BDDC Methods for Discontinuous Galerkin Discretization of Elliptic Problems

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Dedicated to Henryk Woźniakowski on the occasion of his 60th birthday.

Abstract

A discontinuous Galerkin (DG) discretization of Dirichlet problem for second-order elliptic equations with discontinuous coefficients in 2-D is considered. For this discretization, Balancing Domain Decomposition with Constraints (BDDC) algorithms are designed and analyzed as an additive Schwarz method (ASM). The coarse and local problems are defined using special partitions of unity and edge constraints. Under certain assumptions on the coefficients and the mesh sizes across $\partial\Omega_i$, where the Ω_i are disjoint subregions of the original region Ω, a condition number estimate $C(1+\max_i \log(H_i/h_i))^2$ is established with C independent of h_i, H_i and the jumps of the coefficients. The algorithms are well suited for parallel computations and can be straightforwardly extended to the 3-D problems. Results of numerical tests are included which confirm the theoretical results and the necessity of the imposed assumptions.

Key words: interior penalty discretization, discontinuous Galerkin method, elliptic problems with discontinuous coefficients, finite element method, BDDC algorithms, Schwarz methods, preconditioners

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1 Introduction

In this paper, a discontinuous Galerkin approximation of elliptic problems with discontinuous coefficients is considered. The problem is considered in a polygonal region Ω which is a union of disjoint polygonal subregions Ω_i. The discontinuities of the coefficients occur across $\partial\Omega_i$. The problem is approximated by a conforming finite element method (FEM) on matching triangulation in each Ω_i and nonmatching one across $\partial\Omega_i$. Composite discretizations are motivated first of all by the regularity of the solution of the problem being discussed. Discrete problems are formulated using DG methods, symmetric and with interior penalty terms on the $\partial\Omega_i$; see [4,5,8]. A goal of this paper is to design and analyze Balancing Domain Decomposition with Constraints (BDDC) preconditioners for the resulting discrete problem; see [7,17,16] for conforming finite elements. In the first step, the problem is reduced to the Schur complement problem with respect to unknowns on $\partial\Omega_i$ for $i = 1, \ldots, N$. For that, discrete harmonic functions defined in a special way are used. The preconditioners are designed and analyzed using the general theory of ASMs; see [18]. The local spaces are defined on Ω_i and faces of $\partial\Omega_j$ which are common to Ω_i plus zero average values constraints on faces of Ω_i or/and faces of Ω_j. The coarse basis functions follow from local orthogonality with respect to the local spaces and from average constraints across those faces. A special partitioning of unity with respect to the substructures Ω_i is introduced and it is based on master and slave sides of substructures. A side $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a master when ρ_i is larger than ρ_j, otherwise it is a slave, so if $F_{ij} \subset \partial\Omega_i$ is a master side then $F_{ji} \subset \partial\Omega_j$ is a slave side. The h_i- and h_j- triangulations on F_{ij} and F_{ji}, respectively, are built in a way that h_i is coarser where ρ_i is larger. Here h_i and h_j denote the parameters of these triangulations. It is proved that the algorithm is almost optimal and its rate of convergence is independent of h_i and h_j, the number of subdomains Ω_i and the jumps of coefficients. The algorithms are well suited for parallel computations and they can be straightforwardly extended to the problems in the 3-D cases.

DG methods are becoming more and more popular for the approximation of PDEs since they are well suited to dealing with regions with complex geometries or discontinuous coefficients, and local or patch refinements; see [5,4] and the literature therein. The class of DG methods we deal within this paper uses symmetrized interior penalty terms on the boundaries $\partial\Omega_i$. A goal is to design and analyze Balancing Domain Decomposition (BDDC) algorithms for the resulting discrete problem; see [7] and also [17,16]. There are also several papers devoted to algorithms for solving discrete DG problems. In particular in connection with domain decomposition methods, we can mention [15,12,14,1–3] where related discretizations to those discussed here are considered. In these papers Neumann-Dirichlet methods and two-level overlapping and nonoverlapping Schwarz are proposed and analyzed for DG discretization of elliptic problems with continuous coefficients. In [8] for the discontinuous coefficient case, a non optimal multilevel ASM is designed and analyzed. In [6,13], two-level overlapping and nonoverlapping ASMs are proposed and analyzed for DG discretization of fourth order problems. In those works, the coarse problems are based on polynomial coarse basis functions on a coarse triangulation. In addition, ideas of iterative substructuring methods and notions of discrete harmonic extensions are not explored. Condition number estimates of $O(\frac{H}{\delta})$ and $O(\frac{H}{h})$, and $O(\frac{H^3}{\delta^3})$ and $O(\frac{H^3}{h^3})$ are obtained for second and fouth order problems, respectively, where δ is the overlap parameter. In addition, for the cases where the distribution of the coefficients ρ_i is not quasimonotonic, see [10], these methods when extended straightforwardly to 3-D problems have condition number estimates which might deteriorate as the jumps of the coefficients get more severe. To the best of our knowledge, BDDC algorithms for DG discretizations of elliptic problems with continuous and discontinuous coefficients have not been considered in the literature. We note that part of the analysis presented here has previously appeared as a technical report for analyzing several iterative substructuring DG preconditioners of Neumann-Neumann type; see [11]. In [9] we have also successfully extended these preconditioners to the Balancing Domain Decomposition (BDD) method.

The paper is organized as follows. In Section 2 the differential problem and its DG discretization are formulated. In Section 3 the Schur complement problem is derived using discrete harmonic functions in a special way. Some technical tools are presented in Section 4. Sections 5 and 6 are devoted to designing a BDDC algorithm while Section 7 and 8 are devoted to the proof of the main result, Theorem 7.1. In Section 9 we introduce coarse spaces of dimension half smaller than those defined in Section 6. Finally in Section 10 some numerical experiments are presented which confirm the theoretical results. The enclosed numerical results show that the introduced assumption on the coefficients and the parameter steps are necessary and sufficient.

2 Differential and discrete problems

2.1 Differential problem

Consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \;\; \forall v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u,v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u^* \cdot \nabla v dx \quad \text{and} \quad f(v) := \int_{\Omega} f v dx.$$

We assume that $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$ and the substructures Ω_i are disjoint regular polygonal subregions of diameter $O(H_i)$ and form a geometrical conforming partition of Ω, i.e., $\forall i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty, or is a common vertex or an edge of $\partial\Omega_i$ and $\partial\Omega_j$. We assume that $f \in L^2(\Omega)$ and, for simplicity of presentation, let ρ_i be a positive constant.

2.2 Discrete problem

Let us introduce a shape-regular triangulation in each Ω_i with triangular elements and h_i as mesh parameter. The resulting triangulation on Ω is in general nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be the regular finite element (FE) space of piecewise linear continuous functions in Ω_i. Note that we do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Define

$$X_h(\Omega) := X_1(\Omega_1) \times \cdots \times X_N(\Omega_N).$$

The discrete problem obtained by the DG method, see [5,8], is of the form:

Find $u_h^* \in X_h(\Omega)$ such that

$$a_h(u_h^*, v_h) = f(v_h), \quad \forall v_h \in X_h(\Omega) \tag{2}$$

where

$$a_h(u,v) = \sum_{i=1}^{N} \hat{a}_i(u,v) \quad \text{and} \quad f(v) = \sum_{i=1}^{N} \int_{\Omega_i} f v_i dx, \tag{3}$$

$$\hat{a}_i(u,v) := a_i(u,v) + s_i(u,v) + p_i(u,v), \tag{4}$$

$$a_i(u,v) := \int_{\Omega_i} \rho_i \nabla u_i \nabla v_i dx, \tag{5}$$

$$s_i(u,v) := \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial u_i}{\partial n}(v_j - v_i) + \frac{\partial v_i}{\partial n}(u_j - u_i) \right) ds,$$

$$p_i(u,v) := \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} (u_j - u_i)(v_j - v_i) ds, \tag{6}$$

and $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$, $v = \{v_i\}_{i=1}^N \in X_h(\Omega)$. We set $l_{ij} = 2$ when $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a common face (edge) of $\partial\Omega_i$ and $\partial\Omega_j$, and define $\rho_{ij} := 2\rho_i\rho_j/(\rho_i + \rho_j)$ as the harmonic average of ρ_i and ρ_j, and $h_{ij} := 2h_ih_j/(h_i + h_j)$. In order to simplify the notation we include the index $j = \partial$ and put $l_{i\partial} := 1$ when $F_{i\partial} := \partial\Omega_i \cap \partial\Omega$ has a positive measure. We also set $u_\partial = 0$, $v_\partial = 0$ and define $\rho_{i\partial} := \rho_i$ and $h_{i\partial} := h_i$. The $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega_i$, and δ is a positive penalty parameter. We note that when ρ_{ij} is given by the harmonic average, it can be shown that $\min\{\rho_i, \rho_j\} \leq \rho_{ij} \leq 2\min\{\rho_i, \rho_j\}$.

We also define

$$d_i(u,v) := a_i(u,v) + p_i(u,v), \tag{7}$$

and

$$d_h(u,v) := \sum_{i=1}^{N} d_i(u,v). \tag{8}$$

It is known that there exists a $\delta_0 = O(1) > 0$ such that for $\delta \geq \delta_0$, we obtain $|s_i(u,u)| < cd_i(u,u)$ and $\sum_i s_i(u,u) < cd_h(u,u)$, where $c < 1$, and therefore, the problem (2) is elliptic and has a unique solution. A priori error estimates for the method are optimal for the continuous coefficients, see [4,5], and for the discontinuous coefficients if $\rho_i \partial_n u^* - \rho_j \partial_n u^* = 0$ in $L_2(F_{ij})$, see [8]. Note that this condition is satisfied if the solution u^* of (2.1) restricted to the Ω_i and Ω_j is in $H^{3/2+\epsilon}(\Omega_i)$ and $H^{3/2+\epsilon}(\Omega_j)$ with $\epsilon > 0$.

