LIMITS OF SPECIAL WEIERSTRASS POINTS

C. Cumino[a], E. Esteves[b] and L. Gatto[a] [1]

[a] Dipartimento di Matematica, Politecnico di Torino,
C.so Duca degli Abruzzi 24, 10129 Torino, Italy

[b] Instituto Nacional de Matemática Pura e Aplicada,
Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil

1. Introduction

1.1. (*Our main result and its application*) Let $X \cup_P Y$ be the union of two general connected, smooth, nonrational curves X and Y intersecting transversally at a point P. Assume that P is a general point of X or of Y. Our main result is Theorem 5.1, which, in a simplified version, says:

Let $Q \in X$. Then Q is the limit of special Weierstrass points on a family of smooth curves degenerating to $X \cup_P Y$ if and only if $Q \neq P$ and either of the following conditions hold: Q is a special ramification point of the linear system $|K_X + (g_Y + 1)P|$, or Q is a ramification point of the linear system $|K_X + (g_Y + 1 + j)P|$ for $j = \pm 1$ and P is a Weierstrass point of Y.

Above, g_Y stands for the genus of Y and K_X for a canonical divisor of X.

As an application, we use Theorem 5.1 to recover, in a unified and conceptually simpler way, computations made by Diaz and Cukierman of the divisor classes of curves with special Weierstrass points in the moduli spaces of stable curves; see Theorem 9.2.

1.2. (*Motivation*) In order to understand how the above result fits in the literature on the subject, we must recall that in the last two decades several papers on limits of Weierstrass points and linear series on stable curves appeared, from the pioneering [7], [8] and [18] to the more recent [10], [12] and [25]. The investigations about these topics were initially aimed to prove existence theorems (about, e.g., distinguished linear series on smooth curves) or to do enumerative geometry, in the sense of [23], on the moduli space of genus-g stable curves, $\overline{M}_g$. For instance, in the beginning of the eighties, Harris and Mumford [18] proved that the moduli space $\overline{M}_g$ is of general type

[1]Work partially sponsored by MURST (Progetto Nazionale "Geometria delle Varietà Proiettive" Coordinatore Sandro Verra), and supported by GNSAGA-INDAM. The second author was also supported by CNPq, Proc. 478625/03-0, 301117/04-7 and 470761/06-7, and by CNPq/FAPERJ, Proc. E-26/171.174/2003.

for g odd and $g \geq 23$, doing computations on $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, the Picard group of the moduli functor of genus-g stable curves.

The same techniques were successfully used by Diaz [6], for $g \geq 3$, to compute the class $\overline{E}_{g,-1}$ (there named $\overline{D_{g-1}}$) of the closure of the locus of smooth curves having an exceptional (here called *special*) Weierstrass point of type $g-1$; see Subsection 8.1 for the precise definition of $\overline{E}_{g,-1}$. A Weierstrass point Q on a smooth curve of genus g is said to be of type $g-1$ if $\dim|(g-1)Q| \geq 1$, and of type $g+1$ if $\dim|(g+1)Q| \geq 2$. Diaz computed the class $\overline{E}_{g,-1}$ by intersecting it with certain test curves entirely contained in the boundary of $\overline{M}_g$. This way he got relations among the coefficients of the expression of $\overline{E}_{g,-1}$ in terms of the basis for $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ formed by the tautological class λ and the boundary classes $\delta_0, \dots, \delta_{[g/2]}$. These test curves were induced by one-parameter families $\mathcal{F}_i \to X_i$ of curves given as follows: start with a general smooth curve X_i of genus $g-i$, for each $i = 1, \dots, g-1$, and a general smooth pointed curve (Y_i, B_i) of genus i; then the fiber $(\mathcal{F}_i)_P$ over $P \in X_i$ is $X_i \cup_P Y_i$, the point $B_i \in Y_i$ being identified with $P \in X_i$. This can be seen as a curve in $\overline{M}_g$ via a nonconstant map $\gamma_i \colon X_i \to \overline{M}_g$.

The crux of Diaz's method was to evaluate $\int_{X_i} \gamma_i^* \overline{E}_{g,-1}$, which amounts to knowing, with multiplicities, for how many pairs (P, Q) with $P \in X_i$ and $Q \in X_i \cup_P Y_i$ there is a family of smooth curves degenerating to $X_i \cup_P Y_i$ with Weierstrass points of type $g-1$ converging to Q. This was done in [6] by using the theory of admissible coverings introduced and developed in [18]. So half of our Theorem 5.1 is in [6].

After Diaz's work, it was natural to ask what the limits of special Weierstrass points of type $g+1$ are, the other half of Theorem 5.1. In fact, soon afterwards, Cukierman [3] computed the class $\overline{E}_{g,1}$ of the closure of the locus of smooth curves having a Weierstrass point of type $g+1$; again, see Subsection 8.1 for a precise definition. However, his method was not based on test curves, but on a Hurwitz formula with singularities. (He used Diaz's result as well.) Also, the theory of admissible coverings could not be effectively used, as the condition defining $\overline{E}_{g,1}$ is not about the existence of a pencil, but of a net. Of course, once we have an expression for $\overline{E}_{g,1}$ in terms of the generators of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, we can evaluate it along the γ_i. But we cannot infer what the limits of Weierstrass points of type $g+1$ on $X_i \cup_P Y_i$ are just from their number.

Our Theorem 5.1 fills this gap. To show the "only if" part of it is not hard. To show the "if" part we use limit linear series on two-parameter families of curves, instead of admissible coverings.

1.3. (*Application*) Our Theorem 5.1 can be used to compute the classes $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ in a unified and conceptually simpler way. Also, there occur no multiplicity issues, an usual nuisance of the method of test curves.

In brief, here is how. First of all, we consider another divisor class on $\overline{M}_g$, the class $\overline{SW}_g$ of the closure of the locus of smooth curves having a special

Weierstrass point, either of type $g-1$ or of type $g+1$; see Subsection 8.1 for a more precise definition. It turns out that $\overline{SW}_g$ is much easier to compute. An expression for it, in terms of the basis for $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ mentioned above, appeared already in [15], but multiplicity issues exist there, due to the method of test curves.

Here we compute $\overline{SW}_g$ directly, in Theorem 8.4, by intersecting $\overline{SW}_g$ with a general curve in $\overline{M}_g$. No multiplicity issues arise. Of crucial importance in this computation is Theorem 6.1, which, in a way, describes the limits of Weierstrass points on a general irreducible uninodal curve. This description is much finer than that found in [6], Thm. A2.1, p. 60, for instance. For the proof of Theorem 6.1 we use the theory of limit linear series for curves that are not of compact type, developed in [10].

Then we show that $\overline{SW}_g = \overline{E}_{g,-1} + \overline{E}_{g,1}$. This follows from our Proposition 9.1. This is something to be expected, from a purely set-theoretic point of view, but nevertheless, because of multiplicity issues, is not immediate and had to be proven.

Now we use the test curves given by the γ_i. Having the expression for $\overline{SW}_g$ allows us to compute $\int_{X_i} \gamma_i^* \overline{SW}_g$, which gives us the sum

$$\int_{X_i} \gamma_i^* \overline{E}_{g,-1} + \int_{X_i} \gamma_i^* \overline{E}_{g,1}$$

for each $i = 1, \ldots, [g/2]$. For each $j = -1, 1$, let $e_{j,i}$ denote the number of pairs (P, Q) with $P \in X_i$ and $Q \in X_i \cup_P Y_i$ such that there is a family of smooth curves degenerating to $X_i \cup_P Y_i$ with special Weierstrass points of type $g+j$ converging to Q. Theorem 5.1 tells us what these pairs are. Their number, $e_{j,i}$, is computed in [4], Thm. 5.6. And

$$\int_{X_i} \gamma_i^* \overline{E}_{g,j} \geq e_{j,i}. \tag{1}$$

In principle, the inequality may be strict because of multiplicity issues. However, it turns out that

$$\int_{X_i} \gamma_i^* \overline{SW}_g = e_{-1,i} + e_{1,i}.$$

Thus equality holds in (1). From this equality, for each $j = -1, 1$ and each $i = 1, \ldots, [g/2]$, the classes $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ may be computed, as in [6]; see Theorem 9.2 for more details.

1.4. (*Layout*) In Section 2, we present a few preliminaries on ramification schemes, deformations of curves and limit linear series. In Section 3, we introduce *twists*; understanding them is important for studying limit linear series on families whose total space is not regular. In Section 4, we present a few needed results on smoothings of nodal curves and linear series on general smooth curves. In Section 5, we prove our main theorem, Theorem 5.1. In Section 6, we describe the Weierstrass divisors and limit Weierstrass points associated to smoothings of general singular stable curves. In Section 7, we recall some facts about the construction of $\overline{M}_g$ and about its associated Picard groups, and introduce the tautological class λ and the boundary classes δ_i in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$. In Section 8, we define $\overline{SW}_g$, $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$, and find the expression for $\overline{SW}_g$ in terms of λ and the δ_i. Finally, in Section 9, we apply Theorem 5.1 to find the expressions for $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ in terms of λ and the δ_i; see Theorem 9.2.

1.5. (*Notation*) Given a Noetherian scheme X, a coherent sheaf $\mathcal{F}$ on X, and a Cartier divisor D of X, let $\mathcal{F}(D) := \mathcal{F} \otimes \mathcal{O}_X(D)$. If D is effective, there is a natural map $\mathcal{F} \to \mathcal{F}(D)$, which is injective if D does not contain any associated point of $\mathcal{F}$; in this case, we will view $\mathcal{F}$ as a subsheaf of $\mathcal{F}(D)$ via the map.

1.6. (*Acknowledgments*) We wish to thank Nivaldo Medeiros for discussions on related topics and the Referee, who led us to important corrections and a thorough revision of our first version for this article. Also, we acknowledge the use of CoCoA[2] for some of the computations.

2. Preliminaries

2.1. (*Ramification points*) A (*nodal*) *curve* is a connected, reduced, projective scheme of dimension 1 over $\mathbb{C}$ whose only singularities are nodes, i.e. ordinary double points. The canonical sheaf, or dualizing sheaf of a curve C will be denoted ω_C. By the hypothesis on the singularities of C, the sheaf ω_C is a line bundle. The (arithmetic) genus of C, i.e. $h^0(C, \omega_C)$, will be denoted g_C.

Let C be a smooth curve, and V a linear system of dimension $r+1$ of sections of a line bundle L on C, for an integer $r \geq 0$. We say that r is the *rank* of V. For each $P \in C$ and each nonnegative integer a, let $V(-aP) \subseteq V$ be the linear subsystem of sections of V vanishing at P with multiplicity at least a. We call P a *ramification point* of V if $\dim V(-(r+1)P) \geq 1$; otherwise we call P an *ordinary point* of V. A ramification point P of V is said to be *special of type* r if $\dim V(-rP) \geq 2$, and *special of type* $r+2$ if $\dim V(-(r+2)P) \geq 1$. A *special ramification point* of V is a special ramification point of type r or $r+2$. A nonspecial ramification point is also called a *simple ramification point*.

The orders of vanishing at P of the sections of L in V can be ordered increasingly. We call this increasing sequence the *order sequence* of V at P. The order sequence is $0, 1, \dots, r$ if and only if P is an ordinary point of V. The point P is a special ramification point of type $r+2$ if and only if the largest order is at least $r+2$, and of type r if and only if the largest two orders are at least r.

We say that $P \in C$ is ordinary (resp. a *Weierstrass point*) if P is an ordinary point (resp. a ramification point) of the canonical system, i.e. the complete system of sections of ω_C.

2.2. (*Ramification schemes*) Let $p\colon \mathcal{C} \to S$ be a smooth, projective map of schemes whose fibers are curves. For each integer $i \geq 0$, and each invertible sheaf $\mathcal{L}$ on $\mathcal{C}$, let $J^i_p(\mathcal{L})$ denote the relative sheaf of jets, or principal parts, of order i of $\mathcal{L}$. The sheaf $J^i_p(\mathcal{L})$ is locally free of rank $i+1$. Also, there is a natural evaluation map, $e_i\colon p^*p_*\mathcal{L} \to J^i_p(\mathcal{L})$, which locally, after trivializations are taken, is represented by a Wronskian matrix of functions and their derivatives up to order i. The map e_i is functorial on $\mathcal{L}$, that is, a map $\psi\colon \mathcal{L}' \to \mathcal{L}$ between invertible sheaves $\mathcal{L}'$ and $\mathcal{L}$ on $\mathcal{C}$ induces a natural commutative diagram of maps of the form:

$$\begin{array}{ccc} p^*p_*\mathcal{L}' & \xrightarrow{e_i} & J^i_p(\mathcal{L}') \\ {\scriptstyle p^*p_*\psi}\downarrow & & \downarrow{\scriptstyle \psi^i_p} \\ p^*p_*\mathcal{L} & \xrightarrow{e_i} & J^i_p(\mathcal{L}). \end{array}$$

There is a natural identification $J^0_p(\mathcal{L}) = \mathcal{L}$. Furthermore, for each integer $i > 0$ there is a natural exact sequence of the form:

$$0 \to \omega_p^{\otimes i} \otimes \mathcal{L} \longrightarrow J^i_p(\mathcal{L}) \xrightarrow{r_i} J^{i-1}_p(\mathcal{L}) \to 0, \tag{2}$$

where ω_p is the relative dualizing sheaf of p. The truncation maps r_i are compatible with the evaluation maps, that is, $e_{i-1} = r_i \circ e_i$ for each $i > 0$. Also, the truncation sequence (2) is functorial on $\mathcal{L}$, that is, a map $\psi\colon \mathcal{L}' \to \mathcal{L}$ between invertible sheaves $\mathcal{L}'$ and $\mathcal{L}$ on $\mathcal{C}$ induces a natural commutative diagram of exact sequences of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_p^{\otimes i} \otimes \mathcal{L}' & \longrightarrow & J^i_p(\mathcal{L}') & \xrightarrow{r_i} & J^{i-1}_p(\mathcal{L}') & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}\otimes\psi}\downarrow & & {\scriptstyle \psi^i_p}\downarrow & & {\scriptstyle \psi^{i-1}_p}\downarrow & & \\ 0 & \longrightarrow & \omega_p^{\otimes i} \otimes \mathcal{L} & \longrightarrow & J^i_p(\mathcal{L}) & \xrightarrow{r_i} & J^{i-1}_p(\mathcal{L}) & \longrightarrow & 0. \end{array}$$

Sheaves satisfying the same properties as the $J^i_p(\mathcal{L})$ above can be constructed if p is only a flat, projective map whose fibers are (nodal) curves. In addition, they coincide on the smooth locus of p with the corresponding sheaves of jets. Those sheaves appeared in [9], [14], [19] and [20]. We will use the same notation, $J^i_p(\mathcal{L})$, for those sheaves.

So, more generally, let $p\colon \mathcal{C} \to S$ be a flat, projective map whose fibers are curves of genus g. We call p a *family of curves*. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}$, and $\nu\colon \mathcal{V} \to p_*\mathcal{L}$ any map from a locally free sheaf $\mathcal{V}$ of constant positive rank, say $r+1$ for a certain integer $r \geq 0$. For each integer $i \geq 0$, consider the natural evaluation map,

$$u_i : p^*\mathcal{V} \xrightarrow{p^*\nu} p^*p_*\mathcal{L} \longrightarrow J^i_p(\mathcal{L}).$$

We call the degeneracy scheme of u_{r+j}, for $j = -1, 1$, the *special ramification scheme of type* $r+1+j$ of $(\mathcal{V}, \mathcal{L})$, and denote it by $VE_j(\mathcal{V}, \mathcal{L})$. We call the degeneracy scheme of u_r the *ramification scheme* of $(\mathcal{V}, \mathcal{L})$, and denote it by $W(\mathcal{V}, \mathcal{L})$.

If $S := \mathrm{Spec}(\mathbb{C})$, the curve $\mathcal{C}$ is smooth, and ν is injective, then the support of the scheme $W(\mathcal{V},\mathcal{L})$ is the set of ramification points of the linear system $H^0(S,\nu(\mathcal{V})) \subseteq H^0(\mathcal{C},\mathcal{L})$ of sections of $\mathcal{L}$. Also, the support of $VE_j(\mathcal{V},\mathcal{L})$ is the set of special ramification points of type $r+1+j$ of the same linear system, for $j=-1,1$.

The map u_r is a map of locally free sheaves of the same rank $r+1$. Taking determinants, u_r induces a *Wronskian section* w_p of the invertible sheaf

$$\mathcal{W} := \bigwedge^{r+1} J^r_p(\mathcal{L}) \otimes \Big(\bigwedge^{r+1} p^*\mathcal{V}\Big)^{\vee}.$$

Using the truncation sequences (2), we get

$$\mathcal{W} \cong \omega_p^{\otimes\binom{r+1}{2}} \otimes \mathcal{L}^{\otimes r+1} \otimes \Big(\bigwedge^{r+1} p^*\mathcal{V}\Big)^{\vee}.$$

Locally, after trivializations are taken, w_p corresponds to a Wronskian determinant of a sequence of $r+1$ functions. Its zero scheme is the ramification scheme of $(\mathcal{V},\mathcal{L})$.

The formation of the ramification scheme is functorial in the following sense: Suppose there are an invertible sheaf $\mathcal{L}'$ on $\mathcal{C}$, a locally free sheaf $\mathcal{V}'$ of rank $r+1$ on S, a map $\psi\colon \mathcal{L}' \to \mathcal{L}$, and a commutative diagram of maps of the form:

$$\begin{array}{ccc} \mathcal{V}' & \xrightarrow{\nu'} & p_*\mathcal{L}' \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle p_*\psi} \\ \mathcal{V} & \xrightarrow{\nu} & p_*\mathcal{L}. \end{array}$$

Due to the functorial nature of the evaluation maps of jets, the above diagram induces another commutative diagram of maps, where all the sheaves are locally free of rank $r+1$:

$$\begin{array}{ccc} p^*\mathcal{V}' & \xrightarrow{u_r} & J^r_p(\mathcal{L}') \\ {\scriptstyle p^*\mu}\big\downarrow & & \big\downarrow{\scriptstyle \psi^r_p} \\ p^*\mathcal{V} & \xrightarrow{u_r} & J^r_p(\mathcal{L}). \end{array}$$

We may take determinants above, using the truncation sequences (2) and their functoriality for $i=1,\dots,r$. Let $V \subseteq S$ be the degeneration scheme of μ, and $Y \subseteq \mathcal{C}$ that of ψ. (So $\mathrm{Im}(\psi) = \mathcal{I}_{Y/\mathcal{C}} \otimes \mathcal{L}$.) Then

$$\mathcal{I}_{W'/\mathcal{C}}\mathcal{I}^{r+1}_{Y/\mathcal{C}} = \mathcal{I}_{p^{-1}(V)/\mathcal{C}}\mathcal{I}_{W/\mathcal{C}}, \tag{3}$$

where $W := W(\mathcal{V},\mathcal{L})$ and $W' := W(\mathcal{V}',\mathcal{L}')$.

By differentiation, the section w_p induces a global section w'_p of the rank-2 locally free sheaf $J^1_p(\mathcal{W})$. We will call the zero scheme of this section the *special ramification scheme* of $(\mathcal{V},\mathcal{L})$, and denote it by $VSW(\mathcal{V},\mathcal{L})$. Notice that the irreducible components of the ramification scheme have codimension at most 1 in $\mathcal{C}$, while those of the special ramification schemes have codimension at most 2. Also, a local analysis of the matrices representing the maps u_i shows that, set-theoretically,

$$VSW(\mathcal{V},\mathcal{L}) = VE_{-1}(\mathcal{V},\mathcal{L}) \bigcup VE_1(\mathcal{V},\mathcal{L}).$$

Let T be an S-scheme, and let $p_T\colon \mathcal{C}_T \to T$ denote the induced family by base extension. For each coherent sheaf $\mathcal{F}$ on S (resp. $\mathcal{C}$), let $\mathcal{F}_T$ denote its pullback to T (resp. $\mathcal{C}_T$.) Then ν induces a map

$$\nu_T\colon \mathcal{V}_T \to (p_*\mathcal{L})_T \to p_{T*}\mathcal{L}_T,$$

and the (special) ramification scheme(s) of $(\mathcal{V},\mathcal{L})$ pull back to the (special) ramification scheme(s) of $(\mathcal{V}_T,\mathcal{L}_T)$. Furthermore, if $\mathcal{Y} \subseteq \mathcal{C}_T$ is a T-flat closed subscheme whose fibers over T are subcurves of the fibers of p, then the (special) ramification scheme(s) of $(\mathcal{V},\mathcal{L})$ coincide on $\mathcal{Y} - (\mathcal{Y} \cap \overline{\mathcal{C}_T - \mathcal{Y}})$ with the corresponding (special) ramification scheme(s) of $(\mathcal{V}_T, \mathcal{L}_T|_{\mathcal{Y}})$.

In case $\mathcal{L}$ is the relative dualizing sheaf of p, and $\mathcal{V} = p_*\mathcal{L}$, the ramification schemes and special ramification schemes are called *Weierstrass schemes* and *special Weierstrass schemes*. In addition, we set

$$W(p) := W(\mathcal{V},\mathcal{L}), \quad VSW(p) := VSW(\mathcal{V},\mathcal{L}),$$

and $VE_j(p) := VE_j(\mathcal{V},\mathcal{L})$ for $j = -1, 1$.

2.3. (*Smoothings*) Let C be a curve. A *smoothing* of C consists of two data: a flat, projective map $p\colon \mathcal{C} \to S$ to $S := \operatorname{Spec}(\mathbb{C}[[t]])$ with smooth generic fiber, and an isomorphism between the special fiber and C. We will usually identify the special fiber of p with C, "forgetting" the isomorphism. The smoothing is called *regular* if the total space $\mathcal{C}$ is a regular scheme.

Let $p\colon \mathcal{C} \to S$ be a smoothing of C. Since the general fiber is smooth, for each node P of C, there are a nonnegative integer k and a $\mathbb{C}[[t]]$-algebra isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{C},P} \cong \frac{\mathbb{C}[[t,x,y]]}{(xy - t^{k+1})}. \tag{4}$$

We call k the *singularity type* of P in $\mathcal{C}$, and set $k(P) := k$. Notice that $k(P) = 0$ if and only if $\mathcal{C}$ is regular at P.

If $E \subseteq C$ is a subcurve, then E is not necessarily a Cartier divisor of $\mathcal{C}$. However, let m_E be the least common multiple of the $k(P)+1$ for all $P \in E \cap \overline{C - E}$. Then there is a natural effective Cartier divisor on $\mathcal{C}$ whose associated 1-cycle is $m_E[E]$; denote this divisor by E^p.

