,r^{n+1} \\ &\qquad\cdot\Big[(n+2)\,f(r^2)^2 + 4\,r^2\,f(r^2)\,f'(r^2) + 2r^4\,f'(r^2)^2\Big]\,dr \\ &= I(\varepsilon^2)\int_0^\infty (1+r^2)^{2-n}\,r^{n+7}\,dr.\end{aligned}$$

This completes the proof.

In the next result, proved in [7], we compute the Hessian of F at $(0, \varepsilon)$.

Proposition 16. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi_p \, \partial \xi_q} F(0, \varepsilon) \\
&= -\frac{2(n-2)^2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \\
&\qquad \cdot \int_0^\infty \varepsilon^{n-2} \, (\varepsilon^2 + r^2)^{-n} \, r^{n+5} \, \Big[2 \, f(r^2) \, f'(r^2) + r^2 \, f'(r^2)^2 \Big] \, dr \\
&- \frac{(n-2)^2}{2n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \\
&\qquad \cdot \int_0^\infty \varepsilon^{n-2} \, (\varepsilon^2 + r^2)^{-n} \, r^{n+5} \, \Big[2 \, f(r^2) \, f'(r^2) + r^2 \, f'(r^2)^2 \Big] \, dr \\
&+ \frac{(n-2)^2}{4n(n-1)(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \\
&\qquad \cdot \int_0^\infty \varepsilon^{n-2} \, (\varepsilon^2 + r^2)^{1-n} \, r^{n+5} \, f'(r^2)^2 \, dr.
\end{aligned}$$

We now compute:

Proposition 17. *We have*

$$\begin{aligned}
&\int_0^\infty \varepsilon^{n-2} \, (\varepsilon^2 + r^2)^{-n} \, r^{n+5} \, \Big[2 \, f(r^2) \, f'(r^2) + r^2 \, f'(r^2)^2 \Big] \, dr \\
&= J(\varepsilon^2) \int_0^\infty (1 + r^2)^{-n} \, r^{n+9} \, dr,
\end{aligned}$$

where

$$\begin{aligned}
J(s) = 10 \, &\frac{n-10}{n+8} \, \frac{n-8}{n+6} \, \tau \, s^2 + \frac{n-10}{n+8} \, (75 - 3\tau) \, s^3 \\
&- \frac{75}{2} \, s^4 + \frac{9}{2} \, \frac{n+10}{n-12} \, s^5.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&2 \, f(s) \, f'(s) + s \, f'(s)^2 \\
&= 10\tau + (75 - 3\tau) \, s - \frac{75}{2} \, s^2 + \frac{9}{2} \, s^3.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{-n}\,r^{n+5}\Big[2\,f(r^2)\,f'(r^2)+r^2\,f'(r^2)^2\Big]\,dr \\
&= 10\tau\,\varepsilon^4 \int_0^\infty (1+r^2)^{-n}\,r^{n+5}\,dr \\
&+ (75-3\tau)\,\varepsilon^6 \int_0^\infty (1+r^2)^{-n}\,r^{n+7}\,dr \\
&- \frac{75}{2}\,\varepsilon^8 \int_0^\infty (1+r^2)^{-n}\,r^{n+9}\,dr \\
&+ \frac{9}{2}\,\varepsilon^{10} \int_0^\infty (1+r^2)^{-n}\,r^{n+11}\,dr.
\end{aligned}$$

Hence, the assertion follows from the identity

$$\int_0^\infty (1+r^2)^{-n}\,r^{\beta+2}\,dr = \frac{\beta+1}{2n-\beta-3}\int_0^\infty (1+r^2)^{-n}\,r^{\beta}\,dr.$$

Proposition 18. *Assume that $30 \leq n \leq 51$. Then we can choose $\tau \in \mathbb{R}$ such that $I'(1) = 0$, $I''(1) < 0$, and $J(1) < 0$.*

Proof. The condition $I'(1) = 0$ is equivalent to

$$a_n\,\tau^2 + b_n\,\tau + c_n = 0,$$

where

$$\begin{aligned}
a_n &= 4\,\frac{n-12}{n+6}\,\frac{n-10}{n+4}\,(n-8) \\
b_n &= 60\,\frac{n-12}{n+6}\,(n-10) - 12(n-12) \\
c_n &= 200\,\frac{n-12}{n+6}\,(n+8) - 75(n+12) + \frac{27}{4}\,\frac{n+8}{n-14}\,(n+18).
\end{aligned}$$