We use the d_h-norm, also called broken norm, in $X_h(\Omega)$ with weights given by ρ_i and $\frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}}$. For $u = \{u_i\} \in X_h(\Omega)$ we note that

$$d_h(u,u) = \sum_{i=1}^{N} \{\rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (u_i - u_j)^2 ds\}. \tag{9}$$

Lemma 2.1 *There exists $\delta_0 > 0$ such that for $\delta \geq \delta_0$, for all $u \in X_h(\Omega)$ the following inequalities hold:*

$$\gamma_0 d_i(u,u) \leq \hat{a}_i(u,u) \leq \gamma_1 d_i(u,u), \qquad i = 1,\ldots,N, \tag{10}$$

and

$$\gamma_0 d_h(u,u) \leq a_h(u,u) \leq \gamma_1 d_h(u,u), \tag{11}$$

where γ_0 and γ_1 are positive constants independent of the ρ_i, h_i and H_i.

The proof essentially follows from (37), see below, or refer to [8].

3 Schur complement problem

In this section we derive a Schur complement version for the problem (2). We first introduce some auxiliary notations.

Let $u = \{u_i\} \in X_h(\Omega)$ be given. We can represent u_i as

$$u_i = \mathcal{H}_i u_i + \mathcal{P}_i u_i \tag{12}$$

where $\mathcal{H}_i u_i$ is the discrete harmonic part of u_i in the sense of $a_i(.,.)$, see (5), i.e.,

$$a_i(\mathcal{H}_i u_i, v_i) = 0 \qquad \forall v_i \in \overset{o}{X}_i(\Omega_i) \tag{13}$$

$$\mathcal{H}_i u_i = u_i \quad \text{on} \quad \partial\Omega_i, \tag{14}$$

while $\mathcal{P}_i u_i$ is the projection of u_i into $\overset{o}{X}_i\,(\Omega_i)$ in the sense of $a_i(.,.)$, i.e.

$$a_i(\mathcal{P}_i u_i, v_i) = a_i(u_i, v_i) \quad \forall v_i \in \overset{o}{X}_i(\Omega_i). \tag{15}$$

Here $\overset{o}{X}_i\,(\Omega_i)$ is a subspace of $X_i(\Omega_i)$ of functions which vanish on $\partial\Omega_i$, and $\mathcal{H}_i u_i$ is the classical discrete harmonic part of u_i. Let us denote by $\overset{o}{X}_h\,(\Omega)$ the subspace of $X_h(\Omega)$ defined by $\overset{o}{X}_h\,(\Omega) := \{\overset{o}{X}_i(\Omega_i)\}_{i=1}^N$ and consider the global projections $\mathcal{H}u := \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\mathcal{P}u := \{\mathcal{P}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow \overset{o}{X}_h\,(\Omega)$ in the sense of $\sum_{i=1}^N a_i(.,.)$. Hence, a function $u \in X_h(\Omega)$ can therefore be decomposed as

$$u = \mathcal{H}u + \mathcal{P}u. \tag{16}$$

The function $u \in X_h(\Omega)$ can also be represented as

$$u = \hat{\mathcal{H}}u + \hat{\mathcal{P}}u \tag{17}$$

where $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow \overset{o}{X}_h(\Omega)$ is the projection in the sense of $a_h(.,.)$, the original bilinear form of (2), see (3). Since $\hat{\mathcal{P}}_i u_i \in \overset{o}{X}_i(\Omega_i)$ and $v_i \in \overset{o}{X}_i(\Omega_i)$, we have

$$a_i(\hat{\mathcal{P}}_i u, v_i) = a_h(u, v_i).$$

The discrete solution of (2) can be decomposed as $u_h^* = \hat{\mathcal{H}}u_h^* + \hat{\mathcal{P}}u_h^*$. To find $\hat{\mathcal{P}}u_h^*$ we need to solve the following set of standard discrete Dirichlet problems:

Find $\hat{\mathcal{P}}_i u_h^* \in \overset{o}{X}_i(\Omega)$ such that

$$a_i(\hat{\mathcal{P}}_i u_h^*, v_i) = f(v_i) \qquad \forall v_i \in \overset{o}{X}_i(\Omega_i) \tag{18}$$

for $i = 1, \cdots, N$. Note that these problems are local and independent, so they can be solved in parallel. This is a precomputational step.

We now formulate the problem for $\hat{\mathcal{H}}u_h^*$. Let $\hat{\mathcal{H}}_i u$ be the discrete harmonic part of u in the sense of $\hat{a}_i(.,.)$, see (4), where $\hat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$\hat{a}_i(\hat{\mathcal{H}}_i u, v_i) = 0 \qquad \forall v_i \in \overset{o}{X}_i(\Omega_i), \tag{19}$$

$$u_i \quad \text{on} \quad \partial\Omega_i \quad \text{and} \quad u_j \quad \text{on} \quad F_{ji} \subset \partial\Omega_j \quad \text{are given} \tag{20}$$

where u_j are given on $F_{ji} = \partial\Omega_i \cap \partial\Omega_j$. We point out that for $v_i \in \overset{o}{X}_i(\Omega_i)$ we have

$$\hat{a}_i(u_i, v_i) = (\rho_i \nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}. \tag{21}$$

Note that (19) - (20) has a unique solution. To see this, let us rewrite (19) in the form

$$\rho_i(\nabla\hat{\mathcal{H}}_i u, \nabla\varphi_i^k)_{L^2(\Omega_i)} = -\sum_{F_{ij}\subset\partial\Omega_i} \frac{\rho_{ij}}{l_{ij}}(\frac{\partial\varphi_i^k}{\partial n}, u_j - u_i)_{L^2(F_{ij})} \tag{22}$$

where φ_i^k are nodal basis functions of $\overset{o}{X}_i\,(\Omega_i)$ associated with interior nodal points x_k of the h_i-triangulation of Ω_i. Note that $\frac{\partial\varphi_i^k}{\partial n}$ does not vanish on $\partial\Omega_i$ when x_k is a node of an element touching $\partial\Omega_i$. We see that $\hat{\mathcal{H}}_i u$ is a special extension into Ω_i where u is given on $\partial\Omega_i$ and on all the F_{ji}, and therefore, it depends on the values of u_j given on $F_{ji} = \partial\Omega_i \cap \partial\Omega_j$ and on $F_{\partial i}$ (we already have assumed $u_\partial = 0$ for $j = \partial$). Note that $\hat{\mathcal{H}}_i u$ is discrete harmonic except at nodal points close to $\partial\Omega_i$. We will sometimes call $\hat{\mathcal{H}}_i u$ discrete harmonic in a special sense, i.e., in the sense of $\hat{a}_i(.,.)$ or $\hat{\mathcal{H}}_i$. We let $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N \in X_h(\Omega)$.

Note that (19) is obtained from

$$a_h(\hat{\mathcal{H}}u, v) = 0 \tag{23}$$

for $u \in X_h(\Omega)$ and when taking $v = \{v_i\}_{i=1}^N \in \overset{o}{X}_h(\Omega)$. It is easy to see that $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ and $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N$ are orthogonal in the sense of $a_h(.,.)$, i.e.

$$a_h(\hat{\mathcal{H}}u, \hat{\mathcal{P}}v) = 0 \qquad u, v \in X^h(\Omega). \tag{24}$$

In addition,

$$\mathcal{H}\hat{\mathcal{H}}u = \mathcal{H}u, \qquad \hat{\mathcal{H}}\mathcal{H}u = \hat{\mathcal{H}}u \tag{25}$$

since $\hat{\mathcal{H}}u$ and $\mathcal{H}u$ do not change the values of u on any of the nodes on the boundaries of the subdomains Ω_i also denoted by

$$\Gamma := (\cup_i \partial\Omega_{ih_i}), \tag{26}$$

where $\partial\Omega_{ih_i}$ is the set of nodal points of $\partial\Omega_i$. We note that the definition of Γ includes the nodes on both sides of $\cup_i\partial\Omega_i$.

We are now in a position to derive a Schur complement problem for (2). Let us apply the decomposition (17) in (2). We get

$$a_h(\hat{\mathcal{H}}u_h^* + \hat{\mathcal{P}}u_h^*, \hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h) = f(\hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h)$$

or

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) + 2a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{P}}v_h) + a_h(\hat{\mathcal{P}}u_h^*, \hat{\mathcal{P}}v_h) = f(\hat{\mathcal{H}}v_h) + f(\hat{\mathcal{P}}v_h).$$

Using (18) and (23) we have

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) = f(\hat{\mathcal{H}}v_h) \qquad \forall v_h \in X_h(\Omega). \tag{27}$$

This is the Schur complement problem for (2). We denote by $V_h(\Gamma)$ or V, which we will use later, the set of all functions v_h in $X_h(\Omega)$ such that $\hat{\mathcal{P}}v_h = 0$, i.e., the space of discrete harmonic functions in the sense of the $\hat{\mathcal{H}}_i$. We rewrite the Schur complement problem as follows: .

Find $u_h^* \in V_h(\Gamma)$ such that

$$\mathcal{S}(u_h^*, v_h) = g(v_h) \qquad \forall v_h \in V_h(\Gamma); \tag{28}$$

here and below $u_h^* \equiv \hat{\mathcal{H}}u_h^*$, and

$$\mathcal{S}(u_h, v_h) = a_h(\hat{\mathcal{H}}u_h, \hat{\mathcal{H}}v_h) \qquad g(v_h) = f(\hat{\mathcal{H}}v_h). \tag{29}$$

This problem has a unique solution.

4 Technical tools

Our main goal is to design and analyze a BDDC method for solving (28). This will be done in the next section. We now introduce some notations and facts to be used later. Let $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and $v = \{v_i\}_{i=1}^N \in X_h(\Omega)$. Let $d_i(.,.)$ and $d_h(.,.)$ be the bilinear forms defined in (7) and (8).

Note that, for $u, v \in \overset{o}{X}_h(\Omega)$,

$$d_i(u, v) = a_i(u, v) = \rho_i(\nabla u_i, \nabla v_i)_{L^2(\Omega_i)} \tag{30}$$

and, for $u \in X_h(\Omega)$,

$$\gamma_0 d_h(u, u) \le a_h(u, u) \le \gamma_1 d_h(u, u) \tag{31}$$

in view of Lemma 2.1, where γ_0 and γ_1 are positive constants independent of h_i, H_i and ρ_i. The next lemma shows the equivalence between discrete harmonic functions in the sense of $\mathcal{H}$ and in the sense of $\hat{\mathcal{H}}$, and therefore, we can take advantage of all the discrete Sobolev norm results known for $\mathcal{H}$ discrete harmonic extensions.

