A new proof for maximal monotonicity of subdifferential operators

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Abstract

In this paper we present a new proof for maximal monotonicity of subdifferential operators. This result was proved by Rockafellar in [6] where other fundamental results were also proved. The proof presented here is simpler and makes use of classical results from subdifferential calculus as Brønsted-Rockafellar's theorem and Fenchel duality formula.

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1 Introduction

Let X be a real Banach space with dual X^*. A proper convex function on X is a function $f : X \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, such that

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

whenever $x \in X$, $y \in X$ and $0 < \lambda < 1$. The *subdifferential* of f is the point-to-set operator $\partial f : X \rightrightarrows X^*$ defined at $x \in X$ by

$$\partial f(x) = \{u \in X^* \ \mid \ f(y) \geq f(x) + \langle y - x, u\rangle, \ \text{for all} \ y \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical duality product between X and X^*. For each $x \in X$, the elements $u \in \partial f(x)$ are called *subgradients* of f at x.

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A point-to-set operator $A : X \rightrightarrows X^*$ is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \quad \text{whenever} \quad u \in A(x), v \in A(y).$$

It is easy to check that ∂f is monotone. The monotone operator A is called *maximal monotone* if, in addition, its graph

$$G(A) = \{(x, u) \mid u \in A(x)\} \subset X \times X^*$$

is not properly contained in the graph of any other monotone operator $A' : X \rightrightarrows X^*$. This is equivalent to say that

$$\langle x - x_0, u - v_0\rangle \geq 0, \quad \text{for all} \quad (x, v) \in G(A) \Rightarrow (x_0, v_0) \in G(A).$$

Rockafellar proved in a fundamental work [6] that the subdifferential of a proper convex lower semicontinuos (l.s.c. from now on) function is maximal monotone. Beside this result, that paper contain other useful and interesting results (see Theorem 6.1 of [4] for an application). Simpler proofs of Rockafellar's result were given in [5], [7], [1] and [8]. Our aim is to give (another) new and simple proof of the maximal monotonicity of the subdifferential.

For a proper convex function f, the *Fenchel-Legendre conjugate* of f is the function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(u) = \sup\{\langle x, u\rangle - f(x) \mid x \in X\}.$$

If f is also l.s.c., then f^* is proper and from its definition, follows directly the *Fenchel-Young inequality* : for all $x \in X$, $u \in X^*$,

$$f(x) + f^*(u) \geq \langle x, u\rangle, \quad \text{with equality if and only if} \quad u \in \partial f(x). \tag{1}$$

For instance, if we consider $f(x) = \frac{1}{2}\|x\|^2$, it is not difficult to see that $f^*(u) = \frac{1}{2}\|u\|^2$, where $\|\cdot\|$ denotes both norms of vectors spaces X and X^*.

The concept of *ε-subdifferential* of a convex function f was introduced by Brønsted and Rockafellar [3]. It is a point-to-set operator $\partial_\varepsilon f : X \rightrightarrows X^*$ defined at each $x \in X$ as

$$\partial_\varepsilon f(x) = \{u \in X^* \mid f(y) \geq f(x) + \langle y - x, u\rangle - \varepsilon, \quad \text{for all} \quad y \in X\},$$

where $\varepsilon \geq 0$. Note that $\partial f = \partial_0 f$ and $\partial f(x) \subset \partial_\varepsilon f(x)$, for all $\varepsilon \geq 0$. Using the conjugate function f^* of f it is easy to see that

$$u \in \partial_\varepsilon f(x) \quad \Leftrightarrow \quad f^*(u) + f(x) \leq \langle x, u\rangle + \varepsilon. \tag{2}$$

The following fundamental theorem of Brønsted and Rockafellar [3], estimates how well $\partial_\varepsilon f$ approximates ∂f.

Theorem 1.1 *If f is a l.s.c. proper convex function on X and $u \in \partial_\varepsilon f(x)$, for any $\eta > 0$, there exist vectors $z \in X$ and $w \in X^*$ such that $\|z - x\| \leq \eta$, $\|w - u\| \leq \varepsilon/\eta$ and $w \in \partial f(z)$.*

Next we present the classical Fenchel duality formula, which proof can be found in [2, page 11]

Theorem 1.2 *Let us consider two proper and convex functions f and g such that f (or g) is continuous at a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$ and $g(\hat{x}) < \infty$. Then, there exists $u \in X^*$ such that*

$$\inf_{x \in X}\{f(x) + g(x)\} = \max_{u \in X^*}\{-f^*(-u) - g^*(u)\}. \tag{3}$$

These theorems above will be of fundamental importance in the proof of Theorem 2.1, which is presented in the next section.

