COHOMOLOGICAL CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS

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1. Introduction

Projective spaces and hyperquadrics are the simplest projective algebraic varieties, and they can be characterized in many ways. The aim of this paper is to provide a new characterization of them in terms of positivity properties of the tangent bundle (Theorem 1.1).

The first result in this direction was Mori's proof of the Hartshorne conjecture in [Mor79] (see also Siu and Yau [SY80]), that characterizes projective spaces as the only manifolds having ample tangent bundle. Then, in [Wah83], Wahl characterized projective spaces as the only manifolds whose tangent bundles contain ample invertible subsheaves. Interpolating Mori's and Wahl's results, Andreatta and Wiśniewski gave the following characterization:

Theorem [AW01]. *Let X be a smooth complex projective n-dimensional variety. Assume that the tangent bundle T_X contains an ample locally free subsheaf $\mathscr{E}$ of rank r. Then $X \simeq \mathbb{P}^n$ and either $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus r}$ or $r = n$ and $\mathscr{E} = T_{\mathbb{P}^n}$.*

We note that earlier, in [CP98], Campana and Peternell obtained the same result for $r \geq n-2$.

Let $\mathscr{E}$ be an ample locally free subsheaf of $T_{\mathbb{P}^n}$ of rank $p < n$. By taking its determinant, we obtain a non-zero section in $H^0(\mathbb{P}^n, \wedge^p T_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(-p))$. On the other hand, most sections in $H^0(\mathbb{P}^n, \wedge^p T_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(-p))$ do not come from ample locally free subsheaves of $T_{\mathbb{P}^n}$.

This motivates the following characterization of projective spaces and hyperquadrics, which was conjectured by Beauville in [Bea00]. Here Q_p denotes a smooth quadric hypersurface in $\mathbb{P}^{p+1}$, and $\mathscr{O}_{Q_p}(1)$ denotes the restriction of $\mathscr{O}_{\mathbb{P}^{p+1}}(1)$ to Q_p. When $p = 1$, $(Q_1, \mathscr{O}_{Q_1}(1))$ is just $(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2))$.

Theorem 1.1. *Let X be a smooth complex projective n-dimensional variety and $\mathscr{L}$ an ample line bundle on X. If $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p}) \neq 0$ for some positive integer p, then either $(X, \mathscr{L}) \simeq (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$, or $p = n$ and $(X, \mathscr{L}) \simeq (Q_p, \mathscr{O}_{Q_p}(1))$.*

The statement of this theorem can be interpreted in the following way. Let X be a smooth complex projective n-dimensional variety and $\mathscr{L}$ an ample line bundle on X. Consider the sheaf $\mathscr{T}_{\mathscr{L}} := T_X \otimes \mathscr{L}^{-1}$. Then Wahl's theorem [Wah83] says that if $H^0(X, \mathscr{T}_{\mathscr{L}}) \neq 0$ then $X \simeq \mathbb{P}^n$. Theorem 1.1 generalizes this statement to the case when one only assumes that $H^0(X, \wedge^p \mathscr{T}_{\mathscr{L}}) \neq 0$ for some $0 < p \leq n$.

In order to prove Theorem 1.1, first notice that X is uniruled by [Miy87, Corollary 8.6]. Next observe that if the Picard number of X is 1, then it is necessarily a Fano variety. If the Picard number is larger than 1, then we fix a minimal covering family H of rational curves on X, and follow the strategy in [AW01] of looking at the H-rationally connected quotient $\pi : X^\circ \to Y^\circ$ of X (see Section 2 for definitions). We show that any non-zero section $s \in H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p})$ restricts to a non-zero section $s^\circ \in H^0(X^\circ, \wedge^p T_{X^\circ/Y^\circ} \otimes \mathscr{L}^{-p})$, except in the very special case when $p = 2$ and $X \simeq Q_2$. This is achieved in Section 5. Afterwards we need to deal with two cases: the case when X is a Fano manifold with Picard number 1, and the case in which the H-rationally connected quotient $\pi : X^\circ \to Y^\circ$ is either a projective space bundle or a quadric bundle, and $H^0(X^\circ, \wedge^p T_{X^\circ/Y^\circ} \otimes \mathscr{L}^{-p}) \neq 0$.

When X is a Fano manifold with Picard number $\rho(X) = 1$, the result follows from the following.

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Theorem 1.2 (= Theorem 6.3). *Let X be a smooth n-dimensional complex projective variety with $\rho(X) = 1$, $\mathscr{L}$ an ample line bundle on X, and p a positive integer. If $H^0(X, T_X^{\otimes p} \otimes \mathscr{L}^{-p}) \neq 0$, then either $(X, \mathscr{L}) \simeq (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$, or $p = n \geq 3$ and $(X, \mathscr{L}) \simeq (Q_p, \mathscr{O}_{Q_p}(1))$.*

The paper is organized as follows. In Section 2 we gather old and new results about minimal covering families of rational curves and their rationally connected quotients. In Section 3 we show that the relative anticanonical bundle of a generically smooth surjective morphism from a normal projective $\mathbb{Q}$-Gorenstein variety onto a smooth curve is never ample. This will be used to treat the case when the H-rationally connected quotient $\pi : X^\circ \to Y^\circ$ is a quadric bundle. In Section 4, we show that p-derivations can be lifted to the normalization. This technical result will be used in the following section, which is the technical core of the paper. In Section 5, we study the behavior of non-zero global sections of bundles of the form $\wedge^p T_X \otimes \mathscr{M}$ with respect to fibrations $X \to Y$. We also prove some general vanishing results, such as the following.

Theorem 1.3 (= Corollary 5.5). *Let X be a smooth complex projective variety and $\mathscr{L}$ an ample line bundle on X. If $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p-1-k}) \neq 0$ for integers $p \geq 1$ and $k \geq 0$, then $k = 0$ and $(X, \mathscr{L}) \simeq (\mathbb{P}^p, \mathscr{O}_{\mathbb{P}^p}(1))$.*

Finally, in Section 6 we prove Theorem 1.2 and put things together to prove Theorem 1.1.

Notation and definitions. Throughout the present article we work over the field of complex numbers unless otherwise noted. By a vector bundle we mean a locally free sheaf and by a line bundle an invertible sheaf. If X is a variety and $x \in X$, then $\kappa(x)$ denotes the residue field $\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$. Given a variety X, we denote by $\rho(X)$ the Picard number of X. If $\mathscr{E}$ is a vector bundle over a variety X, we denote by $\mathscr{E}^*$ its dual vector bundle, and by $\mathbb{P}(\mathscr{E})$ the Grothendieck projectivization $\mathrm{Proj}_X(\mathrm{Sym}(\mathscr{E}))$. For a morphism $f : X \to T$, the fiber of f over $t \in T$ is denoted by X_t.

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2. MINIMAL RATIONAL CURVES ON UNIRULED VARIETIES

In this section we gather some properties of minimal covering families of rational curves and their corresponding rationally connected quotients. For more details see [Kol96], [Deb01], or [AK03].

Let X be a smooth complex projective uniruled variety and H an irreducible component of $\mathrm{RatCurves}^n(X)$. Recall that only general points in H are in 1:1-correpondence with the associated curves in X.

We say that H is a *covering family of rational curves on* X if the corresponding universal family dominates X. A covering family H of rational curves on X is called *unsplit* if it is proper. It is called *minimal* if, for a general point $x \in X$, the subfamily of H parametrizing curves through x is proper. As X is uniruled, a minimal covering family of rational curves on X always exists. One can take, for instance, among all covering families of rational curves on X one whose members have minimal degree with respect to a fixed ample line bundle.

Fix a minimal covering family H of rational curves on X. Let C be a rational curve corresponding to a general point in H, with normalization morphism $f : \mathbb{P}^1 \to C \subset X$. We denote by $[C]$ or $[f]$ the point in H corresponding to C. We denote by $f^*T_X^+$ the subbundle of f^*T_X defined by

$$f^*T_X^+ = \mathrm{im}\left[H^0\big(\mathbb{P}^1, f^*T_X(-1)\big) \otimes \mathscr{O}_{\mathbb{P}^1}(1) \to f^*T_X\right] \hookrightarrow f^*T_X.$$

By [Kol96, IV.2.9], if $[f]$ is a general member of H, then $f^*T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$, where $d = \deg(f^*T_X) - 2 \geq 0$.

Given a point $x \in X$, we denote by H_x the normalization of the subscheme of H parametrizing rational curves passing through x. By [Kol96, II.1.7, II.2.16], if $x \in X$ is a general point, then H_x is a smooth projective variety of dimension $d = \deg(f^*T_X) - 2$. We remark that a rational curve that is smooth at x is parametrized by at most one element of H_x.

Let $H_1, \dots, H_k$ be minimal covering families of rational curves on X. For each i, let $\overline{H}_i$ denote the closure of H_i in $\mathrm{Chow}(X)$. We define the following equivalence relation on X, which we call $(H_1, \dots, H_k)$-equivalence. Two points $x, y \in X$ are $(H_1, \dots, H_k)$-equivalent if they can be connected by a chain of 1-cycles from $\overline{H}_1 \cup \dots \cup \overline{H}_k$. By [Cam92] (see also [Kol96, IV.4.16]), there exists a proper surjective morphism $\pi^\circ : X^\circ \to Y^\circ$ from a dense open subset of X onto a normal variety whose fibers are $(H_1, \dots, H_k)$-equivalence classes. We call this map the *$(H_1, \dots, H_k)$-rationally connected quotient of X*. When Y° is a point we say that X is $(H_1, \dots, H_k)$-rationally connected.