Lemma 4.1 *For $u \in X_h(\Omega)$ we have*

$$d_i(\mathcal{H}u, \mathcal{H}u) \le d_i(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \le C d_i(\mathcal{H}u, \mathcal{H}u), \quad i = 1, \ldots, N, \tag{32}$$

and

$$d_h(\mathcal{H}u, \mathcal{H}u) \le d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \le C d_h(\mathcal{H}u, \mathcal{H}u) \tag{33}$$

where $\mathcal{H}u = \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ are defined by (13) - (14) and (19) - (20) respectively, and C is a positive constant independent of h_i, u, ρ_i and H_i.

Proof. We note that $\mathcal{P}$ and $\mathcal{H}$ are projections in the sense of $\sum_i a_i(.,.)$ while $\hat{\mathcal{P}}$ and $\hat{\mathcal{H}}$ are projections in the sense of $a_h(.,.)$. Therefore, the left-hand inequality of (33) follows from properties of minimum energy of discrete harmonic extensions in the $\sum_i a_i(.,.)$ sense. To prove the right-hand inequality of (33) note that

$$d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) = d_h(\hat{\mathcal{H}}u, \mathcal{H}\hat{\mathcal{H}}u + \mathcal{P}\hat{\mathcal{H}}u) = d_h(\hat{\mathcal{H}}u, \mathcal{H}u)) + d_h(\hat{\mathcal{H}}u, \mathcal{P}\hat{\mathcal{H}}u) \quad (34)$$

in view of (25). The first term is estimated as

$$d_h(\hat{\mathcal{H}}u, \mathcal{H}u) \leq \varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u), \quad (35)$$

with arbitrary $\varepsilon > 0$. To estimate the second term on the right-hand side of (34) note that, for $v := \mathcal{P}\hat{\mathcal{H}}u \in \overset{o}{X}(\Omega)$ and using (22), we get

$$\begin{aligned} d_h(\hat{\mathcal{H}}u, v) &= \sum_{i=1}^{N} \rho_i (\nabla \hat{\mathcal{H}}_i u_i, \nabla v_i)_{L^2(\Omega_i)} \quad (36) \\ &= -\sum_{i=1}^{N} \sum_{F_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}. \end{aligned}$$

The terms on the right-hand side of (36) are estimated as follows:

$$\begin{aligned} |\rho_{ij}(\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}| &\leq \ \rho_{ij} \parallel \frac{\partial v_i}{\partial n} \parallel_{L^2(F_{ij})} \parallel u_i - u_j \parallel_{L^2(F_{ij})} \quad (37) \\ &\leq C \frac{\rho_{ij}}{h_i^{1/2}} \parallel \nabla v_i \parallel_{L^2(\Omega_i)} \parallel u_i - u_j \parallel_{L^2(F_{ij})} \\ &\leq C \frac{\rho_{ij}}{h_{ij}^{1/2}} \parallel \nabla v_i \parallel_{L^2(\Omega_i)} \parallel u_i - u_j \parallel_{L^2(F_{ij})} \\ &\leq C\{\varepsilon \rho_{ij} \parallel \nabla v_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4\varepsilon h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\} \\ &\leq C\{2\varepsilon \rho_i \parallel \nabla v_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4\varepsilon h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\}, \end{aligned}$$

where we have used that $h_{ij} \leq 2h_i$ and $\rho_{ij} \leq 2\rho_i$. Substituting this into (36), we get

$$d_h(\hat{\mathcal{H}}u, v) \leq C \sum_{i=1}^{N} \{2\varepsilon \rho_i \parallel \nabla \mathcal{P}_i \hat{\mathcal{H}}_i u_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4h_{ij}\varepsilon} \sum_{F_{ij} \subset \partial\Omega_i} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\}, \quad (38)$$

and using

$$\parallel \nabla \mathcal{P}_i \hat{\mathcal{H}}_i u_i \parallel_{L^2(\Omega_i)} \leq \parallel \nabla \hat{\mathcal{H}}_i u_i \parallel_{L^2(\Omega_i)},$$

we obtain

$$d_h(\hat{\mathcal{H}}u, v) \leq C\{\varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u)\}. \quad (39)$$

Substituting (39) and (35) into (34) we get

$$d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq C\{\varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u)\}.$$

Choosing a sufficiently small ε, the right-hand side of (33) follows. $\square$

5 Balancing domain decomposition with constraints method

We design and analyze BDDC methods for solving (28); see [7,17,16] for conforming elements. We use the general framework of ASMs as stated below in Lemma 5.1; see [18]. For $i = 0, \ldots, N$, let V_i be auxiliary spaces and I_i prolongation operators from V_i to V, and define the operators $\tilde{T}_i : V \to V_i$ as

$$b_i(\tilde{T}_i u, v) = a_h(u, I_i v) \quad \forall v \in V_i,$$

where $b_i(\cdot, \cdot)$ is symmetric and positive definite on $V_i \times V_i$, and set $T_i = I_i \tilde{T}_i$. Then the ASMs, in particular the BDDC method, are defined as

$$T = \sum_{i=0}^{N} T_i. \tag{40}$$

The bilinear form a_h is defined in (3). The bilinear forms b_i, the operators I_i, and the spaces V_i, $i = 0, \ldots, N$, are defined in the next subsections.

Lemma 5.1 *Suppose the following three assumptions hold:*

i) There exists a constant C_0 such that, for all $u \in V$, there is a decomposition $u = \sum_{i=0}^{N} I_i u^{(i)}$ with $u^{(i)} \in V_i$, $i = 0, \ldots, N$, and

$$\sum_{i=0}^{N} b_i(u^{(i)}, u^{(i)}) \leq C_0^2 a_h(u, u).$$

ii) There exist constants $\epsilon_{ij}, i, j = 1, \ldots, N$, such that for all $u^{(i)} \in V_i$, $u^{(j)} \in V_j$,

$$a_h(I_i u^{(i)}, I_j u^{(j)}) \leq \epsilon_{ij} a_h(I_i u^{(i)}, I_i u^{(i)})^{1/2} a_h(I_j u^{(j)}, I_j u^{(j)})^{1/2}.$$

iii) There exists a constant ω such that

$$a_h(I_i u, I_i u) \leq \omega b_i(u, u) \quad \forall u \in V_i, \quad i = 0, \ldots, N.$$

Then, T is invertible and

$$C_0^2 a_h(u, u) \leq a_h(Tu, u) \leq (\rho(\epsilon) + 1)\omega a_h(u, u), \quad \forall u \in V.$$

Here, $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

5.1 Notations and the interface condition

Let us denote by Γ_i the set of all nodes on $\partial\Omega_i$ and on the neighboring faces $F_{ji} \subset \partial\Omega_j$. We note that the nodes of ∂F_{ji} (which are vertices of Ω_j) are included in Γ_i. Define W_i as the vector space associated to the nodal values on Γ_i and extended via $\hat{\mathcal{H}}_i$ inside Ω_i. We say that $u^{(i)} \in W_i$ if $u^{(i)}$ is represented as $u^{(i)} := \{u_l^{(i)}\}_{l\in\#(i)}$, where $\#(i) = \{i \text{ and } \cup j : F_{ij} \subset \partial\Omega_i\}$. Here $u_i^{(i)}$ and the $u_j^{(i)}$ stand for the nodal values of $u^{(i)}$ on $\partial\Omega_i$ and the $\bar{F}_{ji}$, respectively. We write $u = \{u_i\} \in V$ to refer to a function defined on all of Γ with each u_i defined (only) on $\partial\Omega_i$. We point out that F_{ij} and F_{ji} are geometrically the same even though the mesh on F_{ij} is inherited from the Ω_i mesh while the mesh on F_{ji} corresponds to the Ω_j mesh.

Denote by $\Lambda_i := \{F_{ij} : F_{ij} \subset \partial\Omega_i\} \cup \{F_{ji} : F_{ji} = F_{ij},\ F_{ji} \subset \partial\Omega_j\}$ the set of all faces of Ω_i and all faces of Ω_j touching Ω_i. Given $u^{(i)} \in W_i$ and $F_{\ell k} \in \Lambda_i$ we use the notation

$$\overline{u}_{\ell k}^{(i)} = \frac{1}{|F_{\ell k}|} \int_{F_{\ell k}} u^{(i)} ds.$$

Let us define the regular zero extension operator $\tilde{I}_i : W_i \to V$ as follows: Given $u^{(i)} \in W_i$, let $\tilde{I}_i u^{(i)}$ be equal to $u^{(i)}$ on nodes Γ_i and zero on $\Gamma \backslash \Gamma_i$.

A face across Ω_i and Ω_j has two sides, the side contained in $\partial\Omega_i$, denoted by F_{ij}, and the side contained in $\partial\Omega_j$, denoted by F_{ji}. In addition, we assign to each pair $\{F_{ij}, F_{ji}\}$ a master and a slave side. If F_{ij} is a slave side then F_{ji} is a master side and vice versa. If F_{ij} is a slave side we will use the notation δ_{ij} (instead of F_{ij}) to emphasize this fact while if F_{ij} is a master side we will use the notation γ_{ij}. The choice of slave-master sides are such that the *interface condition*, stated next, can be satisfied. In this case Theorem 7.1 below holds with a constant C independent of the ρ_i, h_i and H_i.

