ON THE RIGIDITY OF CERTAIN HOLOMORPHIC FIBRATIONS

J. V. PEREIRA AND P. SAD

1. Introduction

Let Γ be a finite set of points in the projective plane $\mathbb{P}^2$ defined as the intersection of two transverse curves of the same degree (we say that Γ is a **complete intersection set**); let also $\pi : S_\Gamma \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points of Γ. The surface S_Γ admits a natural foliation $\widetilde{\mathcal{F}}_\Gamma$: the strict transform of the pencil $\mathcal{F}_\Gamma : F d\,G - G d\,F = 0$ generated by the curves $\{F = 0\}$ and $\{G = 0\}$ that define Γ.

A natural problem is to understand the families of reduced foliations of surfaces (in the sense of [2]) containing $(S_\Gamma, \widetilde{\mathcal{F}}_\Gamma)$; this is related to studying the foliations of $\mathbb{P}^2$, in a neighborhood of $\mathcal{F}_\Gamma$, that have radial singularitied close to the points of Γ.

We consider in this paper the particular situation where the surface S_Γ does not change in the family (or, equivalently, we look at the foliations of $\mathbb{P}^2$ with radial singularities at the points of Γ). The leaves of $\widetilde{\mathcal{F}}_\Gamma$ are fibers of the holomorphic fibration $(F/G)\circ\pi \to \mathbb{P}^1$. In order to study a deformation $\widetilde{\mathcal{F}}$ of this fibration (in the space of foliations of S_Γ) we analyse how a generic fiber $\widetilde{C}$ of $\widetilde{\mathcal{F}}_\Gamma$ is intersected by the leaves of $\widetilde{\mathcal{F}}$. If $\widetilde{C}$ is not $\widetilde{\mathcal{F}}$-invariant then $\mathcal{N}_{\widetilde{\mathcal{F}}}.\widetilde{C} = \mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{C}) + \chi(\widetilde{C})$, where $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is the normal bundle of $\widetilde{\mathcal{F}}$, $\chi(\widetilde{C})$ is the Euler characteristic of $\widetilde{C}$ and $\mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{C})$ is the number of tangency points between $\widetilde{\mathcal{F}}$ and $\widetilde{C}$. In our case $\mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{C}) \geq 0$ since $\widetilde{C}$ is a smooth curve, and also $\mathcal{N}_{\widetilde{\mathcal{F}}}.\widetilde{C} = \mathcal{N}_{\widetilde{\mathcal{F}}_\Gamma}.\widetilde{C}$ by continuity. Since $\widetilde{C}$ is $\widetilde{\mathcal{F}}_\Gamma$-invariant, $\mathcal{N}_{\widetilde{\mathcal{F}}_\Gamma}.\widetilde{C} = Z(\widetilde{\mathcal{F}}, \widetilde{C}) + \widetilde{C}.\widetilde{C}$, where $Z(\widetilde{\mathcal{F}}, \widetilde{C})$ denotes the number of singularities of $\widetilde{\mathcal{F}}_\Gamma$ along $\widetilde{C}$, and we get that $tang(\widetilde{\mathcal{F}}, \widetilde{C}) = -\chi(\widetilde{C})$.

Let $c \in \mathbb{N}$ be the common degree of the polynomials F and G. When $c = 1$ or $c = 2$ we have $\chi(\widetilde{C}) = 2$ and we get a contradiction unless $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_\Gamma$ ([9]). When $c = 3$ we have $tang(\widetilde{\mathcal{F}}, \widetilde{C}) = -\chi(\widetilde{C}) = 0$ and therefore $\widetilde{\mathcal{F}}$ is transverse to the generic fiber of $\widetilde{\mathcal{F}}_\Gamma$, implying that the regular fibers are all isomorphic; this is not possible for a generic choice of F and G, and we conclude again that $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_\Gamma$ in this case (see [10] for a related result). When $c \geq 4$ this type of argument fails, since $tang(\widetilde{\mathcal{F}}, \widetilde{C}) > 0$. Nevertheless we are able to prove for $c \geq 3$:

Theorem 1. *If Γ is a generic complete intersection set then $\mathcal{F}_\Gamma$ is an isolated point of the space of foliations of S_Γ, ie, $\mathcal{F}_\Gamma$ is* **rigid**.

Date: April 13, 2007.
Key words and phrases. algebraic foliation, invariant curves, singulariites, indexes .
The first author is supported by Profix-CNPq.