By inspection, one verifies that $\frac{225}{4}\,a_n - \frac{15}{2}\,b_n + c_n < 0$ for $30 \leq n \leq 51$. Since a_n is positive, there exists a unique real number $\tau < -\frac{15}{2}$ such that $a_n\,\tau^2 + b_n\,\tau + c_n = 0$. Moreover, we have

$$I''(1) - I'(1) = \alpha_n\,\tau + \beta_n$$

and

$$J(1) = \gamma_n\,\tau + \delta_n,$$

where

$$\begin{aligned}
\alpha_n &= 30\,\frac{n-12}{n+6}\,(n-10) - 12\,(n-12) \\
\beta_n &= 200\,\frac{n-12}{n+6}\,(n+8) - \frac{225}{2}\,(n+12) + \frac{27}{2}\,\frac{n+8}{n-14}\,(n+18) \\
\gamma_n &= -3\,\frac{n-10}{n+8} + 10\frac{(n-10)}{n+8}\,\frac{n-8}{n+6} \\
\delta_n &= 75\,\frac{n-10}{n+8} - \frac{75}{2} + \frac{9}{2}\,\frac{n+10}{n-12}.
\end{aligned}$$

By inspection, one verifies that $15\alpha_n > 2\beta_n > 0$ and $15\gamma_n > 2\delta_n > 0$ for $30 \leq n \leq 51$. This implies $I''(1) = \alpha_n\,\tau + \beta_n < -\frac{15}{2}\alpha_n + \beta_n < 0$ and $J(1) = \gamma_n\,\tau + \delta_n < -\frac{15}{2}\gamma_n + \delta_n < 0$. This completes the proof.

Corollary 19. *Assume that τ is chosen such that $I'(1) = 0$, $I''(1) < 0$, and $J(1) < 0$. Then the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0,1)$, and $F(0,1) < 0$.*

Proof. It follows from Proposition 14 that $F(0,1) < 0$.

Since $I'(1) = 0$, we have $\frac{\partial}{\partial\varepsilon}F(0,1) = 0$. Therefore, $(0,1)$ is a critical point of the function $F(\xi,\varepsilon)$. Since $J(1) < 0$, we have

$$\int_0^\infty (1+r^2)^{-n}\,r^{n+5}\left[2\,f(r^2)\,f'(r^2) + r^2\,f'(r^2)^2\right]dr < 0$$

by Proposition 17. Hence, it follows from Proposition 16 that the matrix $\frac{\partial^2}{\partial\xi_p\,\partial\xi_q}F(0,1)$ is positive definite. Finally, the inequality $I''(0) < 0$ implies that $\frac{\partial^2}{\partial\varepsilon^2}F(0,1) > 0$. Consequently, the function $F(\xi,\varepsilon)$ has a strict local minimum at $(0,1)$.

5. Proof of the main theorem

In this section we will use the notation of Section 3.

Proposition 20. *Let g be a smooth metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ such that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1$ for all $x \in \mathbb{R}^n$, $h(x) = 0$ for $|x| \geq 1$, and*

$$h_{ik}(x) = \mu\,\lambda^{2m}\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x) + \beta\,T_{ik}(x)$$

for $|x| \leq \rho$. Suppose f is such that the function $F(\xi,\varepsilon)$ has a strict local minimum at $(0,1)$ with $F(0,1) < 0$. If α, $\rho^{2-n}\mu^{-2}\,\lambda^{n-2-2(2m+2)}$ and $\lambda^{t-(2m+2)}\Big(1+\log(\frac{\rho}{\lambda})\Big)\,\mu^{-1}$ are sufficiently small, then there exists a positive function v such that

$$\Delta_g v - \frac{n-2}{4(n-1)}\,R_g\,v + n(n-2)\,v^{\frac{n+2}{n-2}} = 0,$$

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} < \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}},$$

and $\sup_{|x|\leq\lambda} v(x) \geq c\,\lambda^{\frac{2-n}{2}}$. *Here, c is a positive constant that depends only on n.*