Assumption 1 (The interface condition) *We say that the coefficients $\{\rho_i\}$ and the local mesh sizes $\{h_i\}$ satisfy the interface condition if there exist constants C_0 and C_1, of order $O(1)$, such that for any face F_{ij} the following conditions hold:*

$$\begin{cases} h_i \leq C_0 h_j \quad \textit{and} \quad \rho_i \leq C_1 \rho_j \quad \textit{if } F_{ij} \textit{ is a slave side, or} \\ h_j \leq C_0 h_i \quad \textit{and} \quad \rho_j \leq C_1 \rho_i \quad \textit{if } F_{ij} \textit{ is a master side.} \end{cases} \tag{41}$$

We associate with each Ω_i, $i = 1, \cdots, N$, the weighting diagonal matrices $D^{(i)} = \{D_l^{(i)}\}_{l \in \#(i)}$ on Γ_i defined as follows:

- On $\partial\Omega_i$ $(l = i)$

$$D_i^{(i)}(x) = \begin{cases} 1 \text{ if } x \text{ is a vertex of } \partial\Omega_i, \\ 1 \text{ if } x \text{ is an interior node of a master face } F_{ij}, \\ 0 \text{ if } x \text{ is an interior node of a slave face } F_{ij}, \end{cases} \tag{42}$$

- On F_{ji} $(l = j)$

$$D_j^{(i)}(x) = \begin{cases} 0 \text{ if } x \text{ is an end point of the face } F_{ji}, \\ 1 \text{ if } x \text{ is an interior node and } F_{ji} \text{ is a slave face}, \\ 0 \text{ if } x \text{ is an interior node and } F_{ji} \text{ is a master face}, \end{cases} \tag{43}$$

- For $x \in F_{i\partial}$ we set $D_i^{(i)}(x) = 1$.

Remark 5.1 *We note that two alternatives of weighting diagonal matrices $D^{(i)}$ can also be considered while ensuring that Theorem 7.1 below holds: 1) On faces F_{ij} where h_i and h_j are of the same order, the values of (42) and (43) at interior nodes x of the faces F_{ij} and F_{ji} can be replaced by $\frac{\sqrt{\rho_i}}{\sqrt{\rho_i}+\sqrt{\rho_j}}$; 2) Similarly, on faces F_{ij} where ρ_i and ρ_j are of the same order, we can replace (42) and (43) at interior nodes x of the faces F_{ij} and F_{ji} by $\frac{h_i}{h_i+h_j}$.*

The prolongation operators $I_i : W_i \to V$, $i = 1, \ldots, N$, are defined as

$$I_i = \tilde{I}_i D^{(i)}, \tag{44}$$

and they form a partition of unity on Γ described as

$$\sum_{i=1}^{N} I_i \tilde{I}_i^T = I_\Gamma. \tag{45}$$

6 Local and global spaces

The local spaces $V_i = V_i(\Gamma_i)$, $i = 1, \ldots, N$, are defined as the subspaces of W_i of functions with zero face-average values on all faces F_{ij} and F_{ij} associated to the subdomain Ω_i, i.e., for all $F_{\ell k} \in \Lambda_i$.

For $u^{(i)}, v^{(i)} \in V_i(\Gamma_i)$ we define the local bilinear form b_i as

$$b_i(u^{(i)}, v^{(i)}) := \hat{a}_i(u^{(i)}, v^{(i)}), \tag{46}$$

where the bilinear form $\hat{a}_i$ was defined in (4).

Now we define a BDDC coarse space. As in BDDC methods, here we define the coarse space using local bases and imposing continuity conditions with respect to the primal variables; see [7,17,16].

Recall that $\Lambda_i := \{F_{ij} : F_{ij} \subset \Omega_i\} \cup \{F_{ji} : F_{ji} = F_{ij},\ F_{ji} \subset \Omega_j\}$ is the set of all faces of Ω_i and all faces of Ω_j touching Ω_i. For $F_{\ell k} \in \Lambda_i$ define the local coarse basis function $\Phi^{(i)}_{F_{\ell k}} \in W_i$ by

$$b_i(\Phi^{(i)}_{F_{\ell k}}, v) = 0 \quad \forall v \in V_i(\Gamma_i) \tag{47}$$

with

$$\frac{1}{|F_{\ell k}|} \int_{F_{\ell k}} \Phi^{(i)}_{F_{\ell k}} = 1$$

and

$$\int_{F_{\ell' k'}} \Phi^{(i)}_{F_{\ell k}} = 0, \quad \forall F_{\ell' k'} \neq F_{\ell k} \text{ with } F_{\ell' k'} \in \Lambda_i.$$

Note that $\Phi^{(i)}_{F_{k\ell}} \neq \Phi^{(i)}_{F_{\ell k}}$.

Define $V_{0i} = V_{0i}(\Gamma_i) := \mathrm{Span}\{\Phi^{(i)}_{F_{\ell k}} : F_{\ell k} \in \Lambda_i\} \subset W_i$. Then (47) implies that V_i is $\hat{\mathcal{H}}_i-$orthogonal to V_{0i}, and W_i is a direct sum of V_{0i} and V_i, i.e., $V_{0i} \oplus V_i = W_i$.

The global coarse space V_0 is defined as the set of all $u_0 := \{u_0^{(i)}\} \in \prod_{i=1}^N V_{0i}(\Gamma_i)$ such that, for $i, j = 1, \ldots, N$, we have

$$\overline{u}^{(i)}_{0\ell k} = \overline{u}^{(j)}_{0\ell k} \ \ \forall F_{\ell k} \in \Lambda_i \cap \Lambda_j. \tag{48}$$

The coarse prolongation operator $I_0 : V_0 \to V$ is defined as $I_0 u_0 = \sum_{i=1}^N I_i u_0^{(i)}$ and the bilinear form b_0 is of the form

$$b_0(u_0, v_0) := \sum_{i=1}^N b_i(u_0^{(i)}, v_0^{(i)}). \tag{49}$$

7 Main result

In this section we state and prove our main result.

Theorem 7.1 *Let the Assumption 1 be satisfied. Then, there exists a positive constant C, independent of h_i, H_i and the jumps of ρ_i, such that*

$$a_h(u,u) \le a_h(Tu,u) \le C\left(1+\log\frac{H}{h}\right)^2 a_h(u,u) \qquad \forall u \in V, \tag{50}$$

where T is defined in (40). Here $\log\frac{H}{h} = \max_i \log\frac{H_i}{h_i}$.

Proof. By the general theorem of ASMs we need to check the three key assumptions of Lemma 5.1.

Assumption(i) We prove that for $u = \{u_i\}_{i=1}^N \in V$ there exists $u_0 \in V_0$ and $u^{(i)} \in V_i$ such that

$$I_0 u_0 + \sum_{i=1}^N I_i u^{(i)} = u \tag{51}$$

and

$$b_0(u_0,u_0) + \sum_{i=1}^N b_i(u^{(i)},u^{(i)}) = a(u,u). \tag{52}$$

Let $u = \{u_i\}_{i=1}^N \in V(\Gamma)$. Define $u_0^{(i)} \in V_{0i}(\Gamma_i)$ as

$$u_0^{(i)} = \sum_{F_{\ell k}\in\Lambda_i}\left(\frac{1}{|F_{\ell k}|}\int_{F_{\ell k}} u_\ell ds\right)\Phi^{(i)}_{F_{\ell k}} \tag{53}$$

where functions $\Phi^{(i)}_{F_{ik}}$ were defined in (47). Note that $u_0^{(i)}$ and u have the same face-average values on all faces $F_{\ell k} \in \Lambda_i$, i.e.,

$$\begin{cases} \dfrac{1}{|F_{\ell k}|}\displaystyle\int_{F_{\ell k}} u_\ell ds = \dfrac{1}{|F_{\ell k}|}\displaystyle\int_{F_{\ell k}} u_0^{(i)} ds = \overline{u}^{(i)}_{0\ell k} \\ \\ \dfrac{1}{|F_{\ell k}|}\displaystyle\int_{F_{\ell k}} u_\ell ds = \dfrac{1}{|F_{\ell k}|}\displaystyle\int_{F_{\ell k}} u_0^{(j)} ds = \overline{u}^{(j)}_{0\ell k}, \end{cases} \tag{54}$$

and therefore, for all the faces $F_{\ell k} \in \Lambda_i \cap \Lambda_j$ we have, see (48),

$$\overline{u}^{(i)}_{0\ell k} = \overline{u}^{(j)}_{0\ell k}. \tag{55}$$

Define $u_0 \in V_0$ by $u_0 = \{u_0^{(i)}\}_{i=1}^N$ and set $w = u - I_0 u_0$, where $I_0 u_0 = \sum_{i=1}^N I_i u^{(i)}$. Then we can write

$$w = \sum_{i=1}^N I_i(\tilde{I}_i^T u - u_0^{(i)}) = \sum_{i=1}^N I_i u^{(i)},$$

where we have defined $u^{(i)} = \tilde{I}_i^T u - u_0^{(i)} \in V_i$. Since the prolongation operators I_i form a partition of unity, (51) holds.

To check (52) observe that $u^{(i)}$ has zero face-average values on all faces $F_{\ell k} \in \Lambda_i$, hence it is $\hat{\mathcal{H}}_i-$orthogonal to $u_0^{(i)}$; see (47). Then, from the definition of b_0 we have

$$\begin{aligned} b_0(u_0, u_0) + \sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) &= \sum_{i=1}^N \Big(b_i(u_0^{(i)}, u_0^{(i)}) + b_i(u^{(i)}, u^{(i)})\Big) \\ &= \sum_{i=1}^N b_i(u_0^{(i)} + u^{(i)}, u_0^{(i)} + u^{(i)}) \\ &= \sum_{i=1}^N b_i(\tilde{I}_i^T u, \tilde{I}_i^T u) = a_h(u, u). \end{aligned}$$

This ends the proof of Assumption(i).

Assumption(ii) We need to prove that

$$a_h(I_i u^{(i)}, I_j u^{(j)}) \le C\varepsilon_{ij} a_h^{1/2}(I_i u^{(i)}, I_i u^{(i)})\, a_h^{1/2}(I_j u^{(j)}, I_j u^{(j)}) \tag{56}$$

for $u^{(i)} \in V_i$ and $u^{(j)} \in V_j$, $\;i, j = 1, \cdots, N$, and the spectral radius $\varrho(\varepsilon)$ of $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^N$ is bounded. In our case $\varrho(\varepsilon) \le C$ with constant independent of h_i and H_i. This follows from coloring arguments and the fact that $u^{(i)}$ and $u^{(j)}$ are different from zero only on Ω_i and Ω_j and their neighboring substructures.