The divisor E^p is constructed as the schematic closure of the Cartier divisor with local equations 1 on $\mathcal{C} - E$ and t^{m_E} on $\mathcal{C} - \overline{C - E}$. We need to check that E^p is indeed a divisor at a node $P \in E \cap \overline{C - E}$. Fix an isomorphism of the form (4). Suppose E is defined by the ideal (t,x) in $\widehat{\mathcal{O}}_{\mathcal{C},P}$. Let $I \subset \widehat{\mathcal{O}}_{\mathcal{C},P}$ be the ideal defining E^p at P. ¿From the construction of E^p, we have $I_y = t^{m_E}(\widehat{\mathcal{O}}_{\mathcal{C},P})_y$. On the other hand, the unique associated prime of I is (x,t). So $I = I_y \cap \widehat{\mathcal{O}}_{\mathcal{C},P}$. Now, if $g \in I_y \cap \widehat{\mathcal{O}}_{\mathcal{C},P}$ then there are an integer $\ell > 0$ and $h \in \widehat{\mathcal{O}}_{\mathcal{C},P}$ such that $y^\ell g = t^{m_E} h$. Set $n := m_E/(k+1)$. We may choose $\ell > n$. So $y^{\ell-n} g = x^n h$. But x, y form a regular sequence in $\widehat{\mathcal{O}}_{\mathcal{C},P}$. So $g \in (x^n)$. Conversely, $x^n = t^{m_E}/y^n$. Thus $I = (x^n)$, showing that E^p is indeed a divisor at P, for each $P \in E \cap \overline{C - E}$, and hence a divisor everywhere.

Notice that E itself is a Cartier divisor of $\mathcal{C}$ if and only if $m_E = 1$.

2.4. (*Smoothings of semistable curves*) Let C be a curve and $p\colon \mathcal{C} \to S$ a smoothing of C.

For each integer $d > 0$, let $S \to S$ be the map defined by taking t to t^d, and let $p_d\colon \mathcal{C}_d \to S$ be the smoothing induced by base change. The special fiber of p_d is equal to that of p. But, for a node $P \in C$, if k is the singularity type of P in $\mathcal{C}$, then $(k+1)d - 1$ is the singularity type of P in $\mathcal{C}_d$.

Suppose C has genus at least 2. Suppose as well that C is semistable, that is, each smooth rational component E of C intersects $\overline{C - E}$ in at least two points. Those E that intersect $\overline{C - E}$ in exactly two points are called *exceptional*. Let C^s be the nodal curve obtained by collapsing to a point each and every exceptional component of C. We call C^s the *stable model* of C. The fibers of the map $C \to C^s$ are either points or maximal chains of exceptional components. A *chain of exceptional components* is a connected union of exceptional components. It is possible to order the exceptional components $E_1, \dots, E_r$ of a chain in such a way that

$$\#E_1 \cap E_2 = \#E_2 \cap E_3 = \cdots = \#E_{r-1} \cap E_r = 1,$$

whence the name "chain." There is just another way of doing so, the reverse ordering $E_r, \dots, E_1$. The nodes $P_0, \dots, P_r$ of C in the chain can also be ordered in a *compatible*, *sequential way*, with $P_0 \in E_1$ and $P_r \in E_r$, and $P_i \in E_i \cap E_{i+1}$ for $i = 1, \dots, r-1$.

There are a smoothing $p^s\colon \mathcal{C}^s \to S$ of C^s and an S-map $b\colon \mathcal{C} \to \mathcal{C}^s$ that blows down (collapses) all exceptional components of C. In fact, just let

$$\mathcal{C}^s := \operatorname{Proj}\Big(\bigoplus_{i \geq 0} H^0(\mathcal{C}, \omega_p^{\otimes i})\Big),$$

where ω_p is the relative dualizing sheaf of p. However, the singularity types of $\mathcal{C}^s$ are bigger than those of $\mathcal{C}$: if $P \in C^s$ is a node such that $b^{-1}(P)$ is a chain of r exceptional components, and $k_0, k_1, \dots, k_r$ are the singularity types in $\mathcal{C}$ of the nodes of C on that chain, then the singularity type of P in $\mathcal{C}^s$ is $k_0 + k_1 + \cdots + k_r + r$.

In certain circumstances, it might be interesting to avoid blowing down some of the exceptional components of C in a construction as above. This is possible after base change. With a base change we may produce sections $\Sigma_i \subset \mathcal{C}$ of p through its smooth locus intersecting the components we do not want to blow down. Then just do the above construction with ω_p replaced by $\omega_p(\sum \Sigma_i)$.

So, given a node P of C, and positive integers $m_0, \dots, m_n$, it is possible, with base changes, blowups and blowdowns, to find an integer $d > 0$, a smoothing $\widetilde{p}\colon \widetilde{\mathcal{C}} \to S$ of a semistable curve $\widetilde{C}$, and an S-map $b\colon \widetilde{\mathcal{C}} \to \mathcal{C}_d$ such that $\widetilde{p} = b \circ p_d$ and:

(1) b is an isomorphism off P.
(2) $b^{-1}(P)$ is a chain of n exceptional components of the special fiber $\widetilde{C}$ of $\widetilde{p}$.
(3) The singularity types in $\widetilde{\mathcal{C}}$ of the nodes $P_0, \dots, P_n$ of $\widetilde{C}$ on the chain $b^{-1}(P)$, ordered in a sequential way, are $\ell m_0 - 1, \dots, \ell m_n - 1$ for a certain integer $\ell > 0$. (In fact,

$$\ell(m_0 + \cdots + m_n) = (k+1)d,$$

where k is the singularity type of P in $\mathcal{C}$.)

2.5. (*Limit linear series*) Let C be a curve, and $p\colon \mathcal{C} \to S$ a smoothing of C. Denote by $\mathcal{C}_*$ the general fiber of p.

Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}$. Since p is flat, $H^0(\mathcal{C}, \mathcal{L})$ is a torsion-free $\mathbb{C}[[t]]$-module, whence free. Let $V \subseteq H^0(\mathcal{C}, \mathcal{L})$ be a $\mathbb{C}[[t]]$-submodule. Then also V is free, say of rank $r+1$, for a certain integer $r \geq 0$. Assume V is *saturated*, i.e. $V : (t) = V$. Letting V_* be the subspace of $H^0(\mathcal{C}_*, \mathcal{L}|_{\mathcal{C}_*})$ generated by V, we have that V is saturated if and only if $V = V_* \cap H^0(\mathcal{C}, \mathcal{L})$. In our applications we will actually have $V = H^0(\mathcal{C}, \mathcal{L})$, so saturated.

Let $R \subset \mathcal{C}$ be the ramification scheme of $(V \otimes \mathcal{O}_S, \mathcal{L})$, as defined in Subsection 2.2. Since $\mathcal{C}_*$ is smooth, R is indeed a divisor. But R may not intersect C properly, as R may contain in its support a component of C. Nevertheless, let $\overline{R} := \overline{R \cap \mathcal{C}_*}$. Then $\overline{R}$ intersects C properly. The intersection, $\partial R := \overline{R} \cap C$, is called the *limit ramification scheme*.

In [10] it is shown how to compute the 0-cycle $[\partial R]$ associated to ∂R when p is regular. We review this below.

Let $C_1, \dots, C_n$ be the irreducible components of C. Since C is connected, for each $i = 1, \dots, n$ there is an invertible sheaf $\mathcal{L}_i$ on $\mathcal{C}$ of the form

$$\mathcal{L}_i = \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\textstyle\sum_m a_{i,m} C_m^p), \quad a_{i,m} \in \mathbb{Z},$$

such that the restriction map

$$H^0(\mathcal{C}, \mathcal{L}_i) \longrightarrow H^0(C_i, \mathcal{L}_i|_{C_i}) \tag{5}$$

has kernel $tH^0(\mathcal{C}, \mathcal{L}_i)$. (The divisors C_m^p are as explained in Subsection 2.3.)

There is a natural identification $\mathcal{L}_i|_{\mathcal{C}_*} = \mathcal{L}|_{\mathcal{C}_*}$. Using it, set

$$V_i := H^0(\mathcal{C}, \mathcal{L}_i) \cap V_* \subseteq H^0(\mathcal{C}_*, \mathcal{L}|_{\mathcal{C}_*}).$$

Then also V_i is saturated and free of rank $r+1$. (In fact, $V_{i*} = V_*$.) Let $\overline{V}_i \subseteq H^0(C_i, \mathcal{L}_i|_{C_i})$ be the image of V_i under (5). Since V_i is saturated, and (5) has kernel $tH^0(\mathcal{C}, \mathcal{L}_i)$, the dimension of $\overline{V}_i$ is $r+1$. We call $(\overline{V}_i, \mathcal{L}_i|_{C_i})$ a *limit linear system* on C_i.

Let $R_i \subseteq C_i$ be the ramification scheme of $(\overline{V}_i, \mathcal{L}_i|_{C_i})$, as defined in Subsection 2.2. Put $R'_i := R_i - R_i \cap \overline{C - C_i}$. Then

(6) $[\partial R] - [R'_1] - \cdots - [R'_n]$ is effective and supported on the nodes of C.

Furthermore, if p is regular, then

$$[\partial R] = \sum_{i=1}^{n} [R_i] + \sum_{i<j} \sum_{P \in C_i \cap C_j} (r+1)(r - \ell_{i,j})[P], \tag{7}$$

where $\ell_{i,j} := a_{i,j} + a_{j,i} - a_{i,i} - a_{j,j}$ for each distinct $i, j = 1, \dots, n$.

When $\mathcal{L} = \omega_p$, the relative dualizing sheaf of p, and $V = H^0(\mathcal{C}, \omega_p)$, the limit ramification scheme is called the *limit Weierstrass scheme*, and denoted ∂W_p; also, a limit linear system is called a *limit canonical system*.

Let P be a nonsingular point of C, and $\Gamma \subset \mathcal{C}$ a section of p intersecting C at P. Say, $P \in C_i$. Let P_* be the rational point of $\mathcal{C}_*$ cut out by Γ. Then the behavior of $(V_*, \mathcal{L}|_{\mathcal{C}_*})$ at P_* is partially captured by that of $(\overline{V}_i, \mathcal{L}_i|_{C_i})$ at P. For instance, we have semicontinuity:

$$\dim_{\mathbb{C}} \overline{V}_i(-aP) \geq \dim_{\mathbb{C}((t))} V_*(-aP_*) \quad \text{for each } a = 0, 1, \dots.$$

In fact, let $m := \dim_{\mathbb{C}((t))} V_*(-aP_*)$. Since $V_* = V_{i*}$, we may choose a $\mathbb{C}[[t]]$-basis $\sigma_1, \dots, \sigma_m$ of $V_i \cap V_*(-aP_*)$. The images $\overline{\sigma}_i$ in $\overline{V}_i$ vanish at P with multiplicity at least a as well. If there is a nonzero m-tuple $(c_1, \dots, c_m) \in \mathbb{C}^m$ such that $c_1\overline{\sigma}_1 + \cdots + c_m\overline{\sigma}_m = 0$, then

$$c_1\sigma_1 + \cdots + c_m\sigma_m = t\sigma \tag{8}$$

for some $\sigma \in V_i$, because (5) has kernel $tH^0(\mathcal{C}, \mathcal{L}_i)$ and V_i is saturated. Because of (8), also $\sigma \in V_i \cap V_*(-aP_*)$, and hence $\sigma = b_1\sigma_1 + \cdots + b_m\sigma_m$ for certain $b_i \in \mathbb{C}[[t]]$. Plugging this expression in (8), we get a nontrivial relation for the sections σ_i, an absurd.

In particular, if P_* is a special ramification point of type $r + j$ of V_*, for $j = -1$ or $j = 1$, then so is P with respect to $\overline{V}_i$. When $\mathcal{L} = \omega$ and $V = H^0(\mathcal{C}, \omega)$ we say that P is the limit of a special Weierstrass point of type $g + j$ along p.

3. Twists

3.1. (*Twists*) Let C be a curve, and $p\colon \mathcal{C} \to S$ a smoothing of C. Let $Y \subset C$ be a nonempty, proper subcurve of C. Set $\mathcal{I}_Y^{(0)} := \mathcal{O}_{\mathcal{C}}$, and for each positive integer i, define

$$\mathcal{I}_Y^{(i)} := \ker\Big(\mathcal{I}_Y^{(i-1)} \longrightarrow \frac{\mathcal{I}_Y^{(i-1)}|_Y}{(\text{torsion})}\Big).$$

We call $\mathcal{I}_Y^{(i)}$ the *i-th twist by Y of p*. Clearly, $\mathcal{I}_Y^{(1)}$ is simply the sheaf of ideals $\mathcal{I}_{Y/\mathcal{C}}$ of Y in $\mathcal{C}$. Also, by construction, $\mathcal{I}_Y^{(i)} \supseteq \mathcal{I}_Y^{(i-1)}\mathcal{I}_{Y/\mathcal{C}}$.

Let $\Delta_Y := Y \cap \overline{C - Y}$. The subcurve Y may fail to be a Cartier divisor of $\mathcal{C}$ only at Δ_Y; in fact, only at the singular points of $\mathcal{C}$ in Δ_Y. Away from Δ_Y, we have that $\mathcal{I}_Y^{(i)}$ is the sheaf of ideals of a Cartier divisor. More precisely, $\mathcal{I}_Y^{(i)}=\mathcal{I}_{Y/\mathcal{C}}^i$ away from Δ_Y. Indeed, this is true for $i=1$, and, by induction, if $\mathcal{I}_Y^{(i-1)}=\mathcal{I}_{Y/\mathcal{C}}^{i-1}$ away from Δ_Y, then $\mathcal{I}_Y^{(i-1)}|_Y$ has no torsion on $Y-\Delta_Y$, and hence $\mathcal{I}_Y^{(i)}=\mathcal{I}_Y^{(i-1)}\mathcal{I}_{Y/\mathcal{C}}=\mathcal{I}_{Y/\mathcal{C}}^i$ away from Δ_Y.

Proposition 3.2. *Let C be a curve, and $p\colon \mathcal{C}\to S$ a smoothing of C. Let $Y\subset C$ be a nonempty proper subcurve. Let $Y^c:=\overline{C-Y}$ and $\Delta_Y:=Y\cap Y^c$. Let i be a nonnegative integer. Let $\mathcal{I}_Y^{(i)}$ and $\mathcal{I}_Y^{(i+1)}$ denote the i-th twist and the $(i+1)$-th twist by Y of p. Then the defining exact sequence,*

$$0\longrightarrow \mathcal{I}_Y^{(i+1)}\longrightarrow \mathcal{I}_Y^{(i)}\longrightarrow \frac{\mathcal{I}_Y^{(i)}|_Y}{\text{(torsion)}}\longrightarrow 0,$$

induces, upon restriction to C, a natural exact sequence

$$0\longrightarrow \frac{\mathcal{I}_Y^{(i+1)}|_{Y^c}}{\text{(torsion)}}\longrightarrow \mathcal{I}_Y^{(i)}|_C\longrightarrow \frac{\mathcal{I}_Y^{(i)}|_Y}{\text{(torsion)}}\longrightarrow 0. \tag{9}$$

Furthermore, for each $P\in\Delta_Y$, let k_P be its singularity type in $\mathcal{C}$, and q_P the quotient of the Euclidean division of i by k_P+1. Then

$$\frac{\mathcal{I}_Y^{(i+1)}|_{Y^c}}{\text{(torsion)}}\cong \mathcal{O}_{Y^c}\Big(-\sum_{P\in\Delta_Y}(q_P+1)P\Big)\quad \textit{and}\quad \frac{\mathcal{I}_Y^{(i)}|_Y}{\text{(torsion)}}\cong \mathcal{O}_Y\Big(\sum_{P\in\Delta_Y}q_PP\Big).$$

Proof. Let $P\in\Delta_Y$, and let k be its singularity type in $\mathcal{C}$. Recall that we have an isomorphism (4). Under this isomorphism, $\widehat{\mathcal{I}}_{Y/\mathcal{C},P}$ can be seen as the ideal (y,t). We claim that $\widehat{\mathcal{I}}_{Y,P}^{(i)}$ is the ideal (y^{q+1},y^qt^r) where q and r are the quotient and the remainder of the Euclidean division of i by $k+1$.

Indeed, this description of $\widehat{\mathcal{I}}_{Y,P}^{(i)}$ holds for $i=1$. Suppose by induction that it holds for a certain $i>0$. To describe $\widehat{\mathcal{I}}_{Y,P}^{(i+1)}$, we need to describe the torsion of

$$M:=\frac{(y^{q+1},y^qt^r)}{(y^{q+2},y^{q+1}t^r,y^{q+1}t,y^qt^{r+1})}.$$

Since $r\leq k$, we have $xy^{q+1}=t^{k-r}y^qt^{r+1}$. Thus the class of y^{q+1} is torsion in M. So the torsion submodule of M contains at least the classes of y^{q+1} and y^qt^{r+1}. Notice that, when $r=k$,

$$(y^{q+1},y^qt^{r+1})=(y^{q+1})=(y^{q+2},y^{q+1}t^0).$$

So we need only prove now that

$$\frac{(y^{q+1},y^qt^r)}{(y^{q+1},y^qt^{r+1})}$$

is torsion-free. Indeed, the above module is generated by the class of y^qt^r. Suppose by contradiction that $x^\ell y^qt^r\in(y^{q+1},y^qt^{r+1})$ for a certain integer

$\ell > 0$. Since $\widehat{O}_{\mathcal{C},P}$ is a domain, $x^\ell t^r \in (y, t^{r+1})$, and hence, since $r \leq k$, we have $x^\ell t^k \in (y)$. But

$$\frac{\widehat{O}_{\mathcal{C},P}}{(y)} \cong \frac{\mathbb{C}[[x,t]]}{(t^{k+1})},$$

and thus $x^\ell t^k \notin (y)$, a contradiction.

Now, the inclusion $\mathcal{I}_Y^{(i)} \to \mathcal{O}_{\mathcal{C}}$ is an isomorphism away from Y, and hence its restriction to Y^c, the homomorphism $\mathcal{I}_Y^{(i)}|_{Y^c} \to \mathcal{O}_{Y^c}$, is injective away from Δ_Y. Its image is then isomorphic to $\mathcal{I}_Y^{(i)}|_{Y^c}$ modulo torsion. It is the sheaf of ideals of a subscheme of Y^c supported on Δ_Y. At our chosen P, using our description for $\widehat{\mathcal{I}}_{Y,P}^{(i)}$, we see that this subscheme has multiplicity q if $r = 0$, and $q+1$ otherwise.

An analogous description holds for $\mathcal{I}_Y^{(i+1)}|_{Y^c}$ modulo torsion. From it we get the first isomorphism,

$$\frac{\mathcal{I}_Y^{(i+1)}|_{Y^c}}{(\text{torsion})} \cong \mathcal{O}_{Y^c}\Big(-\sum_{P \in \Delta_Y}(q_P+1)P\Big).$$

Let us prove now the existence of the exact sequence (9). Using the defining sequence of $\mathcal{I}_Y^{(i+1)}$, it is enough to show that the induced map $\mathcal{I}_Y^{(i+1)}|_C \to \mathcal{I}_Y^{(i)}|_C$ factors through an injection as in (9). This is easily seen away from Δ_Y. At $P \in \Delta_Y$, using our description for the modules $\widehat{\mathcal{I}}_{Y,P}^{(j)}$, we need only observe that

$$\mathrm{Im}\left(\frac{(y^{q+1}, y^q t^{r+1})}{(y^{q+1}t, y^q t^{r+2})} \longrightarrow \frac{(y^{q+1}, y^q t^r)}{(y^{q+1}t, y^q t^{r+1})}\right) = \frac{(y^{q+1}, y^q t^{r+1})}{(y^{q+1}t, y^q t^{r+1})}, \tag{10}$$

and prove that the right-hand side is the quotient of

$$N := \frac{(y^{q+1}, y^q t^{r+1})}{(xy^{q+1}, xy^q t^{r+1}, y^{q+1}t, y^q t^{r+2})}$$

modulo torsion. But, since $yy^q t^{r+1} = t^r y^{q+1} t$, the class of $y^q t^{r+1}$ is in the torsion submodule of N. Also, $xy^{q+1} = t^{k-r} y^q t^{r+1}$. Taking the quotient of N modulo $y^q t^{r+1}$, we get the module on the right-hand side of (10). We need only show now that this module is torsion-free. This is clearly true, because if $y^\ell \in (y^{q+1}t, y^q t^{r+1})$ for some integer $\ell > 0$, then $y^\ell \equiv 0 \mod t$, which does not hold.

It remains to identify the sheaf to the right in (9). For each $P \in \Delta_Y$, let r_P denote the remainder of the Euclidean division of i by $k_P + 1$. Set $h := \text{l.c.m.}(k_P + 1 \,|\, P \in \Delta_Y)$. Let m be an integer positive enough that $j := mh - i$ be nonnegative. We claim that

$$t^i \mathcal{I}_{Y^c}^{(j)} = \mathcal{I}^m_{(Y^c)^p/\mathcal{C}} \mathcal{I}_Y^{(i)} \tag{11}$$

as subsheaves of $\mathcal{O}_{\mathcal{C}}$, where $(Y^c)^p$ is the Cartier divisor of $\mathcal{C}$ associated to Y^c; see Subsection 2.3.

Indeed, (11) holds away from Δ_Y because there $(Y^c)^p = hY^c$. To check the equality at each $P \in \Delta_Y$, we use our description of $\widehat{\mathcal{I}}^{(i)}_{Y,P}$ and the analogous description of $\widehat{\mathcal{I}}^{(j)}_{Y^c,P}$.

If $i = q(k+1) + r$, for $0 \leq r \leq k$, where k is the singularity type of P in $\mathcal{C}$, then $j = (mh' - q)(k+1) - r$, where $h' = h/(k+1)$. Also, the equation for $(Y^c)^p$ at P is $x^{h'}$; see Subsection 2.3.

There are two cases to consider. If $r = 0$, then (11) holds at P because

$$t^i x^{mh'-q} = t^{q(k+1)} x^{mh'-q} = (xy)^q x^{mh'-q} = x^{mh'} y^q.$$

On the other hand, if $r > 0$, then $mh' - q - 1$ and $k + 1 - r$ are the quotient and the remainder of the Euclidean division of j by $k+1$. And

$$\begin{aligned} t^i(x^{mh'-q}, x^{mh'-q-1}t^{k+1-r}) =& (xy)^q t^r (x^{mh'-q}, x^{mh'-q-1}t^{k+1-r}) \\ =& (x^{mh'} y^q t^r, x^{mh'-1} y^q t^{k+1}) \\ =& (x^{mh'} y^q t^r, x^{mh'-1} y^q xy) \\ =& x^{mh'} (y^q t^r, y^{q+1}), \end{aligned}$$

showing that (11) holds at P.