Remark 2.1. By [Kol96, IV.4.16], there is a universal constant c, depending only on the dimension of X, with the following property. If $H_1, \dots, H_k$ are minimal covering families of rational curves on X, and $x, y \in X$ are general points on a general $(H_1, \dots, H_k)$-equivalence class, then x and y can be connected by a chain of at most c rational cycles from $\overline{H}_1 \cup \dots \cup \overline{H}_k$.

The next two results are special features of the $(H_1, \dots, H_k)$-rationally connected quotient of X when the families $H_1, \dots, H_k$ are unsplit. The first one says that π° can be extended in codimension 1 to an equidimensional proper morphism with integral fibers, but possibly allowing singular fibers. The second one describes the general fiber of the H-rationally connected quotient of X when H is unsplit and H_x is irreducible for general $x \in X$.

Lemma 2.2. *Let X be a smooth complex projective variety and $H_1, \dots, H_k$ unsplit covering families of rational curves on X. Then there is an open subset X° of X, with $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$, a smooth variety Y°, and a proper surjective equidimensional morphism with irreducible and reduced fibers $\pi^\circ : X^\circ \to Y^\circ$ which is the $(H_1, \dots, H_k)$-rationally connected quotient of X.*

Proof. The fact that the $(H_1, \dots, H_k)$-rationally connected quotient of X can be extended in codimension 1 to an equidimensional proper morphism follows from the proof of [BCD07, Proposition 1]. This holds even in the more general context of quasi-unsplit covering families on $\mathbb{Q}$-factorial varieties. In [BCD07, Proposition 1] this is established for a single quasi-unsplit family, but the same proof works for finitely many quasi-unsplit families. For convenience we review the construction of that extension.

Let $\pi^\circ : X^\circ \to Y^\circ$ be the $(H_1, \dots, H_k)$-rationally connected quotient of X. By shrinking Y° if necessary, we may assume that π° is smooth. Let $Y \to \mathrm{Chow}(X)$ be the normalization of the closure of the image of Y° in $\mathrm{Chow}(X)$, and let $\mathcal{U} \subset Y \times X$ be the restriction of the universal family to Y. Denote by $p : \mathcal{U} \to Y$ and $q : \mathcal{U} \to X$ the induced natural morphisms. Notice that $q : \mathcal{U} \to X$ is birational.

Let $0 \in Y$ and set $\mathcal{U}_0 = p^{-1}(0)$. Then $q(\mathcal{U}_0)$ is contained in an $(H_1, \dots, H_k)$-equivalence class. This follows from taking limits of chains of rational curves from the families $H_1, \dots, H_k$ (see Remark 2.1), observing the assumption that the H_i's are unsplit, and the fact that the image of a general fiber of p in X is an $(H_1, \dots, H_k)$-equivalence class.

Let E be the exceptional locus of q. Since X is smooth, E has pure codimension 1 in $\mathcal{U}$. Set $S = q(E) \subset X$. This is a set of codimension at least 2 in X. We shall show that S is closed with respect to $(H_1, \dots, H_k)$-equivalence. From that it will follow that the morphism $p|_{\mathcal{U} \setminus E} : \mathcal{U} \setminus E \to Y \setminus p(E)$ is proper and induces a proper equidimensional morphism $X \setminus S \to Y \setminus p(E)$ extending π°. Let L be an effective ample divisor on Y. Then there exists an effective q-exceptional divisor F on $\mathcal{U}$ and an effective divisor D on X such that $p^*L + F = q^*D$. First we claim that $\mathrm{supp}\, F = E$. Indeed, let $C \subset E$ be any curve contracted by q. Then C is not contracted by p since $\mathcal{U} \subset Y \times X$. Hence $F \cdot C = q^*D \cdot C - p^*L \cdot C < 0$, and so $C \subset \mathrm{supp}\, F$. This proves the claim. Notice that the general fiber of p does not meet E. Therefore, for any curve $C \subset \mathcal{U}$ contained in a general fiber of p, we have $q^*D \cdot C = 0$. This shows in particular that $D \cdot \ell = 0$ for any curve ℓ from any of the families $H_1, \dots, H_k$. If $\tilde{\ell} \subset \mathcal{U}$ is mapped onto ℓ by q, then $F \cdot \tilde{\ell} = q^*D \cdot \tilde{\ell} - p^*L \cdot \tilde{\ell} \leq 0$. Hence either $\tilde{\ell}$ is contained in $E = \mathrm{supp}\, F$ or it is disjoint from it. Therefore, if ℓ is a curve from any of the families $H_1, \dots, H_k$, then either $\ell \subset S$ or $\ell \cap S = \emptyset$. In other words, S is closed with respect to $(H_1, \dots, H_k)$-equivalence.

Replace X° with $X \setminus S$ and Y° with $Y \setminus p(E)$, obtaining a proper equidimensional morphism $\pi^\circ : X^\circ \to Y^\circ$ with $\mathrm{codim}(X \setminus X^\circ) \geq 2$. Since Y is normal, we may also replace Y° with its smooth locus and we still have the condition $\mathrm{codim}(X \setminus X^\circ) \geq 2$.

The locus B of Y° over which π° has multiple fibers has codimension at least 2 in Y°. To see this, compactify Y° to a smooth projective variety $\bar{Y}$ and take a resolution $\bar{\pi} : \bar{X} \to \bar{Y}$ of the indeterminacies of $X \dashrightarrow \bar{Y}$ with $\bar{X}$ smooth and projective. Let $\bar{C} \subset \bar{Y}$ be a smooth projective curve obtained by intersecting $\dim \bar{Y} - 1$ general very ample divisors on $\bar{Y}$. Let $\bar{\pi}_{\bar{C}} : \bar{X}_{\bar{C}} \to \bar{C}$ be the corresponding morphism. Then $\bar{X}_{\bar{C}}$ is smooth projective and the general fiber of $\bar{\pi}$ is rationally connected. Hence $\bar{\pi}_{\bar{C}}$ has a section by [GHS03], and thus it cannot contain multiple fibers. Now, replace Y° with $Y^\circ \setminus B$ to obtain an equidimensional proper morphism with no multiple fibers.

Let F be a general fiber of π°. For each i, denote by H_i^j, $1 \le j \le n_i$, the unsplit covering families of rational curves on F whose general members correspond to rational curves on X from the family H_i. Let $[H_i^j]$ denote the class of a member of H_i^j in $N_1(F)$ and $\mathcal{H} := \{[H_i^j] \mid i = 1, \ldots, k, j = 1, \ldots, n_i\}$. Then by [Kol96, IV.3.13.3], $N_1(F)$ is generated by $\mathcal{H}$.

Finally we shall show that the locus B' of Y° over which the fibers of π° are not integral has codimension at least 2 in Y°. Let $C \subset Y^\circ$ be a smooth curve obtained by intersecting $\dim Y^\circ - 1$ general very ample divisors on Y°. Let $\pi_C : X_C \to C$ be the corresponding morphism. Then X_C is smooth. We denote the image of the classes $[H_i^j]$'s in $N_1(X_C)$ and their collection $\mathcal{H}$ by the same symbols. By taking limits of chains of rational curves from the families $H_1, \ldots, H_k$ and applying [Kol96, IV.3.13.3] (see Remark 2.1), we see that any curve contained in any fiber of π_C is numerically proportional in $N_1(X_C)$ to a linear combination of the $[H_i^j]$'s. Hence $N_1(X_C/C)$ is generated by $\mathcal{H}$. Therefore, all fibers of π_C are irreducible. Indeed, if F_0' is an irreducible component of a reducible fiber F_0, then F_0' is a Cartier divisor on X_C, and $F_0' \cdot [H_i^j] = 0$ for every H_i^j. On the other hand, there is a curve $\ell \subset F_0$ such that $F_0' \cdot \ell > 0$, contradicting the fact that $N_1(X_C/C)$ is generated by $\mathcal{H}$. Since there are no multiple fibers, the fibers are also reduced. Finally, we replace Y° with $Y^\circ \setminus B'$ and obtain a morphism with the required properties. □

Proposition 2.3. *Let X be a smooth complex projective variety and H an unsplit covering family of rational curves on X. Assume that H_x is irreducible for general $x \in X$. Let $\pi^\circ : X^\circ \to Y^\circ$ be the H-rationally connected quotient of X. Then the general fiber of π° is a Fano manifold with Picard number* 1.

Proof. Let X_t be a general fiber of π°, and suppose $\rho(X_t) \neq 1$. Denote by $[H]$ the class of the members of H in $N_1(X)$. By [Kol96, IV.3.13.3], every proper curve on X_t is numerically proportional to $[H]$ in $N_1(X)$. There exists an irreducible component H_t of $H_{X_t} = \{[C] \in H \mid C \subset X_t\}$ which is an unsplit covering family of rational curves on X_t. Since H_x is irreducible for general $x \in X$, such a component H_t is unique. Since $\rho(X_t) \neq 1$, X_t is not H_t-rationally connected by [Kol96, IV.3.13.3]. Let $\sigma_t : X_t^\circ \to Z_t^\circ$ be the (nontrivial) H_t-rationally connected quotient of X_t. Notice that for every $z \in Z_t^\circ$ there is a curve $C_z \subset X_t$ numerically proportional to $[H]$ in $N_1(X)$, meeting the fiber of σ_t over z, but not contained in it. Since H_t is unique, there is a dense open subset X' of X and a fibration $\sigma : X' \to Z'$ whose fibers are fibers of σ_t for some $t \in Y^\circ$. Moreover, there is a curve $C \subset X$ numerically proportional to $[H]$ in $N_1(X)$, meeting X', and not contracted by σ. But this is impossible. Indeed, let L' be an effective divisor on Z' meeting but not containing the image of C by σ. Let L be the closure of $\sigma^{-1}(L')$ in X. Then $L \cdot C > 0$ while $L \cdot \ell = 0$ for any curve ℓ parametrized by H lying on a fiber of σ. □

Remark 2.4. The statement of Proposition 2.3 does not hold in general if we do not assume that H_x is irreducible for general $x \in X$. Indeed, one may take $\pi^\circ : X^\circ \to Y^\circ$ to be a suitable family of quadric surfaces in $\mathbb{P}^3$ and H to be the family of lines on the fibers of π°.