Assumption(iii). We need to prove that for $i = 1, \cdots, N$,

$$a_h(I_i u^{(i)}, I_i u^{(i)}) \le \omega b_i(u^{(i)} u^{(i)}), \;\; \forall u^{(i)} \in V_i \tag{57}$$

and

$$a_h(I_0 u_0, I_0 u_0) \le \omega b_0(u_0, u_0) \quad \forall u_0 \in V_0 \tag{58}$$

with $\omega \le C(1 + \log \frac{H}{h})^2$ where C is a positive constant independent of h_i, H_i and the jumps of ρ_i.

For the proof of (57) see Lemma 8.1, and for the proof of (58) see Lemma 8.2 in the next section. □

8 Auxiliary lemmas

In this section we complete the proof of Theorem 7.1 by proving two auxiliary lemmas associated with (57) and (58).

Lemma 8.1 *Assume that Assumption 1 holds. Then for $u^{(i)} \in V_i$, $i = 1, \ldots, N$, we have*

$$a_h(I_i u^{(i)}, I_i u^{(i)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}), \tag{59}$$

where C is independent of h_i, H_i and the jumps of ρ_i.

Proof. In order to prove (59) we can replace $a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u)$ by $d_h(\mathcal{H}u, \mathcal{H}u)$ on the left-hand side of (59) and on its right-hand side we can put $d_i(\mathcal{H}\tilde{I}_i u^{(i)}, \mathcal{H}\tilde{I}_i u^{(i)})$ instead of $b_i(u^{(i)}, u^{(i)})$; see Lemma 2.1 and Lemma 4.1.

In order to simplify the notation, all the functions are considered as harmonic extensions in the $\mathcal{H}$ sense. Hence, we denote $\mathcal{H}I_i u$ by $I_i u$ and let $u = \{u_l^{(i)}\}_{l \in \#(i)} \in V_i$. Using (7), (8) and (44) we obtain

$$d_h(I_i u^{(i)}, I_i u^{(i)}) = d_i(\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)}) + \sum_j d_j(\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)}), \tag{60}$$

where the sum is taken over Ω_j which have a common face with Ω_i. The first term on the right-hand side of (60) can be estimated as follows:

$$\begin{aligned} d_i (\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)}) \\ = \rho_i \int_{\Omega_i} |\nabla D_i^{(i)} u_i^{(i)}|^2 dx + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)})^2 dx. \end{aligned} \tag{61}$$

To bound the first term of (61) we use

$$\rho_i \parallel \nabla D_i^{(i)} u_i^{(i)} \parallel_{L^2(\Omega_i)}^2 \leq 2\rho_i \{ \parallel \nabla (D_i^{(i)} u_i^{(i)} - u_i^{(i)}) \parallel_{L^2(\Omega_i)}^2 + \parallel \nabla u_i^{(i)} \parallel_{L^2(\Omega_i)}^2 \}$$

and therefore,

$$\rho_i \parallel \nabla (D_i^{(i)} u_i^{(i)} - u_i^{(i)}) \parallel_{L^2(\Omega_i)}^2 \leq C \sum_{\delta_{ij} \subset \partial\Omega_i} \rho_i \parallel \tilde{u}_i^{(i)} \parallel_{H_{00}^{1/2}(\delta_{ij})}^2 .$$

Here $\tilde{u}_i^{(i)} = u_i^{(i)}$ at the interior nodal points of δ_{ij} and $\tilde{u}_i^{(i)} = 0$ on $\partial\delta_{ij}$. Recall that δ_{ij} denotes F_{ij} when F_{ij} is a slave side. It can be proved, see for example [18], that

$$\rho_i \parallel \tilde{u}_i^{(i)} \parallel^2_{H^{1/2}_{00}(\delta_{ij})} \leq C \left(1 + \log \frac{H_i}{h_i}\right)^2 \rho_i |u_i^{(i)}|^2_{H^1(\Omega_i)}. \tag{62}$$

Here we have used the fact that $u_i^{(i)}$ has zero face-average values.

We now estimate the second term of (61) and (67), see below. Note that for $F_{i\partial}$, i.e. for faces on $\partial\Omega$, the estimates of the terms corresponding to $F_{i\partial}$ follow straightforwardly. On a slave face F_{ij} of $\partial\Omega_i$, i.e. where $h_i \leq C_0 h_j$ and $\rho_i \leq C_1 \rho_j$, we have

$$\parallel D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(F_{ij})} \leq C h_i \max_{F_{ij}} |u_i^{(i)}|^2 \tag{63}$$

and

$$\frac{\rho_{ij}}{h_{ij}} \parallel D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(F_{ij})} \leq C \rho_i \max_{F_{ij}} |u_i^{(i)}|^2 \leq C \left(1 + \log \frac{H_i}{h_i}\right) \rho_i |u_i^{(i)}|^2_{H^1(\Omega_i)},$$

where we have used $\rho_{ij} \leq 2\rho_i$ and $h_i \leq C h_{ij}$ since $h_i < C_0 h_j$. We have also used that $u^{(i)}$ has zero face-average value on any face of Λ_i, therefore, the Poincaré inequality can be used to bound the $H^1(\Omega_i)-$norm by the seminorm.

On a master side F_{ij} of $\partial\Omega_i$, i.e. where $h_j \leq C_0 h_i$ and $\rho_j \leq C_1 \rho_i$, we have

$$\begin{aligned} \parallel D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \parallel_{L^2(F_{ij})} \leq & \parallel u_i^{(i)} - u_j^{(i)} \parallel_{L^2(F_{ij})} \\ & + \parallel \sum_{x_v^j \in \partial F_{ij}} u_j^{(i)}(x_v^j) \varphi_v^j \parallel_{L^2(F_{ij})}, \end{aligned} \tag{64}$$

and using a triangle inequality we obtain

$$\begin{aligned} & \parallel u_j^{(i)}(x_v^j) \varphi_v^j \parallel_{L^2(F_{ij})} \\ & \leq \parallel u_i^{(i)}(x_v^i) \varphi_v^i \parallel_{L^2(F_{ij})} + \parallel u_i^{(i)}(x_v^i) \varphi_v^i - u_j^{(i)}(x_v^j) \varphi_v^j \parallel_{L^2(F_{ij})}, \end{aligned} \tag{65}$$

where φ_v^i and φ_v^j are the nodal basis functions corresponding to x_v^i and x_v^j, respectively. The first term of (65) can be estimated as

$$\parallel u_i^{(i)} \varphi_v^i \parallel^2_{L^2(F_{ij})} \leq C \max_{F_{ij}} |u_i^{(i)}|^2 h_i \leq C h_i \left(1 + \log \frac{H_i}{h_i}\right) |u_i^{(i)}|^2_{H^1(\Omega_i)},$$

while the second term of (65) can be bounded as in (81), see below. Using these estimates in (61) and Lemma 2.1 we get

$$d_i(I_i u^{(i)}, I_i u^{(i)}) \leq C \left(1 + \log \frac{H_i}{h_i}\right)^2 b_i(u^{(i)}, u^{(i)}). \tag{66}$$

We now estimate the second term of (60) by bounding $d_j(\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)})$ by $b_i(u^{(i)}, u^{(i)})$. For $u = \{u_l^{(i)}\} \in V_i$ we have

$$\begin{aligned} d_j \ (\ & \tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)}) \\ &= \rho_j \parallel \nabla D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(\Omega_j)} + \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)})^2 dx. \end{aligned} \tag{67}$$

We need only to estimate the first term of (67) since the second term has been already estimated; see (63), (64) and (65). If F_{ij} is a slave side of $\partial\Omega_i$ then $D_j^{(i)}$ vanishes, and so vanishes $\parallel \nabla D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(\Omega_j)}$. We now consider the case where F_{ij} is a master side of $\partial\Omega_i$ and it is not equal to $F_{i\partial}$. On F_{ji} we decompose $u_j^{(i)} = w_j^{(i)} + \sum_{x_v^j \in \partial F_{ji}} u_j^{(i)}(x_v^j)\varphi_v^j$, where $w_j^{(i)} = D_j^{(i)} u_j^{(i)}$. We have

$$\begin{aligned} \parallel \nabla w_j^{(i)} \parallel^2_{L^2(\Omega_j)} &\leq C \parallel w_j^{(i)} \parallel^2_{H^{1/2}_{00}(F_{ji})} \\ &= C\{|w_j^{(i)}|^2_{H^{1/2}(F_{ji})} + \int_{F_{ji}} \frac{(w_j^{(i)})^2}{\text{dist}(s, \partial F_{ji})} \, ds\}. \end{aligned} \tag{68}$$

We now estimate the first term of (68). Let Q_j be the L_2- projection on the h_j- triangulation of F_{ji}. Then,

$$\begin{aligned} |w_j^{(i)}|^2_{H^{1/2}(F_{ji})} &\leq 2\{|w_j^{(i)} - Q_j u_i^{(i)}|^2_{H^{1/2}(F_{ij})} + |Q_j u_i^{(i)}|^2_{H^{1/2}(F_{ij})}\} \\ &\leq C\{\frac{1}{h_j} \parallel w_j^{(i)} - u_i^{(i)} \parallel^2_{L^2(F_{ij})} + \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)}\} \end{aligned} \tag{69}$$

and

$$\begin{aligned} \parallel & w_j^{(i)} - u_i^{(i)} \parallel^2_{L^2(F_{ij})} \\ & \leq 2 \parallel u_j^{(i)} - u_i^{(i)} \parallel^2_{L^2(F_{ij})} + 2 \parallel \sum_{x_v^j \in \partial F_{ij}} u_j^{(i)}(x_v^j)\varphi_v^j \parallel^2_{L^2(F_{ij})} \end{aligned} \tag{70}$$

where the second term of (70) can be bounded as before, see (64), (65) and (81), and using that $\rho_j \leq C_1 \rho_i$.