In the statement *generic complete intersection set* refers to the set of base points of a generic element of the space of lines of $\mathbb{P}\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(c))$; in other words, the couple (F, G) of polynomials of degree $c \in \mathbb{N}$ is generically chosen in order to define Γ. In §3.2 we exhibit some examples of non-rigidity to show that the hypothesis of genericity is necessary.

We have no results when the surface S_Γ changes in the family of reduced foliations; but still we should mention that for $c = 3$ we can only deform $\widetilde{\mathcal{F}}_\Gamma$ as a fibration (starting with a generic choice of Γ). In fact, $\widetilde{\mathcal{F}}_\Gamma$ has Kodaira dimension equal to 1 and this dimension is constant along the family ([2]). We then apply the Classification Theorem ([1]) to conclude that any foliation in the family is an elliptic fibration.

The proof of Theorem 1 relies on the analysis of the indexes of a plane foliation along a smooth invariant algebraic curve. Let $\{F = 0\}$ be such a curve, of degree $c \in \mathbb{N}$, containing singularities of the foliation at the intersection points with another curve $\{G = 0\}$ of degree $k \leq c$. We prove then that if (F, G) is generically chosen the set of indexes is sufficient to identify completely the foliation (Theorem 2.2). Application of this result in order to prove Theorem 1 is not immediate; we have to show first that the defining curves for the set Γ are invariant curves of the foliation.

Acknowledgements. We are grateful to L.G. Mendes for posing the question that originated this work.

2. Variation of Indexes

2.1. **Division Lemma.** All foliations, unless stated otherwise, are supposed to have isolated singularities.

Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $c \in \mathbb{N}$, invariant by a plane projective foliation $\mathcal{F} \in Fol(d)$ of degree $d \in \mathbb{N}$. The Lemma below can be implicitly found in [4, Proof of Proposition 3]; we assume that $\mathcal{F}$ is defined by $\omega = 0$, ω a homogeneous 1-form of $\mathbb{C}^3$ of degree $d+1$ (or by a homogeneous vector field of $\mathbb{C}^3$ of degree $d \in \mathbb{N}$), and that C is defined by $F = 0$, F a homogeneous polynomial of degree $c \in \mathbb{N}$. Let us denote by R the radial vector field of $\mathbb{C}^3$.

Lemma 2.1. *There exist a polynomial G of degree $d - c + 2$ and a 1-form η of degree $d - c + 1$, both homogeneous, such that*

$$\omega = GdF - \frac{\deg(F)}{\deg(G)} FdG + F\eta \quad \textit{and} \quad i_R(\eta) = 0.$$

Furthermore, the foliation $\mathcal{F}_\eta$ defined by $\eta = 0$ depends only on $\mathcal{F}$ and C when $d \leq 2c - 2$.

Proof. It follows from ([4, Proposition 1]) that there exist a homogenous polynomial G of degree $d - c + 2$ and a homogenous 1-form α of degree $d - c + 1$ such that

$$\omega = GdF + F\alpha \,. \tag{1}$$

After contracting the above expression with the radial vector field we obtain

$$\deg(F)FG + Fi_R\alpha = 0 \,,$$

We rewrite (1) as

$$\omega = GdF - \frac{\deg(F)}{\deg(G)}FdG + F\left(\alpha + \frac{deg(F)}{deg(G)}dG\right).$$

and define $\eta := \alpha + \frac{deg(F)}{deg(G)}dG$; it follows that $i_R(\eta) = 0$.

Let us replace (1) by $\omega' = G'dF' + F'\alpha'$, where $\omega' = \lambda\omega$ and $F' = \mu F$ for $\lambda, \mu \in \mathbb{C}$. Consequently:

$$\omega = \left(\frac{\mu}{\lambda}G'\right)dF + F\left(\frac{\mu}{\lambda}\alpha'\right) = GdF + F\alpha$$

and

$$\left(\frac{\mu}{\lambda}G' - G\right)dF = F\left(\alpha - \frac{\mu}{\lambda}\alpha'\right).$$

From $\left(\frac{\mu}{\lambda}G' - G\right)_{|C} \equiv 0$ we have $\frac{\mu}{\lambda}G' - G = P.F$ for some homogeneous polynomial P; two possibilities arise:

- $d < 2c - 2$; therefore $\frac{\mu}{\lambda}G' = G$, $\frac{\mu}{\lambda}\alpha' = \alpha$ and we get
$$\eta' = \alpha' + \frac{deg(F)}{deg(G)}dG' = \frac{\mu}{\lambda}\eta.$$
- $d = 2c - 2$, so that $\frac{\mu}{\lambda}G' - G = aF$, $\alpha - \frac{\mu}{\lambda}\alpha' = adF$ for $a \in \mathbb{C}$. It follows that $\alpha - \frac{\mu}{\lambda}\alpha' = \frac{\mu}{\lambda}dG' - dG$ and again $\eta' = \frac{\mu}{\lambda}\eta$.

□

We observe that $\mathcal{F}_\eta$ **may have a curve of singularities**.

Our results follow from the analysis of the behaviour of $\mathcal{F}_\eta$ with respect to C when $d\ leq 2c - 2$. For the moment we remark that:

- the singularities of $\mathcal{F}$ contained in C are the points of $\{G = 0\} \cap C$.
- C is not contained in the singular set of $\mathcal{F}_\eta$ (because $deg(\eta) < deg(F)$).
- C is not $\mathcal{F}_\eta$-invariant (because otherwise $deg(C) \leq deg(\mathcal{F}_\eta) + 1$, see [4], or $c \leq d - c + 1$). Let us write $k = deg(G) = d - c + 2$ for simplicity, so that $deg(\mathcal{F}_\eta) = k - 2$. Since $tang(\mathcal{F}_\eta, C) = \mathcal{N}_{\mathcal{F}_\Gamma}.C - \chi(C) = k.c - \left(2 - 2\frac{c-1)(c-2)}{2}\right)$, we find $tang(\mathcal{F}_\eta, C) = c(k + c - 3)$; the tangency points between C and $\mathcal{F}_\eta$ are given by the common solutions of $F = 0$ and $dF(Z_\eta) = 0$ (Z_η is the homogeneous vector field of $\mathbb{C}^3$ of degree $k-2$ which defines $\mathcal{F}_\eta$).

2.2. **Indexes and Foliations.** In [11] we have proved the existence of foliations of sufficiently high degree with prescribed linear holonomy group with respect to a given curve. Here we will consider the opposite situation when the degree of the curve is comparable to the degree of the foliation. More precisely we will consider foliations of degree $d \in \mathbb{N}$ which have an invariant smooth curve of degree $c \in \mathbb{N}$ such that $d \leq 2c-2$ (remark that in all cases $c \leq d+1$). This inequality is equivalent to $Z(\mathcal{F}, C) \leq c^2$. As already pointed out it implies that the decomposition given by Lemma 2.1 is essentially unique.

Let us take a pair of transverse algebraic curves $C = \{F = 0\}$ and E defined by polynomials of degree $c \in \mathbb{N}$ and $k \in \mathbb{N}$ respectively; C is supposed to be a smooth curve and F a reduced polynomial. Denote by $Fol_{C,C\cap E}(d)$ the space of foliations of degree $d = c + k - 2$ which leave C invariant and have $C \cap E$ as the singular set along C. We define the Index Map $\mathcal{I}(C, E) = \mathcal{I}$ as

$$\begin{array}{rcl} \mathcal{I} : \mathbb{F}ol_{C,C\cap E}(d) & \to & \mathcal{A}(C\cap E,\mathbb{C}) \\ \mathcal{F} & \mapsto & (p \mapsto i(\mathcal{F},C,p)) \end{array}$$

where $\mathcal{A}(C\cap E,\mathbb{C})$ is the space of maps from Γ to $\mathbb{C}$ and $i(\mathcal{F},C,p)$ is in the index of $\mathcal{F}$ with respect to C at the point p (cf. [3]).

According to Lemma 2.1 , a foliation $\mathcal{F} \in Fol_{C,C\cap E}(d)$ is defined by a 1-form $\omega = GdF - (c/k)FdG + F\eta = 0$; we may assume that $E = \{G = 0\}$. A simple computation shows that

$$i(\mathcal{F},C,p) = \frac{c}{k} - Res\left(\left(\frac{\eta}{G}\right)_{|C}, p\right), \tag{2}$$

where $\left(\frac{\eta}{G}\right)_{|C}$ means $i^*\left(\frac{\eta}{G}\right)$ for the inclusion $i : C \to \mathbb{P}^2$.