Definition 2.5. Let X be a smooth complex projective variety, and H a minimal covering family of rational curves on X. Let $x \in X$ be a general point. Define the tangent map $\tau_x : H_x \dashrightarrow \mathbb{P}(T_xX^*)$ by sending a curve that is smooth at x to its tangent direction at x. Define $\mathcal{C}_x$ to be the image of τ_x in $\mathbb{P}(T_xX^*)$. This is called the *variety of minimal rational tangents* at x associated to the minimal family H.

The map $\tau_x : H_x \to \mathcal{C}_x$ is in fact the normalization morphism by [Keb02] and [HM04]. If τ_x is an immersion at every point of H_x, then all curves parametrized by H_x are smooth at x by [Kol96, V.3.6] and [Ara06, Proposition 2.7], and, as a consequence, there is a one-to-one corresponcence between points of H_x and the associated curves on X. The variety $\mathcal{C}_x$ comes with a natural projective embedding into $\mathbb{P}(T_xX^*)$. This embedding encodes important geometric properties of X. The following result was proved in [Ara06] and gives a structure theorem for varieties whose variety of minimal rational tangents is linear.

Theorem 2.6 [Ara06]. *Let X be a smooth complex projective variety, H a minimal covering family of rational curves on X, and $\mathcal{C}_x \subset \mathbb{P}(T_xX^*)$ the corresponding variety of minimal rational tangents at $x \in X$. Suppose that for a general $x \in X$, $\mathcal{C}_x$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX^*)$.*

Then there exists an open subset $X^\circ \subset X$ and a $\mathbb{P}^{d+1}$-bundle $\varphi^\circ : X^\circ \to T^\circ$ over a smooth base with the property that every rational curve parametrized by H and meeting X° is a line on a fiber of φ°. In particular, $\varphi^\circ : X^\circ \to T^\circ$ is the H-rationally connected quotient of X. If H is unsplit, then we may take X° such that $\operatorname{codim}(X \setminus X^\circ) \geq 2$.

Proposition 2.7. *Let X be a smooth complex projective variety, H a minimal covering family of rational curves on X, and $\pi^\circ : X^\circ \to Y^\circ$ the H-rationally connected quotient of X. Suppose that the tangent bundle T_X contains a subsheaf $\mathscr{D}$ such that $f^*\mathscr{D}$ is an ample vector bundle for a general member $[f] \in H$. Then π° is a projective space bundle and the inclusion $\mathscr{D}|_{X^\circ} \hookrightarrow T_{X^\circ}$ factors through an inclusion $\mathscr{D}|_{X^\circ} \hookrightarrow T_{X^\circ/Y^\circ}$.*

Proof. Let $\mathcal{C}_x \subset \mathbb{P}(T_xX^*)$ be the variety of minimal rational tangents associated to H at a general point $x \in X$. By [Ara06, Proposition 4.1], $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_xX^*)$ containing $\mathbb{P}(\mathscr{D}^* \otimes \kappa(x))$. In [Ara06, Proposition 4.1] $\mathscr{D}$ is assumed to be ample, but the proof only uses the fact that $f^*\mathscr{D}$ is a subsheaf of $f^*T_X^+$ for general $[f] \in H$.

We shall prove that H_x is irreducible, and thus $\mathcal{C}_x$ is a linear subspace of $\mathbb{P}(T_xX^*)$.

For a general $x \in X$, denote by H_x^i, $1 \leq i \leq k$, the irreducible components of H_x, and by $\mathcal{C}_x^i$ the image of $\tau_x|_{H_x^i}$. Recall that H_x is smooth and hence the H_x^i are disjoint. Furthermore, each $\mathcal{C}_x^i$ is a d-dimensional linear subspace of $\mathbb{P}(T_xX^*)$ containing $\mathbb{P}(\mathscr{D}^* \otimes \kappa(x))$. Thus τ_x is an immersion at every point of H_x, hence all curves parametrized by H_x are smooth at x, and there is a one-to-one corresponcence between points of H_x and the associated curves on X. We shall produce a curve l through x such that, for every $i \in \{1, \dots, k\}$, there exists an element in H_x^i parametrizing l. Since there is a one-to-one corresponcence between points of H_x and the associated curves on X, there exists a unique point in H_x parametrizing l, yielding that H_x is irreducible.

For each $i \in \{1, \dots, k\}$, set $Y_i = \operatorname{locus}(H_x^i)$, and let $\eta_i : \tilde{Y}_i \to Y_i$ be the normalization morphism. Since $f^*\mathscr{D}$ is ample for general $[f] \in H$, one has an injection $\mathscr{D}|_{Y_i^\circ} \hookrightarrow T_{Y_i^\circ}$ over the smooth locus Y_i° of Y_i. By [Ara06, Lemma 3.3], $\tilde{Y}_i \simeq \mathbb{P}^{d+1}$. Under this isomorphism, the rational curves on Y_i parametrized by H_x^i come from the lines on $\mathbb{P}^{d+1}$ passing through a fixed point $x_i \in \tilde{Y}_i$. By [Ara06, Lemma 4.5], the restricted map $\mathscr{D}|_{Y_i} \hookrightarrow T_X|_{Y_i}$ induces an injection of sheaves $\mathscr{D}_i := \eta_i^*\mathscr{D}|_{Y_i} \hookrightarrow T_{\mathbb{P}^{d+1}}$. Furthermore, $\mathscr{D}_i|_l$ is ample for a general line $l \subset \mathbb{P}^{d+1}$. This implies that there is a line $l_i \subset \mathbb{P}^{d+1}$ through x_i and an injection $T_{l_i} \hookrightarrow \mathscr{D}_i|_{l_i}$. If $\mathscr{D}_i$ is not ample, the existence of such l_i follows from [OSS80, Theorem 3.2.1]. If $\mathscr{D}_i$ is ample, it follows by the same argument as in [Ara06, Section 4, pages 946–947].

Fix $i_0 \in \{1, \dots, k\}$. Let $l_{i_0} \subset \mathbb{P}^{d+1}$ be a line through x_{i_0} with $T_{l_{i_0}} \hookrightarrow \mathscr{D}_{i_0}|_{l_{i_0}}$ an injection of sheaves. Set $l = n_{i_0}(l_{i_0})$. Then l is smooth at x and over its smooth locus l° one has an injection $T_{l^\circ} \hookrightarrow \mathscr{D}|_{l^\circ}$. Hence $l \subset Y_i$ for every i, and thus, for every $i \in \{1, \dots, k\}$, l is the image of a line through x_i in $\tilde{Y}_i \simeq \mathbb{P}^{d+1}$, and hence there exists an element in H_x^i parametrizing l. This shows that H_x is irreducible as we noted above.

Now we apply Theorem 2.6 to conclude that π° is a projective space bundle. Moreover, for a general point $x \in X^\circ$, the stalk $\mathscr{D}_x$ is contained in $(T_{X^\circ/Y^\circ})_x$. Since the cokernel of $T_{X^\circ/Y^\circ} \hookrightarrow T_{X^\circ}$ is torsion free, we conclude that there is an inclusion $\mathscr{D}|_{X^\circ} \hookrightarrow T_{X^\circ/Y^\circ}$ factoring $\mathscr{D}|_{X^\circ} \hookrightarrow T_{X^\circ}$. □

3. The relative anticanonical bundle of a fibration

In this section we prove that the relative anticanonical bundle of a generically smooth surjective morphism from a normal projective $\mathbb{Q}$-Gorenstein variety onto a smooth curve cannot be ample. In fact, we prove the following more general result. Note that a similar theorem was proved in [Miy93, Theorem 2].

Theorem 3.1. *Let X be a normal projective variety, $f : X \to C$ a surjective morphism onto a smooth curve, and $\Delta \subseteq X$ a Weil divisor such that (X, Δ) is log canonical over the generic point of C. Then $-(K_{X/C} + \Delta)$ is not ample.*

Proof. Let $X \stackrel{g}{\to} \tilde{C} \stackrel{\sigma}{\to} C$ be the Stein factorization of f. Then $K_{\tilde{C}} = \sigma^*K_C + R_\sigma$ where R_σ is the ramification divisor of σ and so $-(K_{X/\tilde{C}} + \Delta) = -(K_{X/C} + \Delta) + g^*R_\sigma$. Notice that R_σ is effective and hence if $-(K_{X/C} + \Delta)$ is ample, then so is $-(K_{X/\tilde{C}} + \Delta)$.