It remains to estimate the second term of (68). In order to simplify the notation, we take F_{ij} as the interval $[0, H]$. Note that

$$\int_{F_{ji}} \frac{(w_j^{(i)})^2}{\operatorname{dist}(s, \partial F_{ji})} ds \leq C\{\int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds + \int_{H/2}^{H} \frac{(w_j^{(i)})^2}{(H-s)} ds\}. \tag{71}$$

Let us estimate the first term on the right-hand side of (71). We have

$$\begin{aligned}
\int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds &= \int_0^{h_j} \frac{(w_j^{(i)})^2}{s} ds + \int_{h_j}^{H/2} \frac{(u_j^{(i)})^2}{s} ds \\
&\leq C\{\left(u_j^{(i)}(h_j)\right)^2 + \int_{h_j}^{H/2} \frac{(u_i^{(i)})^2 - (u_j^{(i)})^2}{s} ds + \int_{h_j}^{H/2} \frac{(u_i^{(i)})^2}{s} ds\} \\
&\leq C\{\left(u_j^{(i)}(h_j)\right)^2 + \frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|_{L^2(F_{ji})}^2 + \left(1 + \log \frac{H_j}{h_j}\right) \max_{F_{ij}} |u_i^{(i)}|^2\} \\
&\leq C\{\frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|_{L^2(F_{ij})}^2 + \left(1 + \log \frac{H_i}{h_i}\right)\left(1 + \log \frac{H_j}{h_j}\right) \| u_i^{(i)} \|^2)_{H^1(\Omega_i)}\},
\end{aligned}$$

where $u_j^{(i)}(h_j)^2$ has been estimated as in (81). The second term of (71) is estimated similarly. Substituting these estimates into (71) and using that $u_i^{(i)}$ has zero face-average values we get

$$\begin{aligned}
\int_{F_{ji}} \frac{(u_j^{(i)})^2}{\operatorname{dist}(s, \partial F_{ji})} ds \leq C\{\left(1 + \log \frac{H}{h}\right)^2 (\| \nabla u_i^{(i)} \|_{L^2(\Omega_i)}^2 + \\
+ \frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|_{L^2(F_{ij})}^2\}.
\end{aligned} \tag{72}$$

In turn, substituting (69) and (72) into (68), and the resulting estimate into (67), and using Lemma 2.1, we get

$$d_j(\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}). \tag{73}$$

Using (66) and (73) in (60), we get

$$d_h(I_i u^{(i)}, I_i u^{(i)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}).$$

$\square$

Lemma 8.2 *Suppose that Assumption 1 holds. Then, for $u_0 \in V_0$, V_0 defined* *by (48), we have the following inequality*

$$a_h(I_0u_0, I_0u_0) \leq C\left(1+\log\frac{H}{h}\right)^2 b_0(u_0,u_0) \tag{74}$$

where C is independent of h_i, H_i and the jumps of ρ_i.

Proof. By Lemma 2.1 and Lemma 4.1

$$a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq Cd_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq Cd_h(\mathcal{H}u, \mathcal{H}u), \tag{75}$$

where $d_h(.,.)$ is defined by (8). Hence, to prove the result (74) we can replace $a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u)$ by $d_h(\mathcal{H}u, \mathcal{H}u)$ on the left-hand side of (74).

In order to simplify the notation we write u instead of u_0 and put $I_0u_0 = I_0u = \sum_{i=1}^N I_iu^{(i)}$, see (48) and thereafter. We have

$$\begin{aligned} d_i(I_0u, I_0u) = \rho_i \parallel \nabla\{(I_iu^{(i)})_i + \sum_{F_{ij}\subset\partial\Omega_i} (I_ju^{(j)})_i\} \parallel^2_{L^2(\Omega_i)} \\ + \sum_{F_{ij}\subset\partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\Big(\{(I_iu^{(i)})_i + (I_ju^{(j)})_i\} - \{(I_iu^{(i)})_j + (I_ju^{(j)})_j\}\Big)^2 ds. \end{aligned} \tag{76}$$

To bound the second term on the right-hand side of (76) let us consider the case where F_{ij} is a master side. The proof for the case where F_{ij} is a slave side is similar; see also the arguments given in (63) and thereafter. Then using the definition of I_i and $D^{(i)}$, we obtain

$$\begin{aligned} J = &\int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\Big(\{(I_iu^{(i)})_i + (I_ju^{(j)})_i\} - \{(I_iu^{(i)})_j + (I_ju^{(j)})_j\Big)^2 ds \\ = &\int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\left(\{D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)}\} - \{D_j^{(j)}u_j^{(j)} - D_i^{(j)}u_i^{(j)}\}\right)^2 ds \\ = &\int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\left(\{D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)}\} - \{D_j^{(j)}u_j^{(j)} - 0\}\right)^2 ds \\ = &\int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\left(\{D_i^{(i)}u_i^{(i)} - (D_j^{(i)} + D_j^{(j)})u_j^{(i)}\} + D_j^{(j)}\{u_j^{(i)} - u_j^{(j)}\}\right)^2 ds \\ = &\int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}}\frac{\delta}{h_{ij}}\left(\{u_i^{(i)} - u_j^{(i)}\} - \sum_{x_v^j\in\partial F_{ji}} \{u_j^{(i)}(x_v^j) - u_j^{(j)}(x_v^j)\}\varphi_v^j\right)^2 ds \end{aligned} \tag{77}$$

where φ_v^j is the nodal basis function corresponding to x_v^j. Hence,

$$J \leq C \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} \{u_i^{(i)} - u_j^{(i)}\}^2 ds$$
$$+ C h_j \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} \max_{x_v \in \partial F_{ji}} \{u_j^{(i)}(x_v^j) - u_j^{(j)}(x_v^j)\}^2. \quad (78)$$

It remains to estimate the second term of (78). First note that $\overline{u}_{ji}^{(i)} = \overline{u}_{ji}^{(j)}$ since there are primal variables associated to the faces $F_{ji} \in \Lambda_i$ and $F_{ji} \in \Lambda_j$; see (48). Therefore,

$$\begin{aligned} |u_j^{(i)}(x_v^j) - u_j^{(j)}(x_v^j)| &\leq |u_j^{(j)}(x_v^j) - \overline{u}_{ji}^{(j)}| + |u_j^{(i)}(x_v^j) - \overline{u}_{ji}^{(i)}| \\ &\leq C \left(1 + \log \frac{H_j}{h_j}\right)^{\frac{1}{2}} \| \nabla u_j^{(j)} \|_{L^2(\Omega_i)} + |u_j^{(i)}(x_v^j) - \overline{u}_{ji}^{(i)}|. \end{aligned} \quad (79)$$

To deduce the estimate on the first term on the right-hand side of (79) we have used a Poincaré inequality and an L^∞ bound for FEM functions, see [18]. The second term of (79) is estimated as

$$\begin{aligned} |u_j^{(i)}(x_v^j) - \overline{u}_{ji}^{(i)}| &\leq |u_j^{(i)}(x_v^j) - u_i^{(i)}(x_v^i)| + |u_i^{(i)}(x_v^i) - \overline{u}_{ij}^{(i)}| + |\overline{u}_{ij}^{(i)} - \overline{u}_{ji}^{(i)}| \\ &\leq C\{|u_j^{(i)}(x_v^j) - u_i^{(i)}(x_v^i)| + \left(1 + \log \frac{H_i}{h_i}\right)^{\frac{1}{2}} \| \nabla u_i^{(i)} \|_{L^2(\Omega_i)} \\ &\quad + h_j^{-\frac{1}{2}} \| u_i^{(i)} - u_j^{(i)} \|_{L^2(F_{ij})}\}, \end{aligned} \quad (80)$$

where we have used a Poncaré inequality and an L^∞ bound for FEM functions to obtain the second term on the right-hand side of (80) and a Cauchy-Schwarz inequality to obtain the third term of (80). To estimate the first term of (80), let $Q_j u_i^{(i)}$ be the L^2-projection of $u_i^{(i)}$ on the h_j triangulation of F_{ji}. We obtain

$$\begin{aligned} |u_j^{(i)}(x_v^j) - u_i^{(i)}(x_v^i)| &\leq |u_j^{(i)}(x_v^j) - Q_j u_i^{(i)}(x_v^i)| + |Q_j u_i^{(i)}(x_v^i) - u_i^{(i)}(x_v^i)| \\ &\leq C\{h_j^{-\frac{1}{2}} \| u_j^{(i)} - u_i^{(i)} \|_{L^2(F_{ij})} + \left(1 + \log \frac{H_j}{h_j}\right)^{\frac{1}{2}} \| \nabla u_i^{(i)} \|_{L^2(\Omega_i)}\}, \end{aligned} \quad (81)$$

where the first estimate was obtained from an inverse inequality and the second from the approximation properties of the L^2 projection and an L^∞ bound for FEM functions.

By Lemma 2.1 and Lemma 4.1 we can bound the term $d_i(\mathcal{H}\tilde{I}_i u^{(i)}, \mathcal{H}\tilde{I}_i u^{(i)})$ by $b_i(\hat{\mathcal{H}}_i u^{(i)}, \hat{\mathcal{H}}_i u^{(i)})$. Then we conclude that J of (77) can be estimated as

$$J \leq C \left(1 + \log \frac{H}{h}\right) \{b_i(u^{(i)}, u^{(i)}) + b_j(u^{(j)}, u^{(j)}\}, \quad (82)$$

since $\rho_{ij} \le C\rho_i$ and $h_j \le Ch_{ij}$.