Now that the claim (11) is established, we may use that $(Y^c)^p$ is a Cartier divisor of $\mathcal{C}$ intersecting Y at each $P \in \Delta_Y$ with multiplicity $h/(k_P+1)$, to get

$$\frac{\mathcal{I}^{(i)}_Y|_Y}{\text{(torsion)}} \cong \frac{\mathcal{I}^{(j)}_{Y^c}|_Y}{\text{(torsion)}} \otimes \mathcal{O}_Y\Big(\sum_{P\in\Delta_Y} \frac{mh}{k_P+1} P\Big).$$

Using the description of $\mathcal{I}^{(j)}_{Y^c}|_Y$ modulo torsion, analogous to that of $\mathcal{I}^{(i)}_Y|_{Y^c}$ modulo torsion, we have

$$\frac{\mathcal{I}^{(j)}_{Y^c}|_Y}{\text{(torsion)}} = \mathcal{O}_Y\Big(-\sum_{P\in\Delta_Y} (\frac{mh}{k_P+1} - q_P)P\Big).$$

Thus

$$\frac{\mathcal{I}^{(i)}_Y|_Y}{\text{(torsion)}} \cong \mathcal{O}_Y\Big(\sum_{P\in\Delta_Y} q_P P\Big).$$

□

Lemma 3.3. *Let C be a curve. Let $Y \subset C$ be a proper subcurve and set $Z := \overline{C - Y}$. Let k be a nonnegative integer, and $p\colon \mathcal{C} \to S$ a smoothing of C whose singularity type at each $P \in Y \cap Z$ is k. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}$ such that both restriction maps below are surjective:*

$$H^0(\mathcal{C}, \mathcal{L}) \longrightarrow H^0(C, \mathcal{L}|_C) \longrightarrow H^0(Y, \mathcal{L}|_Y)$$

Set

$$m := h^0(Y, \mathcal{L}|_Y) \quad \textit{and} \quad n := h^0(Z, \mathcal{L}|_Z(-\sum_{P\in Y\cap Z} P)).$$

Then the inclusion map $H^0(\mathcal{C},\mathcal{L}(-Y^p))\to H^0(\mathcal{C},\mathcal{L})$ *is a homomorphism of free* $\mathbb{C}[[t]]$*-modules of rank* $m+n$ *whose determinant vanishes at* 0 *with order* $m(k+1)$*. Furthermore, let*

$$W := W(p_*\mathcal{L},\mathcal{L}) \quad \textit{and} \quad W' := W\big(p_*\big(\mathcal{L}(-Y^p)\big),\mathcal{L}(-Y^p)\big).$$

Then W *and* W' *are Cartier divisors and, as such,*

$$W + mZ^p = W' + nY^p.$$

Proof. That W and W' are indeed Cartier divisors was seen in Subsection 2.5. Now, let

$$\mathcal{O}_{\mathcal{C}}(-Y^p) = \mathcal{I}_Y^{(k+1)} \subset \mathcal{I}_Y^{(k)} \subset \cdots \subset \mathcal{I}_Y^{(1)} \subset \mathcal{I}_Y^{(0)} = \mathcal{O}_{\mathcal{C}}$$

be the filtration defined in Subsection 3.1. Tensoring it with $\mathcal{L}$ we obtain a filtration

$$\mathcal{L}(-Y^p) = \mathcal{L}_{k+1} \subset \mathcal{L}_k \subset \cdots \subset \mathcal{L}_1 \subset \mathcal{L}_0 = \mathcal{L}$$

of $\mathcal{L}$. For each $i=0,\dots,k$, consider the inclusion $\nu_i\colon \mathcal{L}_{i+1}\to\mathcal{L}_i$. It follows from Proposition 3.2 that

$$\operatorname{Im}(\nu_i|_C) \cong \mathcal{L}|_Z(-\sum_{P\in Y\cap Z} P) \quad \text{and} \quad \operatorname{Coker}(\nu_i|_C) \cong \mathcal{L}|_Y.$$

So, considering global sections, we have an inequality,

$$h^0(C,\mathcal{L}_i|_C) \leq h^0(Y,\mathcal{L}|_Y) + h^0(Z,\mathcal{L}|_Z(-\sum_{P\in Y\cap Z} P)) = m+n, \tag{12}$$

which is an equality if and only if the induced map

$$H^0(C,\mathcal{L}_i|_C) \to H^0(C,\operatorname{Coker}(\nu_i|_C)) \tag{13}$$

is surjective.

Now, (13) is simply the restriction map $H^0(C,\mathcal{L}|_C)\to H^0(Y,\mathcal{L}|_Y)$ if $i=0$. Thus, by hypothesis, equality holds in (12) for $i=0$. Or, in other words,

$$h^0(C,\mathcal{L}|_C) = m+n.$$

Moreover, since the restriction map $H^0(\mathcal{C},\mathcal{L})\to H^0(C,\mathcal{L}|_C)$ is also surjective,

$$h^0(\mathcal{C}_*,\mathcal{L}|_{\mathcal{C}_*}) = h^0(C,\mathcal{L}|_C) = m+n,$$

where $\mathcal{C}_*$ is the generic fiber of p. This shows that all of the $H^0(\mathcal{C},\mathcal{L}_i)$, for $i=0,\dots,k+1$, have rank $m+n$.

Now, by semicontinuity,

$$h^0(C,\mathcal{L}_i|_C) \geq h^0(\mathcal{C}_*,\mathcal{L}_i|_{\mathcal{C}_*}) = h^0(\mathcal{C}_*,\mathcal{L}|_{\mathcal{C}_*}) = m+n$$

for each $i=0,\dots,k$. Coupling this with (12), we see that equality holds. And so (13) is surjective for each $i=0,\dots,k$.

As a corollary, the map of $\mathbb{C}[[t]]$-modules

$$H^0(\mathcal{C},\mathcal{L}_{i+1}) \longrightarrow H^0(\mathcal{C},\mathcal{L}_i)$$

has determinant vanishing at 0 with order $h^0(Y,\mathcal{L}|_Y)$ for each $i=0,\dots,k$. So the inclusion

$$H^0(\mathcal{C},\mathcal{L}(-Y^p)) \longrightarrow H^0(\mathcal{C},\mathcal{L})$$

has determinant vanishing at 0 with order $m(k+1)$, proving the first statement of the lemma.

As for the second statement, by the functoriality of the ramification scheme, Formula (3) in Subsection 2.2, we have that

$$W' + (m+n)Y^p = W + m(k+1)C.$$

Now, use that $(k+1)C = Y^p + Z^p$ to get the stated equality. □

Lemma 3.4. *Let C be a curve and $p\colon \mathcal{C}\to S$ a smoothing of C. If E and F are subcurves of C with finite intersection, and W_1 and W_2 are subschemes of $\mathcal{C}$ such that*

$$\mathcal{I}_{E^p/\mathcal{C}}\mathcal{I}_{W_1/\mathcal{C}} = \mathcal{I}_{F^p/\mathcal{C}}\mathcal{I}_{W_2/\mathcal{C}}, \tag{14}$$

then there is a subscheme $W\subset W_1\cap W_2$ such that

$$\mathcal{I}_{W_2/\mathcal{C}} = \mathcal{I}_{W/\mathcal{C}}\mathcal{I}_{E^p/\mathcal{C}} \quad \textit{and} \quad \mathcal{I}_{W_1/\mathcal{C}} = \mathcal{I}_{W/\mathcal{C}}\mathcal{I}_{F^p/\mathcal{C}}.$$

Proof. Indeed, since E^p is Cartier, it is enough to find $W\subset W_1$ satisfying the first formula, as then the second formula and the fact that $W\subset W_2$ follow. And it is enough to show that $\mathcal{I}_{W_2/\mathcal{C}}\subseteq \mathcal{I}_{E^p/\mathcal{C}}$. This is a local statement, that holds away from F by (14), and away from E for obvious reasons. So we need only prove it at a point $P\in E\cap F$. At such a point we have an isomorphism of the form (4), where (x,t) and (y,t) define E and F, respectively. The local equations for E^p and F^p are x^r and y^s, respectively, for certain positive integers r and s, as we saw in Subsection 2.3. So, we need only prove that, if J_1 and J_2 are ideals of $\widehat{\mathcal{O}}_{\mathcal{C},P}$ such that $x^rJ_1 = y^sJ_2$, then $J_2\subseteq (x^r)$. But this holds, as x,y form a regular sequence in $\widehat{\mathcal{O}}_{\mathcal{C},P}$. □

4. Preliminaries on general curves

Proposition 4.1. *Let X and Y be two smooth nonrational curves. Let $A\in X$ and $B\in Y$, and let C be the uninodal curve union of X and Y with A identified with B. Let $p\colon\mathcal{C}\to S$ be a smoothing of C. Then the following statements hold:*

(i) If B is at most a simple Weierstrass point of Y, then there is a vector subspace $V\subseteq H^0(X,\omega_X((g_Y+2)A))$ of codimension 1 containing $H^0(X,\omega_X(g_YA))$ such that $(V,\omega_X((g_Y+2)A))$ is a limit canonical system on X. If B is an ordinary point of Y, then A is a base point of this system, i.e. $V=H^0(X,\omega_X((g_Y+1)A))$. On the other hand, if B is a simple Weierstrass point of Y, and if A is an ordinary point of X and p is regular, then $V\neq H^0(X,\omega_X((g_Y+1)A))$.

(ii) If A and B are at most simple Weierstrass points of X and Y, with at least one of them ordinary, then either the limit Weierstrass scheme does not contain the node of C, or contains it with multiplicity 1 and p is nonregular.

Proof. Assume B is at most a simple Weierstrass point of Y. Let ℓ be a positive integer, and set $\mathcal{L} := \omega_p(\ell Y^p)$, where ω_p is the relative dualizing sheaf of p. Then

$$\mathcal{L}|_X \cong \omega_X((\ell+1)A) \quad \text{and} \quad \mathcal{L}|_Y \cong \omega_Y((1-\ell)B). \tag{15}$$

In addition, the following natural sequences are exact:

$$0 \to \mathcal{L}|_X(-A) \to \mathcal{L}|_C \to \mathcal{L}|_Y \to 0, \tag{16}$$

$$0 \to \mathcal{L}|_Y(-B) \to \mathcal{L}|_C \to \mathcal{L}|_X \to 0. \tag{17}$$

By Riemann–Roch, $h^0(X, \mathcal{L}|_X) \geq g$ if and only if $\ell \geq g_Y$. On the other hand, by the hypothesis on B, we have $h^0(Y, \mathcal{L}|_Y) = \max(0, g_Y + 1 - \ell)$ for $\ell \neq g_Y + 1$, whereas for $\ell = g_Y + 1$ either $h^0(Y, \mathcal{L}|_Y) = 0$ if B is an ordinary point of Y, or else $h^0(Y, \mathcal{L}|_Y) = 1$.

Set $\mathcal{M} := \omega_p(g_Y Y^p)$ and $\mathcal{N} := \omega_p((g_Y+1)Y^p)$. From (15) for $\ell := g_Y$, and Riemann–Roch, since $g_Y > 0$ we have $H^1(X, \mathcal{M}|_X(-A)) = 0$. So, the exactness of (16) for $\mathcal{L} := \mathcal{M}$ implies

$$h^0(C, \mathcal{M}|_C) = h^0(X, \mathcal{M}|_X(-A)) + h^0(Y, \mathcal{M}|_Y) = (g-1) + 1 = g.$$

Thus the restriction $H^0(\mathcal{C}, \mathcal{M}) \to H^0(C, \mathcal{M}|_C)$ is surjective.

Consider now the restriction map

$$\alpha\colon H^0(C, \mathcal{M}|_C) \longrightarrow H^0(X, \omega_X((g_Y+1)A)).$$

The exactness of (16) for $\mathcal{L} := \mathcal{M}$ implies that α contains $H^0(X, \omega_X(g_Y A))$ in its image. And the exactness of (17) for $\mathcal{L} := \mathcal{M}$ implies that the kernel of α is isomorphic to $H^0(Y, \omega_Y(-g_Y B))$. Thus α is injective, hence bijective, if and only if B is an ordinary point of Y. In this case, the complete linear system of sections of $\omega_X((g_Y+1)A)$ is a limit canonical system on X. On the other hand, if B is a (simple) Weierstrass point of Y, the image of α is the subspace $H^0(X, \omega_X(g_Y A))$.

Applying (15) for $\ell := g_Y + 1$, as B is at most a simple Weierstrass point of Y, we get $H^0(Y, \mathcal{N}|_Y(-B)) = 0$. So, the natural map

$$\beta\colon H^0(C, \mathcal{N}|_C) \to H^0(X, \omega_X((g_Y+2)A))$$

is injective. The maps α and β fit in a commutative diagram of the form

$$\begin{array}{ccccc} H^0(\mathcal{C}, \mathcal{M}) & \longrightarrow & H^0(C, \mathcal{M}|_C) & \xrightarrow{\alpha} & H^0(X, \omega_X((g_Y+1)A)) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^0(\mathcal{C}, \mathcal{N}) & \longrightarrow & H^0(C, \mathcal{N}|_C) & \xrightarrow{\beta} & H^0(X, \omega_X((g_Y+2)A)), \end{array}$$

where the horizontal maps are induced by restriction, and the vertical maps are induced from the inclusion $\mathcal{M} \to \mathcal{N}$. Since β is injective, the image V of the bottom composition has codimension 1 in $H^0(X, \omega_X((g_Y+2)A))$, and $(V, \omega_X((g_Y+2)A))$ is a limit canonical system on X. From the diagram, V contains the image of the top composition, which is $H^0(X, \omega_X((g_Y+1)A))$ if B is an ordinary point of Y, and is $H^0(X, \omega_X(g_Y A))$ otherwise. In the former case, by dimension considerations, $V = H^0(X, \omega_X((g_Y+1)A))$. This finishes the proof of the first two statements of (i).

We will prove the last statement of (i) and (ii) at the same time. Suppose A is an ordinary point of X. (The proof of (ii) in the case that B is an ordinary point of Y is analogous.) Then, from Plücker formula, the ramification divisor of the complete linear system of sections of $\omega_X((g_Y+1)A)$ has degree

$$(2g_X + g_Y - 1)g + (g_X - 1)g(g-1) - g_X \tag{18}$$

on $X - A$. On the other hand, since B is at most a simple Weierstrass point of Y, also by Plücker formula, the ramification divisor of the complete linear system of sections of $\omega_Y((g_X+1)B)$ has degree

$$(2g_Y + g_X - 1)g + (g_Y - 1)g(g-1) - w_B \tag{19}$$

on $Y - B$, where $w_B = g_Y$ if B is an ordinary point, or else $w_B = g_Y + 1$.

Suppose first that B is an ordinary point of Y. Then, by the already proved second statement of (i), the limit Weierstrass scheme ∂W_p has degree away from the node equal to the sum of (18) and (19), with $w_B = g_Y$. But this sum is $g^3 - g$. So the node of C is not contained in ∂W_p.

Finally, suppose that B is a (simple) Weierstrass point of Y. Then, by the first statement of (i), there is a vector subspace $V \subset H^0(X, \omega_X((g_Y+2)A))$ of dimension g containing $H^0(X, \omega_X(g_Y A))$ such that $(V, \omega_X((g_Y+2)A))$ is a limit canonical system. Since A is an ordinary point of X, Plücker formula yields that the ramification divisor of $(V, \omega_X((g_Y+2)A))$ has degree

$$(2g_X + g_Y)g + (g_X - 1)g(g-1) - w_A \tag{20}$$

on $X - A$, where $w_A = 2g_X + g_Y - 1$ if $V \neq H^0(X, \omega_X((g_Y+1)A))$, and $w_A = 2g_X + g_Y$ otherwise. At any rate, using $w_B = g_Y + 1$, the sum of (19) and (20) is at least $g^3 - g - 1$, with equality if and only if $w_A = 2g_Y + g_X$. So, either ∂W_p does not contain the node of C, and then $w_A = 2g_X + g_Y - 1$ and $V \neq H^0(X, \omega_X((g_Y+1)A))$, or ∂W_p contains it with multiplicity 1.

Now, suppose p is regular. To finish the proof, we need only show that ∂W_p does not contain the node of C. Recall that ∂W_p is the intersection of C and an S-flat closed subscheme $W_p \subset \mathcal{C}$ intersecting the general fiber of p at its Weierstrass points; see Subsection 2.5. Since p is regular, W_p is a Cartier divisor, and the multiplicity of ∂W_p at the node of C is the intersection multiplicity of W_p and C at the node. Since $C = X + Y$, and X and Y contain the node, this multiplicity is either 0 or greater than 1. As it cannot be greater than 1, it follows that ∂W_p does not contain the node. $\square$

Lemma 4.2. *Let X be a general smooth curve, $P \in X$ a general point, and n a positive integer. Let $Q \in X - P$. Then the following statements are equivalent, for each $j = -1, 1$:*

(i) The point Q is a ramification point of the complete system of sections of $\omega_X((n+1+j)P)$.

(ii) There is a unique subspace $V \subseteq H^0(X, \omega_X((n+2)P))$ of codimension 1 containing $H^0(X, \omega_X(nP))$ but different from $H^0(X, \omega_X((n+1)P))$ such that Q is a special ramification point of $(V, \omega_X((n+2)P))$ of type $g_X + n + j$.

Proof. Set $g := g_X + n$. Also, set

$$V' := H^0(\omega_X(nP)) + H^0(\omega_X((n+2)P - gQ)) \subseteq H^0(\omega_X((n+2)P)). \tag{21}$$

Since (X, P) is general, by Prop. 3.1 of [4], all the ramification points but P of the complete linear systems of sections of $\omega_X(nP)$, of $\omega_X((n+1)P)$ and of $\omega_X((n+2)P)$ are simple. Then

$$h^0(X, \omega_X((n+2)P - gQ)) = 1 \quad \text{and} \quad h^0(X, \omega_X(nP - gQ)) = 0. \tag{22}$$

Thus the sum in (21) is direct, and V' has dimension g. In addition,

$$V'(-iQ) = H^0(X, \omega_X(nP - iQ)) \oplus H^0(X, \omega_X((n+2)P - gQ)) \tag{23}$$

for each $i = 0, 1, \dots, g$, and thus

$$\dim V'(-iQ) = h^0(X, \omega_X(nP - iQ)) + 1 \quad \text{for each } i = 0, 1, \dots, g. \tag{24}$$

Suppose first that (i) holds. Then either

$$h^0(X, \omega_X(nP - (g-1)Q)) \geq 1, \tag{25}$$

in which case (24) implies that $\dim V'(-(g-1)Q) \geq 2$, and hence Q is a special ramification point of type $g-1$ of $(V', \omega_X((n+2)P))$; or

$$h^0(X, \omega_X((n+2)P - (g+1)Q)) \geq 1, \tag{26}$$

in which case (21) implies that $V'(-(g+1)Q) \neq 0$, and hence Q is a special ramification point of type $g+1$ of $(V', \omega_X((n+2)P))$. Notice that, in either case, V' cannot be $H^0(X, \omega_X((n+1)P))$, because the complete linear system of sections of $\omega_X((n+1)P)$ has no special ramification points but P.

For the uniqueness, just observe that, if Q is a ramification point of $(V, \omega_X((n+2)P))$, for a subspace V as described in (ii), then (22) implies that $V \supseteq H^0(X, \omega_X((n+2)P - gQ))$, and hence $V \supseteq V'$. Since both V and V' have dimension g, we have $V = V'$.

Finally, suppose (ii) holds. As we saw above, necessarily $V = V'$. So, Q is a special ramification point of $(V, \omega_X((n+2)P))$ of type $g + j$ if and only if $\dim V'(-(g-1)Q) \geq 2$ if $j = -1$ or $\dim V'(-(g+1)Q) \geq 1$ if $j = 1$. Using (24) with $i = g-1$, we see that the former inequality occurs if and only if (25) holds, i.e., if and only if Q is a ramification point of the complete linear system of sections of $\omega_X(nP)$. On the other hand, since $H^0(X, \omega_X(nP - gQ)) = 0$, the latter inequality occurs if and only if (26) holds, i.e., if and only if Q is a ramification point of the complete linear system of sections of $\omega_X((n+2)P)$. □

Proposition 4.3. *Let Y be a smooth nonrational curve, $\Delta \subset Y \times Y$ the diagonal, and p_1 and p_2 the projection maps from $Y \times Y$ onto the indicated factors. Set $\mathcal{L} := (p_2^*\omega_Y)(-(g_Y-1)\Delta)$ and $\mathcal{E} := p_{1*}\mathcal{L}$. Assume Y contains no Weierstrass point of type $g_Y - 1$. Then $\mathcal{E}$ is invertible, and the degeneracy scheme of the evaluation map $p_1^*\mathcal{E} \to \mathcal{L}$ intersects Δ transversally along the set of points (P,P) for P a Weierstrass points of Y.*

Proof. By hypothesis, $h^0(Y, \omega_Y(-(g_Y-1)P)) = 1$ for each $P \in Y$, and hence $\mathcal{E}$ is invertible. Let Z denote the degeneracy scheme of the evaluation map $p_1^*\mathcal{E} \to \mathcal{L}$. For each $P \in Y$, the intersection $Z \cap p_1^{-1}(P)$ is the ramification scheme of the complete linear system of sections of $\omega_Y(-(g_Y-1)P)$, after the natural identification $p_1^{-1}(P) = Y$. Thus $Z \cap p_1^{-1}(P)$ is finite and contains (P,P) if and only if P is a Weierstrass point of Y. We need only show now that Z intersects Δ transversally, what will follow from showing that the intersection number $Z \cdot \Delta$ is $g_Y^3 - g_Y$.

Let $\delta\colon Y \to Y \times Y$ be the diagonal map. We have $\delta^*\mathcal{O}_{Y\times Y}(-\Delta) = \omega_Y$. Thus

$$Z \cdot \Delta = \deg \delta^* Z = \deg(c_1(\omega_Y^{\otimes g_Y}) - c_1(\mathcal{E})).$$

Now, since Y has no Weierstrass point of type $g_Y - 1$, the natural sequence

$$0 \to p_{1*}p_2^*\omega_Y(-(i+1)\Delta) \to p_{1*}p_2^*\omega_Y(-i\Delta) \to \omega_Y \otimes \delta^*\mathcal{O}_{Y\times Y}(-i\Delta) \to 0$$

is exact for $i = 0, \dots, g_Y - 2$. As $c_1(p_{1*}p_2^*\omega_Y) = 0$ and $\delta^*\mathcal{O}_{Y\times Y}(-\Delta) = \omega_Y$, we get

$$c_1(\mathcal{E}) = -\big(c_1(\omega_Y) + c_1(\omega_Y^{\otimes 2}) + \cdots + c_1(\omega_Y^{\otimes g_Y - 1})\big).$$

Thus

$$Z \cdot \Delta = \sum_{i=1}^{g_Y} i \deg(c_1(\omega_Y)) = \binom{g_Y+1}{2}(2g_Y - 2) = g_Y^3 - g_Y.$$

□

Proposition 4.4. *Let a and b be positive integers. Let X be a general smooth curve, and P and Q general points on X. Then the complete linear system of sections of $\omega_X(aP + bQ)$ has only simple ramification points, and P and Q are not among them.*

Proof. Set $g := g_X$. If $g = 0$, all complete linear systems on X have no ramification points. If $g = 1$, the curve X can be any curve, as long as $P - Q$ is neither a-torsion nor b-torsion in its Jacobian variety.