Proof. Since the function $F(\xi,\varepsilon)$ has a strict local minimum at $(0,1)$ with $F(0,1) < 0$, we can find an open set $\Omega' \subset \Omega$ such that $(0,1) \in \Omega'$ and

$$F(0,1) < \inf_{(\xi,\varepsilon)\in\partial\Omega'} F(\xi,\varepsilon) < 0.$$

Using Proposition 12, we obtain

$$\begin{aligned}
&|\mathcal{F}_g(\lambda\xi,\lambda\varepsilon) - \lambda^{2(2m+2)}\,\mu^2\,F(\xi,\varepsilon)| \\
&\leq C\,\lambda^{\frac{(2m+2)2n}{n-2}}\,\mu^{\frac{2n}{n-2}} + C\,\lambda^{2m+2}\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} \\
&+ C\lambda^{t+2m+2}\Big(1+\log(\frac{\rho}{\lambda})\Big)\mu\beta + C\beta^2\lambda^{2t}\Big(1+\log(\frac{\rho}{\lambda})\Big)^2 + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}
\end{aligned}$$

for all $(\xi,\varepsilon) \in \Omega$. This implies

$$\begin{aligned}
&|\lambda^{-2(2m+2)}\mu^{-2}\,\mathcal{F}_g(\lambda\xi,\lambda\varepsilon) - F(\xi,\varepsilon)| \\
&\leq C\,\lambda^{\frac{(2m+2)4}{n-2}}\,\mu^{\frac{4}{n-2}} + C\,\rho^{\frac{2-n}{2}}\,\mu^{-1}\,\lambda^{\frac{n-2}{2}-(2m+2)} + C\,\rho^{2-n}\,\mu^{-2}\,\lambda^{n-2-2(2m+2)} \\
&+ C\,\lambda^{t-(2m+2)}\Big(1+\log(\frac{\rho}{\lambda})\Big)\,\mu^{-1}\,\beta + C\,\mu^{-2}\,\beta^2\,\lambda^{2t-2(2m+2)}\Big(1+\log(\frac{\rho}{\lambda})\Big)^2
\end{aligned}$$

for all $(\xi,\varepsilon) \in \Omega$. If $\rho^{2-n}\mu^{-2}\,\lambda^{n-2-2(2m+2)}$ and $\lambda^{t-(2m+2)}\Big(1+\log(\frac{\rho}{\lambda})\Big)\,\mu^{-1}$ are sufficiently small, then we have

$$\mathcal{F}_g(0,\lambda) < \inf_{(\xi,\varepsilon)\in\partial\Omega'} \mathcal{F}_g(\lambda\xi,\lambda\varepsilon) < 0.$$

Consequently, there exists a point $(\bar{\xi},\bar{\varepsilon}) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\varepsilon}) = \inf_{(\xi,\varepsilon)\in\Omega'} \mathcal{F}_g(\lambda\xi,\lambda\varepsilon) < 0.$$

By Proposition 3, the function $v = v_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}$ is a non-negative weak solution of the partial differential equation

$$\Delta_g v - \frac{n-2}{4(n-1)}\,R_g\,v + n(n-2)\,v^{\frac{n+2}{n-2}} = 0.$$

Using a result of N. Trudinger, we conclude that v is smooth (see [21], Theorem 3 on p. 271). Moreover, we have

$$\begin{aligned}
2(n-2)\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} &= 2(n-2)\,\Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}} + \mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\varepsilon}) \\
&< 2(n-2)\,\Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}}.
\end{aligned}$$

Finally, it follows from Proposition 2 that $\|v - u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\alpha$. This implies

$$|B_\lambda(0)|^{\frac{n-2}{2n}} \sup_{|x|\leq\lambda} v(x) \geq \|v\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} \geq \|u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} - C\,\alpha.$$

Hence, if α is sufficiently small, then we obtain $\lambda^{\frac{n-2}{2}} \sup_{|x|\leq\lambda} v(x) \geq c$.

The Theorem 1.1 will follow from the next result.

Proposition 21. *Let $n, l \in \mathbb{N}$ satisfy one of the following three conditions:*

(1) $n \geq 52$ *and* $l \geq 3$*;*
(2) $30 \leq n \leq 51$ *and* $l \geq 5$*;*
(3) $25 \leq n \leq 29$ *and* $l \geq 7$.