It remains to estimate the first term in (76). We have

$$
\begin{aligned}
&\| \nabla\{(I_i u^{(i)})_i + \sum_{F_{ij}\subset\partial\Omega_i} (I_j u^{(j)})_i\} \|^2_{L^2(\Omega_i)} \\
= &\| \nabla\{(D_i^{(i)} + \sum_{F_{ij}\subset\partial\Omega_i} D_i^{(j)})u_i^{(i)} + \sum_{F_{ij}\subset\partial\Omega_i} D_i^{(j)}(u_i^{(j)} - u_i^{(i)})\} \|^2_{L^2(\Omega_i)} \\
\le &C\{\| \nabla u_i^{(i)} \|^2_{L^2(\Omega_i)} + \sum_{\delta_{ij}\subset\partial\Omega_i} \|D_i^{(j)}(u_i^{(j)} - u_i^{(i)})\|^2_{H^{1/2}_{00}(\delta_{ij})}\}, \qquad (83)
\end{aligned}
$$

where the sum in (83) reduces to the slave sides F_{ij}. From (48) we obtain

$$
\begin{aligned}
&\|D_i^{(j)}(u_i^{(j)} - u_i^{(i)})\|^2_{H^{1/2}_{00}(F_{ij})} \\
&\qquad \le 2\{\|D_i^{(j)}(u_i^{(j)} - \overline{u}_{ij}^{(j)})\|^2_{H^{1/2}_{00}(F_{ij})} + \|D_i^{(j)}(u_i^{(i)} - \overline{u}_{ij}^{(i)})\|^2_{H^{1/2}_{00}(F_{ij})}\} \qquad (84)
\end{aligned}
$$

and therefore, the first term of (84) is estimated as

$$
\begin{aligned}
&\rho_i\|D_i^{(j)}(u_i^{(j)} - \overline{u}_{ij}^{(j))})\|^2_{H^{1/2}_{00}(F_{ij})} \\
\le &2\rho_i\{\|D_i^{(j)}(u_i^{(j)} - u_j^{(j)})\|^2_{H^{1/2}_{00}(F_{ij})} + \|D_i^{(j)}(u_j^{(j)} - \overline{u}_{ji}^{(j)})\|^2_{H^{1/2}_{00}(F_{ij})} \\
&+\|D_i^{(j)}(\overline{u}_{ji}^{(j)} - \overline{u}_{ij}^{(j)})\|^2_{H^{1/2}_{00}(F_{ij})}\} \\
\le &C\rho_i\{\frac{1}{h_j} \| u_i^{(j)} - u_j^{(j)} \|^2_{L^2(F_{ji})} + \left(1 + \log\frac{H_j}{h_j}\right)^2 \| \nabla u_j^{(j)} \|^2_{L^2(\Omega_j)}\} \\
\le &C\left(1 + \log\frac{H_j}{h_j}\right)^2 b_j(u^{(j)}, u^{(j)}), \qquad (85)
\end{aligned}
$$

since $\rho_i \le C_1\rho_j$ when F_{ij} is a slave side, and in view of Lemma 2.1. The second term on the right-hand side of (84) is bounded by

$$
\begin{aligned}
\rho_i\|D_i^{(j)}(u_i^{(i)} - \overline{u}_{ij}^{(i)})\|^2_{H^{1/2}_{00}(F_{ij})} &\le C\rho_i\left(1 + \log\frac{H_i}{h_i}\right)^2 \| \nabla u_i^{(i)} \|_{L^2(\Omega_i)} \\
&\le \left(1 + \log\frac{H_i}{h_i}\right)^2 b_i(u^{(i)}, u^{(i)}). \qquad (86)
\end{aligned}
$$

Using (85) and (86) in (84) and the resulting inequality in (83) we see that

$$\rho_i \parallel \nabla\{(I_i u^{(i)})_i + \sum_{F_{ij}\subset\partial\Omega_i} (I_j u^{(j)})_i\} \parallel^2_{L^2(\Omega_i)} \leq C\left(1+\log\frac{H}{h}\right)^2 \{b_i(u^{(i)},u^{(i)}) + b_j(u^{(j)},u^{(j)})\}.$$

This estimate and (82), see (76), imply that

$$d_i(I_0u_0, I_0u_0) \leq C\left(1+\log\frac{H}{h}\right)^2 \{b_i(u^{(i)},u^{(i)}) + b_j(u^{(j)},u^{(j)})\}.$$

Summing this over i and using Lemma 2.1 and Lemma 4.1 we get (74). □

9 Smaller global spaces

In Section 6 we have defined the coarse space with a primal variable associated to each face $F_{\ell k} \in \Lambda_i$. In this case the number of constraints per subdomain is twice the number of edges of $\partial\Omega_i$ for floating subdomains Ω_i. In this section we discuss choices of subsets of Λ_i which imply smaller coarse problems and still maintain the bound (50) of Theorem 7.1.

Recall that a face across Ω_i and Ω_j has two sides, the side contained in $\partial\Omega_i$, denoted by F_{ij}, and the side contained in $\partial\Omega_j$, denoted by F_{ji}. Let $\tilde{\Lambda}_i$, $i = 1,\ldots,N$, be such that for all pairs of neighboring subdomains Ω_i and Ω_j the subset $\tilde{\Lambda}_i \cap \tilde{\Lambda}_j$ contains one and only one face from each pair $\{F_{ij}, F_{ji}\}$, i.e., F_{ij} or F_{ji}. We denote the chosen face by $\lambda_{ij} = \lambda_{ji}$. For instance, we can choose $\tilde{\Lambda}_i$ as the set of master faces λ_{ij} associated to Ω_i.

After choosing $\tilde{\Lambda}_i$, the local spaces $V_i = V_i(\Gamma_i)$, $i = 1,\ldots,N$, are defined as the subspaces of W_i of functions with zero face-average values on all faces $\lambda_{\ell k} \in \tilde{\Lambda}_i$ while the spaces V_{0i} are defined as $V_{0i} = V_{0i}(\Gamma_i) = \mathrm{Span}\{\Phi^{(i)}_{\lambda_{\ell k}} : \lambda_{\ell k} \in \tilde{\Lambda}_i\} \subset W_i$ where the functions $\Phi^{(i)}_{\lambda_{ik}}$ are defined as in Section 6 replacing Λ_i by $\tilde{\Lambda}_i$ in each subdomain; see (47).

From now on we will use the notation

$$\overline{u}^{(i)}_{\lambda_{\ell k}} = \frac{1}{|\lambda_{\ell k}|}\int_{\lambda_{\ell k}} u^{(i)} ds,$$

where $u^{(i)} \in W_i$. The global coarse space V_0 is now defined as the set of all $u_0 = \{u_0^{(i)}\} \in \prod_{i=1}^N V_{0i}(\Gamma_i)$ such that for $i = 1,\ldots,N$, we have

$$\overline{u}^{(i)}_{0\lambda_{ij}} = \overline{u}^{(j)}_{0\lambda_{ij}} \quad \forall\lambda_{ij} \in \tilde{\Lambda}_i. \tag{87}$$

Recall that $u_0^{(i)}$ is defined locally. Then we have the following possible cases of continuity with respect to the primal variables:
Case 1 $\lambda_{ij} = \lambda_{ji} = F_{ij}$. This case imposes continuity of the face-average values of $u_0^{(i)}$ and $u_0^{(j)}$ on F_{ij}; see (87).
Case 2 $\lambda_{ij} = \lambda_{ji} = F_{ji}$. This case imposes continuity of the face-average values on F_{ji}.

Example 9.1 *Consider the domain $\Omega = (0,1)^2$ and divide it into $N = M \times M$ squares subdomains Ω_i which are unions of fine elements, with $H = 1/M$. We note that for floating subdomains Ω_i, Λ_i has eight coarse basis functions while $\tilde{\Lambda}_i$ has only four coarse basis functions.*

The bilinear forms a_h, b_i and the operators I_i, $i = 1, \ldots, N$, and the operator I_0 are defined in Section 5 and Section 6.

We now show that with these new local and global spaces Theorem 7.1 still holds. The proof is basically the same as the one given in Section 7 and Section 8 with some minor modifications depending on which of the above cases is considered and also on a modification of the Poincaré inequality.

Theorem 9.1 *If Assumption 1 holds, then there exists a positive constant C independent of h_i, H_i and the jumps of ρ_i such that*

$$a_h(u,u) \leq a_h(Tu,u) \leq C\left(1 + \log \frac{H}{h}\right)^2 a_h(u,u) \qquad \forall u \in V, \tag{88}$$

where T is defined in (40), the local spaces V_i, $i = 1, \ldots, N$, are defined above in this section and the global space V_0 is defined using (87). Here $\log \frac{H}{h} = \max_i \log \frac{H_i}{h_i}$.

Proof. We now mention the main modifications of the proof of the three key assumptions of Lemma 5.1.

Assumption(i) Let $u = \{u_i\}_{i=1}^N \in V(\Gamma)$. Define $u_0^{(i)} \in V_{0i}(\Gamma_i)$ by

$$u_0^{(i)} = \sum_{\lambda_{\ell k} \in \tilde{\Lambda}_i} \left(\frac{1}{|\lambda_{\ell k}|}\int_{\lambda_{\ell k}} u ds\right)\Phi_{\lambda_{\ell k}}^{(i)} \tag{89}$$

and proceed as in the proof of Theorem 7.1.

Assumption(ii) It is the same argument given to verify Assumption(ii) in the proof of Theorem 7.1.

Assumption(iii) We modify the proof of Lemma 8.2 and Lemma 8.1 as follows:

For the proof of Lemma 8.2 we consider the following cases to obtain a bound for the left-hand side of (79),

Case 1 $\lambda_{ij} = \lambda_{ji} = F_{ji}$. In this case we use the same argument as in the proof of Lemma 8.2 to estimate the left-hand side of (79).

Case 2 $\lambda_{ij} = \lambda_{ji} = F_{ij}$. In this case we estimate

$$\begin{aligned} &|u_j^{(i)}(x_v^j) - u_j^{(j)}(x_v^j)| \leq \\ &\qquad |u_j^{(i)}(x_v^j) - \overline{u}_{F_{ji}}^{(i)}| + |u_j^{(j)}(x_v^j) - \overline{u}_{F_{ji}}^{(j)}| + |\overline{u}_{F_{ji}}^{(i)} - \overline{u}_{F_{ji}}^{(j)}|. \end{aligned} \tag{90}$$

The first and second term of (90) can be bounded as in **Case 1**. The third term of (90) is bounded as follows: since $\lambda_{ij} = \lambda_{ji} = F_{ij}$ we have that $\overline{u}_{F_{ij}}^{(i)} = \overline{u}_{F_{ij}}^{(j)}$; see (87). Then

$$|\overline{u}_{F_{ji}}^{(i)} - \overline{u}_{F_{ji}}^{(j)}| \leq |\overline{u}_{F_{ji}}^{(i)} - \overline{u}_{F_{ij}}^{(i)}| + |\overline{u}_{F_{ij}}^{(j)} - \overline{u}_{F_{ji}}^{(j)}| \tag{91}$$

and we obtain

$$|\overline{u}_{F_{ji}}^{(i)} - \overline{u}_{F_{ij}}^{(i)}| \leq CH_j^{-\frac{1}{2}} \parallel u_{F_{ji}}^{(i)} - u_{F_{ij}}^{(i)} \parallel_{L^2(F_{ij})} \leq Ch_j^{-\frac{1}{2}} \parallel u_{ji}^{(i)} - u_{ij}^{(i)} \parallel_{L^2(F_{ij})} .$$

An analogous bound holds also for the second term of (91); see (79).