Assume $g > 1$. Let i be any positive integer less than g, and put $j := g - i$. Let Y and Z be two general smooth curves, Y of genus i, and Z of genus j, and let A and M be general points of Y, and B and N general points of Z. By induction on the genus, we may assume the statement of the proposition holds for (Y, A, M) and $\omega_Y(aA + (b + j)M)$, and for (Z, B, N) and $\omega_Z(bB + (a + i)N)$.

Let X_0 be the nodal curve of genus g given as the union of Y and Z, with M identified with N. Since X_0 is nodal, and A and B are nonsingular points of X_0, there are a regular smoothing $p\colon \mathcal{C} \to S$ of X_0, and sections Γ, $\Delta \subset \mathcal{C}$ such that, identifying the closed fiber of p with X_0, we have $\Gamma \cap X_0 = \{A\}$ and $\Delta \cap X_0 = \{B\}$.

Let $\mathcal{C}_*$ denote the general fiber of p. Let P and Q be the points of intersection of Γ and Δ with $\mathcal{C}_*$. The 2-pointed curve $(\mathcal{C}_*, P, Q)$ is defined over a finitely generated field extension of $\mathbb{Q}$, and hence can be viewed as a 2-pointed complex curve. We claim the statement of the proposition holds for this 2-pointed curve (and hence for a general 2-pointed curve).

To prove our claim, let ω_p be the relative dualizing sheaf of p. Let $W_* \subset \mathcal{C}_*$ be the ramification divisor of the complete linear system of sections of $\omega_p(a\Gamma + b\Delta)|_{\mathcal{C}_*}$. We need only show that W_* is reduced, and does not contain P or Q in its support. For this, it is enough to show that the limit ramification scheme ∂W is reduced and does not contain A or B in its support.

Since $\mathcal{C}$ is regular, Y and Z are Cartier divisors. Set

$$\begin{aligned} \mathcal{L}_1 &:= \omega_p(a\Gamma + b\Delta + (b + j - 1)Z),\\ \mathcal{L}_2 &:= \omega_p(a\Gamma + b\Delta + (a + i - 1)Y). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}_1|_Y &= \omega_Y(aA + (b + j)M), & \mathcal{L}_1|_Z &= \omega_Z(bB + (2 - b - j)N),\\ \mathcal{L}_2|_Z &= \omega_Z(bB + (a + i)N), & \mathcal{L}_2|_Y &= \omega_Y(aA + (2 - a - i)M). \end{aligned}$$

Due to the generality of M and N, we have

$$h^0(Y, \mathcal{L}_2|_Y(-M)) = h^0(Z, \mathcal{L}_1|_Z(-N)) = 0,$$

and hence the natural maps

$$\frac{H^0(\mathcal{C}, \mathcal{L}_1)}{tH^0(\mathcal{C}, \mathcal{L}_1)} \longrightarrow H^0(Y, \mathcal{L}_1|_Y) \quad \text{and} \quad \frac{H^0(\mathcal{C}, \mathcal{L}_2)}{tH^0(\mathcal{C}, \mathcal{L}_2)} \longrightarrow H^0(Y, \mathcal{L}_2|_Z)$$

are injective. They are actually isomorphisms, since $H^0(\mathcal{C}, \mathcal{L}_i)$ is free of rank $g + a + b - 1$, and

$$h^0(Y, \mathcal{L}_1|_Y) = h^0(Z, \mathcal{L}_2|_Z) = g + a + b - 1,$$

by the Riemann–Roch theorem.

Since

$$\mathcal{L}_2 \cong \mathcal{L}_1((g + a + b - 2)Y),$$

it follows from Formula (7) that $[\partial W] = [R_1] + [R_2]$, where the divisor R_1, resp. R_2, is the ramification divisor of the complete linear system of sections of $\omega_Y(aA + (b + j)M)$, resp. $\omega_Z(bB + (a + i)N)$; see Subsection 2.5. By induction, R_1 and R_2 are reduced and, viewed as subschemes of X_0, disjoint.

So ∂W is reduced. In addition, A and B are not in the supports of R_1 and R_2, and thus are not in support of ∂W either. □

5. Special Weierstrass points on reducible curves

Theorem 5.1. *Let X and Y be two general smooth nonrational curves. Let $A \in X$ and $B \in Y$, and let C be the uninodal curve union of X and Y with A identified with B. Set $g := g_C$. Suppose that either A is a general point of X or B is a general point of Y. Let $Q \in C$ lying on X. Then, for each $j = -1, 1$, the point Q is the limit of a special Weierstrass point of type $g+j$ along a smoothing of C if and only if Q is not the node of C, and either of the following two situations occur:*

(i) Q is a special ramification point of type $g+j$ of the complete linear system of sections of $\omega_X((g_Y+1)A)$ or

(ii) Q is a ramification point of the complete linear system of sections of $\omega_X((g_Y+1+j)A)$ and B is a Weierstrass point of Y.

Proof. We prove the "only if" part of the statement first. Let $p\colon \mathcal{C} \to S$ be a smoothing of C, as indicated in Figure 1 below, such that Q is the limit of a Weierstrass point of type $g+j$ along p. In particular, Q appears with multiplicity at least 2 in the limit Weierstrass scheme ∂W_p.

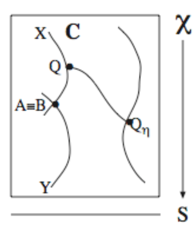

Figure 1. The smoothing.

Since X and Y are general, their Weierstrass points are simple. Also, since either A or B is general, either A or B is ordinary. Thus, it follows from Proposition 4.1, item (ii), that Q is not the node of C.

Suppose first that B is an ordinary point of Y. Then, by Proposition 4.1, item (i), the system of sections of $\omega_X((g_Y+2)A)$ vanishing at A is a limit canonical system, and hence (i) holds.

On the other hand, suppose that B is a Weierstrass point of Y. By Proposition 4.1, item (i), there is a vector subspace $V \subset H^0(X, \omega_X((g_Y+2)A))$ of codimension 1 containing $H^0(X, \omega_X(g_Y A))$ such that $(V, \omega_X((g_Y+2)A))$ is a limit canonical system, and hence admits Q as a special ramification point of type $g+j$. If $V = H^0(X, \omega_X((g_Y+1)A))$, we have (i). Otherwise, (ii) follows from Lemma 4.2.

For the "if" part of the proof, we will construct smoothings as convenient slices of certain 2-parameter families.

Suppose Q is not the node of C. Suppose first that (i) holds. Then $g_X \geq 2$. Also, it follows from Prop. 3.1 in [4] that A is not a general point of X. So, by hypothesis, B is a general point of Y, whence an ordinary point.

We will first deform C by letting A vary to a general point. More precisely, let $\Delta \subseteq X \times X$ be the diagonal, and consider the union U of $X \times X$ with $Y \times X$ with Δ naturally identified with $\{B\} \times X$. Let $q\colon U \to X$ be the projection onto the second factor. Since X is nonsingular, we may identify the complete local ring of X at A with $\mathbb{C}[[t]]$, and let $\widetilde{q}\colon \widetilde{U} \to S$ be the family induced over $S := \operatorname{Spec}(\mathbb{C}[[t]])$ by base change.

Let $V := \mathbb{C}[[t_1, t_2, \dots, t_N]]$ be the base of the universal deformation space of C, where $t_1 = 0$ corresponds to equisingular deformations. The map $\widetilde{q}$ corresponds to a local homomorphism $h\colon V \to \mathbb{C}[[t]]$ such that $h(t_1) = 0$. Since $g_X \geq 2$, the map $\widetilde{q}$ is not, even infinitesimally, a constant family. So there is $j \geq 2$ such that $h(t_j)$ generates $t\mathbb{C}[[t]]$. We may assume that $j = 2$ and, after a harmless reparametrization, that $h(t_2) = t$. Letting $p_i(t) := h(t_i)$ for each $i \geq 3$, we have $h(t_i - p_i(t_2)) = 0$ for each $i \geq 3$. Consider the two-parameter subfamily of the universal deformation of C given precisely by the equations $t_i - p_i(t_2) = 0$ for $i = 3, \dots, N$. Identify the base of this family with $S_2 := \operatorname{Spec}(\mathbb{C}[[t_1, t_2]])$, and let $u\colon T \to S_2$ denote the map giving the family, which is depicted in Figure 2 below.

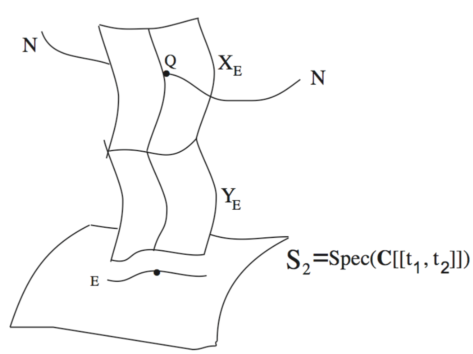

FIGURE 2. The first family.

Notice that T is a regular threefold. Let $E \subset S_2$ be the Cartier divisor given by $t_1 = 0$. The slice $u_E\colon u^{-1}(E) \to E$ is precisely $\widetilde{q}$, under the identification $t_2 = t$. Hence, the pullback u^*E is the sum of two effective Cartier divisors, X_E and Y_E, the first isomorphic to $X \times E$, the second to $Y \times E$, whose intersection on $Y \times E$ is $B \times E$, and on $X \times E$ is the graph Σ of a nonconstant map $E \to X$, sending the special point $o \in E$ to A, and the general point $e \in E$ to the general point of X.

Let $\mathcal{M} := \omega_u(g_Y Y_E)$, where ω_u is the relative dualizing sheaf of u. Then

$$\mathcal{M}|_{X_E} \cong \omega_{X_E/E}((g_Y+1)\Sigma). \tag{27}$$

A fiberwise check, as done in the proof of Proposition 4.1, shows that $u_*\mathcal{M}$ is locally free of rank g, with formation commuting with base change. In addition, since B is an ordinary point of Y, the natural map

$$\gamma\colon (u_*\mathcal{M})|_E \longrightarrow u_{E*}(\mathcal{M}|_{X_E})$$

is an isomorphism.

Form the special ramification scheme $Z \subseteq T$ of type $g+j$ of $(u_*\mathcal{M}, \mathcal{M})$, as explained in Subsection 2.2. Since γ is an isomorphism, Z agrees on $X_E - \Sigma$ with the special ramification scheme of type $g+j$ of $(u_{E*}(\mathcal{M}|_{X_E}), \mathcal{M}|_{X_E})$. Because of (27), the fact that $\Sigma \cap u^{-1}(o) = \{A\}$, and the hypothesis (i), we have that Q is an isolated point of $Z \cap u^{-1}(o)$. Furthermore, since the general point of Σ is the general point of $X \times \{e\}$, Prop. 3.1 in [4] yields $Z \cap u^{-1}(e) \subseteq Y \times \{e\}$.

Since Z is defined locally by two regular functions, there is an irreducible subscheme $N \subseteq Z$ of dimension 1 containing Q. Since Q is an isolated point of $Z \cap u^{-1}(o)$, and $Z \cap u^{-1}(e) \subseteq Y \times \{e\}$, the general point of N must lie on a smooth fiber of u, and hence be a special Weierstrass point of type $g+j$ of that fiber. So Q is the limit of a special Weierstrass point of type $g+j$.

Suppose now that (ii) holds. In particular, B is a Weierstrass point of Y, and hence $g_Y \geq 2$. Letting B vary, we may construct a two-parameter family similar to the one constructed in the first case. Thus we get a family of curves $u\colon T \to S_2$ over $S_2 = \mathrm{Spec}(\mathbb{C}[[t_1, t_2]])$ such that T is a regular threefold, and the pullback u^*E of the Cartier divisor $E \subset S_2$ given by $t_1 = 0$ is the sum of two effective Cartier divisors, X_E and Y_E, the first isomorphic to $X \times E$, the second to $Y \times E$, whose intersection on $X \times E$ is $A \times E$, and on $Y \times E$ is the graph Σ of a nonconstant map $E \to Y$, sending the special point $o \in E$ to B, and the general point $e \in E$ to the general point of Y.

Let $\widetilde{S}_2 \to S_2$ be the blowup map of S_2 at o, and denote by $F \subset \widetilde{S}_2$ the exceptional divisor. Abusing notation, we denote the strict transform of E by E as well, and let o denote the point of intersection of E and F. The fibered product $T \times_{S_2} \widetilde{S}_2$ is singular only at the node of the fiber over o of the second projection $T \times_{S_2} \widetilde{S}_2 \to \widetilde{S}_2$.

Let $\widetilde{T}$ be the blowup of $T \times_{S_2} \widetilde{S}_2$ along the subscheme $Y_E \times_{S_2} E \subset T \times_{S_2} \widetilde{S}_2$. A local analysis shows that $\widetilde{T}$ is smooth. Denote by $\widetilde{X}_E$ and $\widetilde{Y}_E$ the strict transforms in $\widetilde{T}$ of $X_E \times_{S_2} E$ and $Y_E \times_{S_2} E$. Let also $\widetilde{X}_F$ and $\widetilde{Y}_F$ denote the strict transforms of $X \times F$ and $Y \times F$. Let $\widetilde{u}\colon \widetilde{T} \to \widetilde{S}_2$ be the induced map. The fiber $\widetilde{T}_o := \widetilde{u}^{-1}(o)$ consists of three components: two of them disjoint and naturally identified with X and Y, and the remaining, say R, isomorphic to a line and meeting X and Y transversally at A and B. A local analysis shows that $\widetilde{X}_E \cap \widetilde{T}_o = X$ and $\widetilde{Y}_E \cap \widetilde{T}_o = Y \cup R$, while $\widetilde{X}_F \cap \widetilde{T}_o = X \cup R$ and $\widetilde{Y}_F \cap \widetilde{T}_o = Y$. Figure 3 below describes the family given by $\widetilde{u}$.

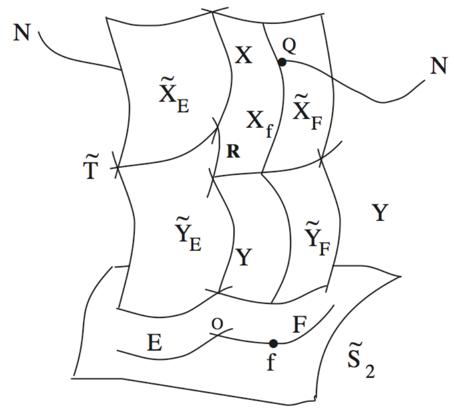

FIGURE 3. The second family.

For each $z \in E \cup F$, let X_z and Y_z denote the components of $\widetilde{u}^{-1}(z)$ that are base extensions of X and Y.

Let $\omega_{\widetilde{u}}$ be the relative dualizing sheaf of $\widetilde{u}$. Let

$$\mathcal{M} := \omega_{\widetilde{u}}(g_Y(\widetilde{Y}_E + \widetilde{Y}_F)), \quad \mathcal{N} := \mathcal{M}(\widetilde{Y}_F), \quad \text{and} \quad \mathcal{P} := \mathcal{N}(\widetilde{Y}_E).$$

Clearly, $\mathcal{M} \subset \mathcal{N} \subset \mathcal{P}$. The restriction $\mathcal{P}|_{\widetilde{X}_F}$ is the pullback of $\omega_X((g_Y+2)A)$ under the composition $\widetilde{X}_F \to X \times F \to X$. Thus

$$\widetilde{u}_*(\mathcal{P}|_{\widetilde{X}_F}) = H^0(X, \omega_X((g_Y + 2)A) \otimes \mathcal{O}_F,$$

and in particular $\widetilde{u}_*(\mathcal{P}|_{\widetilde{X}_F})$ is a locally free $\mathcal{O}_F$-module of rank $g + 1$.

We claim that the natural composition

$$\delta\colon (\widetilde{u}_*\mathcal{N})|_F \longrightarrow (\widetilde{u}_*\mathcal{P})|_F \longrightarrow \widetilde{u}_*(\mathcal{P}|_{\widetilde{X}_F})$$

is injective with invertible cokernel, and that, as f ranges in $F - o$, the image V_f of $\delta(f)$ ranges through all subspaces of $H^0(X, \omega_X((g_Y + 2)A))$ of dimension g containing $H^0(X, \omega_X(g_Y A))$ but for $H^0(X, \omega_X((g_Y + 1)A))$. In particular, $(\widetilde{u}_*\mathcal{N})|_F$ is locally free of rank g.

Once the claim is established, we proceed as in the first case. Indeed, a fiberwise analysis shows that $\widetilde{u}_*\mathcal{N}$ is locally free of rank g on $\widetilde{S}_2 - F$. Thus, from the claim, $\widetilde{u}_*\mathcal{N}$ is locally free of rank g everywhere. Form the special ramification scheme Z of type $g + j$ of $(\widetilde{u}_*\mathcal{N}, \mathcal{N})$. For each $f \in F - o$, since $\delta(f)$ is injective, Z agrees on $X_f - A$ with the special ramification scheme of type $g + j$ of $(V_f, \omega_X((g_Y + 2)A))$.

Now, by Lemma 4.2, there is a subspace $V \subseteq H^0(X, \omega_X((g_Y+2)A))$ of codimension 1 with $V \supset H^0(X, \omega_X(g_Y A))$ but $V \neq H^0(X, \omega_X((g_Y+1)A))$ such that Q is a special ramification point of $(V, \omega_X((g_Y+2)A))$ of type $g+j$. From the claim there is $f \in F - o$ such that $V = V_f$. So, viewing Q as a point of X_f, we have $Q \in Z$.

Since all irreducible components of Z have codimension at most 2 in $\widetilde{T}$, there is an irreducible curve $N \subseteq Z$ passing through $Q \in X_f$. Now, only finitely many points of $X - A$ can be special ramification points of type $g+j$ of a linear system like V, namely, by Lemma 4.2, the ramification points of the complete linear system of sections $\omega_X((g_Y+1+j)A)$. But, again by Lemma 4.2, each of these points is a special ramification point of a unique V. Thus, for all but finitely many $f \in F - o$, the linear system $(V_f, \omega_X((g_Y+2)A))$ has no special ramification points of type $g+j$ on $X - A$. Hence N intersects $\widetilde{X}_F - (\widetilde{X}_F \cap \widetilde{Y}_F)$ in only finitely many fibers over F. So the general point of N must be on a smooth fiber of $\widetilde{u}$, and hence be a special Weierstrass point of type $g+j$ of that fiber. So Q is the limit of a special Weierstrass point of type $g+j$.

Now, let us establish the claim. First, a fiberwise analysis shows that $\widetilde{u}_*\mathcal{M}$ is locally free of rank g, and that $R^1\widetilde{u}_*\mathcal{M}$ is invertible, both with formation commuting with base change. Consider the long exact sequence in higher direct images:

$$0 \to \widetilde{u}_*\mathcal{M} \to \widetilde{u}_*\mathcal{N} \to \widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F}) \to R^1\widetilde{u}_*\mathcal{M} \to R^1\widetilde{u}_*\mathcal{N} \to R^1\widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F}) \to 0.$$

Since $R^1\widetilde{u}_*\mathcal{M}$ is invertible, and $\widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F})$ is supported on F, the middle map above is zero, breaking up the long sequence in two short exact sequences,

$$0 \to \widetilde{u}_*\mathcal{M} \to \widetilde{u}_*\mathcal{N} \to \widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F}) \to 0,$$

$$0 \to R^1\widetilde{u}_*\mathcal{M} \to R^1\widetilde{u}_*\mathcal{N} \to R^1\widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F}) \to 0.$$

The exactness of the first sequence shows the surjectivity of the natural map $(\widetilde{u}_*\mathcal{N})|_F \to \widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F})$. Now, a fiberwise analysis, using that B is a simple Weierstrass point of Y, shows that $R^1\widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_F})$ is a locally free $\mathcal{O}_F$-module of rank 2. So, since $R^1\widetilde{u}_*\mathcal{M}$ is also locally free, the exactness of the second sequence above implies that, for each Cartier divisor $G \subset \widetilde{S}_2$ intersecting F properly, the natural multiplication-by-G map $(R^1\widetilde{u}_*\mathcal{N})(-G) \to R^1\widetilde{u}_*\mathcal{N}$ is injective, and hence the natural map

$$\delta_G \colon \widetilde{u}_*\mathcal{N}|_G \longrightarrow \widetilde{u}_*(\mathcal{N}|_{\widetilde{u}^{-1}(G)})$$

is an isomorphism. This isomorphism allows us to work with slices of the family $\widetilde{u}$ that intersect F properly.

In particular, for each $f \in F - \{o\}$, let $G \subset \widetilde{S}_2$ be a smooth curve passing through f, and whose general point lies on $\widetilde{S}_2 - (E \cup F)$. So we have a smoothing $\widetilde{u}_G \colon \widetilde{u}^{-1}(G) \to G$ of the fiber C, and we can also choose G such that $\widetilde{u}_G$ is regular. Then, by Proposition 4.1, (i), the natural map

$$\delta_{G,f}\colon (\widetilde{u}_{G*}(\mathcal{N}|_{\widetilde{u}^{-1}(G)}))(f) \to H^0(X_f, \mathcal{N}|_{X_f})$$

is injective and, under the isomorphism $\mathcal{N}|_{X_f} \cong \omega_X((g_Y+2)A)$, its image is a g-dimensional subspace of $H^0(X, \omega_X((g_Y+2)A))$ that contains $H^0(X, \omega_X(g_Y A))$ but is different from $H^0(X, \omega_X((g_Y+1)A))$. Now, since $\delta(f) = \delta_{G,f} \circ \delta_G(f)$, and δ_G is an isomorphism, $\delta(f)$ has the same properties.