2 Main result

In this section a new proof for maximal monotonicity of subdifferential of a l.s.c proper convex function is presented as a direct application of Theorems 1.1 and 1.2.

Theorem 2.1 *If f is a l.s.c. proper convex function on X, then ∂f is a maximal monotone operator from X to X^*.*

Proof. Let us suppose $(x_0, v_0) \in X \times X^*$ is such that

$$\langle x - x_0, v - v_0 \rangle \geq 0$$

holds true whenever $v \in \partial f(x)$. We aim to prove that $v_0 \in \partial f(x_0)$.

Define $f_0 : X \to \mathbb{R} \cup \{+\infty\}$,

$$f_0(x) = f(x + x_0) - \langle x, v_0 \rangle. \tag{4}$$

Applying Theorem 1.2 to f_0 and $g(x) = \frac{1}{2}\|x\|^2$ we conclude that there exists $u \in X^*$ such that

$$\inf_{x \in X}\left\{f_0(x) + \frac{1}{2}\|x\|^2\right\} = -f_0^*(u) - \frac{1}{2}\|u\|^2.$$

As f_0 is l.s.c., proper and convex, both sides on the above equation are finite. Therefore, reordering this equation we obtain

$$\inf_{x \in X}\left\{f_0(x) + \frac{1}{2}\|x\|^2\right\} + f_0^*(u) + \frac{1}{2}\|u\|^2 = 0. \tag{5}$$

In particular, there exists a (minimizing) sequence $\{y_n\}$ such that

$$\begin{aligned}\frac{1}{n^2} &\geq f_0(y_n) + \frac{1}{2}\|y_n\|^2 + f_0^*(u) + \frac{1}{2}\|u\|^2 \\ &\geq \langle u, y_n\rangle + \frac{1}{2}\|y_n\|^2 + \frac{1}{2}\|u\|^2 \\ &\geq \frac{1}{2}(\|y_n\| - \|u\|)^2 \geq 0, \qquad (6)\end{aligned}$$

where the second inequality follows from Fenchel-Young inequality. Using the above equation we obtain

$$f_0(y_n) + f_0^*(u) - \langle u, y_n\rangle \leq 1/n^2.$$

Hence, $u \in \partial_{1/n^2} f_0(y_n)$ and by Theorem 1.1 it follows that there exist sequences $\{z_n\}$ in X and $\{w_n\}$ in X^* such that

$$w_n \in \partial f_0(z_n), \quad \|w_n - u\| \leq 1/n \text{ and } \|z_n - y_n\| \leq 1/n. \qquad (7)$$

Using the initial assumption, we also obtain

$$\langle z_n, w_n\rangle \geq 0. \qquad (8)$$

Using (6) we obtain

$$\|y_n\| \to \|u\|, \quad \langle y_n, u\rangle \to -\|u\|^2, \qquad \text{as } n \to \infty, \qquad (9)$$

which, combined with (7) and (8) yields $u = 0$. Therefore, $y_n \to 0$. As f_0 is l.s.c., $x = 0$ minimizes $f_0(x) + \frac{1}{2}\|x\|^2$ and, using (5) we have

$$f_0(0) + f_0^*(0) = 0.$$

Therefore $0 \in \partial f_0(0)$, which is equivalent to $v_0 \in \partial f(x_0)$. □

References

[1] J. M. Borwein. A note on ε-subgradients and maximal monotonicity. *Pacific J. Math.*, 103(2):307–314, 1982.

[2] Haïm Brezis. *Analyse fonctionnelle: Théorie et Applications.* Masson, Paris, 1987.

[3] A. Brøndsted and R. T. Rockafellar. On the subdifferentiability of convex functions. *Proc. Amer. Math. Soc.*, 16:605–611, 1965.

[4] R. S. Burachik and B. F. Svaiter. Maximal monotonicity, conjugation and the duality product. *Proc. Amer. Math. Soc.*, 131(8):2379–2383 (electronic), 2003.

[5] Đinh Thế Lục. On the maximal monotonicity of subdifferentials. *Acta Math. Vietnam.*, 18(1):99–106, 1993.

[6] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific J. Math.*, 33:209–216, 1970.

[7] S. Simons. The least slope of a convex function and the maximal monotonicity of its subdifferential. *J. Optim. Theory Appl.*, 71(1):127–136, 1991.

[8] Peter D. Taylor. Subgradients of a convex function obtained from a directional derivative. *Pacific J. Math.*, 44:739–747, 1973.