For the proof of Lemma 8.1 we can apply Poincaré inequality only in the case which $\lambda_{ij} = F_{ij} \subset \partial\Omega_i$. If this is not the case, i.e., if $\lambda_{ij} = F_{ji} \subset \Omega_j$, we can still bound the $H^1(\Omega_i)$ norm by the seminorm using the following argument: if $u^{(i)} \in V_i$ and $\lambda_{ij} = F_{ji}$ then $u^{(i)}$ has zero face-average value on F_{ji} and therefore,

$$\begin{aligned} \parallel u_i \parallel_{L^2(\Omega_i)} &\leq \parallel u_i - \overline{u}_{F_{ij}}^{(i)} \parallel_{L^2(\Omega_i)} + \parallel \overline{u}_{F_{ij}}^{(i)} - \overline{u}_{F_{ji}}^{(i)} \parallel_{L^2(F_{ij})} \\ &\leq \parallel \nabla u_i \parallel_{L^2(\Omega_i)} + \frac{1}{H_i^{1/2}} \parallel u_{ij}^{(i)} - u_{ji}^{(i)} \parallel_{L^2(F_{ij})} . \end{aligned}$$

Having modified the proof of Lemma 8.2 and Lemma 8.1, then Assumption(iii) follows. □

10 Numerical experiments

In this section we present numerical results for the preconditioner introduced in (40) and show that the bounds of Theorem 7.1 and Theorem 9.1 are reflected in the numerical tests. In particular we show that the Assumption 1, see (41), is necessary and sufficient.

We consider the domain $\Omega = (0,1)^2$ and divide Ω into $N = M \times M$ square subdomains Ω_i which are unions of fine elements, with $H = 1/M$. Inside each subdomain Ω_i we generate a structured triangulation with n_i subintervals in each coordinate direction, and apply the discretization presented in Section 2 with $\delta = 4$. This value $\delta = 4$ was chosen because numerically it was observed that the L^2 approximation error seems to stabilize when δ becomes larger. The minimum value of δ that gives a positive definite system is $\delta_{\min} = 1.565$. In the numerical experiments we use a red-black checkerboard type subdomain partition. On the black subdomains we let $n_i = 2 * 2^{L_b}$ and on the red subdomains we let $n_i = 3 * 2^{L_r}$, where L_b and L_r are integers denoting the number of refinements inside each subdomain Ω_i. Hence, the mesh sizes are $h_b = \frac{2^{-L_b}}{2M}$ and $h_r = \frac{2^{-L_r}}{3M}$, respectively. We solve the second order elliptic problem $-\text{div}(\rho(x)\nabla u^*(x)) = 1$ in Ω with homogeneous Dirichlet boundary conditions. In the numerical experiments, we run PCG until the l_2-norm initial residual is reduced by a factor of 10^6.

In the first test we consider the constant coefficient case $\rho = 1$. We consider different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$, therefore, keeping constant the mesh ratio $h_b/h_r = 3/2$. We place the masters on the black subdomains. We note that the interface condition (41) is satisfied. Table 1 lists the number of PCG iterations and in parenthesis the condition number estimate of the preconditioned system in the case we choose eight coarse functions per subdomain. As expected from the analysis, the condition numbers appear to be independent of the number of subdomains and seem to grow by a logarithmic factor when the size of the local problems increases. Note that in the case of continuous coefficients, the Theorem 7.1 and Theorem 9.1 are valid without any assumptions on h_b and h_r if the master sides are chosen on the larger meshes.
Table 2 is the same as before, however, now we have chosen $\tilde{\Lambda}_i$ as the set of master faces of Ω_i. In this case we have four coarse basis functions in each subdomain. We note that even though the coarse problems are smaller, the results are very similar to the ones presented in Table 1 where the coarse problems are larger. As in the case of Table 2 the smallest eigenvalue of the

preconditioned operator is 1.

We now consider the discontinuous coefficient case where we set $\rho_i = 1$ on the black subdomains and $\rho_i = \mu$ on the red subdomains. The subdomains are kept fixed at 4×4, i.e., 16 subdomains. Table 3 lists the results of computations for different values of μ and for different levels of refinement on the red subdomains. On the black subdomains $n_i = 2$ is kept fixed. The masters are placed on the black subdomains. It is easy to see that the interface condition (41) holds if, and only if, μ is not large, which seems to be in agreement with the results in Table 3.
We repeat the same experiment as in Table 3 but this time with four coarse local basis functions associated to the master sides of the subdomain. The results are presented in Table 4.

$M \downarrow L_r \rightarrow$	0	1	2	3	4	5
2	12 (5.7)	14 (6.7)	15 (7.5)	18 (10.6)	19 (14.5)	19 (19.0)
4	14 (5.8)	18 (8.5)	21 (11.7)	24 (15.2)	27 (19.2)	29 (23.9)
8	15 (5.9)	20 (9.1)	24 (12.3)	27 (15.8)	31 (19.6)	34 (24.0)
16	15 (6.0)	20 (9.4)	25 (12.8)	28 (16.3)	31 (20.1)	35 (24.5)
32	15 (6.0)	20 (9.3)	25 (12.8)	28 (16.3)	32 (20.2)	35 (24.6)

Table 1
PCG/BDDC iteration counts and condition numbers for different sizes of coarse and local problems and constant coefficients ρ_i with 8 coarse basis functions per subdomain.

$M \downarrow L_r$	0	1	2	3	4	5
2	13 (5.7)	15 (6.7)	16 (7.5)	18 (10.7)	19 (14.5)	19 (18.9)
4	15 (5.8)	19 (8.5)	22 (11.7)	24 (15.1)	27 (19.2)	29 (23.8)
8	17 (6.1)	21 (9.1)	25 (12.3)	28 (15.7)	31 (19.6)	34 (24.0)
16	18 (6.1)	23 (9.4)	27 (12.8)	30 (16.3)	32 (20.1)	35 (24.5)
32	18 (6.1)	24 (9.4)	27 (12.8)	30 (16.3)	32 (20.2)	35 (24.6)

Table 2
PCG/BDDC iteration counts and condition numbers for different sizes of coarse and local problems and constant coefficients ρ_i with 4 coarse basis functions per subdomain associated to its master faces.

$\mu \downarrow L_r \rightarrow$	0	1	2	3	4	5
1000	85(2099)	165(2822)	263(3746)	282(4758)	287(5922)	310(7168)
10	28(24.4)	37(32.9)	43(42.3)	47(52.8)	51(64.8)	53(77.7)
0.1	16(6.6)	17(6.8)	16(6.8)	17(6.8)	17(6.9)	17(6.9)
0.001	16(6.96)	16(7.12)	16(7.16)	16(7.25)	17(7.38)	18(7.50)

Table 3
PCG/BDDC iteration counts and condition numbers for different values of coefficients and the local mesh sizes on the red subdomains only. The coefficients and the local mesh sizes on the black subdomains are kept fixed. The subdomains are also kept fixed to 4×4 and 8 coarse basis functions in each subdomain are used.

$\mu \downarrow L_r \rightarrow$	0	1	2	3	4	5
1000	84(2127)	133(2905)	188(3827)	254(4838)	326(5980)	384(7205)
10	32(24.7)	40(33.4)	45(43.0)	49(53.5)	53(65.3)	54(78.0)
0.1	15(6.9)	16 (6.8)	16(6.8)	17 (6.8)	17 (6.9)	17 (7.0)
0.001	15 (7.4)	15 (7.3)	16 (7.2)	17 (7.3)	17 (7.42)	18 (7.52)

Table 4
PCG/BDDC iteration counts and condition numbers for different values of coefficients and the local mesh sizes on the red subdomains only. The coefficients and the local mesh sizes on the black subdomains are kept fixed. The subdomains are also kept fixed to 4×4 and 4 coarse basis functions in each subdomain are used. Master faces are chosen.

11 Conclusions and Extensions

In this paper several BDDC methods with different coarse spaces, for DG discretization of second-order elliptic equations with discontinuous coefficients, have been designed and analyzed. It has been proved that the methods are almost optimal and very well suited for parallel computations. Their rates of convergence are independent of the parameters of the triangulations, the number of substructures and the jumps of the coefficients. The numerical tests confirm the theoretical results.

In 2-D, the methods are based on choosing $D_i^{(i)}$ to be equal to one at the vertices of Ω_i. The methods can be extended to 3-D by considering $D_i^{(i)}$ to be equal to one at nodal points of edges and vertices of the Ω_i. In this case the Theorem 7.1 and Theorem 9.1 hold. The methods also can be generalized to the case where $\kappa_i = \frac{\mathrm{max_x \rho_i(x)}}{\mathrm{mix_x \rho_i(x)}}$ is not large. In this case, define constants $\bar{\rho}_i$ as the integral average of the $\rho_i(x)$ over the Ω_i. The $\bar{\rho}_i$ are used to determine the mortar and slave sides, and can be used to define the weighting matrices $D^{(i)}$ as well. For the bilinear forms $b_i(\cdot,\cdot)$ we exact solvers where $\rho_i(x)$ are considered rather than $\bar{\rho}_i$. In this case, the Theorem 7.1 and Theorem 9.1 are valid, with lower bound equal to one, and upper bound now involving a constant C depending linearly on κ_i. The case where the $\rho_i(x)$ have large variations inside the Ω_i will be discussed elsewhere. Finally, we remark that the condition number of the preconditioned systems deteriorates as we increase the penalty parameters δ to large values.

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