When C and E are transverse to each other at $p \in C\cap E$, we have

$$i(\mathcal{F},C,p) = \frac{c}{k} \Leftrightarrow i^*\eta\,(p) = 0 \tag{3}$$

The equality $i^*\eta\,(p) = 0$ means that $\mathcal{F}_\eta$ is tangent to C at p.

If the foliation is the pencil $\mathcal{F}_\Gamma : GdF - (c/k)FdG = 0$, all $k.c$ indexes at the points of $C\cap E$ are equal to c/k; a natural question to ask is whether the converse is true. This is not always the case (see [12] for a counterexample). Before stating the main result of this Section, we need a Lemma; set $S_l = \mathrm{H}^0(\mathbb{P}^2,\mathcal{O}(l))$ and $\mathbb{S}_l = \mathbb{P}\mathrm{H}^0(\mathbb{P}^2,\mathcal{O}(l)))$ for $l > 0$.

Lemma 2.2. *Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_0(c,k) \subset \mathbb{S}_c \times \mathbb{S}_k$ such that if $(C,E) \in \mathcal{U}_0(c,k)$ then C and E are transverse to each other and no foliation of degree $k-2$ is tangent to C at the points of $C\cap E$.*

Proof. Let $X_h(n)$ be the set of homogeneous vector fields of $\mathbb{C}^3$ of degree n, and H the set

$$\{(F,G) \in S_c \times S_k; \exists (Z,A,B) \in X_h(k-2) \times S_{k-3} \times S_{c-3}; dF(Z) = A.F + B.G\}$$

Then H is an algebraic subvariety of $S_c \times S_k$. Let us show that H is a strict subvariety. For that we take $F_0 \in S_c$ as the equation of a plane rational curve of degree c with nodal singularities and G_0 defining a plane curve of degree k which is transverse to $\{F_0 = 0\}$. We know from the genus formula that $\{F_0 = 0\}$ has $\frac{(c-1).(c-2)}{2}$ nodal singularities. If $(F_0, G_0) \in H$, one has $Df_0(Z_0) = A_0.F_0 + B_0.G_0$ for a $(Z_0, A_0, B_0) \in X_h(k-2) \times S_{c-3} \times S_{k-3}$. Let us compute the number of intersection points between $\{dF_0(Z_0) = 0\}$ and $\{F_0 = 0\}$:

- $k.c$ points of $\{F_0 = 0\} \cap \{G_0 = 0\}$, which are smooth points of $\{F_0 = 0\}$.
- $(c-1)(c-2)$ points corresponding to the nodal singularities of $\{F_0 = 0\}$

We have then $k.c + (c-1).(c-2) = (k+c-3).c = (k+c-3).c$, contradiction.

Let now $\mathcal{U}(c,k)$ be the open subset of $S_c \times S_k$ of pairs of curves (C,E) such that C and E are transverse to each other; finally we set $\mathcal{U}_0(c,k) = \mathcal{U}(c,k) \cap (S_c \times S_k \setminus H)$. Consider $(\bar{C},\bar{E}) = (\{\bar{F} = 0\},\{\bar{G} = 0\}) \in \mathcal{U}_0(c,k)$; let $d\bar{F}(\bar{Z})(p) = 0$ at all points in $\bar{C}\cap\bar{E}$ and some $\bar{Z} \in X_h(k-2)$. By Noether's Theorem ([13]), $d\bar{F}(\bar{Z}) = \bar{A}.\bar{F} + \bar{B}.\bar{G}$ for some $(\bar{A},\bar{B}) \in S_{c-3} \times S_{k-3}$, so that $(\bar{C},\bar{E}) \in H$, contradiction unless $\bar{Z}$=0. □

Remark. The argument above is inspired in Severi's idea to prove the Brill-Noether Theorem ([14],pg. 240-244).

We have as a consequence:

Theorem 2. *Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_1(c,k) \subset \mathbb{S}_c \times \mathbb{S}_k$ such that if $(C,E) \in \mathcal{U}_1(c,k)$ then C is smooth, C and E are transverse to each other and $\mathcal{I}(C,E)$ is injective.*

Proof. It is enough to define $\mathcal{U}_1(c,k) \subset \mathcal{U}(c,k)$ (obtained in Lemma 2.2) as the set of pairs $(C,E) \in \mathcal{U}(c,k)$ such that C is a smooth curve, and use (3). □

Before proceeding let us take a closer look at the case E is a **conic**.