Thus, in order to prove the statement, we may assume that f has connected fibers. Let us now assume to the contrary that $-(K_{X/C}+\Delta)$ is ample. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities of (X,Δ), A an ample divisor on C, and $m \gg 0$ such that $D = -m(K_{X/C}+\Delta) - f^*A$ is very ample. Then

$$K_{\tilde{X}} + \pi_*^{-1}\Delta \sim_{\mathbb{Q}} \pi^*(K_X+\Delta) + E_+ - E_-,$$

where E_+ and E_- are effective π-exceptional divisors with no common components and such that the support of $\pi_*^{-1}\Delta + E_+ + E_-$ is an snc divisor. By the log canonical assumption, E_- can be decomposed as $E_- = E + F$ where $\lceil E \rceil$ is reduced and E_- agrees with E over the generic point of C. Set $\tilde{f} = f \circ \pi$ and let $\tilde{D} \in |\pi^* D|$ be a general member. Setting $\tilde{\Delta} = \pi_*^{-1}\Delta + \frac{1}{m}\tilde{D} + E$, we obtain that $(\tilde{X}, \tilde{\Delta})$ is log canonical and that

$$K_{\tilde{X}} + \tilde{\Delta} + F \sim_{\mathbb{Q}} \tilde{f}^* K_C + E_+ - \frac{1}{m}\tilde{f}^*A. \tag{3.1.1}$$

Furthermore, since E_+ is π-exceptional, $\pi_*\mathscr{O}_{\tilde{X}}(lE_+)$ is an ideal sheaf in $\mathscr{O}_X$ for any $l \in \mathbb{Z}$ (see for instance [Deb01, Lemma 7.11]). Then for any $l \in \mathbb{N}$ sufficiently divisible,

$$\begin{aligned} \tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C}+\tilde{\Delta})) &\overset{\iota}{\hookrightarrow} \tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C}+\tilde{\Delta}+F)) \simeq \\ &\simeq \tilde{f}_*\mathscr{O}_{\tilde{X}}(l(mE_+ - \tilde{f}^*A)) \simeq \tilde{f}_*\mathscr{O}_{\tilde{X}}(lmE_+) \otimes \mathscr{O}_C(-lA) \subseteq \mathscr{O}_C(-lA). \end{aligned}$$

Finally, observe that

- $\tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C}+\tilde{\Delta}+F))$ is nonzero by (3.1.1) and because E_+ is effective,
- $\tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C}+\tilde{\Delta}))$ is semi-positive by [Cam04, Thm. 4.13], and
- ι is an isomorphism over a nonempty open subset of C.

Therefore, $\tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C}+\tilde{\Delta}))$ is a non-zero semi-positive sheaf contained in $\mathscr{O}_C(-lA)$, but that contradicts the fact that A is ample. □

4. LIFTING p-DERIVATIONS TO THE NORMALIZATION

In this section we show that p-derivations (see definition 4.4 below) can be lifted to the normalization. This is a generalization of Seidenberg's theorem in [Sei66]. The proofs in this section follow closely the proof of Theorem 2.1.1 in [Käl06] and we also use the following result from [Käl06].

Lemma 4.1 [Käl06, Lemma 2.1.2]. *Let $(A,\mathfrak{m},k)$ be a local Noetherian domain and ∂ a derivation of A. Let ν be a discrete valuation on the fraction field $K(A)$ with center in A. Then there exists a $c \in \mathbb{Z}$ such that $\nu\left(\frac{\partial(x)}{x}\right) \geq c$ for any $x \in K(A) \setminus \{0\}$.*

Definition 4.2. Let R be a ring, A an R-algebra and M an A-module. Denote by $\Omega_{A/R}$ the module of relative differentials of A over R. Given a positive integer p, we denote by $\Omega^p_{A/R}$ the p-th wedge power of $\Omega_{A/R}$. A *p-derivation of A over R with values in M* is an A-linear map $\partial : \Omega^p_{A/R} \to M$. Such a map ∂ induces a skew symmetric map $K(A)^{\oplus p} \to M \otimes_A K(A)$, where $K(A)$ denotes the fraction field of A. We use the same symbol ∂ to denote this induced map. When $M = A$ and R is clear from the context, we call ∂ simply a *p-derivation of A*.

Lemma 4.3. *Let $(A,\mathfrak{m},k)$ be a local Noetherian domain, p a positive integer, and ∂ a p-derivation of A. Let ν be a discrete valuation on the fraction field $K(A)$ with center in A. Then there exists $c \in \mathbb{Z}$ such that $\nu\left(\frac{\partial(x_1,\ldots,x_p)}{x_1\cdots x_p}\right) \geq c$ for any $x_1,\ldots,x_p \in K(A)\setminus\{0\}$.*

Proof. We use induction on p. If $p=1$, this is Lemma 4.1. Suppose now that $p \geq 2$ and let $(A,\mathfrak{m},k)$ be a local Noetherian domain, ∂ a p-derivation of A, and ν a discrete valuation on the fraction field $K(A)$ with center in A. Let $m_1,\ldots,m_r$ be generators of the maximal ideal $\mathfrak{m}$.

Using the formula

$$\frac{\partial(x_{1,1}x_{1,2},\ldots,x_{p,1}x_{p,2})}{x_{1,1}x_{1,2}\cdots x_{p,1}x_{p,2}} = \sum \frac{\partial(x_{1,i_1},\ldots,x_{p,i_p})}{x_{1,i_1}\cdots x_{p,i_p}},$$

we get

$$\nu\left(\frac{\partial(x_{1,1}x_{1,2},\dots,x_{p,1}x_{p,2})}{x_{1,1}x_{1,2}\cdots x_{p,1}x_{p,2}}\right)\geq \min\left\{\nu\left(\frac{\partial(x_{1,i_1},\dots,x_{p,i_p})}{x_{1,i_1}\cdots x_{p,i_p}}\right)\right\}$$

for $x_{1,1},x_{1,2},\dots,x_{p,1},x_{p,2}\in A\setminus\{0\}$. Further observe that

$$\frac{\partial(x_1^{-1},x_2,\dots,x_p)}{x_1^{-1}x_2\cdots x_p}=-\frac{\partial(x_1,\dots,x_p)}{x_1\cdots x_p}.$$

Also, if $a\in A$, then a may be written as a sum of products $m_{i_1}\cdots m_{i_k}u$ with $u\in A\setminus\mathfrak{m}$. Therefore we only have to check that the required inequality holds for $x_1,\dots,x_p\in\{m_1,\dots,m_r\}\cup(A\setminus\mathfrak{m})$.

If $x_1,\dots,x_p\in A\setminus\mathfrak{m}$ then

$$\nu\left(\frac{\partial(x_1,\dots,x_p)}{x_1\cdots x_p}\right)=\nu(\partial(x_1,\dots,x_p))\geq 0.$$

Suppose now that at least one of the x_i's is in $\mathfrak{m}$. For simplicity we assume that $x_1,\dots,x_l\in A\setminus\mathfrak{m}$ and $x_{l+1},\dots,x_p\in\{m_1,\dots,m_r\}$, $0\leq l<p$. We may view $\partial(\cdot,\cdots,\cdot,x_{l+1},\dots,x_p)$ as an l-derivation of A. The result then follows by induction. □

Definition 4.4. Let S be a scheme, X a scheme over S, and $\mathscr{L}$ a line bundle on X. Denote by $\Omega_{X/S}$ the sheaf of relative differentials of X over S, and by $\Omega^p_{X/S}$ its p-th wedge power for $p\in\mathbb{N}$. A *p-derivation of X over S with values in $\mathscr{L}$* is a morphism of sheaves $\partial:\Omega^p_{X/S}\to\mathscr{L}$. When S is the spectrum of a field and $\mathscr{L}$ is clear from the context, we drop S and $\mathscr{L}$ from the notation and call ∂ simply a *p-derivation on X*.

Proposition 4.5. *Let X be a Noetherian integral scheme over a field k of characteristic zero and $\eta:\widetilde{X}\to X$ its normalization. Let $\mathscr{L}$ be a line bundle on X, p a positive integer, and ∂ a p-derivation with values in $\mathscr{L}$. Then ∂ extends to a unique p-derivation $\bar{\partial}$ on $\widetilde{X}$ with values in $\eta^*\mathscr{L}$.*

Proof. The uniqueness of $\bar{\partial}$ is clear since $\mathscr{L}$ is torsion free and η is birational. The existence of the lifting can be established locally. So we may assume that X is the spectrum of an integral k-algebra A, $\mathscr{L}$ is trivial, and ∂ is a p-derivation of A. Let A' denote the integral closure of A in its fraction field $K(A)$. There exists a unique extension of ∂ to a p-derivation of $K(A)$, which we also denote by ∂. We must prove that $\partial(A',\dots,A')\subset A'$.

First we reduce the problem to the case when A is a 1-dimensional local ring and A' is a DVR. Since A' is integrally closed in $K(A)$, A' is the intersection of its localizations at primes of height one [Mat80, 2. Theorem 38]. Let $\mathfrak{p}'$ be a prime of height one of A', and set $\mathfrak{p}=\mathfrak{p}'\cap A$. Notice that $\partial(A_\mathfrak{p},\dots,A_\mathfrak{p})\subset A_\mathfrak{p}$, and the result follows if we prove that $\partial(A'_{\mathfrak{p}'},\dots,A'_{\mathfrak{p}'})\subset A'_{\mathfrak{p}'}$. Hence we may assume that A is a 1-dimensional local ring and A' is a DVR. Denote by $\mathfrak{m}$ and $\mathfrak{m}'$ the maximal ideals of A and A' respectively.