Then there exists a smooth metric g on $\mathbb{R}^n$ with the following properties:

(i) $g_{ik}(x) = \delta_{ik}$ *for* $|x| \geq \frac{1}{2}$
(ii) *There exists a sequence of non-negative smooth functions v_ν ($\nu \in \mathbb{N}$) such that*
$$\Delta_g v_\nu - \frac{n-2}{4(n-1)}\, R_g\, v_\nu + n(n-2)\, v_\nu^{\frac{n+2}{n-2}} = 0$$
for all $\nu \in \mathbb{N}$,
$$\int_{\mathbb{R}^n} v_\nu^{\frac{2n}{n-2}} < \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}}$$
for all $\nu \in \mathbb{N}$
(iii) $0 \in \mathbb{R}^n$ *is a blow-up point of v_ν*
(iv) $\nabla_g^j W_g(0) = 0$ *for all $0 \leq j < l$, but $\nabla_g^l W_g(0) \neq 0$.*

Proof. It follows from [5], [7], and Section 4 of the present paper that, for every $n \geq 25$, there exists a polynomial f with

$$\deg(f) = \begin{cases} 1 & \text{if} \quad n \geq 52, \\ 2 & \text{if} \quad 30 \leq n \leq 51, \\ 3 & \text{if} \quad 25 \leq n \leq 29, \end{cases}$$

and such that the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0, 1)$ with $F(0, 1) < 0$.

Let $\overline{W} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a multi-linear form with all the algebraic properties of the Weyl tensor such that

$$\sum_{i,k,p,q=1}^n (\overline{W}_{ipkq} + \overline{W}_{iqkp})^2 > 0.$$

We will further assume that

$$\overline{W}_{1pkq} = 0 \quad \text{for all } 1 \leq k, p, q \leq n$$

if l is even, and

$$\overline{W}_{1pkq} = \overline{W}_{2pkq} = 0 \quad \text{for all } 1 \leq k, p, q \leq n$$

if l is odd.

We define

$$T_{ik}(x) = |x'|^l \sum_{p,q=2}^{n} \overline{W}_{ipkq} x_p x_q$$

if l is even, and

$$T_{ik}(x) = x_2 |x'|^{l-1} \sum_{p,q=3}^{n} \overline{W}_{ipkq} x_p x_q$$

if l is odd, where

$$x' = (x_2, \ldots, x_n).$$

It is not difficult to check that $T_{ik}(x) = T_{ki}(x)$, $\sum_{i=1}^n x_i T_{ik}(x) = 0$, $\sum_{i=1}^n T_{ii}(x) = 0$, $\sum_{i=1}^n \partial_i T_{ik}(x) = 0$, and

$$T_{ik}(x_1, x') = T_{ik}(0, x'),$$

for all $x = (x_1, x') \in \mathbb{R}^n$.

Here $t = l + 2$.

Choose a smooth cutoff function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(s) = 1$ for $s \leq 1$ and $\eta(s) = 0$ for $s \geq 2$. We define a trace-free symmetric two-tensor on $\mathbb{R}^n$ by

$$\begin{aligned} h_{ik}(x) = \sum_{N=N_0}^{\infty} \eta(4N^2\,|x - y_N|)\, 2^{-(m+\frac{1}{8})N}\, f(2^N\,|x - y_N|^2)\, H_{ik}(x - y_N) \\ + \beta\, \eta(4|x|)\, T_{ik}(x), \end{aligned}$$

where $y_N = (\frac{1}{N}, 0, \ldots, 0) \in \mathbb{R}^n$. It is straightforward to verify that $h(x)$ is C^∞ smooth.

Moreover, if N_0 is sufficiently large and β is sufficiently small, then we have $h(x) = 0$ for $|x| \geq \frac{1}{2}$ and $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha$ for all $x \in \mathbb{R}^n$. (Here, α is the constant that appears in Proposition 20.)

We now define a Riemannian metric g by $g(x) = \exp(h(x))$. Since $T_{ik}(x) = T_{ik}(x - y_N)$ and $l > 2\deg(f)$, the metric

$$g^{(N)}(z) = g(z + y_N)$$

satisfies the hypotheses of Proposition 20 with $\rho_N = \frac{1}{4N^2}$, $\mu_N = 2^{-\frac{N}{8}}$, and $\lambda_N = 2^{-\frac{N}{2}}$. The assertion follows from Proposition 20 and the Taylor expansion

$$g_{ik}(x) = \delta_{ik} + \beta\, T_{ik}(x) + O(|x|^{2l+4})$$

at the origin. We are using that, in conformal normal coordinates, the Weyl tensor and the Riemann curvature tensor have the same order of vanishing at a point (see [11]).

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