To understand what happens at o, consider the slice of $\widetilde{u}$ over E. We claim the natural map

$$\eta\colon \widetilde{u}_*(\mathcal{N}|_{\widetilde{u}^{-1}(E)}) \longrightarrow \widetilde{u}_*(\mathcal{N}|_{\widetilde{X}_E})$$

is an isomorphism. Indeed, let $\Sigma_E := \widetilde{X}_E \cap \widetilde{Y}_E$. Since δ_E is an isomorphism, applying the long exact sequence in higher direct images to the exact sequence

$$0 \to \mathcal{N}|_{\widetilde{X}_E}(-\Sigma_E) \to \mathcal{N}|_{\widetilde{u}^{-1}(E)} \to \mathcal{N}|_{\widetilde{Y}_E} \to 0$$

we get that the natural map $(\widetilde{u}_*\mathcal{N})|_E \to \widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_E})$ is surjective, and that the image of η contains $\widetilde{u}_*(\mathcal{N}|_{\widetilde{X}_E}(-\Sigma_E))$. Thus, to show our last claim we need only show that the natural map

$$\epsilon\colon \widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_E}) \longrightarrow \widetilde{u}_*(\mathcal{N}|_{\Sigma_E})$$

is an isomorphism.

Since the map $\Sigma_E \to E$ is an isomorphism, $\widetilde{u}_*(\mathcal{N}|_{\Sigma_E})$ is locally free of rank 1. Also $\widetilde{u}_*(\mathcal{N}|_{\widetilde{Y}_E})$ is locally free of rank 1, because it is so over the generic point $e \in E$. Since the point in the intersection $\Sigma_E \cap \widetilde{u}^{-1}(e)$ is not a Weierstrass point of Y_e, the map $\epsilon(e)$ is an isomorphism.

To show that also $\epsilon(o)$ is an isomorphism, it amounts to show that the point in $\Sigma_o := \Sigma_E \cap \widetilde{u}^{-1}(o)$ of intersection of X_o and R is not a limit ramification point of $(\widetilde{u}_*\mathcal{N}|_{\widetilde{Y}_E}, \mathcal{N}|_{\widetilde{Y}_E})$. This is indeed the case, since $\widetilde{Y}_E$ is the blowup of $Y \times E$ at (B, o), and Σ_E is the strict transform of the graph of the map $E \to Y$ obtained by considering the identity map of Y locally analytically at B. So, the transversality stated in Proposition 4.3 shows that Σ_o is not a limit ramification point.

Finally, since η and δ_E are isomorphisms, it follows that $\delta(o)$, which is the composition of the isomorphism $\eta(o) \circ \delta_E(o)$ with the inclusion

$$H^0(X, \omega_X((g_Y+1)A)) \to H^0(X, \omega_X(g_Y+2)A),$$

is injective of rank g. So, $\delta(f)$ is injective of rank g for every $f \in F$, and hence δ is injective with invertible cokernel. Moreover, as the image of $\delta(o)$ is different from that of $\delta(f)$ for $f \in F - o$, then, as f varies in $F - o$, the image V_f of $\delta(f)$ varies through all the codimension-1 subspaces of $H^0(X, \omega_X((g_Y+2)A))$ containing $H^0(X, \omega_X(g_Y A))$, with the exception of $H^0(X, \omega_X((g_Y+1)A))$. The proof of our claim on δ is complete. $\square$

6. Weierstrass schemes of smoothings

Theorem 6.1. *Let C be the semistable curve of genus $g \geq 2$ that is union of a smooth curve X of genus $g-1$ and a chain E of $g-1$ exceptional components $E_1, \dots, E_{g-1}$ ordered in the usual way. Let $A \in X \cap E_1$ and $B \in X \cap E_{g-1}$ be the unique points of intersection. Suppose X is general, and A and B are general points of X. Let $p : \mathcal{C} \to S$ be a smoothing of C and $W(p)$ its Weierstrass scheme. Suppose that the singularity types of $\mathcal{C}$ at the nodes of C are equal. Then $W(p)$ is a Cartier divisor, and the difference*

$$W(p) - \sum_{i=1}^{g-1} \frac{i(g-i)g}{2} E_i^p \tag{28}$$

is effective and intersects each fiber of p transversally. In particular, the limit Weierstrass scheme of p is reduced and contains no node of C in its support.

Proof. Figure 4 below describes the curve C.

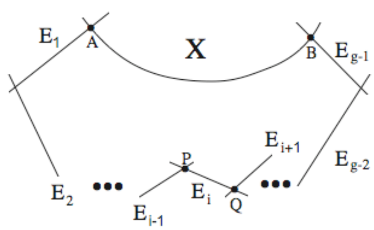

Figure 4. The nodal curve.

Identify the closed fiber of p with C. We will prove the second statement first. Let ω_p be the relative dualizing sheaf of p. By adjunction, ω_p restricts to the trivial sheaf on each E_i and to $\omega_X(A+B)$ on X. So, each global section of ω_p that vanishes on X vanishes on the whole fiber C, and hence the restriction map

$$H^0(\mathcal{C}, \omega_p) \longrightarrow H^0(X, \omega_X(A+B))$$

has image of dimension g. Since $h^0(X, \omega_X(A+B)) = g$ by the Riemann–Roch theorem, the restriction map is surjective. So the complete linear system of sections of $\omega_X(A+B)$ is a limit canonical system on X.

By Proposition 4.4, the ramification points of $H^0(X, \omega_X(A+B))$ are simple. So, by (6), the limit Weierstrass scheme ∂W_p of p is reduced on X away from A and B. Also, since neither A nor B is a ramification point of $H^0(X, \omega_X(A+B))$, again by Proposition 4.4, the Plücker formula yields

$$\deg(\partial W_p \cap X - \{A, B\}) = g^3 - g^2. \tag{29}$$

Now, for each $i = 1, \dots, g-1$, set

$$\mathcal{L}_i := \omega_p\Big(-i(g-i)E_i^p - \sum_{m=1}^{i-1} m(g-i)E_m^p - \sum_{m=i+1}^{g-1} (g-m)iE_m^p\Big). \tag{30}$$

Notice that $\mathcal{L}_i$ has degree g on E_i and zero on each E_m for $m \neq i$. Also,

$$\mathcal{L}_i|_X \cong \omega_X\big(-(g-i-1)A - (i-1)B\big).$$

Since A and B are general points of X, we have $h^0(X, \mathcal{L}_i|_X) = 1$ and

$$h^0\big(X, \mathcal{L}_i|_X(-A)\big) = h^0\big(X, \mathcal{L}_i|_X(-B)\big) = h^0\big(X, \mathcal{L}_i|_X(-A-B)\big) = 0. \tag{31}$$

Let V_i be the image of

$$H^0(\mathcal{C}, \mathcal{L}_i) \longrightarrow H^0(E_i, \mathcal{L}_i|_{E_i})$$

Since $h^0(X, \mathcal{L}_i|_X(-A-B)) = 0$, and $\deg \mathcal{L}_i|_{E_m} = 0$ for $m \neq i$, each global section of $\mathcal{L}_i$ that vanishes on E_i vanishes on the whole C. Thus $(V_i, \mathcal{L}_i|_{E_i})$ is a limit canonical system on E_i.

Let P and Q be the nodes of C in E_i. A section in V_i that vanishes at P (or Q) is the restriction of a global section of $\mathcal{L}_i$ that vanishes at P (or Q). Since $\deg \mathcal{L}_i|_{E_m} = 0$ for $m \neq i$, it follows from (31) that this global section vanishes on all components of C but possibly E_i. In particular, its restriction in V_i vanishes at P and Q. In other words,

$$V_i(-P) = V_i(-Q) = V_i(-P-Q) = H^0(E_i, \mathcal{L}_i|_{E_i}(-P-Q)), \tag{32}$$

where the last equality follows from dimension considerations.

By (32), the system $(V_i, \mathcal{L}_i|_{E_i})$ contains a complete subsystem of codimension 1, namely $H^0(E_i, \mathcal{L}_i|_{E_i}(-P-Q))$. Since complete systems on the projective line have no ramification points, the order sequence of $(V_i, \mathcal{L}_i|_{E_i})$ at each point of E_i starts with $0, \dots, g-2$. The last order can only be $g-1$ or g, since $\mathcal{L}_i|_{E_i}$ has degree g. Thus all ramification points of $(V_i, \mathcal{L}_i|_{E_i})$ are simple. In addition, since P and Q are not ramification points of $H^0(E_i, \mathcal{L}_i|_{E_i}(-P-Q))$, it follows from (32) that they are not ramification points of $(V_i, \mathcal{L}_i|_{E_i})$ either.

Thus, by (6), the scheme ∂W_p is reduced on $E_i - \{P, Q\}$. Also, since neither P nor Q is a ramification point of $(V_i, \mathcal{L}_i|_{E_i})$, Plücker formula yields

$$\deg(\partial W_p \cap E_i - \{P, Q\}) = g. \tag{33}$$

Finally, since there are $g-1$ rational components in C, we get

$$g^3 - g = \deg(\partial W_p \cap C) \geq g^3 - g^2 + (g-1)g = g^3 - g,$$

where the inequality follows from combining (29) and (33) for $i = 1, \dots, g-1$. The inequality is thus an equality, showing that we accounted for all points in the support of ∂W_p, and hence that all of them appear with multiplicity 1. The second statement is proved.

To prove the first statement, we will consider a filtration $\mathcal{L}_{i,j}$ of subsheaves of ω_p containing $\mathcal{L}_i$, defined below.

For each $i = 1, \dots, g-1$ and each $j = 0, 1, \dots, i(g-i)-1$, let k, k', ℓ, ℓ' be integers such that

$$j = ki + \ell = k'(g-i) + \ell', \quad 0 \le k, \ell' \le g-i-1, \quad 0 \le k', \ell \le i-1, \tag{34}$$

and put

$$c_{i,j,m} := km + \max(0, \ell - i + m + 1), \quad m = 1, \dots, i, \tag{35}$$

$$c'_{i,j,m} := k'(g-m) + \max(0, \ell' + i - m + 1), \quad m = i, \dots, g-1. \tag{36}$$

Notice that $c_{i,j,i} = c'_{i,j,i} = j+1$. Finally, set

$$D_{i,j} := \sum_{m=1}^{i} c_{i,j,m} E^p_m + \sum_{m=i+1}^{g-1} c'_{i,j,m} E^p_m, \tag{37}$$

and put $\mathcal{L}_{i,j} := \omega_p(-D_{i,j})$. Notice that

$$\mathcal{L}_i = \mathcal{L}_{i,i(g-i)-1}.$$

For each $i = 1, \dots, g-1$ set $D_{i,-1} := 0$ and $\mathcal{L}_{i,-1} := \omega_p$. And for each $j = 0, 1, \dots, i(g-i)-1$ set $F_{i,j} := D_{i,j} - D_{i,j-1}$. Then $\mathcal{L}_{i,j} = \mathcal{L}_{i,j-1}(-F_{i,j})$. It follows from (34), (35) and (36) that

$$F_{i,j} = E^p_{i-\ell} + E^p_{i-\ell+1} + \cdots + E^p_i + E^p_{i+1} + \cdots + E^p_{i+\ell'}. \tag{38}$$

Using this, it can also be shown, by induction on j, that

$$\mathcal{L}_{i,j}|_X \cong \begin{cases} \omega_X((1-k)A + (1-k')B) & \text{if } \ell \neq i-1 \text{ and } \ell' \neq g-i-1, \\ \omega_X(-kA + (1-k')B) & \text{if } \ell = i-1 \text{ and } \ell' \neq g-i-1, \\ \omega_X((1-k)A - k'B) & \text{if } \ell \neq i-1 \text{ and } \ell' = g-i-1, \\ \omega_X(-kA - k'B) & \text{if } \ell = i-1 \text{ and } \ell' = g-i-1. \end{cases}$$

and

$$\deg \mathcal{L}_{i,j}|_{E_m} = \begin{cases} k + k' + 2 & \text{if } m = i, \\ -1 & \text{if } m = i-\ell-1 \text{ or } m = i+\ell'+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F^r_{i,j} \subset C$ be the reduced subscheme with same support as $F_{i,j}$. It follows that

$$h^0(F^r_{i,j}, \mathcal{L}_{i,j-1}|_{F^r_{i,j}}) = k + k' + 1, \tag{39}$$

and, setting $\widehat{F}^r_{i,j} := \overline{C - F^r_{i,j}}$, that

$$h^0\big(\widehat{F}^r_{i,j}, \mathcal{L}_{i,j}|_{\widehat{F}^r_{i,j}}\big) = h^0\big(X, \omega_X(-kA - k'B)\big).$$

Since A and B are general points of X, and $k + k' \le g-2$, it follows that

$$h^0(\widehat{F}^r_{i,j}, \mathcal{L}_{i,j}|_{\widehat{F}^r_{i,j}}) = g - 1 - k - k'. \tag{40}$$

Notice that

$$\mathcal{L}_{i,j}|_{\widehat{F}^r_{i,j}} \cong \mathcal{L}_{i,j-1}|_{\widehat{F}^r_{i,j}}(-\sum_{P\in F^r_{i,j}\cap\widehat{F}^r_{i,j}} P).$$

Thus we have the following short exact sequence:

$$0 \to \mathcal{L}_{i,j}|_{\widehat{F}^r_{i,j}} \to \mathcal{L}_{i,j-1}|_C \to \mathcal{L}_{i,j-1}|_{F^r_{i,j}} \to 0. \tag{41}$$

Putting together (39) and (40), we get

$$h^0(C,\mathcal{L}_{i,j-1}|_C) \leq h^0(F^r_{i,j},\mathcal{L}_{i,j-1}|_{F^r_{i,j}}) + h^0(\widehat{F}_{i,j},\mathcal{L}_{i,j}|_{\widehat{F}^r_{i,j}}) = g.$$

By semicontinuity, since $\mathcal{L}_{i,j-1}$ restricts to the canonical sheaf on the generic fiber of p, and this sheaf has g linearly independent sections, equality holds above. It follows that each of the restriction maps in the composition below is surjective:

$$H^0(\mathcal{C},\mathcal{L}_{i,j-1}) \longrightarrow H^0(C,\mathcal{L}_{i,j-1}|_C) \longrightarrow H^0(F^r_{i,j},\mathcal{L}_{i,j-1}|_{F^r_{i,j}}). \tag{42}$$

We may now apply Lemma 3.3. Let u be the common singularity type of $\mathcal{C}$ at the nodes of C. Because of (39), we get that the inclusion map

$$H^0(\mathcal{C},\mathcal{L}_{i,j}) \longrightarrow H^0(\mathcal{C},\mathcal{L}_{i,j-1}) \tag{43}$$

is a map of free $\mathbb{C}[[t]]$-modules of the same rank g and determinant vanishing at 0 with order $(u+1)(k+k'+1)$. Summing these orders for each integer $j=0,1,\dots,i(g-i)-1$, we get that the determinant of the inclusion

$$H^0(\mathcal{C},\mathcal{L}_i) \longrightarrow H^0(\mathcal{C},\omega_p)$$

has order $(u+1)i(g-i)g/2$.

Now, using (30), and using the functoriality of the ramification scheme, Formula (3) in Subsection 2.2, we get

$$t^{(u+1)i(g-i)g/2}\mathcal{I}_{W(p)/\mathcal{C}} = \mathcal{I}^{i(g-i)g}_{E^p_i/\mathcal{C}}\mathcal{I}_{N_i/\mathcal{C}}\mathcal{I}_{W_i/\mathcal{C}},$$

where $W_i := W(p_*\mathcal{L}_i,\mathcal{L}_i)$ and N_i is an effective Cartier divisor of $\mathcal{C}$ not containing E_i in its support. Since all global sections of $\mathcal{L}_i$ that vanish on E_i vanish as well on the whole C, it follows that W_i does not contain E_i.

Observe that $\operatorname{div}(t^{u+1}) = X^p + E^p_1 + \cdots + E^p_{g-1}$. Then

$$(\mathcal{I}_{X^p/\mathcal{C}}\prod_{j\neq i}\mathcal{I}_{E^p_j/\mathcal{C}})^{i(g-i)g/2}\mathcal{I}_{W(p)/\mathcal{C}} = \mathcal{I}^{i(g-i)g/2}_{E^p_i/\mathcal{C}}\mathcal{I}_{N_i/\mathcal{C}}\mathcal{I}_{W_i/\mathcal{C}}.$$

Using Lemma 3.4 repeatedly, we get that

$$\mathcal{I}_{W(p)/\mathcal{C}} = \mathcal{I}^{i(g-i)g/2}_{E^p_i/\mathcal{C}}\mathcal{I}_{Z_i/\mathcal{C}}$$

where Z_i is a Cartier divisor contained in W_i+N_i, and thus not containing E_i in its support. Putting these together for $i=1,\dots,g-1$, and using Lemma 3.4 repeatedly, we get that

$$\mathcal{I}_{W(p)/\mathcal{C}} = \mathcal{I}_{Z/\mathcal{C}}\prod_{i=1}^{g-1}\mathcal{I}^{i(g-i)g/2}_{E^p_i/\mathcal{C}},$$

where Z is a Cartier divisor contained in all the Z_i, and thus not containing any of the E_i in its support. Since X is not in the support of $W(p)$, it is not in that of Z either. So Z intersects C properly, and hence is flat over S. That the intersection is transversal follows from the fact that the limit Weierstrass scheme is reduced. □

Theorem 6.2. *Let g be a positive integer. Let X be a general smooth curve of genus $g-1$, and A and B general points of X. Let C be the nodal curve of genus g obtained from X by identifying A and B. Then no point of C is a limit of special Weierstrass points on a family of smooth curves degenerating to C.*

Proof. The statement is true if $g = 1$, because an elliptic curve has no Weierstrass points. Suppose $g > 1$ now. Let $p\colon \mathcal{C} \to S$ be a smoothing of C. We claim that the geometric general fiber has no special Weierstrass point. It is enough to show, after base changes, blowups and blowdowns with center in the special fiber, that the limit Weierstrass divisor is reduced.

So, as observed in Subsection 2.4, we may replace p by a smoothing $\widetilde{p}\colon \widetilde{\mathcal{C}} \to S$ whose special fiber is the curve C described in Theorem 6.1, and whose nodes have equal singularity types in $\widetilde{\mathcal{C}}$. Then, by Theorem 6.1, the limit Weierstrass divisor of $\widetilde{p}$ is reduced. □

Proposition 6.3. *Let i and g be integers with $0 < i < g$. Let Y and Z be two general smooth curves of genera i and $g-i$, respectively. Let $A \in Y$ and $B \in Z$ be general points. Let C be the stable curve union of Y and Z with A and B identified. Let $p\colon \mathcal{C} \to S$ be a smoothing of C, and $W(p)$ its Weierstrass scheme. Then $W(p)$ is a Cartier divisor, and the difference*

$$W(p) - \binom{g-i+1}{2} Y^p - \binom{i+1}{2} Z^p \tag{44}$$

is effective and intersects each fiber of p transversally. In particular, the limit Weierstrass scheme of p is reduced and does not contain the node of C in its support.

(Cukierman showed in [3], Prop. 2.0.8, p. 325, that the difference (44) is effective and intersects properly each fiber of p, when $\mathcal{C}$ is regular. His proof can be easily adapted to show our proposition. We give the proof below just for the sake of completeness.)

Proof. Identify the closed fiber of p with C. Let ω_p be the relative dualizing sheaf of p, and set $\mathcal{L}_j := \omega_p(-jY^p)$ for $j = 0, \dots, g-i+1$. We have $\mathcal{L}_j|_Y \cong \omega_Y((j+1)A)$ and $\mathcal{L}_j|_Z \cong \omega_Z((1-j)B)$. So

$$\begin{cases} h^0(Y, \mathcal{L}_j|_Y) = & i + j, \\ h^0(Z, \mathcal{L}_j|_Z) = & g - i + 1 - j, \end{cases} \qquad \text{for each } j = 0, \dots, g-i+1, \tag{45}$$

where in the second equality we used that B is a general point of Z. From the natural exact sequence

$$0 \to \mathcal{L}_j|_Z(-B) \to \mathcal{L}_j|_C \to \mathcal{L}_j|_Y \to 0$$

it follows that

$$h^0(C, \mathcal{L}_j|_C) \leq h^0(Y, \mathcal{L}_j|_Y) + h^0(Z, \mathcal{L}_{j+1}|_Z) = (i+j) + (g-i-j) = g$$

for $j = 0, \dots, g-i$. By semicontinuity, the equality holds, and it follows that both restriction maps below are surjective, for each $j = 0, \dots, g-i$:

$$H^0(\mathcal{C}, \mathcal{L}_j) \longrightarrow H^0(C, \mathcal{L}_j|_C) \longrightarrow H^0(Y, \mathcal{L}_j|_Y). \tag{46}$$

Since $h^0(\mathcal{L}_{g-i}|_Z(-B)) = 0$, because B is general, the surjection

$$H^0(\mathcal{C}, \mathcal{L}_{g-i}) \to H^0(Y, \mathcal{L}_{g-i}|_Y)$$

has kernel $tH^0(\mathcal{C}, \mathcal{L}_{g-i})$. So, the complete linear system of sections of $\omega_Y((1+g-i)A)$ is a limit canonical system. Let R be its ramification scheme. By [4], Prop. 3.1, the intersection $R \cap (Y - A)$ is reduced, and R contains A with multiplicity i.

Let ∂W be the limit Weierstrass scheme of p. Then, it follows from (6) that $\partial W \cap (C - Z)$ is reduced and, by Plücker Formula applied to R,

$$\#\partial W \cap (C - Z) = \deg R - i = g(g-1)(i-1) + g(g+i-1) - i.$$

Analogously, $\partial W \cap (C - Y)$ is reduced and

$$\#\partial W \cap (C - Y) = g(g-1)(g-i-1) + g(g+(g-i)-1) - (g-i).$$

Summing up, we see that, excluding the node of C, there are $g^3 - g$ points in ∂W. Since $\deg \partial W = g^3 - g$, it follows that ∂W is reduced, and does not contain the node in its support. This proves the second statement of the proposition.

Now, let us prove the first statement. Since the restriction maps in (46) are surjective, we may apply Lemma 3.3. Using (45), we get that the determinant of the map of $\mathbb{C}[[t]]$-modules

$$H^0(\mathcal{C}, \mathcal{L}_{j+1}) \longrightarrow H^0(\mathcal{C}, \mathcal{L}_j)$$

vanishes at 0 with order $(u+1)(i+j)$ for each $j = 0, \dots, g-i-1$, where u is the singularity type of $\mathcal{C}$ at the node of C. Summing up, the determinant of the map of $\mathbb{C}[[t]]$-modules

$$H^0(\mathcal{C}, \mathcal{L}_{g-i}) \longrightarrow H^0(\mathcal{C}, \omega_p)$$

vanishes at 0 with order

$$(u+1)\Big(i(g-i) + \binom{g-i}{2}\Big) = \frac{(u+1)(g-i)(g+i-1)}{2}.$$

Let $W' := W(p_*\mathcal{L}_{g-i}, \mathcal{L}_{g-i})$. By the functoriality of the ramification scheme, Formula (3) in Subsection 2.2,

$$W(p) + \frac{(u+1)(g-i)(g+i-1)}{2} C = (g(g-i))Y^p + W'.$$

Observing that $(u+1)C = Y^p + Z^p$, we get that

$$W(p) + \frac{(g-i)(g+i-1)}{2} Z^p = W' + \binom{g-i+1}{2} Y^p.$$

Since $(p_*\mathcal{L}_{g-i}, \mathcal{L}_{g-i})$ restricts to a nondegenerate linear system on Y, we have that W' does not contain Y. Using Lemma 3.4, we get that

$$W(p) = \binom{g-i+1}{2} Y^p + D,$$

where D is a Cartier divisor contained in W', and thus not containing Y.