Next we further reduce the problem to the case when A and A' are complete local rings. Let $\bar{R}$ be the completion of A' with respect to the $\mathfrak{m}'$-adic topology. Let $\bar{A}$ be the completion of A with respect to the $\mathfrak{m}$-adic topology. Since $\bar{A}$ is 1-dimensional, there is an inclusion of local rings $\bar{A}\subset\bar{R}$. Let ν be a discrete valuation of $K(A')$ whose valuation ring is A'. By Lemma 4.3, ∂ is a continuous p-derivation of R with values in $K(A')$. Hence it has a unique extension to a continuous p-derivation $\bar{\partial}$ of $K(\bar{R})$. Notice that the condition $\partial(A,\dots,A)\subset A$ implies that $\partial(\bar{A},\dots,\bar{A})\subset\bar{A}$ by the Artin-Rees Lemma. Since $K(A)\cap\bar{R}=A'$, the result then follows if we prove that $\partial(\bar{R},\dots,\bar{R})\subset\bar{R}$. Therefore we may assume that A and A' are complete 1-dimensional local rings.

Now we use induction on p. If $p=1$, this is Seidenberg's theorem [Sei66], so we may assume that $p\geq 2$. Let k_A be a coefficient field in A, and $k_{A'}$ a coefficient field in A' containing k_A [Eis95, Theorem 7.8]. The extension $k_{A'}|k_A$ is finite. Let $t\in\mathfrak{m}'$ be a uniformizing parameter. It suffices to show that $\partial(x_1,\dots,x_p)\in A'$ for $x_1,\dots,x_p\in k_{A'}\cup\{t\}$. Since ∂ is skew symmetric and $p\geq 2$, we have $\partial(t,\dots,t)=0$. So we may assume that $x_1\in k_{A'}$. Since $k_{A'}|k_A$ is finite and separable, there exists $P(X)=\sum a_iX^i\in k_A[X]$ such that $P(x_1)=0$ and $P'(x_1)\neq 0$. Thus

$$0=\partial(P(x_1),x_2,\dots,x_p)=P'(x_1)\partial(x_1,\dots,x_p)+\sum\partial(a_i,x_2,\dots,x_p)x_1^i.$$

Finally, $\partial(a_i,\_,\dots,\_)$ may be viewed as a $p-1$ derivation of A and so $\partial(x_1,\dots,x_p)\in A'$ by the induction hypothesis. □

5. SECTIONS OF $\wedge^p T_X \otimes \mathscr{M}$

The following lemma will be used several times in this section.

Lemma 5.1. *Let Y be a smooth variety, $\pi : X \to Y$ a smooth morphism, $\mathscr{M}$ a line bundle on X, and $p \geq 2$ an integer. Suppose that for a general fiber, F, of π, $H^0(F, \wedge^i T_F \otimes \mathscr{M}|_F) = 0$ for $0 \leq i \leq p-2$. Then there exists an exact sequence:*

$$0 \to H^0(X, \wedge^p T_{X/Y} \otimes \mathscr{M}) \to H^0(X, \wedge^p T_X \otimes \mathscr{M}) \to H^0(X, \wedge^{p-1} T_{X/Y} \otimes \pi^* T_Y \otimes \mathscr{M}).$$

Proof. The short exact sequence

$$0 \to T_{X/Y} \to T_X \to \pi^* T_Y \to 0$$

yields a filtration $\wedge^p T_X \otimes \mathscr{M} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \mathscr{F}_2 \supseteq \cdots \supseteq \mathscr{F}_p \supseteq \mathscr{F}_{p+1} = 0$ such that

$$\mathscr{F}_i / \mathscr{F}_{i+1} \simeq \wedge^i T_{X/Y} \otimes \pi^* \wedge^{p-i} T_Y \otimes \mathscr{M}$$

for each i. In particular, one has the short exact sequence,

$$0 \to \wedge^p T_{X/Y} \otimes \mathscr{M} \to \mathscr{F}_{p-1} \to \wedge^{p-1} T_{X/Y} \otimes \pi^* T_Y \otimes \mathscr{M} \to 0. \tag{5.1.1}$$

The assumption that $H^0(F, \wedge^i T_F \otimes \mathscr{M}|_F) = 0$ for $0 \leq i \leq p-2$ for a general fiber of π implies that $H^0(X, \mathscr{F}_i/\mathscr{F}_{i+1}) = 0$ for $0 \leq i \leq p-2$, thus $H^0(X, \wedge^p T_X \otimes \mathscr{M}) = H^0(X, \mathscr{F}_0) = \cdots = H^0(X, \mathscr{F}_{p-1})$ and the result follows from (5.1.1). □

The condition that $H^0(F, \wedge^i T_F \otimes \mathscr{M}|_F) = 0$ for $0 \leq i \leq p-2$ and F a general fiber of π is easily verified when π is a projective space bundle and $\mathscr{M}|_F$ is sufficiently negative. In this case we get the following.

Lemma 5.2. *Let Y be a smooth projective variety of dimension ≥ 1, $\mathscr{E}$ an ample vector bundle of rank $r+1 \geq 2$ and $\mathscr{N}$ a nef line bundle on Y. Consider the projective bundle $\pi : X = \mathbb{P}(\mathscr{E}) \to Y$ with tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let $p, q \in \mathbb{N}$ and assume that $p \geq 2$. Then*

$$H^0(X, \wedge^p T_{X/Y} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^* \mathscr{N}^{-1}) = 0. \tag{5.2.1}$$

Proof. First observe, that if $p > r$ then the statement is trivially true, so we will assume that $p \leq r$. Let $i \in \mathbb{N}$, $i < p$. After twisting by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^* \mathscr{N}^{-1}$, the short exact sequence

$$0 \to \wedge^{p-i-1} T_{X/Y} \to \wedge^{p-i} (\pi^* \mathscr{E}^*(1)) \to \wedge^{p-i} T_{X/Y} \to 0$$

yields the exact sequence

$$\begin{aligned} \cdots \to H^i(X, \wedge^{p-i}(\pi^*\mathscr{E}^*)(-i-q) \otimes \pi^*\mathscr{N}^{-1}) \to H^i(X, \wedge^{p-i} T_{X/Y}(-p-q) \otimes \pi^* \mathscr{N}^{-1}) \to \\ \to H^{i+1}(X, \wedge^{p-i-1} T_{X/Y}(-p-q) \otimes \pi^* \mathscr{N}^{-1}) \to \dots \end{aligned} \tag{5.2.2}$$

Since $i < p \leq r$ and $R^j \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(l) = 0$ for $0 < j < r$ and for any $l \in \mathbb{Z}$, the Leray spectral sequence implies that

$$H^i(X, \wedge^{p-i}(\pi^*\mathscr{E}^*)(-i-q) \otimes \pi^*\mathscr{N}^{-1}) = H^i(Y, \wedge^{p-i}\mathscr{E}^* \otimes \mathscr{N}^{-1} \otimes \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-i-q)).$$

The sheaf $\pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-i-q)$ is zero unless $i = q = 0$, in which case it is isomorphic to $\mathscr{O}_Y$. Furthemore, $H^0(Y, \wedge^p \mathscr{E}^* \otimes \mathscr{N}^{-1}) = 0$ since $\mathscr{E}$ is ample and $\mathscr{N}$ is nef, and hence

$$H^i(X, \wedge^{p-i}(\pi^*\mathscr{E}^*)(-i-q) \otimes \pi^*\mathscr{N}^{-1}) = 0$$

for $0 \leq i \leq p-1$. Therefore, by (5.2.2), one has a series of injections,

$$\begin{aligned} H^0(X, \wedge^p T_{X/Y}(-p-q) \otimes \pi^*\mathscr{N}^{-1}) \hookrightarrow H^1(X, \wedge^{p-1} T_{X/Y}(-p-q) \otimes \pi^*\mathscr{N}^{-1}) \hookrightarrow \dots \\ \dots \hookrightarrow H^i(X, \wedge^{p-i} T_{X/Y}(-p-q) \otimes \pi^*\mathscr{N}^{-1}) \hookrightarrow \dots \hookrightarrow H^p(X, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^*\mathscr{N}^{-1}). \end{aligned}$$

By the Kodaira vanishing theorem $H^p(X, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^*\mathscr{N}^{-1}) = 0$, and the statement follows. □

Corollary 5.3. *Let Y be a smooth projective variety of dimension ≥ 1 and $\mathscr{E}$ an ample vector bundle of rank $r+1 \geq 2$ on Y. Consider the projective bundle $\pi : X = \mathbb{P}(\mathscr{E}) \to Y$ with tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Suppose that $H^0(X, \wedge^p T_X \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^*\mathscr{N}^{-1}) \neq 0$ for some integers $p \geq 2$, $q \geq 0$, and some nef line bundle $\mathscr{N}$ on Y. Then $Y \simeq \mathbb{P}^1$, $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, $p = 2$, $q = 0$, and $\mathscr{N} \simeq \mathscr{O}_{\mathbb{P}^1}$.*