Example 2.1. When E is a conic then η induces a degree 0 foliation of $\mathbb{P}^2$. These foliations are pencils of lines and, as such, are completely determined by the base point of the pencil.

If η_p is the degree 0 foliation corresponding to the pencil of lines through $p = [a:b:c] \in \mathbb{P}^2$ then the tangency points between $\mathcal{F}_{\eta_p}$ and C are the points of intersection of C with its polar curve centered at p, i.e.,

$$T_qC \subset \ker \eta_p(q) \iff \left(a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z} \right)(q) = 0\,,$$

where F is an irreducible polynomial defining C. It follows from Noether's Theorem that the map $\mathcal{I}(C,E)$ is not injective if, and only if, there exists $[a:b:c] \in \mathbb{P}^2$ such that

$$G \text{ divides } \left(a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z} \right)\,,$$

where G is quadratic polynomial which defines E.

When C is a conic then this never happens since a polar curve of C has degree 1. Thus for any C and any E the map $\mathcal{I}(C,E)$ is always injective.

When C is a cubic then the map $\mathcal{I}(C,E)$ is not injective if, and only if, E is a polar curve of C.

Suppose now that C is a quartic; let us identify the set of polar curves of C with a projective plane Λ, linearly embedded in the projective space $\mathbb{S}_3$. If $\mathcal{I}(C,E)$ is not injective then there exists a point in Λ intersecting $W_{1,2}$, the image of the multiplication map $\mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_3$. Since $W_{1,2}$ has codimension 2 in $\mathbb{S}_3$ then a generic Λ will intersect $W_{1,2}$ is a finite set of points. Moreover it can be easily verified that $W_{1,2}$ is a linear projection of the Segre Variety $S_{2,5} \subset \mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^6) \cong \mathbb{P}(S_1 \otimes S_2)$ to $\mathbb{S}_3$ from a center that does not intersect $S_{2,5}$. Since the degree of $S_{2,5}$ (cf. [6, page 233]) is $\binom{5+2}{2} = 21$ then the degree of $W_{1,2}$ is also 21. Thus a generic Λ will intersect $W_{1,2}$ in 21 points counted with multiplicity. Translating to our situation we obtain that for a generic $C \in \mathbb{S}_4$ the cardinality of

$$\{E \in \mathbb{S}_2 \mid \mathcal{I}(C,E) \text{ is not injective}\}$$

is 21.

For C of degree at least 5 we can argue as follows. Let $\Delta = \mathbb{P}(\mathbb{C}\frac{\partial}{\partial x} \oplus \mathbb{C}\frac{\partial}{\partial y} \oplus \mathbb{C}\frac{\partial}{\partial z})$ and $\Sigma \subset \mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta \times \mathbb{S}_c$ be defined by the relation

$$([B],[G],[\partial],[F]) \in \Sigma \iff [\partial(F)] = [B \cdot G]$$

Notice that every $\partial \in \mathbb{C}\frac{\partial}{\partial x} \oplus \mathbb{C}\frac{\partial}{\partial y} \oplus \mathbb{C}\frac{\partial}{\partial z}$ acting as a derivation induces a surjective linear map $\partial : S_c \to S_{c-1}$. Thus if $\pi_1 : \Sigma \to \mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta$ is the natural projection to $\mathbb{S}_{c-3} \times \mathbb{S}_2 \times \Delta$ then π_1 induces a structure of $\mathbb{P}^{c+1}$-bundle on Σ. In particular

$$\dim \Sigma = c + 8 + \frac{c(c-3)}{2} \, .$$

Since $c \geq 5$, $\dim \Sigma < \dim \mathbb{S}_c$ and consequently $U = \mathbb{S}_c \backslash \pi_2(\Sigma) \neq \emptyset$ where $\pi_2 : \Sigma \to \mathbb{S}_c$ is the natural projection to $\mathbb{S}_c$.

We conclude that for every $C \in U$ and every $E \in \mathbb{S}_2$ the map $\mathcal{I}(C, E)$ is injective.