Analogously,

$$W(p) = \binom{i+1}{2} Z^p + D',$$

where D' is a Cartier divisor not containing Z. Applying Lemma 3.4 again, we get that

$$W(p) = \binom{g-i+1}{2} Y^p + \binom{i+1}{2} Z^p + D'',$$

where D'' is a Cartier divisor intersecting C properly. That the intersection is actually transversal follows from the fact that the limit Weierstrass scheme is reduced. □

7. The moduli space of stable curves and its Picard groups

7.1. (*A versal family of stable curves*) Let g be an integer at least 2, and let $\overline{M}_g$ denote the coarse moduli space of stable curves. As this will be important for us, we will recall how $\overline{M}_g$ is constructed; see [16]. Given a Deligne–Mumford stable curve X, its dualizing sheaf ω_X is ample; in fact $\omega_X^{\otimes n}$ is very ample for each $n \geq 3$. Also, by Riemann-Roch,

$$h^0(X, \omega_X^{\otimes n}) = (2n-1)(g-1) \quad \text{for each } n \geq 2.$$

Fix an integer $n \geq 3$. Let $N := (2n-1)(g-1) - 1$. Choosing a basis for $H^0(X, \omega_X^{\otimes n})$, we may view X as a closed subscheme of degree $2n(g-1)$ of $\mathbf{P}^N$. The subscheme depends on the choice of the basis only, so any other choice would yield a projectively equivalent subscheme. We say that all these subschemes are n-canonically embedded.

More generally, a stable curve X of genus g in $\mathbf{P}^N$ is said to be *n-canonically embedded* if $\omega_X^{\otimes n} \cong \mathcal{O}_X(1)$.

Let H be the Hilbert scheme parametrizing closed subschemes of $\mathbf{P}^N$ with Hilbert polynomial $2n(g-1)T + 1 - g$, and $\mathcal{U} \subseteq \mathbf{P}^N \times H$ the universal closed subscheme. Then there is a (locally closed) subscheme $K \subseteq H$ parametrizing n-canonically embedded stable curves of genus g. In fact, let $H' \subset H$ be the open subscheme parametrizing (connected, reduced, nodal) curves X in $\mathbf{P}^N$. The induced subfamily $\mathcal{U}_{H'} \subseteq H' \times \mathbf{P}^N$ admits a Picard scheme over H', actually an algebraic space, $\mathrm{Pic}_{\mathcal{U}_{H'}/H'}$. The sheaves $\omega^{\otimes n}_{\mathcal{U}_{H'}/H'}$ and $\mathcal{O}_{\mathcal{U}_{H'}}(1)$ induce a map

$$H' \longrightarrow \mathrm{Pic}_{\mathcal{U}_{H'}/H'} \times_{H'} \mathrm{Pic}_{\mathcal{U}_{H'}/H'}.$$

The scheme K is simply the inverse image of the diagonal under this map.

Notice that $\mathrm{Pic}_{\mathcal{U}_{H'}/H'}$ may not be separated over H', so the diagonal is only a locally closed subscheme of $\mathrm{Pic}_{\mathcal{U}_{H'}/H'} \times_{H'} \mathrm{Pic}_{\mathcal{U}_{H'}/H'}$, and hence so is K inside H'. Also, compare the definition of K with that of $\widetilde{\mathcal{K}}$ on p. 102 of [17]; they should be the same subschemes, but $\widetilde{\mathcal{K}}$ is not correctly defined.

Let $\mathcal{V} := \mathcal{U}_K \subset \mathbf{P}^N \times K$ be the induced subscheme and $v\colon \mathcal{V} \to K$ the family induced by the second projection $\mathbf{P}^N \times K \to K$. First, K is smooth. The proof of this fact uses the infinitesimal criterion of smoothness, and is essentially the proof given to [17], Lemma 3.35, p. 103.

Second, the family $v\colon \mathcal{V} \to K$ is versal. Indeed, let $p\colon \mathcal{C} \to S$ be any family of stable curves of genus g, and denote by ω_p its dualizing sheaf. The pushforward $p_*(\omega_p^{\otimes n})$ is locally free of rank $N+1$. So, for each point $s \in S$ there is an open neighborhood $S_s \subseteq S$ of s such that $p_*(\omega_p^{\otimes n})|_{S_s}$ is free. Choose an isomorphism $\mathcal{O}_{S_s}^{\oplus N+1} \to p_*(\omega_p^{\otimes n})|_{S_s}$. Since $\omega_p^{\otimes n}$ is globally generated, we get a map $\rho\colon \mathcal{C}_s \to \mathbf{P}^N \times S_s$, where $\mathcal{C}_s := p^{-1}(S_s)$. Since the fibers of p are stable, and $n \geq 3$, the map ρ is an embedding. The images of the fibers of p over S_s are n-canonically embedded curves, by construction, so we get a map $S_s \to K$. By the universal property of the Hilbert scheme, we get a Cartesian diagram,

$$\begin{array}{ccc} \mathcal{C}_s & \longrightarrow & \mathcal{V} \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle v}\big\downarrow \\ S_s & \longrightarrow & K, \end{array}$$

thus showing that v is indeed versal.

Finally, the group of automorphisms $\mathrm{PGL}(N)$ of $\mathbf{P}^N$ acts in a natural way on H, and hence there is an induced action $\sigma\colon \mathrm{PGL}(N) \times K \to K$ on K. Gieseker [16] constructs $\overline{M}_g$ as a geometric GIT quotient of K under this action for any n sufficiently large. The quotient map, $\Phi\colon K \to \overline{M}_g$, is also the map induced by the family $v\colon \mathcal{V} \to K$.

7.2. (*The Picard groups of* $\overline{M}_g$) Let $g \geq 2$. The moduli space $\overline{M}_g$ is neither fine nor smooth. There are actually two Picard groups associated to it, both important. Keep the notation of Subsection 7.1. Assume n is sufficiently large, so that $\overline{M}_g$ is the GIT quotient of K. The first Picard group we define is the so-called *Picard group of the moduli functor*: $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$. Roughly speaking, an element ξ of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ consists of the equivalence class of the assignment of an invertible sheaf ξ_p on the base S of each family of genus-g stable curves $p\colon \mathcal{C} \to S$ and of an isomorphism $t_\alpha\colon f^*\xi_p \to \xi_q$ for each

Cartesian diagram

$$\alpha\colon\quad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{C} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow{\ f\ } & S. \end{array} \tag{47}$$

where p and q are families of genus-g stable curves. The isomorphisms must satisfy natural compatibility requirements, when two Cartesian diagrams as above are juxtaposed. Two assignments $(\xi_p, t_\alpha)_{p,\alpha}$ and $(\xi'_p, t'_\alpha)_{p,\alpha}$ are called equivalent if there are isomorphisms $u_p\colon \xi_p \to \xi'_p$ for all p such that $u_q t_\alpha = t'_\alpha f^* u_p$ for all α.

According to [18], p. 50, there is an isomorphism

$$\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \longrightarrow \mathrm{Pic}(K)^{\mathrm{PGL}(N)}, \tag{48}$$

where the target is the group of isomorphism classes of invertible sheaves on K invariant under the action of $\mathrm{PGL}(N)$. (Recall that an invertible sheaf $\mathcal{L}$ on K is said to be invariant under $\mathrm{PGL}(N)$ if there is an isomorphism $\sigma^*\mathcal{L} \to p_2^*\mathcal{L}$ satisfying the cocycle condition (see (*) on p. 30 of [24]), where $p_2\colon \mathrm{PGL}(N) \times K \to K$ is the second projection.)

It is not difficult to understand how (48) works: It carries an element ξ of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ to the isomorphism class of ξ_v. That there is an isomorphism $\sigma^*\xi_v \to p_2^*\xi_v$ follows from the Cartesian diagrams below:

$$\begin{array}{ccccccc} \mathcal{V} & \longleftarrow & \mathcal{V}_2 & \xrightarrow{\ h\ } & \mathcal{V}_1 & \longrightarrow & \mathcal{V} \\ {\scriptstyle v}\big\downarrow & & {\scriptstyle v_2}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & {\scriptstyle v}\big\downarrow \\ K & \xleftarrow{\ p_2\ } & \mathrm{PGL}(N)\times K & \xrightarrow{\ \mathrm{id}\ } & \mathrm{PGL}(N)\times K & \xrightarrow{\ \sigma\ } & K \end{array} \tag{49}$$

where $\mathcal{V}_1 := (\mathrm{id}_{\mathbf{P}^N}, \sigma)^{-1}(\mathcal{V})$ and $\mathcal{V}_2 := (\mathrm{id}_{\mathbf{P}^N}, p_2)^{-1}(\mathcal{V})$ in $\mathbf{P}^N \times \mathrm{PGL}(N) \times K$, and h is the restriction of the automorphism of $\mathbf{P}^N \times \mathrm{PGL}(N) \times K$ induced by the natural action of $\mathrm{PGL}(N)$ on $\mathbf{P}^N$. (The automorphism h composed with the third projection to K is simply the third projection.) The fact that the isomorphism $\sigma^*\xi_v \to p_2^*\xi_v$ satisfies the cocycle condition follows from the compatibility requirements that ξ must satisfy.

Roughly, the inverse to (48) is defined as follows: Let $\mathcal{L}$ be a $\mathrm{PGL}(N)$-invariant invertible sheaf on K. Let $p\colon \mathcal{C} \to S$ be any family of stable sheaves, and ω_p its dualizing sheaf. As we saw in Subsection 7.1, there is an open covering $S = \bigcup S_i$ such that $p_*(\omega_p^{\otimes n})|_{S_i}$ is free for each i, and, from the choice of an isomorphism $\mathcal{O}_{S_i}^{\oplus N+1} \to p_*(\omega_p^{\otimes n})|_{S_i}$, we get an induced map $u_i\colon S_i \to K$. For each i, let $\mathcal{M}_i := u_i^*\mathcal{L}$. The maps $u_i|_{S_i\cap S_j}$ and $u_j|_{S_i\cap S_j}$ may not be equal, but since they differ because of the choice of a trivialization for $p_*(\omega_p^{\otimes n})|_{S_i\cap S_j}$, they differ by the action of a map $S_i \cap S_j \to \mathrm{PGL}(N)$. Since there is an isomorphism $\sigma^*\mathcal{L} \to p_2^*\mathcal{L}$, we get an isomorphism $\varphi_{i,j}\colon \mathcal{M}_i|_{S_i\cap S_j} \to \mathcal{M}_j|_{S_i\cap S_j}$. Using that the isomorphism $\sigma^*\mathcal{L} \to p_2^*\mathcal{L}$ satisfies the cocycle condition, we can show that also the $\varphi_{i,j}$

satisfy the cocycle condition, and thus give rise to an invertible sheaf ξ_p on S such that $\xi_p|_{S_i} \cong \mathcal{M}_i$ for each i. The isomorphisms t_α for Cartesian diagrams α may be constructed locally, in a similar fashion. And the compatibility requirements, for juxtapositions of Cartesian diagrams, may be checked locally.

The second Picard group of interest is $\mathrm{Pic}(\overline{M}_g)$. According to [22], Lemma 5.8, p. 100, there is an injection

$$\mathrm{Pic}(\overline{M}_g) \longrightarrow \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \tag{50}$$

whose cokernel is torsion. Thus we get an isomorphism

$$\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \otimes \mathbb{Q} \tag{51}$$

Again, it is not difficult to understand how (50) works: Given an invertible sheaf $\mathcal{L}$ on $\overline{M}_g$ and a family of stable curves $p\colon \mathcal{C} \to S$, let $u_p\colon S \to \overline{M}_g$ be the induced map and define $\xi_p := u_p^*\mathcal{L}$. Given a Cartesian diagram α as in (47), we just observe that $u_q = u_p f$ to get an isomorphism $t_\alpha\colon f^*\xi_p \to \xi_q$. The compatibility of the isomorphisms t_α with juxtapositions of Cartesian diagrams is immediate.

To understand the inverse of (51), first observe that, since $\overline{M}_g$ has only finite quotient singularities, every codimension-1 subvariety Y of $\overline{M}_g$ is $\mathbb{Q}$-Cartier, i.e. there is a Cartier divisor D of $\overline{M}_g$ such that $[D] = n[Y]$ for some integer $n > 0$. So the natural injection $\mathrm{Pic}(\overline{M}_g) \to A^1(\overline{M}_g)$ from the Picard group of $\overline{M}_g$ to the Chow group of codimension-1 cycle classes of $\overline{M}_g$, taking the isomorphism class of an invertible sheaf $\mathcal{L}$ to $c_1(\mathcal{L}) \cdot [\overline{M}_g]$, becomes an isomorphism,

$$\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} \longrightarrow A^1(\overline{M}_g) \otimes \mathbb{Q}, \tag{52}$$

upon tensoring with $\mathbb{Q}$.

Now, by a result of Looijenga's [21], and Pikaart's and de Jong's [26], there is a family of stable curves $b\colon \mathcal{Y} \to B$ of genus g over a smooth, projective scheme B such that the induced map $u_b\colon B \to \overline{M}_g$ is finite and surjective. (The fibers parametrize the so-called "non-Abelian level structures.")

So, given $\xi \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, let $\mathcal{L}$ be the assigned invertible sheaf on B. Define

$$\zeta := (1/d)u_{b*}(c_1(\mathcal{L}) \cdot [B]) \quad \text{in } A^1(\overline{M}_g) \otimes \mathbb{Q},$$

where d is the degree of u_b. And let $\rho \in \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ be the element corresponding to ξ under the isomorphism (51). We claim that ρ corresponds to ζ under the isomorphism (52). Indeed, there is an integer m such that $m\rho$ arises from an invertible sheaf $\mathcal{M}$ on $\overline{M}_g$. Since ρ corresponds to ξ under (51), we have that $\mathcal{L}^{\otimes m} = u_b^*\mathcal{M}$. Now, it follows from the projection formula that

$$m\zeta = (1/d)u_{b*}(c_1(u_b^*\mathcal{M}) \cdot [B]) = (1/d)dc_1(\mathcal{M}) \cdot [\overline{M}_g] = c_1(\mathcal{M}) \cdot [\overline{M}_g].$$

So $m\zeta$ corresponds to $\mathcal{M}$, or $m\rho$, under (52), and hence ζ corresponds to ρ.

7.3. (*Tautological and boundary classes*) Keep the notation used in Subsections 7.1 and 7.2. There is a natural element λ in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$: For each family $p\colon \mathcal{C} \to S$ of genus-g stable curves, let $\lambda_p := \det p_*\omega_p$, where ω_p is the dualizing sheaf of p. Since the formation of ω_p and $p_*\omega_p$ commutes with base change, it follows that λ is well-defined. The element λ is called a *tautological class*.

There are boundary classes as well. First, some terminology is useful. Given a connected nodal curve X, we say that a node P of X is *disconnecting* if $X - P$ is not connected. (In this case, $X - P$ has exactly two connected components.) Otherwise, we say that P is *connecting*.

For each $i = 0, \dots, [g/2]$, define subsets $\Delta'_i \subset K$, where Δ'_0 is the set of points $s \in K$ such that the fiber $\mathcal{V}_s$ has a connecting node, and Δ'_i, for $i > 0$, is the set of points $s \in K$ such that $\mathcal{V}_s$ has a disconnecting node P, and the closure in $\mathcal{V}_s$ of one of the connected components of $\mathcal{V}_s - P$ has (arithmetic) genus i. These are closed subsets of K of codimension 1. Give them their reduced induced scheme structures. Then they are Cartier divisors, because K is smooth. If s belongs to Δ'_i for a certain i, so does t, for any $t \in K$ such that $\mathcal{V}_s$ and $\mathcal{V}_t$ are projectively equivalent. So the Δ'_i are invariant under the action of $\mathrm{PGL}(N)$, and hence so are their associated invertible sheaves. Let $\delta_0, \dots, \delta_{[g/2]}$ denote the corresponding elements of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$. We call these elements *boundary classes*. By construction, $\delta_{i,v}$ is the invertible sheaf associated to Δ'_i for each $i = 0, \dots, [g/2]$.

We will also view λ and the δ_i as elements of $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ under the natural isomorphism (51).

The group $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ is generated by λ and the δ_i, and these elements form a basis if $g \geq 3$; see [1], Thm. 1, p. 154. If $g = 2$, Mumford showed that δ_0 and δ_1 form a basis for $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$; see [23], Thm. 10.1, p. 320. Furthermore, λ is expressed in terms of δ_0 and δ_1 by Mumford's relation (Eq. (8.4) on p. 317 of loc. cit.):

$$10\lambda - \delta_0 - 2\delta_1 = 0. \tag{53}$$

8. Special Weierstrass classes

8.1. (*Special Weierstrass classes*) Fix an integer $g \geq 2$. Let $p\colon \mathcal{C} \to S$ be a family of genus-g stable curves over a smooth, connected, quasi-projective scheme S. Assume that the general fiber of p is smooth and has no special Weierstrass points. Let $S' \subseteq S$ be the open subscheme over which p is smooth, and set $\mathcal{C}' := p^{-1}(\mathcal{C})$ and $p' := p|_{\mathcal{C}'}\colon \mathcal{C}' \to S'$. Set

$$SW(p') := p'_*[VSW(p')] \quad \text{and} \quad E_j(p') := p'_*[VE_j(p')]$$

for $j = -1, 1$. Since the general fiber of p' has no special Weierstrass points, and any smooth curve has only finitely many Weierstrass points, the schemes $VSW(p')$ and the $VE_j(p')$ are pure of codimension 2 in $\mathcal{C}'$. Thus $SW(p')$ and the $E_j(p')$ are cycles of pure codimension 1 in S'. Let $\overline{SW}(p)$ and $\overline{E}_j(p)$, for $j=-1,1$, denote the closures in S of $SW(p')$ and $E_j(p')$, for $j=-1,1$. Notice that, since S is smooth, we may and will view $SW(p')$, $\overline{SW}(p)$ and the $E_j(p')$ and $\overline{E}_j(p)$ as Cartier divisors.

Recall now the notation in Section 7. Let K be the scheme of n-canonically embedded genus-g stable curves, for n sufficiently large, $v\colon \mathcal{V}\to K$ the versal family, and $\Phi\colon K\to \overline{M}_g$ the induced map. Since v is versal, Φ is surjective. The fibers of Φ are $\mathrm{PGL}(N)$-orbits of points with finite stabilizers, so they are irreducible of the same dimension. Since $\overline{M}_g$ is irreducible, so is K.

Now, a general smooth curve has no special Weierstrass points; see [4], Cor. 3.3. So the general fiber of v is smooth and has no special Weierstrass points. We may thus consider the Cartier divisors $\overline{SW}(v)$ and $\overline{E}_j(v)$, for $j=-1,1$, of K. We claim that their associated invertible sheaves are $\mathrm{PGL}(N)$-invariant.

Indeed, recall diagram (49). Since p_2 and σ are smooth, and since the formation of $VSW(v')$ commutes with base change, it follows from [13], Lemma 1.7.1 and Prop. 1.7, p. 18, that

$$[p_2^{-1}(SW(v'))]=v'_{2*}[VSW(v'_2)] \quad \text{and} \quad [\sigma^{-1}(SW(v'))]=v'_{1*}[VSW(v'_1)],$$

and hence $p_2^{-1}(\overline{SW}(v))=\overline{SW}(v_2)$ and $\sigma^{-1}(\overline{SW}(v))=\overline{SW}(v_1)$. Now, $h(VSW(v'_2))=VSW(v'_1)$, and thus $\overline{SW}(v_2)=\overline{SW}(v_1)$. It follows that

$$p_2^{-1}(\overline{SW}(v))=\sigma^{-1}(\overline{SW}(v)),$$

and hence the invertible sheaf associated to $\overline{SW}(v)$ is $\mathrm{PGL}(N)$-invariant. An analogous reasoning works for the $\overline{E}_j(v)$ instead of $\overline{SW}(v)$.

Now, using the isomorphism (48), the invertible sheaves associated to $\overline{SW}(v)$, $\overline{E}_{-1}(v)$ and $\overline{E}_1(v)$ define elements of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, which we denote by $\overline{SW}_g$, $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$, respectively. We call these the *special Weierstrass classes* of $\overline{M}_g$. We will also view them as elements in $\mathrm{Pic}(\overline{M}_g)\otimes\mathbb{Q}$, under the identification given by (51).

Lemma 8.2. *Let*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{C} \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ T & \xrightarrow{f} & S \end{array}$$

be a Cartesian diagram of maps, where p and q are families of smooth curves, and T and S are smooth, connected, quasi-projective schemes. Assume that the general fiber of q has no special Weierstrass points. Then the general fiber of p has no special Weierstrass points either, and

(54) $\quad f^{-1}(SW(p))=SW(q)$ *and* $f^{-1}(E_j(p))=E_j(q)$ *for* $j=-1,1$.

Proof. The first statement holds because having no special Weierstrass points is an open property.

As for (54), we will only show that $f^{-1}(SW(p))=SW(q)$, as the proofs of the remaining equalities are analogous. To ease notation, let $V:=VSW(p)$

and $W := VSW(q)$. Since p and q are smooth, and so are S and T, we need to show that

$$f^*p_*[V] = q_*[W], \tag{55}$$

where we view $p_*[V]$ and $q_*[W]$ as Cartier divisors, and f^* as the pullback map on Cartier divisors.

Let $D := p(V)$, scheme-theoretically, and put $E := p^{-1}(D)$. Since the general fiber of p has no special Weierstrass points, V has pure codimension 2 in $\mathcal{C}$, and D has pure codimension 1 in S. Since S is smooth, D is an effective Cartier divisor. So is $E := f^{-1}(D)$, because the general fiber of q has no special Weierstrass points. Consider the induced sequence of Cartesian diagrams:

$$\begin{array}{ccc} W & \xrightarrow{f_2} & V \\ {\scriptstyle q_1}\downarrow & & {\scriptstyle p_1}\downarrow \\ E & \xrightarrow{f_1} & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S, \end{array}$$

where $p_1 := p|_V$ and $q_1 := q|_W$, where the bottom vertical maps are inclusions, and where $f_1 := f|_E$ and $f_2 := h|_W$.