Proof. Let $F \simeq \mathbb{P}^r$ denote a general fiber of π and set $\mathscr{M} = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q) \otimes \pi^*\mathscr{N}^{-1}$. Then by Bott's formula $H^0(F, \wedge^i T_F \otimes \mathscr{M}|_F) = 0$ for every $0 \le i \le p-2$. Then Lemma 5.1 and Lemma 5.2 imply that $H^0(X, \wedge^{p-1}T_{X/Y} \otimes \pi^*(T_Y \otimes \mathscr{N}^{-1}) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-p-q)) \neq 0$. By Bott's formula again $H^0(F, \wedge^{p-1}T_F(-p-q)) \neq 0$ implies that $q = 0$ and $r = p-1$. Therefore we have

$$0 \neq H^0(X, \wedge^r T_{X/Y} \otimes \pi^*(T_Y \otimes \mathscr{N}^{-1}) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-r-1)) = H^0(X, \pi^*\left(T_Y \otimes \det \mathscr{E}^* \otimes \mathscr{N}^{-1}\right)) \simeq H^0(Y, \pi_*\pi^*\left(T_Y \otimes \det \mathscr{E}^* \otimes \mathscr{N}^{-1}\right)) \simeq H^0(Y, T_Y \otimes \big(\underbrace{\det \mathscr{E} \otimes \mathscr{N}}_{\text{ample}}\big)^{-1}).$$

Now Wahl's theorem [Wah83] yields that $Y \simeq \mathbb{P}^m$ for some $m > 0$. Then we immediately obtain that $\deg(\det \mathscr{E} \otimes \mathscr{N}) \le 2$. Since $\mathscr{E}$ is ample on a projective space,

$$2 \le r+1 = \operatorname{rk} \mathscr{E} \le \deg \mathscr{E} \le \deg(\det \mathscr{E} \otimes \mathscr{N}) - \deg \mathscr{N} \le 2 - \deg \mathscr{N} \le 2.$$

Therefore all of these inequalities must be equalities and we have that $r+1 = p = 2$, $q = 0$ and $\mathscr{N} \simeq \mathscr{O}_Y$. Furthermore, this implies that then $\mathscr{O}_{\mathbb{P}^m}(2) \simeq \det \mathscr{E} \hookrightarrow T_{\mathbb{P}^m}$ and hence $m = 1$. ☐

Proposition 5.4. *Let X be a smooth projective variety, $H \subset \operatorname{RatCurves}^n(X)$ a minimal covering family of rational curves on X, $\mathscr{L}$ an ample line bundle on X, and $\mathscr{M}$ a nef line bundle on X such that $c_1(\mathscr{M}) \cdot C > 0$ for every $[C] \in H$. Suppose that $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1}) \neq 0$ for some integer $p \ge 1$. Then $(X, \mathscr{L}, \mathscr{M}) \simeq (\mathbb{P}^p, \mathscr{O}_{\mathbb{P}^p}(1), \mathscr{O}_{\mathbb{P}^p}(1))$.*

Proof. Let $[f] \in H$ be a general member and write $f^*T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-d-1}$. The condition that both $f^*\mathscr{L}$ and $f^*\mathscr{M}$ are ample and that $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1}) \neq 0$ implies that $f^*\mathscr{L} \simeq \mathscr{O}_{\mathbb{P}^1}(1) \simeq f^*\mathscr{M}$, and thus H is unsplit. A non-zero section $s \in H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1})$ and the contraction

$$\mathscr{C}_\theta : \wedge^p T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1} \to \wedge^{p-1} T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1}$$

induced by a differential form $\theta \in \Omega_X$, gives rise to a non-zero map

$$\Omega_X \to \wedge^{p-1} T_X \otimes \mathscr{L}^{-p} \otimes \mathscr{M}^{-1}$$
$$\theta \mapsto \mathscr{C}_\theta(s),$$

the dual of which is the non-zero map

$$\varphi : \Omega_X^{p-1} \otimes \mathscr{L}^p \otimes \mathscr{M} \to T_X. \tag{5.4.1}$$

The sheaf $f^*(\Omega_X^{p-1} \otimes \mathscr{L}^p \otimes \mathscr{M})$ is ample. Thus, by Proposition 2.7 and Theorem 2.6, there is an open subset $X^\circ \subset X$, with $\operatorname{codim}_X(X \setminus X^\circ) \ge 2$, a smooth variety Y°, and a $\mathbb{P}^{d+1}$-bundle $\pi^\circ : X^\circ \to Y^\circ$ such that any rational curve from H meeting X° is a line on a fiber of π°. Moreover, the restriction of s to X° lies in $H^0(X^\circ, \wedge^p T_{X^\circ/Y^\circ} \otimes \mathscr{L}|_{X^\circ}^{-p} \otimes \mathscr{M}|_{X^\circ}^{-1})$, and its restriction to a general fiber F yields a non-zero section in $H^0(F, \wedge^p T_F \otimes \mathscr{L}|_F^{-p} \otimes \mathscr{M}|_F^{-1})$. On the other hand, by Bott's formula, $H^0(\mathbb{P}^{d+1}, \wedge^p T_{\mathbb{P}^{d+1}}(-p-1)) = 0$ unless $p = d+1$.

Suppose $\dim(Y^\circ) > 0$. Since $\operatorname{codim}_X(X \setminus X^\circ) \ge 2$, Y° contains a complete curve through a general point. Let $g : B \to Y^\circ$ be the normalization of a complete curve passing through a general point of Y°. Set $X_B := X^\circ \times_{Y^\circ} B$, and denote by $\mathscr{L}_{X_B}$ and $\mathscr{M}_{X_B}$ the pullbacks of $\mathscr{L}$ and $\mathscr{M}$ to X_B respectively. Then $X_B \to B$ is a $\mathbb{P}^p$-bundle, and the section s induces a non-zero section in $H^0(X_B, \wedge^p T_{X_B/B} \otimes \mathscr{L}_{X_B}^{-p} \otimes \mathscr{M}_{X_B}^{-1})$. But this is impossible by Corollary 5.3. Thus $\dim(Y^\circ) = 0$ and $X \simeq \mathbb{P}^p$. ☐

Corollary 5.5. *Let X be a smooth projective variety and $\mathscr{L}$ an ample line bundle on X. If $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p-1-k}) \neq 0$ for integers $p \ge 1$ and $k \ge 0$, then $k = 0$ and $(X, \mathscr{L}) \simeq (\mathbb{P}^p, \mathscr{O}_{\mathbb{P}^p}(1))$.*

Proof. Note that X is uniruled by [Miy87]. The result follows easily from Proposition 5.4. ☐

Here is how we are going to apply these results under the assumptions of Theorem 1.1. Suppose that $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p}) \neq 0$ for some ample line bundle $\mathscr{L}$ on X and integer $p \ge 2$. Then X is uniruled by [Miy87] and we fix a minimal covering family H of rational curves on X. Let $\pi : X^\circ \to Y^\circ$ be the H-rational quotient of X. By shrinking Y° if necessary, we may assume that Y° and π are smooth. Corollary 5.5 provides the vanishing required to apply Lemma 5.1 to $\pi : X^\circ \to Y^\circ$, yielding the following.

Lemma 5.6. *Let Y be a smooth variety, $\pi : X \to Y$ a smooth morphism with connected fibers, and $\mathscr{L}$ a line bundle on X. Let F be a general fiber of π. Suppose that F is projective and that the restriction $\mathscr{L}|_F$ is ample. If $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p}) \neq 0$ for some integer $p \geq 2$, then either $(F, \mathscr{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathscr{O}_{\mathbb{P}^{p-1}}(1))$ and $H^0(X, \wedge^{p-1} T_{X/Y} \otimes \pi^* T_Y \otimes \mathscr{L}^{-p}) \neq 0$, or $\dim(F) \geq p$ and $H^0(X, \wedge^p T_{X/Y} \otimes \mathscr{L}^{-p}) \neq 0$.*

Proof. Corollary 5.5 implies that $H^0(F, \wedge^i T_F \otimes \mathscr{L}|_F^{-p}) = 0$ for $0 \leq i \leq p-2$. So we may apply Lemma 5.1 with $\mathscr{M} = \mathscr{L}^{-p}$ to conclude that either $H^0(X, \wedge^{p-1} T_{X/Y} \otimes \pi^* T_Y \otimes \mathscr{L}^{-p}) \neq 0$, or $\dim F \geq p$ and $H^0(X, \wedge^p T_{X/Y} \otimes \mathscr{L}^{-p}) \neq 0$. In the first case we have $H^0(F, \wedge^{p-1} T_F \otimes \mathscr{L}|_F^{-p}) \neq 0$, and Corollary 5.5 implies that $(F, \mathscr{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathscr{O}_{\mathbb{P}^{p-1}}(1))$ and so the desired statement follows. □

Let X, H, and $\pi : X^\circ \to Y^\circ$ be as in the above discussion. If we are under the first case of Lemma 5.6, then Theorem 2.6 implies that the $\mathbb{P}^{p-1}$-bundle $\pi : X^\circ \to Y^\circ$ can be extended in codimension 1. Next we show that in this case we must have $X \simeq Q_2$.