We summarize the discussion above in the following table.

degree of C	**type of C**	$\{\mathbf{E} \in \mathbb{S}_\mathbf{k} \,\vert\, \mathcal{I}(\mathbf{C}, \mathbf{E})$ **is not injective** $\}$
2	arbitrary	empty
3	arbitrary	$\{aF_x + bF_y + cF_z = 0\}_{[a:b:c] \in \mathbb{P}^2}$
4	generic	finite with 21 elements
≥ 5	generic	empty

3. The Rigidity of a generic $\mathcal{F}_\Gamma$: Proof of Theorem 1

The proof of Theorem 1 will follow from the above results. We start with a simple lemma:

Lemma 3.1. *Let Γ be the intersection of two transversal curves $C = \{F = 0\}$ and $E = \{G = 0\}$ of degree c and k respectively and let $\mathcal{F}$ be a holomorphic foliation of degree $c + k - 2$ with singular set containing Γ. Then $\mathcal{F}$ is induced by a 1-form*

$$\omega = GdF - \frac{\deg(F)}{\deg(G)} FdG + F\alpha + G\beta \, ,$$

where α and β are homogenous 1-form satisying $i_R\alpha = i_R\beta \equiv 0$. In particular α and β define foliations of $\mathbb{P}^2$ of degrees $k - 2$ and $c - 2$ respectively.

Proof. A direct application of Noether's Theorem ([13]) gives that $\mathcal{F}$ is induced by a 1-form $\omega = F\alpha_0 + G\beta_0$. Thus $i_R\omega = 0$ implies that $Fi_R\alpha_0 = -Gi_R\beta_0$. To conclude it is sufficient to take $\alpha = \alpha_0 + \frac{\deg(F)}{\deg(G)} FdG$ and $\beta = \beta_0 + (-GdF)$. $\square$

Let $C = \{F = 0\}$ and $E = \{G = 0\}$ be transverse curves of degree c in $\mathbb{P}^2$ and let $\Gamma = C \cap E$. Recall from the introduction that we denote by $\pi : S_\Gamma \to \mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ at the points of Γ and by $\widetilde{\mathcal{F}}_\Gamma$ the strict transform of the foliation $\mathcal{F}_\Gamma$ induced by $FdG - GdF = 0$. If $\widetilde{\mathcal{F}}$ is a foliation *close* to $\widetilde{\mathcal{F}}_\Gamma$ then both $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_\Gamma$ are transversal to the exceptional divisor of S_Γ. Thus $\mathcal{F} = \pi_* \widetilde{\mathcal{F}}$ is a foliation of $\mathbb{P}^2$ with radial singularities on Γ. Let now $\mathcal{U}_2(c,c) \subset \mathcal{U}_1(c,c) \subset \mathcal{U}_0(c,c)$ be the Zariski open subset of $\mathbb{S}_c \times \mathbb{S}_c$ with the property that if $(C, E) \in \mathcal{U}_1(c,c)$ then both C and E are smooth curves.

Using Lemma 3.1 we see that $\mathcal{F}$ is induced by

$$\omega = GdF - FdG + F\alpha + G\beta$$

where both α and β induce foliations of degree $c - 2$. We write this equality using homogeneous vector fields in $\mathbb{C}^3$:

$$Z = Z_\Gamma + F.Z_\alpha + G.Z_\beta$$

where Z defines $\mathcal{F}$, Z_Γ defines the pencil $\mathcal{F}_\Gamma$ and Z_α, Z_β define the foliations associated to $\alpha = 0$ and $\beta = 0$, respectively.

We claim that $\beta = 0$, that is, C is $\mathcal{F}$-invariant. If not, we observe that $tang(\mathcal{F}, C, p) \geq 2$ at any point $p \in \Gamma$; therefore the points of Γ contribute at least $2c^2$ to $tang(\mathcal{F}, C)$. The points of tangency between $\mathcal{F}$ and C are the common solutions to $dF(Z) = 0$ and $F = 0$, or $G.dF(Z_\beta) = 0$ and $F = 0$. Since we have already c^2 solutions to $G = 0$ and $F = 0$, it follows that the points of Γ are also solutions to $dF(Z_\beta) = 0$ and $F = 0$. Consequently $\mathcal{F}_\beta$ is tangent to C along the points of Γ, which is impossible since $(C, E) \in \mathcal{U}_2(c, c) \subset \mathcal{U}_0(c, c)$ (Lemma 2.2). Therefore $\beta = 0$.

Finally we may apply Theorem 2 to $(C, E) \in \mathcal{U}_2(c, c) \subset \mathcal{U}_1(c, c)$ to conclude that $\alpha = 0$. This concludes the proof of Theorem 1.