Now, f is the composition of a regular embedding, its graph, followed by a smooth map, the projection $T \times S \to S$. So f is a local complete intersection (l.c.i.) map, in the sense of Fulton [13], p. 112. So there are well-defined Gysin maps $f^!\colon A^0(V) \to A^0(W)$ and $f^!\colon A^0(D) \to A^0(E)$, denoted the same by Fulton. According to [13], Prop. 6.6, p. 113, the maps $f^!$ satisfy all the properties asserted in loc. cit., Thm. 6.2, p. 98, including the compatibility with pushforwards. Thus

$$f^!p_{1*}[V] = q_{1*}f^![V] \tag{56}$$

as classes in $A^0(E)$.

The subscheme V of $\mathcal{C}$ is determinantal, and has pure codimension 2, which is the expected codimension. As $\mathcal{C}$ is smooth, V is Cohen–Macaulay; see [13], Thm. 14.4, p. 254. Analogously, W is a Cohen–Macaulay subscheme of $\mathcal{Y}$ of pure codimension 2. Thus, since the above diagram is Cartesian, f_2 is l.c.i. of the same codimension as f. Analogously, since D and E are Cartier divisors of S and T, respectively, also f_1 is l.c.i. of the same codimension as f.

By [13], Prop. 6.6, p. 113 and Thm. 6.2, p. 98,

$$f^![V] = f_2^![V] \quad \text{and} \quad f^!p_{1*}[V] = f_1^!p_{1*}[V].$$

Now, $f_2^![V] = [W]$; see [13], Lemma 1.7.1, p. 18 and Ex. 6.1.7, p. 97. Analogously, writing $p_{1*}[V] = \sum_i n_i[D_i]$, where the D_i are the irreducible components of D, we get

$$f_1^! p_{1*}[V] = \sum_i n_i f_1^! [D_i] = \sum_i n_i [f^{-1}(D_i)].$$

So, from (56), it follows that

$$q_{1*}[W] = \sum_i n_i [f^{-1}(D_i)]$$

as classes in $A^0(E)$. But both sides above are cycles of codimension 0 in E. Thus equality holds as cycles. Or, as codimension-1 cycles of T,

$$q_*[W] = \sum_i n_i [f^{-1}(D_i)].$$

It is now enough to observe that the right-hand side is simply the cycle associated to the Cartier divisor $f^* p_*[V]$. □

Lemma 8.3. *Let Y be a smooth, connected, projective variety of dimension at least 1. Let $Z_1, Z_2 \subset Y$ be closed subschemes of codimension 1 and 2, respectively. Let $\mathcal{L}$ be an invertible sheaf on Y. Then there is a smooth subcurve $C \subseteq Y$ such that $\# C \cap Z_1 < \infty$ and $C \cap Z_2 = \emptyset$, and such that $\mathcal{L}|_C \cong \mathcal{O}_C$ if and only if $\mathcal{L} \cong \mathcal{O}_Y$.*

Proof. If Y has dimension 1, set $C := Y$. Suppose $\dim Y > 1$, and let us argue by induction on $\dim Y$. Since Y is smooth and projective, and $\dim Y > 1$, Serre Duality can be applied to deduce that $H^1(Y, \mathcal{L}(-n)) = 0$ for a certain integer n. Also, by Bertini Theorem, a general section of $\mathcal{O}_Y(n)$ has smooth, connected zero scheme. Call H this zero scheme. By its generality, $Z_1 \cap H$ and $Z_2 \cap H$ have codimension 1 and 2, respectively, in H.

Since $H^1(Y, \mathcal{L}(-H)) = 0$, the restriction map $H^0(Y, \mathcal{L}) \to H^0(H, \mathcal{L}|_H)$ is surjective. Suppose there is an isomorphism $\mathcal{L}|_H \cong \mathcal{O}_H$. This isomorphism yields a section of $\mathcal{L}|_H$, and hence lifts to a section s of $\mathcal{L}$ that is nonzero along H. So the zero scheme Z of s is contained in $Y - H$, and since H is ample, Z must be finite. However, if nonempty, Z would have codimension 1 in Y, and hence would be infinite. So $Z = \emptyset$, and thus s induces an isomorphism $\mathcal{L} \cong \mathcal{O}_Y$.

By induction, there is a smooth subcurve $C \subseteq H$ such that $\# C \cap Z_1 < \infty$ and $C \cap Z_2 = \emptyset$, and such that $\mathcal{L}|_C \cong \mathcal{O}_C$ if and only if $\mathcal{L}|_H \cong \mathcal{O}_H$. But, as we saw, $\mathcal{L}|_H \cong \mathcal{O}_H$ if and only if $\mathcal{L} \cong \mathcal{O}_Y$. □

Theorem 8.4. *Let $g \geq 2$. The following formula holds in* $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$*:*

$$\begin{aligned}\overline{SW}_g \quad = \quad & (3g^4 + 4g^3 + 9g^2 + 6g + 2)\lambda - \frac{g(g+1)(2g^2+g+3)}{6}\delta_0 \\ & -(g^3 + 3g^2 + 2g + 2) \sum_{i=1}^{[g/2]} i(g-i)\delta_i.\end{aligned}$$

Proof. The above formula was shown in [15], Thm. 5.1, p. 44, using the method of test curves. Here we will show how to obtain it directly.

To show the formula is equivalent to showing that a power of a certain invertible sheaf $\mathcal{N}$ on $\overline{M}_g$ is trivial. As mentioned in Subsection 7.2, there is a family $b\colon \mathcal{Y} \to B$ of genus-g stable curves over a smooth, projective variety B such that the induced map $u_b\colon B \to \overline{M}_g$ is finite and surjective. By the projection formula, to show that a power of $\mathcal{N}$ is trivial is equivalent to show that a power of the pullback $u_b^*\mathcal{N}$ is trivial. Now, using Lemma 8.3, there is a smooth subcurve $S \subset B$ such that a power of $\mathcal{N}$ is trivial if and only if a power of $u_b^*\mathcal{N}|_S$ is trivial.

Let $p\colon \mathcal{C} \to S$ be the base change of the family $b\colon \mathcal{Y} \to B$ under the inclusion $S \to B$. Since S may be chosen "general," intersecting properly certain subschemes of B, and since the map $B \to \overline{M}_g$ is finite, we may assume that p has finitely many singular fibers, all of them uninodal. Also, we may assume the general fiber of p is a general curve of genus g, and in particular has no special Weierstrass points, and each of the singular fibers of p is general, among singular curves. In particular, it follows from Theorem 5.1 and Theorem 6.2 that no singular fiber of p has a point that is a limit of special Weierstrass points on a family of smooth curves degenerating to the fiber.

Let $\nu := u_b|_S$. We claim that $\nu^*\overline{SW}_g = \overline{SW}(p)$; see the notation introduced in Subsection 8.1. Indeed, let K be the scheme of n-canonically embedded stable curves of genus g, for n sufficiently large, and let $v\colon \mathcal{V} \to K$ be the versal family; see Subsection 7.1. Then there are an open covering $S = \bigcup S_i$ and maps $u_i\colon S_i \to K$ for each i such that $p_i := p|_{p^{-1}(S_i)}\colon p^{-1}(S_i) \to S_i$ is obtained from v by base change under u_i, for every i. From the construction of $\overline{SW}_g$, it is enough to show that $\overline{SW}(p_i) = u_i^{-1}(\overline{SW}(v))$ for each i. Since none of the singular fibers of p has limits of special Weierstrass points, it remains to show that $SW(p_i') = u_i^{-1}(SW(v'))$. But this follows from Lemma 8.2.

For each $s \in S$ such that $\mathcal{C}_s := p^{-1}(s)$ is singular, let P_s denote the unique node of $\mathcal{C}_s$, and let k_s be the singularity type of P_s in $\mathcal{C}$. Let S_0 be the set of $s \in S$ such that $\mathcal{C}_s$ is singular and irreducible. In addition, for each $i = 1, \ldots, [g/2]$, let

$$S_i := \{s \in S \,|\, \mathcal{C}_s \text{ contains a component of genus } i\}.$$

Let $\lambda' := c_1(p_*\omega_p)$, where ω_p is the relative dualizing sheaf of p, and set

$$\delta_i' := \sum_{s \in S_i} (k_s + 1)[s] \tag{57}$$

for $i = 0, 1, \ldots, [g/2]$. Then $\lambda' = \nu^*\lambda$ and $\delta_i' = \nu^*\delta_i$ for $i = 0, 1, \ldots, [g/2]$; see [17], Section 3D, especially p. 146. Since also $\nu^*\overline{SW}_g = \overline{SW}(p)$, to show the statement of the theorem we need only show that in $\mathrm{Pic}(S) \otimes \mathbb{Q}$ the class of $\overline{SW}(p)$ satisfies an equation similar to that of $\overline{SW}_g$, but with λ and the δ_i replaced by λ' and the δ'_i.

To show the formula for $\overline{SW}(p)$ we may replace S by any finite covering $T \to S$, as the induced pullback map $\mathrm{Pic}(S) \otimes \mathbb{Q} \to \mathrm{Pic}(T) \otimes \mathbb{Q}$ is injective. The new formula to be shown is completely analogous. So, up to replacing S by a finite covering, we may assume that $k_s + 1$ is divisible by g for each $s \in S_0$. (The finite covering can be any of degree g that is totally ramified at the points of S_0.)

By considering blowups and blowdowns, we may find a map of schemes $\beta\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ such that

(1) β is an isomorphism away from the points P_s for $s \in S_0$;
(2) for each $s \in S_0$, the fiber $\widetilde{\mathcal{C}}_s := (p \circ \beta)^{-1}(s)$ is the nodal curve that is the union of the normalization of $\mathcal{C}_s$ and a chain of $g-1$ rational curves connecting the branches over P_s, and $\beta\colon \widetilde{\mathcal{C}}_s \to \mathcal{C}_s$ is the map collapsing the chain to P_s;
(3) the singularity type in $\widetilde{\mathcal{C}}$ of each of the nodes of $\widetilde{\mathcal{C}}_s$ is $(k_s+1)/g - 1$ for each $s \in S_0$.

Let $\widetilde{p} := p \circ \beta$. For each $s \in S_0$, let

$$\widetilde{k}_s := \frac{k_s + 1}{g} - 1, \tag{58}$$

let $\widetilde{\mathcal{C}}_s^n \subset \widetilde{\mathcal{C}}_s$ be the normalization of $\mathcal{C}_s$, and let $E_s = E_{s,1} \cup \cdots \cup E_{s,g-1}$ be the chain of rational components of $\widetilde{\mathcal{C}}_s$, where the $E_{s,i}$ are ordered sequentially. Also, for each $i \geq 1$ and each $s \in S_i$, let Y_s denote the component of the fiber $\widetilde{\mathcal{C}}_s$ of genus i and Z_s that of genus $g-i$. Notice that $Y_s^{\widetilde{p}}$ and $Z_s^{\widetilde{p}}$ are Cartier divisors of $\widetilde{\mathcal{C}}$ such that $Y_s^{\widetilde{p}} + Z_s^{\widetilde{p}} = (k_s+1)\widetilde{p}^*(s)$ and $Y_s^{\widetilde{p}} \cdot Z_s = 1$. Thus

$$Y_s^{\widetilde{p}} \cdot Z_s^{\widetilde{p}} = k_s + 1 = -Y_s^{\widetilde{p}} \cdot Y_s^{\widetilde{p}} = -Z_s^{\widetilde{p}} \cdot Z_s^{\widetilde{p}}. \tag{59}$$

Similarly, for $s \in S_0$, we have that $(\widetilde{\mathcal{C}}_s^n)^{\widetilde{p}}$ and the $E_{s,i}^{\widetilde{p}}$ are Cartier divisors of $\widetilde{\mathcal{C}}$ such that

$$(\widetilde{\mathcal{C}}_s^n)^{\widetilde{p}} + \sum_{i=1}^{g-1} E_{s,i}^{\widetilde{p}} = (\widetilde{k}_s + 1)\widetilde{p}^*(s),$$

and, for all $i, j = 1, \ldots, g-1$,

$$E_{s,i}^{\widetilde{p}} \cdot E_{s,j}^{\widetilde{p}} = \begin{cases} \widetilde{k}_s + 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1, \\ -2(\widetilde{k}_s + 1) & \text{if } i = j. \end{cases} \tag{60}$$

Consider the Weierstrass divisor $W_{\widetilde{p}}$ of $\widetilde{p}$. Let

$$W := W_{\widetilde{p}} - \sum_{i=1}^{[g/2]} \sum_{s\in S_i} \left(\binom{g-i+1}{2} Y_s^{\widetilde{p}} + \binom{i+1}{2} Z_s^{\widetilde{p}} \right) - \sum_{s\in S_0} \sum_{i=1}^{g-1} \frac{i(g-i)g}{2} E_{s,i}^{\widetilde{p}}. \tag{61}$$

We claim that W is effective and intersects transversally each singular fiber of $\widetilde{p}$.

Indeed, the claim can be checked formally around each $s \in S$ for which $\widetilde{p}^{-1}(s)$ is singular. So, it is possible to treat the fibers over S_0 and over the S_i for $i > 0$ independently. Now, our Theorem 6.1 shows that W is effective on a neighborhood of the fiber over any $s \in S_0$, and intersects that fiber transversally. Furthermore, our Proposition 6.3 shows that W is effective on a neighborhood of the fiber over any $s \in S_1 \cup \cdots \cup S_{[g/2]}$, and intersects that fiber transversally as well. The claimed is proved.

Since W intersects transversally each singular fiber of $\widetilde{p}$, its branch locus over S is simply $VSW(p')$, where p' is the restriction of p to its smooth locus, under the identification given by β. Since $VSW(p')$ has codimension 2 in $\widetilde{\mathcal{C}}$, we have that, in $\mathrm{Pic}(\widetilde{\mathcal{C}})$:

$$[VSW(p')] = c_2(J^1_{\widetilde{p}}(\mathcal{O}_{\widetilde{\mathcal{C}}}(W))) = c_1(W)\big(c_1(W) + c_1(\omega_{\widetilde{p}})\big). \tag{62}$$

In addition, $c_1(W)$ can be computed from the definition of W, since, by Plücker formula,

$$c_1(W_{\widetilde{p}}) = \binom{g+1}{2} c_1(\omega_{\widetilde{p}}) - \widetilde{p}^*\lambda'. \tag{63}$$

We used above that $\beta_*\omega_{\widetilde{p}} = \omega_p$, where $\omega_{\widetilde{p}}$ is the relative dualizing sheaf of $\widetilde{p}$.

Since $\overline{SW}(p) = \widetilde{p}_*[VSW(p')]$, to compute $\overline{SW}(p)$ we use (61) and (63) in (62), and expand the product. To simplify the resulting expression for $\overline{SW}(p)$, let us first identify terms that vanish. First, observe that Cartier divisors supported on different fibers have zero product. So, for instance, $Y_s^{\widetilde{p}} Z_{s'}^{\widetilde{p}} = 0$ if $s \neq s'$. Also, $(\widetilde{p}^*\lambda')^2 = \widetilde{p}^*((\lambda')^2) = 0$ by reason of dimension, and

$$\widetilde{p}_*(c_1(Y_s^{\widetilde{p}})\widetilde{p}^*\lambda') = \widetilde{p}_*(c_1(Z_s^{\widetilde{p}})\widetilde{p}^*\lambda') = \widetilde{p}_*(c_1(E_{s',j}^{\widetilde{p}})\widetilde{p}^*\lambda') = 0$$

for each $s \in S_1 \cup \ldots S_{[g/2]}$, each $s' \in S_0$ and each $j = 1, \ldots, g-1$, by the projection formula, since the $Y_s^{\widetilde{p}}$, $Z_s^{\widetilde{p}}$ and $E_{s',j}^{\widetilde{p}}$ are collapsed to points under $\widetilde{p}$. Taking these vanishings in consideration, we have

$$\overline{SW}(p) = u - a + b + c - (g^2 + g + 1)(d + e), \tag{64}$$

where

$$\begin{aligned}
u &:= \binom{g+1}{2}\left(\frac{g^2+g+2}{2}\right)\widetilde{p}_*(c_1(\omega_{\widetilde{p}})^2),\\
a &:= (g^2+g+1)\widetilde{p}_*(c_1(\omega_{\widetilde{p}})\widetilde{p}^*\lambda'),\\
b &:= \widetilde{p}_*\left(\sum_{i=1}^{[g/2]}\sum_{s\in S_i}\left(\binom{g-i+1}{2}c_1(Y_s^{\widetilde{p}})+\binom{i+1}{2}c_1(Z_s^{\widetilde{p}})\right)^2\right),\\
c &:= \widetilde{p}_*\left(\sum_{s\in S_0}\Big(\sum_{i=1}^{g-1}\frac{i(g-i)g}{2}c_1(E_{s,i}^{\widetilde{p}})\Big)^2\right),\\
d &:= \sum_{i=1}^{[g/2]}\sum_{s\in S_i}\widetilde{p}_*\left(c_1(\omega_{\widetilde{p}})\left(\binom{g-i+1}{2}c_1(Y_s^{\widetilde{p}})+\binom{i+1}{2}c_1(Z_s^{\widetilde{p}})\right)\right),\\
e &:= \sum_{s\in S_0}\sum_{i=1}^{g-1}\widetilde{p}_*\left(c_1(\omega_{\widetilde{p}})\Big(\frac{i(g-i)g}{2}c_1(E_{s,i}^{\widetilde{p}})\Big)\right)
\end{aligned}$$

First, to compute u, we use a formula derived from the Grothendieck–Riemann–Roch formula (see [17], Formula 3.110, p. 158):

$$\widetilde{p}_*(c_1(\omega_{\widetilde{p}})^2) = 12\lambda' - \delta'_0 - \delta'_1 - \cdots - \delta'_{[g/2]}.$$

Second, by the projection formula, $\widetilde{p}_*(c_1(\omega_{\widetilde{p}})\widetilde{p}^*\lambda') = (2g-2)\lambda'$. So

$$a = 2(g^2+g+1)(g-1)\lambda'.$$

Third, from (59) we get

$$\widetilde{p}_*(c_1(Y_s^{\widetilde{p}})c_1(Z_s^{\widetilde{p}})) = (k_s+1)[s] = -\widetilde{p}_*(c_1(Y_s^{\widetilde{p}})c_1(Y_s^{\widetilde{p}})) = -\widetilde{p}_*(c_1(Z_s^{\widetilde{p}})c_1(Z_s^{\widetilde{p}}))$$

for each $i = 1, \ldots, [g/2]$ and $s \in S_i$. Thus

$$\begin{aligned}
b &= \sum_{i=1}^{[g/2]}\sum_{s\in S_i}(k_s+1)\left(2\binom{g-i+1}{2}\binom{i+1}{2} - \binom{g-i+1}{2}^2 - \binom{i+1}{2}^2\right)[s]\\
&= -\sum_{i=1}^{[g/2]}\left(\binom{g-i+1}{2}-\binom{i+1}{2}\right)^2\delta'_i = -\sum_{i=1}^{[g/2]}\frac{(g+1)^2(g-2i)^2}{4}\delta'_i.
\end{aligned}$$

Fourth, from (60), for each $s \in S_0$ and all $i, j = 1, \ldots, g-1$,

$$\widetilde{p}_*(E_{s,i}^{\widetilde{p}}\cdot E_{s,j}^{\widetilde{p}}) = \begin{cases} (\widetilde{k}_s+1)[s] & \text{if } |i-j| = 1,\\ 0 & \text{if } |i-j| > 1,\\ -2(\widetilde{k}_s+1)[s] & \text{if } i = j.\end{cases}$$

Recalling (58), we have

$$\begin{aligned}
c =&\frac{g^2}{2}\left(\sum_{i=1}^{g-1}(i(g-i)(i+1)(g-i-1)-i^2(g-i)^2)\right)\sum_{s\in S_0}(\widetilde{k}_s+1)[s]\\
=&\frac{g}{2}\left(\sum_{i=1}^{g-1}i(g-i)(g-2i-1)\right)\delta_0'\\
=&\frac{g}{2}\left(g(g-1)\sum_{i=1}^{g-1}i-(3g-1)\sum_{i=1}^{g-1}i^2+2\sum_{i=1}^{g-1}i^3\right)\delta_0'\\
=&\frac{g}{2}\left(\frac{g^2(g-1)^2}{2}-\frac{(3g-1)(g-1)g(2g-1)}{6}+\frac{(g-1)^2g^2}{2}\right)\delta_0'\\
=&-\frac{g^4-g^2}{12}\delta_0'.
\end{aligned}$$

Fifth, since $\omega_{\widetilde{p}}|_{Y_s}=\omega_{Y_s}(P_s)$ and $\omega_{\widetilde{p}}|_{Z_s}=\omega_{Z_s}(P_s)$ for each $s\in S_i$ with $i=1,\dots,[g/2]$, we have

$$\begin{aligned}
\widetilde{p}_*(c_1(\omega_{\widetilde{p}})c_1(Y_s^{\widetilde{p}})) =&(2i-1)(k_s+1)[s],\\
\widetilde{p}_*(c_1(\omega_{\widetilde{p}})c_1(Z_s^{\widetilde{p}})) =&(2(g-i)-1)(k_s+1)[s].
\end{aligned}$$

So

$$\begin{aligned}
d =&\sum_{i=1}^{[g/2]}\left(\binom{g-i+1}{2}(2i-1)+\binom{i+1}{2}(2(g-i)-1)\right)\delta_i'\\
=&\sum_{i=1}^{[g/2]}\left(i(g-i)(g+3)-\binom{g+1}{2}\right)\delta_i'.
\end{aligned}$$

Sixth, since $\omega_{\widetilde{p}}|_{E_{s,j}}$ is trivial for each $s\in S_0$ and each $j=1,\dots,g-1$, we have that $e=0$.

Now, put together the expressions for u, a, b, c, d and e in (64) to get

$$\begin{aligned}
\overline{SW}(p) =&\big(3g(g+1)(g^2+g+2)-2(g^2+g+1)(g-1)\big)\lambda'\\
&-\left(\frac{g^4-g^2}{12}+\frac{g(g+1)(g^2+g+2)}{4}\right)\delta_0'\\
&-\sum_{i=1}^{[g/2]}\frac{(g+1)^2(g-2i)^2}{4}\delta_i'\\
&-\sum_{i=1}^{[g/2]}(g^2+g+1)\left(i(g-i)(g+3)-\binom{g+1}{2}\right)\delta_i'\\
&-\sum_{i=1}^{[g/2]}\frac{g(g+1)(g^2+g+2)}{4}\delta_i'.
\end{aligned}$$

It is now easy to check that the coefficients of λ, δ_0 and the δ_i in the expression for $\overline{SW}_g$ in the statement of the theorem match the coefficients of λ', δ'_0 and the δ'_i in the above expression for $\overline{SW}(p)$. □

Remark 8.5. For $g \geq 3$, since $\lambda, \delta_0, \dots, \delta_{[g/2]}$ form a $\mathbb{Z}$-basis of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, and $\overline{SW}_g \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, the formula in Theorem 8.4 holds in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$. On the other hand, smooth curves of genus 2 have no special Weierstrass points. Thus $\overline{SW}_g = 0$, and the formula in Theorem 8.4 simply says that

$$130\lambda - 13\delta_0 - 26\delta_1 = 0,$$

which is a multiple of Mumford's relation, (53). For yet another way to obtain this relation, and a generalization, see [11].