Lemma 5.7. *Let X be a smooth projective variety and $\mathscr{L}$ an ample line bundle on X. Let $X^\circ \subset X$ be an open subset whose complement has codimension at least 2 in X. Let $\pi : X^\circ \to Y^\circ$ be a smooth projective morphism with connected fibers onto a smooth quasi-projective variety. If $H^0(X^\circ, \wedge^{p-1} T_{X^\circ/Y^\circ} \otimes \pi^* T_{Y^\circ} \otimes \mathscr{L}|_{X^\circ}^{-p}) \neq 0$ for some integer $p \geq 2$, then $p = 2$, $X^\circ = X \simeq Q_2$, and $Y^\circ \simeq \mathbb{P}^1$.*

Proof. Suppose that for some $p \geq 2$ there is a non-zero section

$$s \in H^0(X^\circ, \wedge^{p-1} T_{X^\circ/Y^\circ} \otimes \pi^* T_{Y^\circ} \otimes \mathscr{L}|_{X^\circ}^{-p}) \neq 0.$$

By Corollary 5.5, the fibers of π are isomorphic to $\mathbb{P}^{p-1}$, and the restriction of $\mathscr{L}$ to each fiber is isomorphic to $\mathscr{O}_{\mathbb{P}^{p-1}}(1)$. Since π has relative dimension $p-1$, there exists an inclusion $\wedge^{p-1} T_{X^\circ/Y^\circ} \otimes \pi^* T_{Y^\circ} \subseteq \wedge^p T_{X^\circ}$, and thus s, as in (5.4.1), yields a map $\varphi : \Omega_{X^\circ}^{p-1} \otimes \mathscr{L}|_{X^\circ}^p \to T_{X^\circ}$ of rank p at the generic point. Since $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$, s extends to a section $\tilde{s} \in H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p})$. Denote by

$$\widetilde{\varphi} : \Omega_X^{p-1} \otimes \mathscr{L}^p \to T_X$$

the associated map, which has rank p at the generic point.

Let $\mathscr{E} = \pi_* \mathscr{L}$. By [Fuj75, Corollary 5.4], $X^\circ \simeq \mathbb{P}(\mathscr{E})$ over Y° and then $\wedge^{p-1} T_{X^\circ/Y^\circ} \otimes \mathscr{L}^{-p} \simeq \pi^*(\det \mathscr{E}^*)$, and s is the pullback of a global section $s_{Y^\circ} \in H^0(Y^\circ, T_{Y^\circ} \otimes \det \mathscr{E}^*)$. This implies that the distribution $\mathscr{D}$ defined by s is integrable. Moreover, its leaves are the pullbacks of the leaves of the foliation $\mathscr{F}^\circ$ defined by the map $\det \mathscr{E} \hookrightarrow T_{Y^\circ}$ associated to s_{Y°.

Since $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$, we can find complete curves sweeping out a dense open subset of Y°. Let C be a general complete curve on Y°. Compactify Y° to a smooth variety Y, and let $\mathscr{F}$ be an invertible subsheaf of T_Y extending $\mathscr{F}^\circ$. Then $\mathscr{F}|_C = \det \mathscr{E}|_C$ is ample. By [BM01, Theorem 0.1] (see also [KSCT07, Theorem 1]), the leaf of the foliation $\mathscr{F}$ through any point of C is rational. We conclude that the leaves of $\mathscr{F}^\circ$ are (possibly noncomplete) rational curves. Thus the closures of the leaves of the distribution $\widetilde{\mathscr{D}}$ defined by $\widetilde{\varphi}$ are algebraic.

Let $F \subset X$ be the closure of a leaf of $\widetilde{\mathscr{D}}$ that meets X° and let $\eta : \widetilde{F} \to F$ be its normalization. Then there exists a morphism $\widetilde{F} \to B$ onto a smooth rational curve. The general fiber of this morphism is isomorphic to $\mathbb{P}^{p-1}$ and the restriction of $\mathscr{L}$ to the general fiber is isomorphic to $\mathscr{O}_{\mathbb{P}^{p-1}}(1)$. The fibers are thus generically reduced and finally reduced since fibers satisfy Serre's condition S_1. By [Fuj75, Corollary 5.4], $\widetilde{F} \to B$ is a $\mathbb{P}^{p-1}$-bundle and, in particular, $\widetilde{F}$ is smooth.

The section $\tilde{s} \in H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p})$ defines a non-zero map $\Omega_X^p \to \mathscr{L}^{-p}$. Since F is the closure of a leaf of $\widetilde{\mathscr{D}}$ and $\mathscr{L}|_F$ is torsion free, the restriction of this map to F factors through a map $\Omega_F^p \to \mathscr{L}|_F^{-p}$. By Lemma 4.5, this map extends to a map $\Omega_{\widetilde{F}}^p \to \eta^* \mathscr{L}|_F^{-p}$. Corollary 5.3 then implies that $p = 2$ and $\widetilde{F} \simeq Q_2$. Moreover $\eta^* \mathscr{L}|_F \simeq \mathscr{O}_{Q_2}(1)$. In particular, $\pi : X^\circ \to Y^\circ$ is a $\mathbb{P}^1$-bundle. Denote by H the unsplit covering family of rational on X whose general member corresponds to a fiber of π.

We claim that the general leaf of $\mathscr{F}^\circ$ is a complete rational curve. From this it follows that the general leaf of $\widetilde{\mathscr{D}}$ is compact, and contained in X°. Let $\widetilde{F}$ denote the normalization of the closure of a general leaf of

$\widetilde{\mathscr{D}}$. Since $\widetilde{F} \simeq Q_2$ and $\eta^*\mathscr{L}|_F \simeq \mathscr{O}_{Q_2}(1)$, X admits an unsplit covering family H' of rational curves whose general member corresponds to a ruling of $\widetilde{F} \simeq Q_2$ that is not contracted by π. Since $\operatorname{codim}(X \setminus X^\circ) \geq 2$, the general member of H' corresponds to a complete rational curve contained in X°. Its image in Y° is a complete leaf of $\mathscr{F}^\circ$. As we noted above, this implies that $F = \widetilde{F} \simeq Q_2$. Notice that the section $\tilde{s}$ does not vanish anywhere on a general leaf $F \simeq Q_2$ of $\mathscr{F}^\circ$.

Let $\varphi : X' \to Z'$ be the (H, H')-rationally connected quotient of X. Then the general fiber of φ is a leaf $F \simeq Q_2$ of $\mathscr{F}^\circ$. By Lemma 2.2, we may assume that $\operatorname{codim}_X(X \setminus X') \geq 2$, Z' is smooth, and φ is a proper surjective equidimensional morphism with irreducible and reduced fibers. Therefore $\varphi : X' \to Z'$ is a quadric bundle by [Fuj75, Corollary 5.5]. Since the families H and H' are distinct, φ is in fact a smooth quadric bundle.

We claim that in fact $X = F$ and Z' is a point. Suppose otherwise, and let $g : C \to Z'$ be the normalization of a complete curve passing through a general point of Z'. Set $X_C = X' \times_{Z'} C$, denote by $\varphi_C : X_C \to C$ the corresponding (smooth) quadric bundle, and write $\mathscr{L}_{X_C}$ for the pullback of $\mathscr{L}$ to X_C. The section $\tilde{s}$ induces a non-zero section in $H^0(X_C, \omega_{X_C/C}^{-1} \otimes \mathscr{L}_{X_C}^{-2})$ that does not vanish anywhere on a general fiber of π_C. Thus $\omega_{X_C/C}^{-1}$ is ample, contradicting Proposition 3.1. □

6. PROOF OF THEOREM 1.1

In order to prove the main theorem, we shall reduce it to the case when X has Picard number $\rho(X) = 1$. To treat that case, we will recall some facts about slopes of vector bundles that will be used later.

Definition 6.1. Let X be an n-dimensional projective variety and $\mathscr{H}$ an ample line bundle on X. Let $\mathscr{E}$ be a torsion free sheaf on X. We define the slope of $\mathscr{E}$ with respect to $\mathscr{H}$ to be $\mu_{\mathscr{H}}(\mathscr{E}) = \frac{c_1(\mathscr{E})\cdot c_1(\mathscr{H})^{n-1}}{\operatorname{rk}(\mathscr{E})}$. We say that a vector bundle $\mathscr{F}$ on X is *$\mu_{\mathscr{H}}$-semistable* if for any torsion free subsheaf $\mathscr{E}$ of $\mathscr{F}$ we have $\mu_{\mathscr{H}}(\mathscr{E}) \leq \mu_{\mathscr{H}}(\mathscr{F})$. Given a vector bundle $\mathscr{F}$ on X, there exists a filtration of $\mathscr{F}$ by torsion free subsheaves

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \ldots \subsetneq \mathscr{E}_k = \mathscr{F},$$

with $\mu_{\mathscr{H}}$-semistable quotients $\mathcal{Q}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$, and such that $\mu_{\mathscr{H}}(\mathcal{Q}_1) > \mu_{\mathscr{H}}(\mathcal{Q}_2) > \ldots > \mu_{\mathscr{H}}(\mathcal{Q}_k)$. This is called the *Harder-Narasimhan filtration* of $\mathscr{F}$ (see [MR82], [HN75, 1.3.9]).