Remarks. When $c > 3$ we do not really understand for which pair of curves the conclusion of the Theorem holds. For instance we do not know if the conclusion holds if we suppose that the pencil is a Lefschetz pencil, i.e., all singularities have multiplicity one and every element of the pencil has at most one singularity.

Theorem 1 is also true for for generic complete intersection sets defined as the the intersection of curves $\{F = 0\}$ and $\{G = 0\}$ of degrees k and c with $k < c$; the fibration which is the desingularisation of $GdF - \frac{c}{k} FdG = 0$ is rigid. The same proof as above applies with minor modifications.

3.1. **Fermat Curves and Non-Rigid Foliations.** In order to conclude we exhibit below a family of examples showing that Theorem 1 does not hold for arbitrary Γ when $c \geq 3$.

Example 3.1. *For every $c \geq 3$ there exists a complete intersection $\Gamma \subset \mathbb{P}^2$ of degree c^2 such that $\mathcal{F}_\Gamma$ is not rigid.*

Proof. If $C = \{x^c - y^c = 0\}$ and $E = \{y^c - z^c = 0\}$ then the pencil generated by C and E is a pencil whose generic element is isomorphic to the Fermat curve of degree c and three singular elements: C, E and $\{x^c - z^c = 0\}$. Let ω_{2c-2} be a 1-form which defines the associated foliation.

The pencil generated by $\{x^c(y^c - z^c) = 0\}$ and $\{y^c(x^c - z^c) = 0\}$ defines a foliation of degree

$$c + 1 = \underbrace{4c - 2}_{\text{is a pencil of degree } 2c \text{ curves}} - \underbrace{3(c-1)}_{\text{with 3 singular fibers of degree 1 and multiplicity } c}$$

and has radial singularities at $\Gamma = C \cap E$. Denote by η_{c+1} the 1-form which induces this foliation. Thus for arbitrary $P_{c-3} \in S_{c-3}$ the strict transform of the foliation associated to the 1-form

$$\omega_{2c-2} + P_{c-3}\eta_{c+1}$$

is a deformation of $\mathcal{F}_\Gamma$. □

References

[1] M. Brunella, *Birational geometry of foliations.* Publicações do IMPA, 2000.

[2] M. Brunella, *Invariance par dformations de la dimension de Kodaira d'un feuilletage sur une surface.* Essays on geometry and related topics, Vol. 1, 2, 113–132, Monogr. Enseign. Math., 38, Enseignement Math., Geneva, 2001.

[3] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphics vector fields*, Annals of Mathematics **115**, (1982), 579–595.

[4] D. Cerveau and A. Lins Neto, *Holomorphic foliations in* $C\mathrm{P}(2)$ *having an invariant algebraic curve.* Ann. Inst. Fourier (Grenoble) **41** (1991), no. 4, 883–903.

[5] S. C. Coutinho and J. V. Pereira, *On the density of foliations without algebraic invariant sets.* J.reine angew Math. **557** (2003), 53-80.

[6] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics **133**, Springer (1992).

[7] J. P. Jouanolou, *Équations de Pfaff algébriques*, Lect. Notes in Math., 708, Springer (1979).

[8] A. Lins Neto, *Foliations on* $\mathbb{C}\mathrm{P}(2)$ *with invariant algebraic curve and radial singularities.* Rev. Semin. Iberoam. Mat. Singul. Tordesillas **2** (1998), no. 3, 29–35.

[9] L. G. Mendes and M. Sebastiani, *Sur la densité des systèmes de Pfaff sans solution algébrique.* Ann. Inst. Fourier (Grenoble) **44** (1994), no. 1, 271–276.

[10] L. G. Mendes and P. Sad, *On dicritical foliations and Halphen pencils.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1 (2002), no. 1, 93–109.

[11] J. V. Pereira and P. Sad, *On the holonomy group of algebraic curves invariant by holomorphic foliations.*Annali di Matematica Pura ed Applicata, 185, pp. 257-271, 2006.

[12] P.Sad, *Regular Foliations along Curves.*Ann. Fac. Sci. Toulouse 8 (1999), 661-675.

[13] W. Fulton, *Algebraic Curves*, Mathematics Lecture Note Series, Benjamin.

[14] J. Harris and Ian Morrison, *Moduli of Curves.* Graduate Texts in Mathematics 187, Springer.