9. The special ramification loci of type $g + j$

Proposition 9.1. *Let $g \geq 2$. Then*

$$\overline{SW}_g = \overline{E}_{g,-1} + \overline{E}_{g,1} \quad \textit{in } \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g).$$

Proof. Let K be the scheme of n-canonically embedded stable curves of genus g, for n sufficiently large, $v\colon \mathcal{V} \to K$ the versal family, and $\Phi\colon K \to \overline{M}_g$ the induced map; see Subsection 7.1. Let $K' \subset K$ be the open locus over which v is smooth, and set $\mathcal{V}' := v^{-1}(\mathcal{V})$ and $v' := v|_{\mathcal{V}'}\colon \mathcal{V}' \to K'$. From the constructions of $\overline{SW}_g$, $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ in Subsection 8.1, it suffices to show that

$$[VSW(v')] = [VE_{-1}(v')] + [VE_1(v')], \tag{65}$$

as codimension-2 cycles in $\mathcal{V}'$. This is a local statement, that can be checked on a neighborhood of a point $P \in \mathcal{V}'$. Let $C := v^{-1}(v(P))$.

Locally around P, the scheme $VE_1(v')$ is given by all maximal minors of a (Wronskian) matrix of regular functions of the form

$$M = \begin{bmatrix} A \\ c \\ d \end{bmatrix},$$

where A is a matrix with g columns and $g-1$ rows, and c and d are row vectors of size g. Furthermore, $VE_{-1}(v')$ is given by all maximal minors of the matrix A, and $VSW(v')$ is given by the determinants of the square submatrices

$$M_1 := \begin{bmatrix} A \\ c \end{bmatrix} \quad \text{and} \quad M_2 := \begin{bmatrix} A \\ d \end{bmatrix}.$$

The equality (65) will follow now from Lemma 5.3 in [4], if we show that M_1 has rank at least $g-1$ at P, when P is a general point of an irreducible component of $VSW(v')$. Equivalently, we claim that $h^0(C, \omega_C(-gP)) \leq 1$ for such P. Now, let M_g (resp. $M_{g,1}$) be the moduli space of smooth curves (resp. pointed smooth curves) of genus g. By [5], Thm. 4.13, p. 918, and Claim 3 on p. 920, the subset $Z \subseteq M_{g,1}$ parametrizing pointed curves (X, Q) such that $h^0(X, \omega_X(-gQ)) \geq 2$ has codimension at least 3. So the subset of M_g parametrizing curves X having a point Q such that $h^0(X, \omega_X(-gQ)) \geq 2$ has codimension at least 2. On the other hand, since the irreducible components of $VSW(v')$ have codimension 2 in $\mathcal{V}'$, they dominate codimension-1 subvarieties of K' under v. Our claim follows by comparing codimensions, using that Φ has irreducible fibers of the same dimension; see Subsection 8.1. □

Theorem 9.2. *Let $g \geq 3$. Let*

$$a_{-1} = \frac{g^2(g-1)(3g-1)}{2} \quad \text{and} \quad a_1 = \frac{(g+1)(g+2)(3g^2+3g+2)}{2}, \tag{66}$$

$$b_{-1,0} = \frac{g(g-1)^2(g+1)}{6} \quad \text{and} \quad b_{1,0} = \frac{g(g+1)^2(g+2)}{6}, \tag{67}$$

and

$$b_{j,i} = \frac{1}{2} i(g-i)\Big((g+j)^2(g+3) + 2(g+j) - (g+1)\Big) \tag{68}$$

for each $j = -1, 1$ and $i = 1, \dots, [g/2]$. Then

$$\overline{E}_{g,j} = a_j\lambda - b_{j,0}\delta_0 - b_{j,1}\delta_1 - \cdots - b_{j,[g/2]}\delta_{[g/2]} \text{ in } \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$$

for $j = -1, 1$.

Proof. Since $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ is freely generated by λ and the classes δ_i, by [1], Thm. 1, p. 154, we may write

$$\overline{E}_{g,j} = a_j\lambda - b_{j,0}\delta_0 - b_{j,1}\delta_1 - \cdots - b_{j,[g/2]}\delta_{[g/2]} \text{ in } \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$$

for $j = -1, 1$, where the coefficients a_j and $b_{j,\ell}$ are integers. It remains to determine these integers, and verify that (66), (67) and (68) hold.

The coefficients a_{-1} and a_1 were determined by Diaz using Porteous formula on a general one-parameter family of smooth curves in [6], Thm. 4.33, p. 21 and Thm. A1.4, p. 59. We recall Diaz's reasoning here, applied to a different family.

Let K be the scheme of n-canonically embedded stable curves of genus g, for n sufficiently large, and $v\colon \mathcal{V} \to K$ the universal family of embedded curves; see Subsection 7.1. Let $K' \subseteq K$ be the open locus over which v is smooth, and set $\mathcal{V}' := v^{-1}(\mathcal{V})$ and $u := v|_{\mathcal{V}'}\colon \mathcal{V}' \to K'$. Let ω_u be the relative dualizing sheaf of u.

Now, $VE_j(u)$ is the degeneration scheme of the natural evaluation map of vector bundles:

$$e_r\colon u^*u_*\omega_u \longrightarrow J^r_u(\omega_u),$$

where $r := g - 1 + j$; see Subsection 2.2. The expected codimension of the degeneration scheme, which is 2, is achieved because a general smooth curve has no special Weierstrass points; see [4], Cor. 3.3. Thus, by Porteous formula,

$$[VE_{-1}(u)] = c_2(E-F) \quad \text{and} \quad [VE_1(u)] = c_1(E-F)^2 - c_2(E-F),$$

where $E := u^*u_*\omega_u$ and $F := J^r_u(\omega_u)$. Expanding,

$$[VE_{-1}(u)] = c_1(F)^2 - c_2(F) + c_2(E) - c_1(E)c_1(F),$$
$$[VE_1(u)] = c_2(F) - c_1(E)c_1(F) + c_1(E)^2 - c_2(E).$$

Clearly, $c_1(E) = u^*\lambda_u$, where λ_u is the tautological class of u defined in Subsection 7.3. On the other hand, using (2) for $p := u$ and $\mathcal{L} := \omega_u$, repeatedly for $i = 1, \dots, r$, and using Whitney formula, we get

$$c_1(F) = \binom{r+2}{2} c_1(\omega_u),$$
$$c_2(F) = \sum_{\ell=1}^{r} (\ell+1)\binom{\ell+1}{2} c_1(\omega_u)^2.$$

Computing,

$$c_2(F) = \frac{r(r+1)(r+2)(3r+5)}{24} c_1(\omega_u)^2,$$
$$c_1(F)^2 - c_2(F) = \frac{(r+1)(r+2)(r+3)(3r+4)}{24} c_1(\omega_u)^2.$$

Now, use that $u_*(c_1(\omega_u)u^*\lambda_u) = (2g-2)\lambda_u$ by the projection formula, while $u_*(c_1(\omega_u)^2) = 12\lambda_u$, from the Grothendieck–Riemann–Roch formula, as stated in [13], Thm. 15.2, p. 286. Also, by the projection formula and dimensional reasons, $u_*c_2(E) = u_*(c_1(E)^2) = 0$, as both $c_2(E)$ and $c_1(E)$ are pullbacks of classes from K'. Thus

$$u_*[VE_{-1}(u)] = \frac{(r+1)(r+2)}{2}\Big((r+3)(3r+4) - 2(g-1)\Big)\lambda_u,$$
$$u_*[VE_1(u)] = \frac{(r+1)(r+2)}{2}\Big(r(3r+5) - 2(g-1)\Big)\lambda_u,$$

where $r = g-2$ in the first formula and $r = g$ in the second one. Replacing r and computing, we get

$$u_*[VE_{-1}(u)] = \frac{g^2(g-1)(3g-1)}{2}\lambda_u,$$
$$u_*[VE_1(u)] = \frac{(g+1)(g+2)(3g^2+3g+2)}{2}\lambda_u.$$

Recall from Subsection 8.1 that $\overline{E}_j(v)$ is the closure of $u_*[VE_j(u)]$, and is, by definition, the invertible sheaf on K corresponding to $\overline{E}_{g,j}$. So

$$a_j\lambda_v|_{K'} = \overline{E}_j(v)|_{K'} = u_*[VE_j(u)]$$

for $j = -1, 1$, and thus

$$a_{-1}\lambda_v|_{K'} = \frac{g^2(g-1)(3g-1)}{2}\lambda_v|_{K'},$$
$$a_1\lambda_v|_{K'} = \frac{(g+1)(g+2)(3g^2+3g+2)}{2}\lambda_v|_{K'}.$$

Since λ is not a linear combination of the boundary classes, $\lambda_v|_{K'} \neq 0$, and hence the formulas in (66) follow.

To compute the remaining numbers, we use test families. Our first family, $p_0\colon \mathcal{C}_0 \to \mathbf{P}^1$, is constructed by fixing a general curve of genus $g-1$ and identifying a fixed general point of that curve with a base point of a general pencil of plane cubics; see [17], Ex. 3.140, p. 173. For this family, $p_{0*}\omega_{p_0}$ is free, where ω_{p_0} is its relative dualizing sheaf, and hence there is a map $h_0\colon \mathbf{P}^1 \to K$ such that p_0 is the base extension of v under h_0. It follows from Theorem 5.1 and [4], Prop. 3.1, that for a nonsingular member of the pencil, the resulting stable curve does not contain any limit of special Weierstrass points. Thus $\int_{\mathbf{P}^1} h_0^*\overline{SW}(v) \geq 0$, with strict inequality if and only if there is a fiber of p_0 containing limits of special Weierstrass points.

On the other hand,

$$\int_{\mathbf{P}^1} h_0^*\lambda_v = 1, \quad \int_{\mathbf{P}^1} h_0^*\delta_{0,v} = 12, \quad \int_{\mathbf{P}^1} h_0^*\delta_{1,v} = -1$$

and $\int_{\mathbf{P}^1} h_0^*\delta_{i,v} = 0$ for each $i \geq 2$. Indeed, the first formula holds because of the linearity of the pencil; the second because there are exactly 12 singular members in the pencil, each an irreducible nodal cubic; the third because the total space of the pencil of cubics is the blowup of $\mathbf{P}^2$ at the 9 base points, resulting in 9 exceptional divisors, each with self-intersection -1; and the fourth because none of the fibers of p_0 are represented in the subset $\Delta_i' \subset K$ for any $i \geq 2$; see [17], pp. 146–147 for more details.

Now, we may use the formulas for $\int_{\mathbf{P}^1} h_0^*\lambda_v$ and the $\int_{\mathbf{P}^1} h_0^*\delta_{i,v}$, and the formula for $\overline{SW}_g$ in Theorem 8.4 to compute

$$\int_{\mathbf{P}^1} h_0^*\overline{SW}(v) = 0. \tag{69}$$

So no fiber of p_0 contains limits of special Weierstrass points, and hence also

$$\int_{\mathbf{P}^1} h_0^*\overline{E}_j(v) = 0 \quad \text{for } j = -1, 1.$$

Using again the formulas for $\int_{\mathbf{P}^1} h_0^*\lambda_v$ and the $\int_{\mathbf{P}^1} h_0^*\delta_{i,v}$, we get

$$a_j - 12b_{j,0} + b_{j,1} = 0 \quad \text{for } j = -1, 1. \tag{70}$$

Thus the formulas for the $b_{j,0}$ may be derived from those for the a_j and the $b_{j,i}$ for $j = -1, 1$ and $i \geq 1$.

(The relations (70) were obtained directly by Diaz [6], Lemma 7.2, p. 40, for $j = -1$, and by Gatto [15], p. 67, for $j = 1$, and from them Gatto concluded (69). Here we proceeded in the opposite way.)

To compute the $b_{j,i}$ for each $i = 1, \ldots, [g/2]$ we do the following. For each $i = 1, \ldots, [g/2]$, let X be a general smooth curve of genus $g-i$, and let Y be a general smooth curve of genus i, and $B \in Y$ a general point. Identifying the diagonal $\Delta \subset X \times X$ with $\{B\} \times X \subset Y \times X$ in the natural way, we get a flat, projective map $p_i\colon \mathcal{F}_i \to X$ whose fiber over each $P \in X$ is the uninodal stable curve union of X and Y with P and B identified; denote by $X \cup_P Y$ this fiber. Again for this family, $p_{i*}\omega_{p_i}$ is free, where ω_{p_i} is the dualizing sheaf of p_i, because the normalization of $\mathcal{F}_i$ is a constant family over X. So there are maps $h_i\colon X \to K$ and $h_i'\colon \mathcal{F}_i \to \mathcal{V}$ making the diagram below Cartesian:

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{h_i'} & \mathcal{V} \\ {\scriptstyle p_i}\downarrow & & {\scriptstyle v}\downarrow \\ X & \xrightarrow{h_i} & K. \end{array}$$

Now, observe that $\overline{E}_j(v) = v_*[\overline{VE}_j(v)]$, where $\overline{VE}_j(v)$ is the schematic closure of $VE_j(u)$. Since $VE_j(u)$ is of codimension 2 in $\mathcal{V}'$, so is $\overline{VE}_j(v)$ in $\mathcal{V}$. Now, X and K are smooth, so h_0 is a l.c.i. map. By [13], Prop. 6.6, p. 113,

$$h_i^*\overline{E}_j(v) = h_i^* v_*[\overline{VE}_j(v)] = p_{i*}h_i^![\overline{VE}_j(v)],$$

where $h_i^!\colon A^2(\mathcal{V}) \to A^2(\mathcal{F}_i)$ is the Gysin map. So

$$\int_X h_i^*\overline{E}_j(v) = \int_X p_{i*}h_i^![\overline{VE}_j(v)] = \int_{\mathcal{F}_i} h_i^![\overline{VE}_j(v)]. \tag{71}$$

Now, $(h_i')^{-1}(\overline{VE}_j(v))$ is the set of points $Q \in X \cup_P Y$ for all $P \in X$ which are limits of special Weierstrass points along smooth curves degenerating to $X \cup_P Y$. By Theorem 5.1, the set $(h_i')^{-1}(\overline{VE}_j(v))$ intersects $X \cup_P Y$ at Q if and only if one of the following four situations occur:

(1) $Q \in X - P$, and Q is a special ramification point of type $g+j$ of the complete linear system of sections of $\omega_X((i+1)P)$;
(2) B is a Weierstrass point of Y, and Q is a ramification point different from P of the complete linear system of sections of $\omega_X((i+1+j)P)$;
(3) $Q \in Y - B$, and Q is a special ramification point of type $g+j$ of the complete linear system of sections of $\omega_Y((g-i+1)B)$;
(4) P is a Weierstrass point of X, and Q is a ramification point of the complete linear system of sections of $\omega_Y((g-i+1+j)B)$ different from B .

However, B is general, whence an ordinary point of Y. Moreover, by [4], Prop. 3.1, the complete linear system of sections of $\omega_Y((g-i+1)B)$ has no special ramification points other than B. So neither (2) nor (4) occurs. In addition, by [4], Thm. 5.6, the number of points $(P,Q) \in X \times X$ off the diagonal such that Q is a special ramification point of type $g+j$ of the complete linear system of sections of $\omega_X((i+1)P)$ is finite, and equal to

$$d_{j,i} := (g-i)(g-i-1)\Big((g+j)^2(i+1)^2 - (g-i+j)^2\Big).$$

Finally, since X is general, X has no special Weierstrass points by [4], Cor. 3.3, and hence the number of Weierstrass points of X is finite, and equal to

$$d'' := (g-i-1)(g-i)(g-i+1),$$

by the Plücker formula. Furthermore, since Y and B are general, by [4], Prop. 3.1, all the ramification points of the complete linear system of sections of $\omega_Y((g-i+1+j)B)$ are simple, except from B, which has weight i. So, by the Plücker formula, the number of ramification points different from B of this linear system is finite, and equal to

$$d'_{j,i} := (g+j)(g+i+j-1)+(i-1)(g+j)(g+j-1)-i = i(g+j)^2 - i.$$

So the number of points in $(h'_i)^{-1}(\overline{VE}_j(v))$ is finite, and equal to

$$e_{j,i} := d_{j,i} + d'' d'_{j,i}.$$

In particular, $h_i^![\overline{VE}_j(v)]$ is represented by an effective 0-cycle with support $(h'_i)^{-1}(\overline{VE}_j(v))$ and, using (71),

$$\int_X h_i^* \overline{E}_j(v) \geq e_{j,i}. \tag{72}$$

Computing,

$$e_{j,i} = i(g-i)(g-i-1)\Big((g+j)^2(g+3)+2(g+j)-(g+1)\Big). \tag{73}$$

On the other hand, we claim that

$$\int_X h_i^* \overline{SW}(v) = e_{-1,i} + e_{1,i}. \tag{74}$$

Indeed, $h_i^*\delta_{j,v} = 0$ for every $j \neq i$, since none of the fibers of p_i are represented in $\Delta'_j \subset K$. Also, $h_i^*\lambda_v = 0$ because $p_{i*}\omega_{p_i}$ is free, and

$$\int_X h_i^*\delta_{i,v} = 2(1-g+i),$$

because X has genus $g-i$, and thus the self-intersection of the diagonal in $X \times X$ is $2(1-g+i)$. (Again, see [17], pp. 146–147 for more details.) So, using the formula for $\overline{SW}_g$ in Theorem 8.4, a simple computation yields (74).

Using Proposition 9.1, and using (72) for $j=-1,1$ and (74), we get

$$e_{-1,i}+e_{1,i} = \int_X h_i^*\overline{SW}(v) = \int_X h_i^*\overline{E}_{-1}(v) + \int_X h_i^*\overline{E}_1(v) \geq e_{-1,i}+e_{1,i}.$$

Thus $\int_X h_i^*\overline{E}_j(v) = e_{j,i}$ for $j=-1,1$. Using the formulas for $\int_X h_i^*\lambda_v$ and the $\int_X h_i^*\delta_{j,v}$, and the expression (73), we obtain the formula (68) for $b_{j,i}$. Finally, using (66), (68) and the relations (70), we get the formulas in (67) for $b_{-1,0}$ and $b_{1,0}$. □

References

[1] E. Arbarello and M. Cornalba, *The Picard groups of the moduli spaces of curves.* Topology **26**, no. 2, (1987), 153–171.

[2] CoCoATeam, CoCoA*: a system for doing Computations in Commutative Algebra.* Available at `http://cocoa.dima.unige.it`.

[3] F. Cukierman, *Families of Weierstrass points.* Duke Math. J. **58**, no. 2, (1989), 317–346.
[4] C. Cumino, E. Esteves and L. Gatto, *Special ramification loci on the double product of a general curve.* Available at math.AG/0701662. To appear in Q. J. Math..
[5] S. Diaz, *Tangent spaces in moduli via deformation with applications to Weierstrass points.* Duke Math. J. **51**, no. 4, (1984), 905–922.
[6] S. Diaz, *Exceptional Weierstrass points and the divisor on moduli space that they define.* Mem. Amer. Math. Soc., vol. 56, no. 327; Amer. Math. Soc., Providence, 1985.
[7] D. Eisenbud and J. Harris, *Limit linear series: Basic theory.* Invent. Math. **85**, no. 2, (1986), 337–371.
[8] D. Eisenbud and J. Harris, *Existence, decomposition, and limits of certain Weierstrass points.* Invent. Math. **87**, no. 3, (1987), 495–515.
[9] E. Esteves, *Wronski algebra systems on families of singular curves.* Ann. Sci. Éc. Norm. Sup. (4) **29**, no. 1, (1996), 107–134.
[10] E. Esteves, *Linear systems and ramification points on reducible nodal curves.* In: "Algebra Meeting" (ed. E. Esteves, A. Garcia and A. Pacheco) Mat. Contemp., vol. 14, pp. 21–35; Soc. Bras. Mat., Rio de Janeiro, 1998.
[11] E. Esteves and L. Gatto, *A geometric interpretation and a new proof of a relation by Cornalba and Harris.* Comm. Algebra **31**, no. 8, (2003), 3753–3770.
[12] E. Esteves and N. Medeiros, *Limit canonical systems on curves with two components.* Invent. Math. **149**, no. 2, (2002), 267–338.
[13] W. Fulton, *Intersection theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 2; Springer-Verlag, Berlin Heidelberg, 1984.
[14] L. Gatto, *Weight sequences versus gap sequences at singular points of Gorenstein curves.* Geom. Dedicata **54**, no. 3, (1995), 267–300.
[15] L. Gatto, *On the closure in $\overline{M}_g$ of smooth curves having a special Weierstrass point.* Math. Scand. **88**, no. 1, (2001), 41–71.
[16] D. Gieseker, *Lectures on moduli of curves.* Tata Institute of Fundamental Research; Springer-Verlag, Heidelberg, 1982.
[17] J. Harris and I. Morrison, *Moduli of curves.* Graduate Texts in Mathematics, vol. 187; Springer-Verlag, New York, 1998.
[18] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–86.
[19] D. Laksov and A. Thorup, *The algebra of jets.* Michigan Math. J. **48**, (2000), 393–416.
[20] D. Laksov and A. Thorup, *Wronski systems for families of local complete intersection curves.* Comm. Algebra **31**, no. 8, (2003), 4007–4035.
[21] E. Looijenga, *Smooth Deligne–Mumford compactifications by means of Prym level structures.* J. Alg. Geom. **3**, (1994), 283–293.

[22] D. Mumford, *Stability of projective varieties.* L'Enseignement Math., II. Ser. **23**, (1977), 39–110.

[23] D. Mumford, *Towards an enumerative geometry of the moduli space of curves.* In: "Arithmetic and Geometry, Vol. II" (ed. M. Artin and J. Tate) Progress in Math., vol. 36, pp. 271–328; Birkhäuser, Boston, 1983.

[24] D. Mumford and J. Fogarty, *Geometric Invariant Theory.* Springer-Verlag, 2$^{\text{nd}}$ ed., Berlin Heidelberg, 1982.

[25] B. Osserman, *A limit linear series moduli scheme.* Ann. Inst. Fourier (Grenoble) **56**, no. 4, (2006), 1165–1205.

[26] M. Pikaart and A. J. de Jong, *Moduli of curves with non-abelian level structure.* In "The moduli space of curves" (Texel Island, 1994), pp. 483–509, Progr. Math., vol. 129, Birkhäuser Boston, Boston, 1995.