Lemma 6.2. *Let X be a projective variety and $\mathscr{H}$ an ample line bundle on X. Let $\mathscr{F}$ be a vector bundle on X, p a positive integer, and $\mathscr{N}$ an invertible subsheaf of $\mathscr{F}^{\otimes p}$. Then $\mathscr{F}$ contains a torsion free subsheaf $\mathscr{E}$ such that $\mu_{\mathscr{H}}(\mathscr{E}) \geq \frac{\mu_{\mathscr{H}}(\mathscr{N})}{p}$.*

Proof. Consider the Harder-Narasimhan filtration of $\mathscr{F}$:

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \ldots \subsetneq \mathscr{E}_r = \mathscr{F},$$

with $\mathcal{Q}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$, $\mu_{\mathscr{H}}$-semistable for $1 \leq i \leq r$, and $\mu_{\mathscr{H}}(\mathcal{Q}_1) > \mu_{\mathscr{H}}(\mathcal{Q}_2) > \ldots > \mu_{\mathscr{H}}(\mathcal{Q}_k)$. For each $1 \leq i \leq r$ there exists a filtration

$$\mathscr{E}_{i-1}^{\otimes p} = \mathscr{G}_{i,0} \subsetneq \mathscr{G}_{i,1} \subsetneq \ldots \subsetneq \mathscr{G}_{i,p} = \mathscr{E}_i^{\otimes p},$$

with quotients $\mathscr{G}_{i,j}/\mathscr{G}_{i,j-1} \simeq \mathscr{E}_{i-1}^{\otimes p-j} \otimes \mathcal{Q}_i^{\otimes j}$. From the filtrations described above, we see that the inclusion $\mathscr{N} \hookrightarrow \mathscr{F}^{\otimes p}$ induces an inclusion $\mathscr{N} \hookrightarrow \mathcal{Q}_1^{\otimes i_1} \otimes \ldots \otimes \mathcal{Q}_k^{\otimes i_k}$, for suitable non-negative integers i_j's such that $\sum i_j = p$. Since each $\mathcal{Q}_i$ is $\mu_{\mathscr{H}}$-semistable, so is the tensor product $\mathcal{Q}_1^{\otimes i_1} \otimes \ldots \otimes \mathcal{Q}_k^{\otimes i_k}$ (see [HL97, Theorem 3.1.4]). Hence

$$\mu_{\mathscr{H}}(\mathscr{N}) \leq \mu_{\mathscr{H}}(\mathcal{Q}_1^{\otimes i_1} \otimes \ldots \otimes \mathcal{Q}_k^{\otimes i_k}) = \sum i_j \mu_{\mathscr{H}}(\mathcal{Q}_j) \leq p\mu_{\mathscr{H}}(\mathcal{Q}_1),$$

and $\mathscr{E} = \mathscr{E}_1 = \mathcal{Q}_1$ is the required subsheaf of $\mathscr{F}$. □

Now we can prove our main theorems.

Theorem 6.3. *Let X be a smooth n-dimensional projective variety with $\rho(X) = 1$, $\mathscr{L}$ an ample line bundle on X, and p a positive integer. Suppose that $H^0(X, T_X^{\otimes p} \otimes \mathscr{L}^{-p}) \neq 0$. Then either $(X, \mathscr{L}) \simeq (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$, or $p = n \geq 3$ and $(X, \mathscr{L}) \simeq (Q_p, \mathscr{O}_{Q_p}(1))$.*

Proof. First notice that X is uniruled by [Miy87], and hence a Fano manifold with $\rho(X) = 1$. The result is clear if $\dim X = 1$, so we assume that $n \geq 2$. Fix a minimal covering family H of rational curves on X. By Lemma 6.2, T_X contains a torsion free subsheaf $\mathscr{E}$ such that $\mu_{\mathscr{L}}(\mathscr{E}) \geq \frac{\mu_{\mathscr{L}}(\mathscr{L}^p)}{p} = \mu_{\mathscr{L}}(\mathscr{L})$. This implies that $\frac{\deg f^*\mathscr{E}}{\operatorname{rk}\mathscr{E}} \geq \deg f^*\mathscr{L}$ for a general member $[f] \in H$. If $r = \operatorname{rk}(\mathscr{E}) = 1$, then $\mathscr{E}$ is ample and we are done by Wahl's theorem. Otherwise, as $f^*\mathscr{E}$ is a subsheaf of $f^*T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$, we must have $\deg f^*\mathscr{L} = 1$ and either $f^*\mathscr{E}$ is ample, or $f^*\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}$ for a general $[f] \in H$. If $f^*\mathscr{E}$ is ample, then $X \simeq \mathbb{P}^n$ by Proposition 2.7, using the fact that $\rho(X) = 1$. If $f^*\mathscr{E}$ is not ample, then $\mathscr{O}_{\mathbb{P}^1}(2) \subset f^*\mathscr{E}$ for general $[f] \in H$, and so $\mathcal{C}_x \subset \mathbb{P}(\mathscr{E}^* \otimes \kappa(x))$ for a general $x \in X$. Thus by [Ara06, 2.6] $(f^*T_X^+)_o \subset (f^*\mathscr{E})_o$ for a general $o \in \mathbb{P}^1$ and a general $[f] \in H$. Since $f^*T_X^+$ is a subbundle of f^*T_X, we have an inclusion of sheaves $f^*T_X^+ \hookrightarrow f^*\mathscr{E}$, and thus $\det(f^*\mathscr{E}) = f^*\omega_X^{-1}$. Since $\rho(X) = 1$, this implies that $\det \mathscr{E}^{**} = \omega_X^{-1}$, and thus $0 \neq h^0(X, \wedge^r T_X \otimes \omega_X) = h^{n-r}(X, \mathscr{O}_X)$. The latter is zero unless $n = r$ since X is a Fano manifold. If $n = r$, then we must have $\omega_X^{-1} \simeq \mathscr{L}^{\otimes n}$. Hence $X \simeq Q_n$ by [KO73]. □

Proof of Theorem 1.1. Let X be a smooth projective variety and $\mathscr{L}$ an ample line bundle on X such that $H^0(X, \wedge^p T_X \otimes \mathscr{L}^{-p}) \neq 0$. By Theorem 6.3, we may assume that $\rho(X) \geq 2$. We may also assume that $p \geq 2$ as the case $p = 1$ is just Wahl's theorem. We shall proceed by induction on n.

Notice that X is uniruled by [Miy87]. Let $H \subset \operatorname{RatCurves}^n(X)$ be a minimal covering family of rational curves on X, and $[f] \in H$ a general member. By analyzing the degree of the vector bundle $f^*(\wedge^p T_X \otimes \mathscr{L}^{-p})$, we conclude that $f^*\mathscr{L} \simeq \mathscr{O}_{\mathbb{P}^1}(1)$, and thus H is unsplit. Let $\pi^\circ : X^\circ \to Y^\circ$ be the H-rational quotient of X. By shrinking Y° if necessary, we may assume that π° is smooth. Since $\rho(X) \geq 2$, we must have $\dim Y^\circ \geq 1$ by [Kol96, IV.3.13.3].

Let F be a general fiber of π° and set $k = \dim F$. By Lemma 5.6, either

- $k = p-1$, $(F, \mathscr{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathscr{O}_{\mathbb{P}^{p-1}}(1))$, and $H^0(X^\circ, \wedge^{p-1} T_{X^\circ/Y^\circ} \otimes \pi^* T_{Y^\circ} \otimes \mathscr{L}^{-p}) \neq 0$, or
- $k \geq p$ and $H^0(X^\circ, \wedge^p T_{X^\circ/Y^\circ} \otimes \mathscr{L}^{-p}) \neq 0$.

In the first case $\pi : X^\circ \to Y^\circ$ is a $\mathbb{P}^{p-1}$-bundle and we may assume that $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$ by Theorem 2.6. Then we apply Lemma 5.7 and conclude that $X \simeq Q_2$.

In the second case, the induction hypothesis implies that either $(F, \mathscr{L}|_F) \simeq (\mathbb{P}^k, \mathscr{O}_{\mathbb{P}^k}(1))$, or $k = p$ and $(F, \mathscr{L}|_F) \simeq (Q_p, \mathscr{O}_{Q_p}(1))$. If $F \simeq \mathbb{P}^k$, again by Theorem 2.6, $\pi : X^\circ \to Y^\circ$ is a $\mathbb{P}^k$-bundle, and we may assume that $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$. As in the end of the proof of Proposition 5.4, we reach a contradiction by applying Corollary 5.3 to $X^\circ \times_{Y^\circ} B \to B$, where $B \to Y^\circ$ is the normalization of a complete curve passing through a general point of Y°.

Suppose now that $F \simeq Q_p$. Then, by Lemma 2.2 and [Fuj75, Corollary 5.5], π° can be extended to a quadric bundle $\pi : X' \to Y'$ with irreducible and reduced fibers, where X' is an open subset of X with $\operatorname{codim}_X(X \setminus X') \geq 2$, and Y' is smooth. Denote by X'' the open subset of X' where π is smooth. Notice that $\operatorname{codim}_{X'}(X' \setminus X'') \geq 2$. A non-zero global section of $\wedge^p T_X \otimes \mathscr{L}^{-p}$ restricts to a non-zero global section of $\wedge^p T_{X''/Y'} \otimes \mathscr{L}|_{X''}^{-p}$, which, in turn, extends to a non-zero global section $s \in H^0(X', \omega_{X'/Y'}^{-1} \otimes \mathscr{L}|_{X'}^{-p})$ since X' is smooth. The section s does not vanish anywhere on a general fiber of π.

Let $g : C \to Y'$ be the normalization of a complete curve passing through a general point of Y'. Set $X_C = X' \times_{Y'} C$, denote by $\pi_C : X_C \to C$ the corresponding quadric bundle, and write $\mathscr{L}_{X_C}$ for the pullback of $\mathscr{L}$ to X_C. The general fiber of π_C is smooth. Now notice that X_C is a local complete intersection variety, and nonsingular in codimension one, since the fibers of π are reduced. In particular, X_C is a normal Gorenstein variety, and the morphism π_C is generically smooth. The section s induces a non-zero section in $H^0(X_C, \omega_{X_C/C}^{-1} \otimes \mathscr{L}_{X_C}^{-p})$ that does not vanish anywhere on the general fiber of π_C. Thus $\omega_{X_C/C}^{-1}$ is ample, contradicting Proposition 3.1. □

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